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A P-ADIC MEASURE ATTACHED TO THE ZETA FUNCTIONS ASSOCIATED WITH TWO ELLIPTIC MODULAR FORMS II

by **Haruzo HIDA**

0. Introduction.

Let f be a cusp form for $\Gamma_0(N)$ of weight $k \geq 2$ with character $\psi : (\mathbf{F}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, and let g be another cusp form for $\Gamma_0(N)$ of weight $\ell < k$ with character ξ. Write their Fourier expansion as

$$f = \sum_{n=1}^{\infty} a(n)e(nz) \text{ and } g = \sum_{n=1}^{\infty} b(n)e(nz) \text{ for } e(z) = \exp(2\pi i z),$$

and define Dirichlet series of f and g by

$$\mathcal{D}_N(s,f,g) = \Big(\sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \psi\xi(n) n^{k+\ell-2s-2} \Big) \Big(\sum_{n=1}^{\infty} a(n)b(n)n^{-s} \Big).$$

As in the first part [11], our object of study is the p-adic nature of the algebraic numbers :

$$(0.1) \qquad \frac{\mathcal{D}_N(\ell+m,f,g)}{\pi^{\ell+2m+1} < f,f >} \text{ for integers } m \text{ with } 0 \leq m < k-\ell.$$

In particular, we shall construct a p-adically analytic L-function of three variables, which interpolates the values (0.1) by regarding all the ingredients m, f and g as variables.

Key-words : p-adic measure – zeta function – Modular form – p-adic L-function.

Let $p \geq 5$ be a prime number. Let $\mathcal{O}$ be a valuation ring finite flat over $\mathbf{Z}_p$. We have constructed in [13] and [14] the universal Hecke algebra $\mathbf{h}(N;\mathcal{O})$, for each positive integer N prime to p, as a subalgebra of the endomorphism algebra of p-adic cusp forms of level N with coefficients in $\mathcal{O}$, topologically generated by Hecke operators $T(n)(n > 0)$. Then, the ordinary part $\mathbf{h}^o(N;\mathcal{O})$ is shown to be finite flat over the Iwasawa algebra $\Lambda = \mathcal{O}[[\Gamma]]$ of the topological group $\Gamma = 1 + p\mathbf{Z}_p$. We fix an algebraic closure $\overline{\mathbf{Q}}_p$ of the p-adic field $\mathbf{Q}_p$ and let $\overline{\mathbf{Q}}$ denote the algebraic closure of $\mathbf{Q}$ inside $\mathbf{C}$. We shall assume $\mathcal{O}$ to be a subring of $\overline{\mathbf{Q}}_p$ and fix once and for all an embedding $i : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. For each finite order character $\varepsilon : \Gamma \to \overline{\mathbf{Q}}^\times$ and for each integer k, the continuous character of Γ into $\overline{\mathbf{Q}}_p$ given by $\gamma \mapsto \gamma^k\varepsilon(\gamma)$ induces an $\mathcal{O}$-algebra homomorphism $P_{k,\varepsilon}$ of Λ into $\overline{\mathbf{Q}}_p$. Let $\mathcal{I}$ be a normal integral domain finite flat over Λ. Replacing $\mathcal{O}$ by its finite extension in $\overline{\mathbf{Q}}_p$ if necessary, we may assume that $\mathcal{O}$ is integrally closed in $\mathcal{I}$. Let $\mathcal{X}(\mathcal{I}) = \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathcal{I}, \overline{\mathbf{Q}}_p)$; i.e., $\mathcal{X}(\mathcal{I})$ is the space of all $\overline{\mathbf{Q}}_p$-valued points of $\mathrm{Spec}(\mathcal{I})_{/\mathcal{O}}$. Let $\mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ be the dense subset of $\mathcal{X}(\mathcal{I})$ (under the Zariski topology) consisting of points of $\mathcal{X}(\mathcal{I})$ whose restriction to Λ is of the form $P_{k,\varepsilon}$ with $k \geq 0$. We put

$$\mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}) = \mathcal{X}_{\mathrm{alg}}(\mathcal{I}) \cap \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathcal{I},\mathcal{O})$$
$$\text{and } \mathcal{X}(\mathcal{I};\mathcal{O}) = \mathcal{X}(\mathcal{I}) \cap \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathcal{I},\mathcal{O}).$$

For $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$, the integer k and the character ε defined by $P|_\Lambda = P_{k,\varepsilon}$ will be called the weight of P and the character of P. The weight (resp. the character) of P will be denoted by $k(P)$ (resp. ε_P). The exponent in p of the conductor of ε_P will be denoted by $r(P)$ (when ε_P is trivial, we shall agree to put $r(P) = 1$). We fix a Λ-algebra homomorphism $\lambda : \mathbf{h}^o(N;\mathcal{O}) \to \mathcal{I}$. Then $\lambda(T(n))$ is an element of $\mathcal{I}$, and thus we can consider it as a function on $\mathcal{X}(\mathcal{I})$ with values in $\overline{\mathbf{Q}}_p$. We shall write its value $\lambda(T(n))(P)$ as $a_P(n) \in \overline{\mathbf{Q}}_p$. Then it is seen in [14] that there is a family of cusp forms $f_P \in S_{k(P)}(\Gamma_1(Np^{r(P)}))$ parametrized by the points $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})(k(P) \geq 2)$ such that the image under $i : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ of each n-th coefficient of f_P is given by $a_P(n)$. Moreover, f_P is a common eigenform of all Hecke operators satisfying $f|T(n) = a_P(n)f$ and $|a_P(p)|_p = 1$. The function f_P is called the cusp form belonging to λ at P.

For simplicity, we shall suppose that $N = 1$. We take another Λ-algebra homomorphism $\lambda' : \mathbf{h}^o(1;\mathcal{O}) \to \mathcal{I}$. We write

$$g_Q = \sum_{n=1}^{\infty} b_Q(n)q^n \in S_{k(Q)}(\Gamma_1(p^{r(Q)}))$$

for the cusp form belonging to λ' at $Q \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$. Let ψ_P and ξ_Q be the characters of f_P and g_Q, respectively. Then it is known that there exist characters $\psi, \xi : (\mathbf{Z}/p\mathbf{Z})^\times \to \overline{\mathbf{Q}}^\times$ such that $\psi_P = \varepsilon_P \psi \omega^{-k(P)}$ and $\xi_P = \varepsilon_P \xi \omega^{-k(P)}$ for all $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$, where ω is the Teichmüller character modulo p.

For each normalized common eigenform $f \in S_k(\Gamma_1(p^r))$, as is known by the theory of new forms, there exists a unique primitive form f_0 whose eigenvalue for $T(\ell)$ coincides with that of f for almost all primes ℓ. This f_0 is called the primitive form associated with f. We define complex numbers $\alpha(f)$, $\alpha'(f)$ and $W(f)$ by

$$\sum_{n=0}^{\infty} a(p^n, f_0)p^{-ns} = [(1 - \alpha(f)p^{-s})(1 - \alpha'(f)p^{-s})]^{-1}$$

and

$$f_0|_k \begin{pmatrix} 0 & -1 \\ C & 0 \end{pmatrix} = W(f)f_0^\rho,$$

where $f_0|T(n) = a(n, f_0)f_0$ and C is the conductor of f (i.e. the smallest possible level of f_0) and $f_0^\rho(z) = \overline{f_0(-\bar{z})}$. For two normalized common eigenforms f and g, we take the associated primitive form f_0 and g_0 and define the primitive Rankin product of f and g by

$$\mathcal{D}(s, f, g) = \mathcal{D}_{C(f,g)}(s, f_0, g_0),$$

where $C(f, g)$ is the least common multiple of the conductors of f and g. We may suppose that $\alpha(g_Q) = b_Q(p)$, $\alpha(f_P) = a_P(p)$ and $\alpha(g_Q^\rho) = \alpha(g_Q)^\rho$ for the complex conjugation ρ. Then we define some Euler factors by

$$\begin{aligned}
S(P) &= \left(\frac{\psi_P(p)p^{k(P)-1}}{\alpha(f_P)^2}\right)\left(1 - \frac{\psi_P(p)p^{k(P)-2}}{\alpha(f_P)^2}\right), \\
E'_{P,Q}(s) &= \left(1 - \frac{\xi_P(p)p^{s-1}}{\alpha(g_Q^\rho)\alpha(f_P)}\right) \\
&\qquad \times (1 - \alpha'(f_P)\alpha(g_Q^\rho)p^{-s})(1 - \overline{\xi}_P(p)\alpha'(f_P)\alpha'(g_Q^\rho)p^{-s}), \\
E''_{P,Q}(s) &= \left(1 - \frac{p^{s-1}}{\alpha(g_Q^\rho)\alpha(f_P)}\right) = 1 - (\alpha(g_Q)/\alpha(f_P))p^{s-k(Q)}, \\
E_{P,Q}(s) &= E'_{P,Q}(s)E''_{P,Q}(s).
\end{aligned}$$

Let $\mathcal{I}\hat{\otimes}_{\mathcal{O}}\mathcal{I}\hat{\otimes}_{\mathcal{O}}\Lambda$ be the profinite completion of the tensor product $\mathcal{I}\otimes_{\mathcal{O}}\mathcal{I}\otimes_{\mathcal{O}}\Lambda$. Any element $F \in \mathcal{I}\hat{\otimes}\mathcal{I}\hat{\otimes}\Lambda$ can be considered as a p-adic analytic function on $\mathcal{X}(\mathcal{I}) \times \mathcal{X}(\mathcal{I}) \times \mathcal{X}(\Lambda)$. We shall construct an element D of the quotient field of $\mathcal{I}\hat{\otimes}\mathcal{I}\hat{\otimes}\Lambda$ whose value at $(P,Q,R) \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}) \times \mathcal{X}_{\mathrm{alg}}(\mathcal{I}) \times \mathcal{X}_{\mathrm{alg}}(\Lambda)$ gives essentially the algebraic number

$$\mathcal{D}(k(Q)+k(R), f_P, g_Q^\rho)/\Omega(P,Q,R).$$

Here, as the transcendental factor of $\mathcal{D}$, we shall take

$$\Omega(P,Q,R) = (2\pi i)^{k(Q)+2k(R)-1}(2i)^{k(P)+1}\pi^2 < f_P^0, f_P^0 > \overline{G(\xi_Q)},$$

where f_P^0 is the primitive form associated with f_P and $G(\xi_Q)$ is the Gauss sum for ξ_Q. (We understand that $G(\xi_Q) = 1$ if $\xi_Q = \mathrm{id}$.) This transcendental factor looks complicated but has an intrinsic meaning, which will be explained in § 4 in the text in the language of motives of Deligne. More precisely, our result in a simplest case is as follows :

THEOREM I. — *There exists a unique function D in the quotient field of $\mathcal{I}\hat{\otimes}\mathcal{I}\hat{\otimes}\Lambda$ on $X(\mathcal{I}) \times \mathcal{X}(\mathcal{I}) \times \mathcal{X}(\Lambda)$ with the following interpolation property : For a point $(P,Q,R) \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}) \times \mathcal{X}_{\mathrm{alg}}(\mathcal{I}) \times \mathcal{X}_{\mathrm{alg}}(\Lambda)$, we suppose that $\psi_p = \xi_Q = \varepsilon_R = \mathrm{id}$, and $k(P) - k(Q) > k(R) \geq 0$, $k(Q) > 2$ and $k(P) \geq 2$. Then $D(P,Q,R)$ is finite and*

$$(0.2)\quad D(P,Q,R) = \\ cwS(P)^{-1}E(P,Q,R)\mathcal{D}(k(Q)+k(R), f_P, g_Q^\rho|\omega^{k(R)})/\Omega(P,Q,R),$$

where $E(P,Q,R) = E_{P,Q}(k(Q)+k(R))$, $c = (-1)^{k(Q)}\Gamma(k(Q)+k(R))\Gamma(k(R)+1)$ and $w = W(g_Q)W(f_P)^{-1}$. Moreover, if $H \in \mathcal{I}$ annihilates the module of congruence of λ (for definition of this module, see § 4 in the text), *then $H(P)D(P,Q,R)$ is p-adic analytic; namely, $HD \in \mathcal{I}\hat{\otimes}\mathcal{I}\hat{\otimes}\Lambda$ as a function on $\mathcal{X}(\mathcal{I}) \times \mathcal{X}(\mathcal{I}) \times \mathcal{X}(\Lambda)$.*

Now we restrict $D(P,Q,R)$ to $\mathcal{X}(\mathcal{I}) \times \mathcal{X}(\mathcal{I}) \times P_0$ for $P_0 = P_{0,\mathrm{id}} \in \mathcal{X}(\Lambda)$ and write this function as $D(P,Q)$. We know that

$$E''(P,Q) = E''_{P,Q}(k(Q)) = 1 - \alpha(g_Q)/\alpha(f_P) \\ = 1 - \lambda'(T(p))(Q)/\lambda(T(p))(P)$$

gives an element in $\mathcal{I} \otimes \mathcal{I}$ (since $\lambda(T(p)) \in \mathcal{I}^\times$). When $\lambda = \lambda'$, $E''(P,Q)$ has a trivial zero at the diagonal divisor $\Delta = \{(P,P) \in \mathcal{X}(\mathcal{I})^2 | P \in \mathcal{X}(\mathcal{I})\}$. Now we ask whether the function $D'(P,Q) = D(P,Q)/E''(P,Q)$ has a pole at Δ or not. To give an answer to this question, we fix a topological generator u of Γ and identify $\Lambda\hat{\otimes}\Lambda$ with the power series ring $\mathcal{O}[[X,Y]]$ naturally. We also regard $\Lambda\hat{\otimes}\Lambda$ as a subalgebra of $\mathcal{I}\hat{\otimes}\mathcal{I}$. Then Δ is defined by the equation $X = Y$.

THEOREM II. — *Suppose that $\lambda = \lambda'$. Then D' has a simple pole at Δ; namely, we have*

$$((X-Y)D')(P,Q)|_{P=Q} = (1+Y(P))\frac{p-1}{p}\log(u)$$

if P is non-critical.

(We say that P is non-critical if P is outside the support of the module of congruence of λ in $\mathcal{X}(\mathcal{I})$).

Here are some remarks about the theorems : The p-adic interpolation along the cyclotomic line (i.e. the line of the variable R) of our type of zeta functions was first obtained by Panciskin [24] in a different method from ours; thus, our result extends the domain of the interpolation to the spectrum of the Hecke algebras. We shall give a formulation of Theorems I and II in § 5 Example d in full generality, where we shall state the result valid for λ and λ' with arbitrary level and give the similar evaluation of the function $D(P,Q,R)$ at any algebraic point without assuming that $\psi_P = \xi_Q = \varepsilon_R = \mathrm{id}$. Since every primitive from f with $|a(p)|_p = 1$ belongs to a homomorphism λ of above type, Theorem I is general enough to give a p-adic interpolation of the values (0.1) for any pair of primitive forms f and g with $|a(p)|_p = |b(p)|_p = 1$ of different weight. If $\mathcal{I} = \Lambda$, we may identify $\Lambda\hat{\otimes}\Lambda\hat{\otimes}\Lambda$ with $\mathcal{O}[[X,Y,Z]]$ by fixing a topological generator u of Γ. Then the function $D(P,Q,R)$ is given by a quotient of power series $F(X,Y,Z)/H(X)$ so that

$$D(P_{k,\varepsilon}, P_{\ell,\delta}, P_{m,\gamma}) = F(\varepsilon(u)u^k - 1, \delta(u)u^\ell - 1, \gamma(u)u^m - 1)/H(\varepsilon(u)u^k - 1).$$

Thus in this case, the p-adic L-function D is a usual Iwasawa function of three variables. However, there are examples of λ whose values cannot be contained in Λ [12, § 4]. Thus, in general, D may not be an Iwasawa function but is a function on a covering space (of finite degree) over $\mathrm{Spec}(\mathcal{O}[[X,Y,Z]])$. Theorem II may be considered as a p-adic analogue of the well known residue formula [30, 2.5] :

$$\mathrm{Res}_{s=k}\mathcal{D}(s,f,f^\rho) = \Gamma(k)^{-1}C^{-2}2^{2k}\pi^{k+1} < f,f >_{\Gamma_1(C)} \quad (\text{if } C > 2),$$

where C is the conductor of the primitive form f.

We shall now give a summary of the content of each section. In § 1, we give a short account of several operations in the space of p-adic modular forms which will be used to define the convolution product of p-adic measures. After that we shall prove the duality between the space of p-adic modular forms and Hecke algebras (Th. 1.3). In § 2, we shall give a characterization of the space of p-adic cusp forms (Th. 2.2) which may be viewed as a p-adic analogue of the classical cuspidal condition. In § 3, we shall generalize the measure theory over Λ to that over the integral extension $\mathcal{I}$. After giving a brief summary of the theory of modules of congruence in § 4, we shall state our main result in full generality in § 5. In fact, our method is applicable to a rather wide class of measures μ, satisfying certain algebraicity conditions (cf. (5.1 a,b,c)) on a p-adic space X with values in the space of p-adic modular forms. We shall establish a p-adic interpolation of the values $\mathcal{D}(m, f_P, \mu(\phi))$ by varying m, locally constant functions ϕ on X and $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ (Th. 5.1). From this general result, we shall deduce Theorems I and II as well as a generalization of the result in the first part [11] on theta measures. After giving in § 6 some facts on real analytic Eisenstein series for our later use, we develop a theory of p-adic Rankin convolution of measures in §§ 7 and 8. The final section § 9 is devoted to the proof of the main results.

In the first draft of this paper, the values of the p-adic L-function $D(P, Q, R)$ as in Th. I was given in a much more complicated form without uniformity. The simplification of the expression, especially the introduction of the Euler factor $E(P, Q, R)$, is due to B. Perrin-Riou. The author is very much thankful for her careful reading of the manuscript.

Notation. — We shall use the notation introduced in [11] and [14]. For each matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{R})$ with $\det(\alpha) > 0$, we define an operation (of weight $k \in \mathbf{Z}$) on functions on the Poincaré upper half plane $\mathfrak{H}$ with values in $\mathbf{C}$ by $(f|_k\alpha)(z) = \det(\alpha)^{k/2} f\left(\dfrac{az+b}{cz+d}\right)$ $(cz+d)^{-k}$. For each congruence subgroup Δ of $SL_2(\mathbf{Z})$, we denote by $\mathcal{M}_k(\Delta)$ (resp. $S_k(\Delta)$) the space of holomorphic modular forms (resp. holomorphic cusp forms) for Δ of weight k. For each character $\psi : \Delta \to \mathbf{C}^\times$ of finite order, we put

$$\mathcal{M}_k(\Delta, \psi) = \{f \in \mathcal{M}_k(\mathrm{Ker}(\psi)) \;\big|\; f|_k\gamma = \psi(\gamma) f \text{ for all } \gamma \in \Delta\},$$
$$S_k(\Delta, \psi) = \mathcal{M}_k(\Delta, \psi) \cap S_k(\mathrm{Ker}(\psi)).$$

For each $f \in S_k(\Delta, \psi)$ and $g \in \mathcal{M}_k(\Delta, \psi)$, the Petersson inner product is defined by

$$< f, g >_\Delta = \int_{\Delta\backslash\mathfrak{H}} \overline{f(z)} g(z) y^{k-2} dx dy.$$

When $\Delta = \Gamma_0(N)$, we write $< f, g >_N$ for $< f, g >_{\Gamma_0(N)}$. The Fourier expansion of each $f \in \mathcal{M}_k(\Gamma_1(N))$ is always written as

$$f = \sum_{n=0}^{\infty} a(n, f) q^n \quad \text{for} \quad q = \exp(2\pi i z).$$

Throughout the paper, we fix an embedding $i : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Thus every algebraic number can be viewed as a complex number as well as a p-adic number in $\overline{\mathbf{Q}}_p$ uniquely. The normalized p-adic absolute value of $x \in \overline{\mathbf{Q}}_p$ will be written as $|x|_p$.

Contents

1. Hecke algebras and p-adic modular forms.

We shall give here a brief account of the theory of p-adic modular forms and their Hecke algebras. We shall refer to our previous papers [11], [13] and [14] for the results states here without proof.

Let Δ be a congruence subgroup of $SL_2(\mathbf{Z})$ and for any subalgebra A of $\mathbf{C}$, we put

$$\begin{aligned} \mathcal{M}_k(\Delta; A) &= \{f \in \mathcal{M}_k(\Delta) \mid a(n, f) \in A \text{ for all } n\} \\ S_k(\Delta; A) &= S_k(\Delta) \cap \mathcal{M}_k(\Delta; A) . \end{aligned}$$

For each (abelian) character of finite order $\psi : \Delta \to A^\times$, put

$$\mathcal{M}_k(\Delta,\psi : A) = \mathcal{M}_k(\mathrm{Ker}(\psi); A) \cap \mathcal{M}_k(\Delta,\psi)$$
$$S_k(\Delta,\psi; A) = S_k(\Delta,\psi) \cap \mathcal{M}_k(\Delta,\psi; A) \ .$$

Now we shall suppose that Δ is of the following form :

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \bmod N,\ b \equiv 0 \bmod M,\ a \equiv d \equiv 1 \bmod t \right\}$$

(1.1)

for positive integers N and M and a divisor t of $M \cdot N$. Let us take a finite extension $K_0/\mathbf{Q}$ in $\overline{\mathbf{Q}}$, and let K be the topological closure of K_0 in $\overline{\mathbf{Q}}_p$. We put

$$\begin{aligned} \mathcal{M}_k(\Delta; K) &= \mathcal{M}_k(\Delta; K_0) \otimes_{K_0} K, \\ \mathcal{M}_k(\Delta,\psi; K) &= \mathcal{M}_k(\Delta,\psi; K_0) \otimes_{K_0} K, \\ S_k(\Delta; K) &= S_k(\Delta; K_0) \otimes_{K_0} K, \\ S_k(\Delta,\psi; K) &= S_k(\Delta,\psi; K_0) \otimes_{K_0} K. \end{aligned}$$

These spaces depend only on K and are independent of the choice of the dense subfield K_0. By using q-expansions, we may consider these spaces inside the formal power series ring $K[[q^{1/M}]]$ (if Δ is of the form (1.1)). For each $j > 0$, put

$$\mathcal{M}^j(\Delta; K) = \bigoplus_{k=0}^{j} \mathcal{M}_k(\Delta; K),\ \ S^j(\Delta; K) = \bigoplus_{k=1}^{j} S_k(\Delta; K) \ .$$

One may take these sums inside $K[[q^{1/M}]]$, and we shall take inductive limits inside $K[[q^{1/M}]]$:

$$\begin{aligned} \mathcal{M}(\Delta; K) &= \mathcal{M}^\infty(\Delta; K) = \varinjlim_j \mathcal{M}^j(\Delta; K) \simeq \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Delta; K) \ , \\ S(\Delta; K) &= S^\infty(\Delta; K) = \varinjlim_j S^j(\Delta; K) \simeq \bigoplus_{k=1}^{\infty} S_k(\Delta; K) \ . \end{aligned}$$

We shall define a p-adic norm on these spaces by

$$|f|_p = \sup_n |a(n,f)|_p \text{ if } f = \sum_{n=0}^{\infty} a(n,f) q^{n/M} \ .$$

Let $\overline{\mathcal{M}}(\Delta;K)$ (resp. $\overline{S}(\Delta;K)$) be the completion of $\mathcal{M}(\Delta;K)$ (resp. $S(\Delta;K)$) under this norm inside $K[[q^{1/M}]]$. They are K-Banack spaces. Let $\mathcal{O}_k$ be the p-adic integer ring of K and put

$$\mathcal{M}_k(\Delta;\mathcal{O}_K) = \mathcal{M}_k(\Delta;K) \cap \mathcal{O}_K[[q^{1/M}]],$$

$$\begin{aligned} S_k(\Delta;\mathcal{O}_K) &= S_k(\Delta;K) \cap \mathcal{O}_K[[q^{1/M}]], \\ \mathcal{M}^j(\Delta;\mathcal{O}_K) &= \mathcal{M}^j(\Delta;K) \cap \mathcal{O}_K[[q^{1/M}]], \\ S^j(\Delta;\mathcal{O}_K) &= S^j(\Delta;K) \cap \mathcal{O}_K[[q^{1/M}]] \text{ for } j = 1, 2, \cdots, \infty\ , \\ \overline{\mathcal{M}}(\Delta;\mathcal{O}_K) &= \overline{\mathcal{M}}(\Delta;K) \cap \mathcal{O}_K[[q^{1/M}]], \\ \overline{S}(\Delta;\mathcal{O}_K) &= \overline{S}(\Delta;K) \cap \mathcal{O}_K[[q^{1/M}]]\ . \end{aligned}$$

The space $\overline{S}(\Delta;\mathcal{O}_K)$ (resp. $\overline{\mathcal{M}}(\Delta;\mathcal{O}_K)$) is the completion of $S^\infty(\Delta;\mathcal{O}_K)$ (resp. $\mathcal{M}^\infty(\Delta;\mathcal{O}_K)$) under the norm $|\ |_p$. Let M and N be positive integers prime to p, and put

$$\Gamma_0(Np^r, M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid b \equiv 0 \bmod M,\ c \equiv 0 \bmod Np^r \right\}$$

$$\Gamma_1(Np^r, M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(Np^r, M) \mid a \equiv d \equiv 1 \bmod MNp^r \right\}$$

$\Gamma_0(Np^r) = \Gamma_0(Np^r, 1), \Gamma_1(Np^r) = \Gamma_1(Np^r, 1)$ and

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N, N) \mid a \equiv d \equiv 1 \bmod N \right\}\ .$$

It is known by Katz that as subspaces of $\mathcal{O}_K[[q^{1/M}]]$

$$\begin{aligned} \overline{S}(\Delta \cap \Gamma_1(p^r);\mathcal{O}_K) &= \overline{S}(\Delta;\mathcal{O}_K), \\ \overline{\mathcal{M}}(\Delta \cap \Gamma_1(p^r);\mathcal{O}_K) &= \overline{\mathcal{M}}(\Delta : \mathcal{O}_K) \end{aligned} \tag{1.2}$$

for $\Delta = \Gamma_1(N), \Gamma_1(N, M)$ and $\Gamma(N)$. Proof of (1.2) may be found in [13, (1.9b) and Cor. 1.2] for $\Gamma_1(N)$ and in [20, 5.6.3] for $\Gamma(N)$. The operator $[M] : \sum_{n=0}^{\infty} a_n q^{\frac{n}{M}} \mapsto \sum_{n=0}^{\infty} a_n q^n$ induces an isomorphism : $\overline{S}(\Gamma_1(N, M);\mathcal{O}_K) \simeq \overline{S}(\Gamma_1(MN);\mathcal{O}_K)$ (resp. $\overline{\mathcal{M}}(\Gamma_1(N, M);\mathcal{O}_K) \simeq \overline{\mathcal{M}}(\Gamma_1(MN);\mathcal{O}_K)$); hence, (1.2) is true for $\Delta = \Gamma_1(N, M)$. We shall write $\overline{\mathcal{M}}(N;\mathcal{O}_K)$ and $\overline{S}(N, \mathcal{O}_K)$ for $\overline{\mathcal{M}}(\Gamma_1(N);\mathcal{O}_K)$ and $\overline{S}(\Gamma_1(N) : \mathcal{O}_K)$. We put, for $A \subset \mathbf{C}$ or $A = K, \mathcal{O}_K$,

$$\begin{aligned} S_k(Np^\infty; A) &= \varinjlim_r S_k(\Gamma_1(Np^r); A), \\ \mathcal{M}_k(Np^\infty; A) &= \varinjlim_r \mathcal{M}_k(\Gamma_1(Np^r); A)\ . \end{aligned}$$

By (1.2), we have natural inclusions

(1.3) $\quad S_k(Np^\infty;\mathcal{O}_K) \subset \overline{S}(N;\mathcal{O}_K),\ \mathcal{M}_k(Np^\infty;\mathcal{O}_K) \subset \overline{\mathcal{M}}(N;\mathcal{O}_K)$.

We shall now introduce several operations on $\overline{\mathcal{M}}(N;\mathcal{O}_K)$:

I. *The action of* Z_N :

Put $Z_n = \varprojlim_r (\mathbf{Z}/Np^r\mathbf{Z})^\times = \mathbf{Z}_p^\times \times (\mathbf{Z}/N\mathbf{Z})^\times$. For each element $z \in Z_N$, we shall write z_p (resp. z_0) for its projection in $\mathbf{Z}_p^\times$ (resp. $(\mathbf{Z}/N\mathbf{Z})^\times$). We let $z \in Z_N$ act on $\mathcal{M}_k(\Gamma_1(Np^r);K)$ and $\mathcal{M}_k(\Gamma(N) \cap \Gamma_1(p^r);K)$ by $f \mapsto f|z = z_p^k f|_k\sigma_z$, where $\sigma_z \in SL_2(\mathbf{Z})$ is a matrix with $\sigma_z \equiv \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$ mod Np^r. We let z act on $\mathcal{M}(\Delta;K)$ as follows : for $f = \sum_k f_k$ with $f_k \in \mathcal{M}_k(\Delta;K)$, $f|z = \sum_k z_p^k f_k|_k\sigma_z$. This action preserves $\mathcal{M}(\Delta;\mathcal{O}_K)$, extends to $\overline{\mathcal{M}}(\Delta;\mathcal{O}_K)$ and $\overline{S}(\Delta;\mathcal{O}_K)$ for $\Delta = \Gamma_1(N)$ or $\Gamma(N)$ by continuity and induces the original action on $\mathcal{M}_k(\Gamma_1(Np^r);\mathcal{O}_K)$ and $S_k(\Gamma_1(Np^r);\mathcal{O}_K)$ under the inclusion (1.3) (this fact is due to Katz; see [20, 5.3] and [13, § 3]). Via the natural projection : $\mathbf{Z} \to (\mathbf{Z}/Np^r\mathbf{Z})$, any integer n prime to Np can be considered as an element of Z_N.

II. *The action of Hecke operators* $T(n)$:

We shall define, for each positive integer n, the Hecke operator $T(n)$ by its effect on q-expansion coefficients :

(1.4a) $$a(m, f|T(n)) = \sum_{\substack{0<\ell|m,\ell|n \\ (\ell,Np)=1}} \ell^{-1} a(mn/\ell^2, f|\ell) \text{ for } f \in \overline{\mathcal{M}}(N;\mathcal{O}_K)\ ,$$

where $f|\ell$ is the image of f under the action of the integer ℓ as an element of Z_N. We can easily deduce from (1.4a)

(1.4b) $$f|(T(\ell)^2 - T(\ell^2)) = \ell^{-1}(f|\ell) \quad \text{for each prime } \ell \nmid Np\ .$$

By definition, $T(n)$ induces the usual Hecke operator on $\mathcal{M}_k(\Gamma_1(Np^r);K)$ for each k and $r \geq 1$ and preserves $\mathcal{M}(\Gamma_1(Np^r);\mathcal{O}_K)$, since the action of Z_N respects $\mathcal{O}_K$-integral forms. Then by continuity, $T(n)$ extends automatically on $\overline{\mathcal{M}}(N;\mathcal{O}_K)$. Of course, $T(n)$ respects $\overline{S}(N;\mathcal{O}_K)$ and $S^j(\Gamma_1(Np^r);\mathcal{O}_K)$ for $r = 1,\ldots,\infty$ and $j = 1,\ldots,\infty$.

III. *The operator* $[t]$ *for* $0 < t \in \mathbf{Q}^\times$:

If Δ is of type (1.1), $\Delta' = \left(\begin{smallmatrix} t & 0 \\ 0 & s \end{smallmatrix}\right)^{-1} \Delta \left(\begin{smallmatrix} t & 0 \\ 0 & s \end{smallmatrix}\right) \cap SL_2(\mathbf{Z})$ is again of type (1.1) for each pair of positive integers t and s. Define for each $f \in \overline{M}(\Delta; \mathcal{O}_K)$ with $f = \sum_{n=0}^{\infty} a(n,f)\, q^{\frac{n}{M}}$

$$f|[t/s] = \sum_{n=0}^{\infty} a(n,f)\, q^{\frac{nt}{Ms}} .$$

Obviously this operation preserves $\mathcal{O}_K$-integral forms and induces a linear map of $\mathcal{M}_k(\Delta; K)$ into $\mathcal{M}_k(\Delta'; K)$ for each k. Thus by continuity, we have an $\mathcal{O}_K$-linear map :

$$[t/s] : \overline{\mathcal{M}}(\Delta; \mathcal{O}_K) \to \overline{\mathcal{M}}(\Delta'; \mathcal{O}_K) .$$

This map $[t/s]$ is obviously injective.

IV. *The action of* $SL_2(\mathbf{Z}/L\mathbf{Z})$:

Let M and N be positive integers prime to p, and put $L = MN$. We shall fix a primitive L-th root of unity ζ_L, and suppose that $\zeta_L \in K$. Then it is known that the action : $f \mapsto f|_k\gamma$ with $\gamma \in SL_2(\mathbf{Z})$ leaves $\mathcal{M}_k(\Gamma(L); K)$ stable and factors through $SL_2(\mathbf{Z}/L\mathbf{Z})$. We let $SL_2(\mathbf{Z}/L\mathbf{Z})$ act on $\mathcal{M}(\Gamma(L); K)$ and $\mathcal{S}(\Gamma(L); K)$ diagonally; namely, for $f = \sum_k f_k$ with $f_k \in \mathcal{M}_k(\Gamma(L); K)$, $f|\gamma = \sum_k f_k|_k\gamma$. This action preserves $\mathcal{O}_K$-integral forms [13, § 1] and extends to an action on $\overline{\mathcal{M}}(\Gamma(L); \mathcal{O}_K)$ and $\overline{\mathcal{S}}(\Gamma(L); \mathcal{O}_K)$ by continuity. This action is given on $\mathcal{M}_k(\Gamma_1(Np^r; M); K)$ under the embedding (1.2) as follows : For each $\overline{\gamma} \in SL_2(\mathbf{Z}/L\mathbf{Z})$, we choose $\gamma \in SL_2(\mathbf{Z})$ such that $\gamma \equiv \overline{\gamma} \bmod L$ and $\gamma \equiv 1 \bmod p^r$. Then we have $f|\overline{\gamma} = f|_k\gamma$ for $f \in \mathcal{M}_k(\Gamma_1(Np^r; M); K)$.

V. *Trace morphisms :*

Let M, N and L as above, and let $\psi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}_K^\times$ be a character. Since $Z_L \cong \Gamma \times (\mathbf{Z}/Lp\mathbf{Z})^\times$ and $L = MN$, we may consider ψ as a character of $(\mathbf{Z}/Lp\mathbf{Z})^\times$ or Z_L. Let Δ denote either of $\Gamma_1(N)$, $\Gamma(N)$ or $\Gamma_1(N, M)$. Since Z_L acts naturally on $\overline{\mathcal{M}}(\Delta; \mathcal{O}_K)$, the subgroup $(\mathbf{Z}/Lp\mathbf{Z})^\times$ acts on $\overline{\mathcal{M}}(\Delta; \mathcal{O}_K)$. Put

$$\overline{\mathcal{M}}(\Delta; \mathcal{O}_K)[\psi] = \left\{ f \in \overline{\mathcal{M}}(\Delta; \mathcal{O}_K) \,\middle|\, f|\zeta = \psi(\zeta) f \quad \text{for } \zeta \in (\mathbf{Z}/Lp\mathbf{Z})^\times \subset Z_L \right\},$$

$$\overline{\mathcal{S}}(\Delta : \mathcal{O}_K)[\psi] = \overline{\mathcal{M}}(\Delta; \mathcal{O}_K)[\psi] \cap \overline{\mathcal{S}}(\Delta; \mathcal{O}_K) .$$

Let ω be the Teichmüller character defined on $\mathbf{Z}_p^\times$ by $\omega(x) = \lim_{n\to\infty} x^{p^n}$ and let $\varepsilon : \Gamma \to \mathcal{O}_K^\times$ be a finite order character of $\Gamma = 1 + p\mathbf{Z}_p$ of conductor p^r for $r \geq 1$; that is, $\mathrm{Ker}(\varepsilon) = 1 + p^r\mathbf{Z}_p$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(Np^r)$ and each character $\alpha : (\mathbf{Z}/Np^r\mathbf{Z})^\times \to \mathcal{O}_K^\times$, write

$$\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha(d) .$$

Then α gives a character of $\Gamma_0(Np^r)$ and $\Gamma_0(Np^r, M)$. For each $s \geq r$, let Δ be either of $\Gamma_0(Np^s)$ or $\Gamma_0(Np^s, M)$. Note that if $f \in \mathcal{M}_k(\Delta, \varepsilon\psi\omega^{-k}; \mathcal{O}_K)$, then for $\zeta \in (\mathbf{Z}/Lp\mathbf{Z})^\times \subset Z_L$,

$$f|\zeta = \zeta_p^k\psi\omega^{-k}(\zeta)f = \psi(\zeta)f ,$$

since $\omega(\zeta) = \zeta_p$ and $\varepsilon(\zeta) = 1$ as characters of Z_L. This shows

$$\begin{aligned}(1.5)\quad \mathcal{M}_k(\Gamma_0(Np^s), \varepsilon\psi\omega^{-k}; \mathcal{O}_K) &\subset \overline{\mathcal{M}}(N; \mathcal{O}_K)[\psi], \\ \mathcal{M}_k(\Gamma_0(Np^s, M), \varepsilon\psi\omega^{-k}; \mathcal{O}_K) &\subset \overline{\mathcal{M}}(\Gamma_1(N, M); \mathcal{O}_K)[\psi] .\end{aligned}$$

Let R be a representative set for $\Gamma_0(Np^r, M) \setminus \Gamma_0(Np^r)$; thus,

$$\Gamma_0(Np^r) = \coprod_{\gamma\in R} \Gamma_0(Np^r, M)\gamma .$$

We may suppose that $\gamma \equiv 1 \bmod p^r$ for all $\gamma \in R$. Write simply α for $\varepsilon\psi\omega^{-k}$. Then we see that $\alpha(\gamma) = \psi(\gamma) = \psi_L(\gamma)$ for $\gamma \in R$, where ψ_L is the restriction of ψ to $(\mathbf{Z}/L\mathbf{Z})^\times$. We shall define a trace map

$$\mathrm{Tr}_{L/N} : \mathcal{M}_k(\Gamma_0(Np^r, M), \alpha; \mathcal{O}_K) \to \mathcal{M}_k(\Gamma_0(Np^r), \alpha; \mathcal{O}_K)$$

by $f|\mathrm{Tr}_{L/N} = \sum_{\gamma\in R} \psi_L^{-1}(\gamma)f|_k\gamma$.

Recall that z_p (resp. z_0) is the projection of $z \in Z_L$ in $\mathbf{Z}_p^\times$ (resp. $(\mathbf{Z}/L\mathbf{Z})^\times$). Then the action of $z_0 \in (\mathbf{Z}/L\mathbf{Z})^\times$ on $\overline{\mathcal{M}}(\Gamma(L); \mathcal{O}_K)$ coincides with that of $\begin{pmatrix} z_0^{-1} & 0 \\ 0 & z_0 \end{pmatrix} \in SL_2(\mathbf{Z}/L\mathbf{Z})$ by definition. For any $\Delta \subset SL_2(\mathbf{Z})$, we denote by $\overline{\Delta}$ the image of Δ in $SL_2(\mathbf{Z}/L\mathbf{Z})$. Note that for every r, $\overline{\Gamma}_1(Np^r, M) = \overline{\Gamma}_1(N, M)$ and $\overline{\Gamma}_0(Np^r, M) = \overline{\Gamma}_0(N, M)$. If we write $\overline{\Gamma}$ for the image of $\gamma \in R$ in $SL_2(\mathbf{Z}/L\mathbf{Z})$, we see that $\overline{R} = \{\overline{\gamma}|\gamma \in R\}$ gives a complete representative set for $\overline{\Gamma}_0(N, M) \setminus \overline{\Gamma}_0(N)$.

LEMMA 1.1. — *Put*

$$m(\Gamma_1(Np,M),\psi;K)=\bigoplus_{k=0}^{\infty}\mathcal{M}_k(\Gamma_0(Np,M),\psi\omega^{-k};K)$$

and

$$m(\Gamma_1(Np,M),\psi;\mathcal{O}_K)=m(\Gamma_1(Np,M),\psi;K)\cap\mathcal{O}_K[[q^{1/M}]].$$

Let $\overline{m}(\Gamma_1(Np,M),\psi;\mathcal{O}_K)$ *be the completion in* $\mathcal{O}_K[[q^{1/M}]]$ *of* $m(\Gamma_1(Np,M),\psi;\mathcal{O}_K)$ *for the norm* $|\ |_p$. *Then we have that* $\overline{m}(\Gamma_1(Np,M),\psi;\mathcal{O}_K)=\overline{M}(\Gamma_1(N,M),\mathcal{O}_K)[\psi]$ *in* $\mathcal{O}_K[[q^{1/M}]]$.

Proof. — Put $\Delta=\Gamma_1(N,M)\cap\Gamma_0(p)$ and $\mu=\{\zeta\in\mathbf{Z}_p^\times|\zeta^{p-1}=1\}$.

Then, we can decompose, according to the action of μ, for $A=\mathcal{O}_K$ and K,

$$\mathcal{M}(\Gamma_1(Np,M);A)=\bigoplus_{a=0}^{p-2}\mathcal{M}(\Gamma_1(Np,M);A)[\omega^a]\ ,$$

$$\overline{\mathcal{M}}(\Gamma_1(Np,M);A)=\bigoplus_{a=0}^{p-2}\overline{\mathcal{M}}(\Gamma_1(Np,M);A)[\omega^a]\ ,$$

where on the subspace indicated by $[\omega^a]$, μ acts via the character ω^a. Note that $\mathcal{M}(\Gamma_1(Np,M);K)[\omega^a]=\bigoplus_{k=0}^{\infty}\mathcal{M}_k(\Delta,\omega^{a-k};K)$ and

$$\mathcal{M}(\Gamma_1(Np,M);\mathcal{O}_K)[\omega^a]=\mathcal{M}(\Gamma_1(Np,M);K)[\omega^a]\cap\mathcal{O}_K[[q^{1/M}]]\ .$$

Let ψ_p be the restriction of ψ to $(\mathbf{Z}/p\mathbf{Z})^\times$. Then $\psi_p=\omega^a$ for a suitable a. This shows that $\mathcal{M}(\Gamma_1(Np,M);\mathcal{O}_K)[\psi_p]$ is dense in $\overline{\mathcal{M}}(\Gamma_1(Np,M);\mathcal{O}_K)[\psi_p]$. We may define a trace map

$$\mathrm{Tr}:\overline{\mathcal{M}}(\Gamma_1(Np,M);K)[\psi_p]\to\overline{\mathcal{M}}(\Gamma(L),K)$$

by $f|\mathrm{Tr}=\sum_{\gamma\in\overline{\Delta}\backslash\overline{\Gamma}_0(N,M)}\psi_L(\gamma)^{-1}f|\gamma$.

This map sends $\overline{\mathcal{M}}(\Gamma_1(Np,M);A)[\psi_p]$ into $\overline{\mathcal{M}}(\Gamma_1(N,M);A)[\psi]$ for $A=\mathcal{O}_K$ and K and is uniformely continuous. On the other hand, Tr includes the trace map, defined before the lemma, of $\mathcal{M}_k(\Delta,\omega^{a-k};K)$ onto $\mathcal{M}_k(\Gamma_0(Np,M),\psi\omega^{-k};K)$ for each k and hence induces a map of $\mathcal{M}(\Gamma_1(Np,M);K)[\psi_p]$ into $m(\Gamma_1(Np,M),\psi;K)$. By continuity, we have a map

$$\mathrm{Tr}:\overline{\mathcal{M}}(\Gamma_1(Np,M);K)[\psi_p]\to\overline{m}(\Gamma_1(Np,M),\psi;K)\ .$$

On $\overline{m}(\Gamma_1(Np,M),\psi;K)$, Tr is merely the multiplication by $[\overline{\Gamma}_0(Np,M):\overline{\Delta}]$ and hence is surjective. If one considers Tr as a map of $\overline{\mathcal{M}}(\Delta;K)[\psi_p]$ into $\overline{\mathcal{M}}(\Gamma(L);K)$, it is surjective onto $\overline{\mathcal{M}}(\Gamma_1(N,M);K)[\psi]$ by the same argument. This shows that

$$\overline{m}(\Gamma_1(Np,M),\psi;K)=\overline{\mathcal{M}}(\Gamma_1(N,M);K)[\psi]$$

and therefore, $\overline{m}(\Gamma_1(Np,M),\psi;\mathcal{O}_K)=\overline{\mathcal{M}}(\Gamma_1(N,M):\mathcal{O}_K)[\psi]$.

By this lemma, we can define

$$\mathrm{Tr}_{L/N}:\overline{\mathcal{M}}(\Gamma_1(N,M);\mathcal{O}_K)[\psi]\to\overline{\mathcal{M}}(N;\mathcal{O}_K)[\psi]$$

by $f|\mathrm{Tr}_{L/N}=\sum_{\gamma\in\overline{R}}\psi_L(\gamma)^{-1}f|\gamma$, where $\overline{R}$ is a complete representative set for $\overline{\Gamma}_0(N,M)\setminus\overline{\Gamma}_0(N)$ in $SL_2(\mathbf{Z}/L\mathbf{Z})$. We have a commutative diagram for each pair (k,ε)

$$(1.6)\quad \begin{array}{ccc}\mathrm{Tr}_{L/N}:\overline{\mathcal{M}}(\Gamma_1(N,M),\mathcal{O}_K)[\psi] & \to & \overline{\mathcal{M}}(N;\mathcal{O}_K)[\psi]\\ \uparrow & & \uparrow\\ \mathrm{Tr}_{L/N}:\mathcal{M}_k(\Gamma_0(Np^r,M),\varepsilon\psi\omega^{-1};\mathcal{O}_K) & \to & \mathcal{M}_k(\Gamma_0(Np^r),\varepsilon\psi\omega^{-k};\mathcal{O}_K)\ ,\end{array}$$

where the vertical arrows indicate the embeddings of (1.5).

VI. *The twisted trace operator $T_{L/N}$:*

Let L be a positive integer prime to p and N be a divisor of L. Note that $\begin{pmatrix}N&0\\0&L\end{pmatrix}^{-1}\Gamma_1(L)\begin{pmatrix}N&0\\0&L\end{pmatrix}=\Gamma_1(N,L/N)$ and we know from III that the map $[N/L]$ sends $\overline{\mathcal{M}}(L;\mathcal{O}_K)[\psi]$ into $\overline{\mathcal{M}}(\Gamma_1(N,L/N);\mathcal{O}_K)[\psi]$ for each character $\psi:(\mathbf{Z}/Np\mathbf{Z})^\times\to\mathcal{O}_K^\times$. We shall define the operator $T_{L/N}:\overline{\mathcal{M}}(L;\mathcal{O}_K)[\psi]\to\overline{\mathcal{M}}(N;\mathcal{O}_K)[\psi]$ as the composition of $[N/L]:\overline{\mathcal{M}}(L;\mathcal{O}_K)[\psi]\to\overline{\mathcal{M}}(\Gamma_1(N,L/N);\mathcal{O}_K)[\psi]$ and of $\mathrm{Tr}_{L/N}:\overline{\mathcal{M}}(\Gamma_1(N,L/N);\mathcal{O}_K)[\psi]\to\overline{\mathcal{M}}(N;\mathcal{O}_K)[\psi]$. When we denote by R a complete representative set for $\Gamma_0(Np^r,L/N)\setminus\Gamma_0(Np^r)$, we have a disjoint decomposition

$$\Gamma_0(Lp^r)\begin{pmatrix}1&0\\0&L/N\end{pmatrix}\Gamma_0(Np^r)=\coprod_{\gamma\in R}\Gamma_0(Lp^r)\begin{pmatrix}1&0\\0&L/N\end{pmatrix}\gamma\,.$$

From this combined with (1.6), we have a commutative diagram for each pair (k,ε) :

$$(1.7)\quad \begin{array}{ccc}T_{L/N}:\overline{\mathcal{M}}(L;\mathcal{O}_K)[\psi] & \to & \overline{\mathcal{M}}(N;\mathcal{O}_K)[\psi]\\ \uparrow & & \uparrow\\ \mathcal{M}_k(\Gamma_0(Lp^r),\varepsilon\psi\omega^{-k};\mathcal{O}_K) & \to & \mathcal{M}_k(\Gamma_0(Np^r),\varepsilon\psi\omega^{-k};\mathcal{O}_K)\ ,\end{array}$$

where the operator of the lower line is defined by

$$f \mapsto (L/N)^{\frac{k}{2}} \sum_{\gamma \in R} f|_k \begin{pmatrix} 1 & 0 \\ 0 & L/N \end{pmatrix} \gamma .$$

VII. *The twisting operator for each Dirichlet character* χ :

Let N and M be positive integers prime to p and let $\chi : (\mathbf{Z}/Mp^r\mathbf{Z})^\times \to \mathcal{O}_K^\times$ be a character. For each $f \in \mathcal{M}_k(\Gamma_1(N); K)$, put

$$f|\chi = \sum_{n=0}^{\infty} \chi(n) a(n, f) q^n, \tag{1.8}$$

where we put $\chi(n) = 0$ if $(n, Mp^r) > 1$. Then it is well known that $f|\chi \in \mathcal{M}_k(\Gamma_1(NM^2p^{2r}); K)$ and this induces an operator

$$\chi : \overline{\mathcal{M}}(N; \mathcal{O}_K) \to \overline{\mathcal{M}}(NM^2; \mathcal{O}_K)$$

given by the formula (1.8).

VIII. *Differentiation* :

It is known that the differential operator $d = q\dfrac{d}{dq} : \mathcal{O}_K[[q]] \to \mathcal{O}_K[[q]]$ preserves the spaces $\overline{\mathcal{M}}(N; \mathcal{O}_K)$ (in fact, it takes $\overline{\mathcal{M}}(N; \mathcal{O}_K)$ into $\overline{S}(N; \mathcal{O}_K)$; cf. Cor. 2.3 in the text). We note the following relations :

$$\chi \circ d = d \circ \chi, nd \circ T(n) = T(n) \circ d, d \circ [t] = t[t] \circ d \tag{1.9}$$
$$\text{(for } 0 < t \in \mathbf{Z}\text{) and } Z_p^2 d \circ [z] = [z] \circ d \text{ for } z \in Z_N.$$

Let V be a subspace of $\overline{\mathcal{M}}(N; \mathcal{O}_K)$ (of finite rank over $\mathcal{O}_K$) stable under the Hecke operators in II. We shall define *the Hecke algebra* $\mathbf{h}(V)$ of V by the $\mathcal{O}_K$-subalgebra of the $\mathcal{O}_K$-linear endomorphism algebra of V generated by $T(n)$ for all $n > 0$. We write $\mathbf{h}_k(\Delta, \psi; \mathcal{O}_K)$ for $\mathbf{h}(S_k(\Delta, \psi; \mathcal{O}_K))$ for Δ with $\Gamma_1(N) \supset \Delta \supset \Gamma_1(Np^r)$) and $\mathbf{h}^j(\Gamma_1(Np^r); \mathcal{O}_K)$ for $\mathbf{h}(S^j(\Gamma_1(Np^r); \mathcal{O}_K))$. The restriction of operators in $\mathbf{h}^i(\Gamma_1(Np^r); \mathcal{O}_K)$ to the subspace $S^j(\Gamma_1(Np^r); \mathcal{O}_K)$ of $S^i(\Gamma_1(Np^r); \mathcal{O}_K)$ for each pair $i > j$ gives a surjective $\mathcal{O}_K$-algebra homomorphism : $\mathbf{h}^i(\Gamma_1(Np^r); \mathcal{O}_K) \to \mathbf{h}^j(\Gamma_1(Np^r); \mathcal{O}_K)$. Define

$$\mathbf{h}(N; \mathcal{O}_K) = \varprojlim_j \mathbf{h}^j(\Gamma_1(Np); \mathcal{O}_K) .$$

Naturally $\mathbf{h}(N;\mathcal{O}_K)$ acts on $S^\infty(\Gamma_1(Np);\mathcal{O}_K)$, and by the uniform continuity, its action extends to $\overline{S}(N;\mathcal{O}_K)$. The restriction of operators in $\mathbf{h}(N;\mathcal{O}_K)$ to the subspace $V \subset \overline{S}(N;\mathcal{O}_K)$ gives a surjective $\mathcal{O}_K$-algebra homomorphism

$$\rho_V : \mathbf{h}(N;\mathcal{O}_K) \to \mathbf{h}(V) .$$

Especially, by (1.2), $\mathbf{h}^j(\Gamma_1(Np^r);\mathcal{O}_K)$, $\mathbf{h}_k(\Gamma_1(Np^r);\mathcal{O}_K)$ and $\mathbf{h}_k(\Gamma_0(Np^r),\varepsilon\psi\omega^{-k};\mathcal{O}_K)$ are residue algebras of $\mathbf{h}(N;\mathcal{O}_K)$. By (1.4b), each element of Z_N induces an endomorphism of $\overline{S}(N:\mathcal{O}_K)$ which belongs to $\mathbf{h}(N;\mathcal{O}_K)$; thus, we have a continuous character : $Z_N \to \mathbf{h}(N;\mathcal{O}_K)$, which induces an $\mathcal{O}_K$-algebra homomorphism of the continuous group algebra $\mathcal{O}_K[[Z_N]] = \varprojlim_r \mathcal{O}_K[(\mathbf{Z}/Np^r\mathbf{Z})^\times]$ into $\mathbf{h}(N;\mathcal{O}_K)$. Let $\Lambda_K = \mathcal{O}_K[[\Gamma]]$ denote the continuous group algebra of $\Gamma = 1 + p\mathbf{Z}_p$. Then the canonical decomposition $Z_N \simeq \Gamma \times (\mathbf{Z}/Np\mathbf{Z})^\times$ induces an algebra isomorphism :

$$\mathcal{O}_K[[Z_N]] \cong \Lambda_K \otimes_{\mathcal{O}_K} \mathcal{O}_K[(\mathbf{Z}/Np\mathbf{Z})^\times] .$$

In particular, $\mathbf{h}(N;\mathcal{O}_K)$ is a Λ_K-algebra.

We shall now define a pairing

$$(1.10) \qquad < , >: \mathbf{h}(V) \times V \to \mathcal{O}_K \text{ by } <h,f>= a(1,f|h) .$$

In particular, we include the case $V = \overline{S}(N;\mathcal{O}_K)$, and in this case, we just put $\mathbf{h}(V) = \mathbf{h}(N;\mathcal{O}_K)$.

PROPOSITION 1.2. — *Let V be a subspace of $\overline{S}(N;\mathcal{O}_K)$, and suppose that :*

(i) *V is free of finite rank over $\mathcal{O}_K$;*

(ii) $V(K) \cap \mathcal{O}_K[[q]] = V$ *for* $V(K) = V \otimes_{\mathcal{O}_K} K \subset L[[q]]$;

(iii) *V is stable under $T(n)$ for all $n > 0$.*

Then the pairing (1.10) *is perfect; namely, it induces isomorphisms :* $\mathbf{h}(V) \cong \mathrm{Hom}_{\mathcal{O}_K}(V,\mathcal{O}_K)$ *and* $V \cong \mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}(V),\mathcal{O}_K)$.

Proof. — We shall firstly show the non-degeneracy of the pairing on $\mathbf{h}(V;K)$ and $V(K)$ for $\mathbf{h}(V;K) = \mathbf{h}(V) \otimes_{\mathcal{O}_K} K$. By (1.4a), we have a formula : $a(m,f) =< T(m),f >$ for all $m > 0$. If $< h,f >= 0$ for all $h \in \mathbf{h}(V;K)$, then $a(m,f) = 0$ for all $m > 0$ and $f = 0$. Conversely, if $< h,f >= 0$ for all $f \in V(K)$, then $0 =< h,f|T(m) >= a(1,f|T(m)h) = a(1,f|hT(m)) =< T(m),f|h >= a(m,f|h)$ for all $m > 0$. This shows that $f|h = 0$ for all $f \in V(K)$ and thus $h = 0$ as operator on $V(K)$. We shall next show that $V \simeq \mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}(V), \mathcal{O}_K)$, which is sufficient for the assertion since $\mathcal{O}_K$ is a discrete valuation ring. By the non-degeneracy already proven, the induced map by the pairing from V to $\mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}(V); \mathcal{O}_K)$ is injective; so, we shall prove the surjectivity. For each linear form $\phi : \mathbf{h}(V) \to \mathcal{O}_K$, we can find $f \in V(K)$ so that $\phi(h) =< h, f >$ for all $h \in \mathbf{h}(V)$. Note that for all $m > 0$, $a(m, f) =< T(m), f >= \phi(T(m)) \in \mathcal{O}_K$ and thus $f \in V(K) \cap \mathcal{O}_K[[q]] = V$. This finishes the proof.

THEOREM 1.3 — *The pairing* (1.10) *induces isomorphisms :* $\mathbf{h}(N; \mathcal{O}_K) \cong \mathrm{Hom}_{\mathcal{O}_K}(\overline{S}(N : \mathcal{O}_K), \mathcal{O}_K)$ and $\overline{S}(N; \mathcal{O}_K) \cong \mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}(N; \mathcal{O}_K), \mathcal{O}_K)$.

Proof. — By Prop. 1.2, the pairing induces :

$$\mathrm{Hom}_{\mathcal{O}_K}(S^j(\Gamma_1(Np); \mathcal{O}_K), \mathcal{O}_K) \cong \mathbf{h}^j(\Gamma_1(Np); \mathcal{O}_K).$$

We then know that

$$\begin{aligned}\mathbf{h}(N; \mathcal{O}_K) &= \varprojlim_j \mathrm{Hom}_{\mathcal{O}_K}(S^j(\Gamma_1(Np); \mathcal{O}_K), \mathcal{O}_K)\\ &\cong \mathrm{Hom}_{\mathcal{O}_K}(\varinjlim_j S^j(\Gamma_1(Np); \mathcal{O}_K), \mathcal{O}_K)\\ &= \mathrm{Hom}_{\mathcal{O}_K}(S^\infty(\Gamma_1(Np); \mathcal{O}_K), \mathcal{O}_K)\\ &\cong \mathrm{Hom}_{\mathcal{O}_K}(\overline{S}(\Gamma_1(Np); \mathcal{O}_K), \mathcal{O}_K).\end{aligned}$$

([1, II.6.6, Prop, 11]).

The last equality follows from the uniform continuity of every linear form on $S^\infty(\Gamma_1(Np); \mathcal{O}_K)$ (with values in $\mathcal{O}_K$). On the other hand, by Prop. 1.2 the pairing induces : for every $m > 0$,

(1.11)

$$\mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}(N; \mathcal{O}_K), \mathcal{O}_K/p^m\mathcal{O}_K) \cong S^\infty(\Gamma_1(Np); \mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^m\mathcal{O}_K.$$

In fact, by Prop. 1.2, we know that

$$\mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}^j(\Gamma_1(Np); \mathcal{O}_K), \mathcal{O}_K/p^m\mathcal{O}_K) \cong S^j(\Gamma_1(Np); \mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^m\mathcal{O}_K.$$

Note that

$$\mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}^j(\Gamma_1(Np); \mathcal{O}_K), \mathcal{O}_K/p^m\mathcal{O}_K) \cong \mathrm{Hom}_{\mathcal{O}_K/p^m\mathcal{O}_K}(\mathbf{h}^j(\Gamma_1(Np); \mathcal{O}_K/p^m\mathcal{O}_K), \mathcal{O}_K/p^m\mathcal{O}_K).$$

The above module is nothing but the Pontryagin dual module of $\mathbf{h}^j(\Gamma_1(Np); \mathcal{O}_K/p^m\mathcal{O}_K)$. Then the perfectness of the Pontryagin duality shows that

$$\begin{aligned}
&\varinjlim_j \mathrm{Hom}_{\mathcal{O}_K/p^m\mathcal{O}_K}(\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K/p^m\mathcal{O}_K),\mathcal{O}_K/p^m\mathcal{O}_K)\\
&\qquad\cong \mathrm{Hom}_{\mathcal{O}_K/p^m\mathcal{O}_K}(\varprojlim_j \mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K/p^m\mathcal{O}_K),\mathcal{O}_K/p^m\mathcal{O}_K)\\
&\qquad\cong \mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}(N;\mathcal{O}_K),\mathcal{O}_K/p^m\mathcal{O}_K).
\end{aligned}$$

This shows (1.11). By definition, we know that

$$\begin{aligned}
\overline{S}(N;\mathcal{O}_K) &= \varprojlim_m (S^\infty(\Gamma_1(Np);\mathcal{O}_K)\otimes_{\mathcal{O}_K}\mathcal{O}_K/p^m\mathcal{O}_K)\\
&\cong \varprojlim \mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}(N;\mathcal{O}_K),\mathcal{O}_K/p^m\mathcal{O}_K)\\
&\cong \mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}(N;\mathcal{O}_K),\varprojlim_m \mathcal{O}_K/p^m\mathcal{O}_K) \quad ([1,\ \text{II.6.3, Prop. 5}])\\
&= \mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}(N;\mathcal{O}_K),\mathcal{O}_K).
\end{aligned}$$

This shows the assertion.

2. The projection to the ordinary part.

We shall begin with the definition of the projection to the ordinary part, which is an idempotent e in $\mathbf{h}(N;\mathcal{O}_K)$. Since $\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K)$ is free of finite rank over $\mathcal{O}_K$ for each $j>0$, it is a product of local rings; so we write

$$\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K)=\prod_R R$$

for local rings R. Let $\mathfrak{m}(R)$ be the unique maximal ideal of R. We say that R is ordinary if $T(p)\notin\mathfrak{m}(R)$ (i.e. the projection of $T(p)$ in R is a unit in R). Let $\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K)^o$ be the product of all ordinary local factors of $\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K)$. We denote by e_j the idempotent in $\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K)$ corresponding to $\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K)^o$. By definition, $\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K)^o$ is the biggest factor of $\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K)$ on which the image of $T(p)$ is a unit, and thus the natural projection of $\mathbf{h}^i(\Gamma_1(Np);\mathcal{O}_K)$ onto $\mathbf{h}^j(\Gamma_1(Np);\mathcal{O}_K)$ for each pair $i>j$ sends e_i to e_j. Put

$$e=\varprojlim_j e_j\in\mathbf{h}(N;\mathcal{O}_K)\ ,$$

which is an idempotent characterized by the fact that $eT(p)$ is a unit in $e\mathbf{h}(N;\mathcal{O}_K)$ and $(1-e)T(p)$ is topologically nilpotent. Let us write

$\mathbf{h}^o(N;\mathcal{O}_K)$ for $e\mathbf{h}(N;\mathcal{O}_K)$, which is a Λ_K-algebra as well as an $\mathcal{O}_K[[Z_N]]$-algebra, and also write $\overline{S}^o(N;\mathcal{O}_K)$ for $e\overline{S}(N;\mathcal{O}_K)$. Similarly, we shall define the ordinary parts $\mathbf{h}_k^o(\Gamma_1(Np^r);\mathcal{O}_K)$ and $\mathbf{h}_k^o(\Delta,\psi;\mathcal{O}_K)$ for Δ such that $\Gamma_1(N)\cap\Gamma_0(p)\supset\Delta\supset\Gamma_1(Np^r)$ by the biggest direct factor of the original Hecke algebra on which the image of $T(p)$ is a unit.

If γ is an element of $1+p\mathbf{Z}_p$, we denote by $\iota(\gamma)$ its image in the group Γ embedded tautologically in Λ_K. We choose and fix a topological generator u of $1+p\mathbf{Z}_p$. We shall simply write "u" for the corresponding p-adic number. Put

$$\omega_{k,r}=\iota(u^{p^{r-1}})-u^{kp^{r-1}}\in\Lambda_K\quad\text{for } 0\le k\in\mathbf{Z}.$$

Then the correspondence : $\iota(u)\mapsto u^k\iota(u)$ gives an isomorphism of $\mathcal{O}_K$-algebra : $\Lambda_K/\omega_{k,r}\Lambda_K\cong\mathcal{O}_K[\Gamma/\Gamma_r]$, where $\Gamma_r=\Gamma^{p^r}\cong 1+p^r\mathbf{Z}_p$. For each finite order character $\varepsilon:\Gamma\to\mathcal{O}_K^\times$ of conductor p^r and for each integer k, put $P_{k,\varepsilon}=\iota(u)-u^k\varepsilon(u)\in\Lambda_K$. Then the assignment : $\iota(u)\mapsto u^k\varepsilon(u)$ induces an isomorphism : $\Lambda_K/P_{k,\varepsilon}\Lambda_K\cong\mathcal{O}_K$. We define a congruence subgroup for each $0<r\in\mathbf{Z}$ by $\Phi_r=\Gamma_0(p^r)\cap\Gamma_1(Np)$. Then if we put $\varepsilon\begin{pmatrix}a&b\\c&d\end{pmatrix}=\varepsilon(d)$ for $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Phi_r,\varepsilon:\Phi_r\to\mathcal{O}_K^\times$ gives a character. The following fact is proven in [13, § 3] and [14, § 1] :

THEOREM 2.1. — *Suppose that $p\ge 5$. Then $\mathbf{h}^o(N;\mathcal{O}_K)$ is free of finite rank over Λ_K, and the natural morphisms of $\mathbf{h}(N;\mathcal{O}_K)$ onto $\mathbf{h}_k(\Gamma_1(Np^r);\mathcal{O}_K)$ and onto $\mathbf{h}_k(\Phi_r,\varepsilon;\mathcal{O}_K)$ induce isomorphisms for each $k\ge 2$, $\varepsilon:\Gamma\to\mathcal{O}_K^\times$ and $r\ge 1$:*

$$\mathbf{h}^o(N;\mathcal{O}_K)/\omega_{k,r}\mathbf{h}^o(N;\mathcal{O}_K)\cong\mathbf{h}_k^o(\Gamma_1(Np^r);\mathcal{O}_K),$$
$$\mathbf{h}^o(N;\mathcal{O}_K)/P_{k,\varepsilon}\mathbf{h}^o(N;\mathcal{O}_K)\cong\mathbf{h}_k^o(\Phi_r,\varepsilon;\mathcal{O}_K)$$

which take $T(n)$ of $\mathbf{h}(N;\mathcal{O}_K)$ to $T(n)$ of the right-hand side.

We shall now discuss about the p-adic cuspidal condition. Let $U=\{\pm\begin{pmatrix}1&u\\0&1\end{pmatrix}\mid u\in\mathbf{Z}/N\mathbf{Z}\}$ and $S=U\setminus SL_2(\mathbf{Z}/N\mathbf{Z})/U$. By the map : $SL_2(\mathbf{Z}/N\mathbf{Z})\ni\gamma\mapsto\gamma(\infty)$, we can identify S with the set of all cusps of $X_1(N)$. Fix a primitive root of unity $\zeta_N\in\overline{\mathbf{Q}}$ and suppose that $\zeta_N\in K$. As seen in § 1.IV, $SL_2(\mathbf{Z}/N\mathbf{Z})$ naturally acts on $\overline{\mathcal{M}}(\Gamma(N);\mathcal{O}_K)$. It is known that $\overline{\mathcal{M}}(N;\mathcal{O}_K)=\overline{\mathcal{M}}(\Gamma(N);\mathcal{O}_K)^U=H^0(U,\overline{\mathcal{M}}(\Gamma(N);\mathcal{O}_K))$ (e.g. [13, § 1]). Thus for any $\gamma\in SL_2(\mathbf{Z}/N\mathbf{Z})$, the number $a(0,f|\gamma)$ for each $f\in\overline{\mathcal{M}}(N;\mathcal{O}_K)$ only depends on the double coset $U\gamma U$. Thus for each

$(s,z) \in S \times \mathbf{Z}_p^\times$, we may define

$$\delta(f)(s,z) = a(0, f|z\gamma),$$

where $z \in \mathbf{Z}_p^\times$ acts on f as an element of Z_N (§ 1.I) and $s = U\gamma U$ with $\gamma \in SL_2(\mathbf{Z}/N\mathbf{Z})$. Note that the action of $z \in \mathbf{Z}_p^\times$ and $\gamma \in SL_2(\mathbf{Z}/N\mathbf{Z})$ commutes each other, because z has no component on $(\mathbf{Z}/N\mathbf{Z})^\times$. Since the action of $\mathbf{Z}_p^\times$ is continuous under the norm $| \ |_p$, the function $\delta(f) : S \times \mathbf{Z}_p^\times \to \mathcal{O}_K$ is also continuous. Let $C(S \times \mathbf{Z}_p^\times; \mathcal{O}_K)$ be the p-adic Banach space of all continuous function on $S \times \mathbf{Z}_p^\times$ with the uniform norm $\| \phi \| = \operatorname{Sup}_{s,z} |\phi(s,z)|_p$. We let $\mathbf{Z}_p^\times$ act on $C(S \times \mathbf{Z}_p^\times; \mathcal{O}_K)$ by $(\phi|z)(s,x) = \phi(s, zx)$. Then, we have a continuous morphism of $\mathcal{O}_K[[\mathbf{Z}_p^\times]]$-modules $\delta : \overline{\mathcal{M}}(N; \mathcal{O}_K) \to C(S \times \mathbf{Z}_p^\times; \mathcal{O}_K)$. Consider the tower of modular curves : $\to X_1(Np^r) \to X_1(Np^{r-1}) \to \cdots \to X_1(N)$. Then the set of all unramified cusps over S forms a homogeneous space under the action of $\mathbf{Z}_p^\times (\subset Z_N)$; namely, we have a natural isomorphism :

$$\{\text{unramified cusps of } X_1(Np^r)\} \cong (\mathbf{Z}/p^r\mathbf{Z})^\times \times S.$$

Roughly speaking, the set of all cusps on the irreducible component of the reduction of $X_1(Np^r)$ mod p corresponding to the valuation : $f \mapsto |f|_p$ of the modular function field $\mathbf{Q}(X_1(Np^r))$ is exactly the image of the set of unramified cusps on $X_1(Np^r)$. Thus the above map δ is nothing but the evaluation of p-adic modular forms at these cusps, and naturally, we have the following result :

THEOREM 2.2. — *Suppose that $\zeta_N \in K$. Then, we have a natural exact sequence of $\mathcal{O}_K[[\mathbf{Z}_p^\times]]$-modules :*

$$0 \to \overline{S}(N; \mathcal{O}_K) \to \overline{\mathcal{M}}(N; \mathcal{O}_K) \to C(S \times \mathbf{Z}_p^\times; \mathcal{O}_K) \to 0.$$

Moreover, the action of e on $C(S \times \mathbf{Z}_p^\times; \mathcal{O}_K)$ induced by the above exact sequence is the identity action; namely, we have that

$$\overline{\mathcal{M}}(N; \mathcal{O}_K)|(1-e) \subset \overline{S}(N; \mathcal{O}_K).$$

Proof. — Since $\overline{\mathcal{M}}(N; \mathcal{O}_K) = \overline{\mathcal{M}}(N; \mathbf{Z}_p[\zeta_N]) \otimes_{\mathbf{Z}_p[\zeta_N]} \mathcal{O}_K$ and $C(S \times \mathbf{Z}_p^\times; \mathcal{O}_K) = C(S \times \mathbf{Z}_p^\times; \mathbf{Z}_p[\zeta_N]) \otimes_{\mathbf{Z}_p[\zeta_N]} \mathcal{O}_K$, we may suppose that $\mathcal{O}_K = \mathbf{Z}_p[\zeta_N]$. Let $\mathbf{F} = \mathcal{O}_K/p\mathcal{O}_K$, which is a finite field and $\mathcal{O}_K$ is isomorphic to the ring of Witt vectors W of $\mathbf{F}$. Thus, we write W for $\mathcal{O}_K$. We also write for any torsion W-module A

$$\mathcal{M}(\Gamma(N), A) = \overline{\mathcal{M}}(\Gamma(N); W) \otimes_W A, \ S(N; A) = \overline{S}(N; W) \otimes_W A$$

and $\mathcal{M}(N;A) = \overline{\mathcal{M}}(N;W) \otimes_W A$. Put $S' = SL_2(\mathbf{Z}/N\mathbf{Z})/U$, and define, in exactly the same manner as δ, the map

$$\delta' : \overline{\mathcal{M}}(\Gamma(N);W) \to C(S' \times \mathbf{Z}_p^\times; W).$$

We let $SL_2(\mathbf{Z}/N\mathbf{Z})$ acts on $C(S' \times \mathbf{Z}_p^\times; W)$ by $\phi|\gamma)(s,z) = \phi(\gamma s, z)$. Then δ' is equivariant under $SL_2(\mathbf{Z}/N\mathbf{Z})$. Note that

$$\overline{\mathcal{M}}(\Gamma(N);W)^U = \overline{\mathcal{M}}(N;W), C(S' \times \mathbf{Z}_p^\times; W)^U = C(S \times \mathbf{Z}_p^\times; W).$$

As the order of U is prime to p, we have $H^1(U, \operatorname{Ker}(\delta')) = 0$, and therefore the surjectivity of δ follows from that of δ'. We have that

$$\overline{\mathcal{M}}(\Gamma(N);W) = \varprojlim_r \mathcal{M}(\Gamma(N);W_r),$$

where $W_r = W/p^r W$ and $\mathcal{M}(\Gamma(N);W_r) = \overline{\mathcal{M}}(\Gamma(N);W) \otimes_W W_r$. Similarly, $C(S' \times \mathbf{Z}_p^\times; W) = \varprojlim_r C(S' \times \mathbf{Z}_p^\times; W_r)$. If the induced map

$$\delta'_r : \mathcal{M}(\Gamma(N), W_r) \to C(S' \times \mathbf{Z}_p^\times; W_r)$$

is surjective for every r, then the image of δ' is dense and compact, and hence δ' is surjective (as a Banach space under the norm $|\ |_p, \overline{\mathcal{M}}(\Gamma(N), W)$ is compact). The surjectivity of δ'_r follows from that of δ'_1 by Nakayama's lemma. (The finiteness assumption over W_r is not necessary for applying Nakayama's lemma, since the maximal ideal of W_r is nilpotent [2, II.3.2]). Write

$$V_{1,n} = \mathcal{M}(\Gamma(N), \mathbf{F})^{\Gamma_n} \text{ for } \Gamma_n = 1 + p^n\mathbf{Z}_p.$$

Note that we have natural isomorphisms :

$$C(S' \times \mathbf{Z}_p^\times; \mathbf{F}) \cong C(S' \times \mathbf{Z}_p; W) \otimes_W \mathbf{F}$$
$$\text{and } c(S' \times \mathbf{Z}_p^\times; \mathbf{F})^{\Gamma_n} \cong C(S' \times (\mathbf{Z}/p^n\mathbf{Z})^\times; \mathbf{F}).$$

Thus δ'_1 induces $\delta'_{1,n} : V_{1,n} \to C(S' \times (\mathbf{Z}/p^n\mathbf{Z})^\times; \mathbf{F})$. Since $C(S' \times \mathbf{Z}_p^\times; \mathbf{F}) = \varinjlim_n C(S' \times (\mathbf{Z}/p^n\mathbf{Z})^\times; \mathbf{F})$ and $\mathcal{M}(\Gamma(N); \mathbf{F}) = \varinjlim_n V_{1,n}$, the surjectivity of δ'_1 follows from that of $\delta'_{1,n}$ for each n. We are now reduced to prove that $\delta'_{1,n} : V_{1,n} \to C(S' \times (\mathbf{Z}/p^n\mathbf{Z})^\times; \mathbf{F})$ is surjective. For a while, we suppose that $N \geq 3$. Then the ring $V_{1,n}$ for $n \geq 1$ coincides with that defined by Katz in [19, 1.4] under the same symbol. Put $R = V_{1,0} = \mathcal{M}(\Gamma(N), \mathbf{F})^{\mathbf{Z}_p^\times}$ and $S_1 = \operatorname{Spec}(R)$, $T_{1,n} = \operatorname{Spec}(V_{1,n})$. Then R is a Dedekind domain and $T_{1,n}$ is a Galois (étale) covering of S_1 with group $(\mathbf{Z}/p^n\mathbf{Z})^\times$. The evaluation : $R \ni f \mapsto a(0, f|\gamma)$ for each $s = \gamma U$ gives distinct $\mathbf{F}$-valued points of S_1 for each $s \in S'$. We identify S' with the set of these points. By [19, (1.2.1)], each point of S' is decomposed completely in $T_{1,n}$, and the set of all points of $T_{1,n}$ over each $s \in S'$ is isomorphic to $(\mathbf{Z}/p^n\mathbf{Z})^\times$, and the Galois group $\mathrm{Gal}(T_{1,n}/S_1) \cong (\mathbf{Z}/p^n\mathbf{Z})^\times$ acts on this set via the natural permutation [19, (1.2.3)]. Therefore, the set of points of $T_{1,n}$ over S' is isomorphic to $S' \times (\mathbf{Z}/p^n\mathbf{Z})^\times$. By the q-expansion principle, $q^{1/N}$ gives the local parameter of each point of $T_{1,n}$ over S'. Thus the map $\delta'_{1,n} : V_{1,n} \to C(S' \times (\mathbf{Z}/p^n\mathbf{Z})^\times; \mathbf{F})$ coincides with the evaluation of functions in $V_{1,n}$ at the points of $T_{1,n}$ over S'. Since $T_{1,n}$ is étale irreducible over S_1, $V_{1,n}$ is a Dedekind domain. Thus the approximation theorem of Dedekind domains ([2, VII.2.4]) affirms the surjectivity of $\delta'_{1,n}$. This shows the surjectivity of δ when $N \geq 3$. Even when $N = 1$ or 2, under the assumption that $p \geq 5$, we can recover the surjectivity of $\delta'_{1,n}$ by the technique as in [19, § 4] (see also a remark in [13] after Th. 1.1 there). Thus hereafter, we shall not suppose that $N \geq 3$ and prove the triviality of the action of e on $C(S \times \mathbf{Z}_p^\times; W)$. If $p^m \equiv 1 \bmod N$, then $T(p^m)$ commutes with the action of $SL_2(\mathbf{Z}/N\mathbf{Z})$ on $\overline{\mathcal{M}}(N; W)$. In fact, it is known (cf. [27, chap. 3]) that $\Gamma_1(Np)\begin{pmatrix} 1 & 0 \\ 0 & p^m \end{pmatrix}\Gamma_1(Np) = \coprod_{u \bmod p^m} \Gamma_1(Np)\beta_{uN}$, where $\beta_{uN} \in M_2(\mathbf{Z})$ is such that $\det(\beta_{uN}) = p^m$ and $\beta_{uN} \equiv \begin{pmatrix} 1 & uN \\ 0 & p^m \end{pmatrix} \bmod Np^m$. For any $\gamma \in \Gamma_0(p^m), \gamma\beta_{uN}\gamma^{-1}$ satisfies the same condition as above, and thus, $T(p^m)$ commutes with $\overline{\gamma} \in SL_2(\mathbf{Z}/N\mathbf{Z})$ on $\mathcal{M}_k(\Gamma_1(Np); K)$ for all k. This shows the commutativity on $\overline{\mathcal{M}}(N; W)$. Thus, for any $\gamma \in SL_2(\mathbf{Z}/N\mathbf{Z})$,

$$a(n, f|T(p^m)\gamma) = a(n, f|\gamma T(p^m)) = a(np^m, f|\gamma)\ ;$$

especially, $a(0, f|T(p^m)\gamma) = a(0, f|\gamma)$. Thus e acts on $C(S \times \mathbf{Z}_p^\times; W)$ trivially. Now we shall show the exactitude of the sequence at $\overline{\mathcal{M}}(N; W)$. From the triviality of the action of e on $C(S \times \mathbf{Z}_p^\times; W)$, we conclude that (i) the sequence :

$$0 \to \mathrm{Ker}(\delta)|e \xrightarrow{\alpha} \overline{\mathcal{M}}(N; W)|e \xrightarrow{\delta} C(S \times \mathbf{Z}_p^\times; W) \to 0$$

is exact, and (ii) $\mathrm{Ker}(\delta)|(1 - e) = \overline{\mathcal{M}}(N; W)|(1 - e)$. By our argument which proves the surjectivity of δ, the following sequence is also exact : $0 \to ((\mathrm{Ker}(\delta)|e) \otimes_W \mathbf{F})^\Gamma \xrightarrow{\alpha} (\mathcal{M}(N; \mathbf{F})|e)^\Gamma \xrightarrow{\delta} C(S; \mathbf{F}) \to 0$. Note that $(\mathcal{M}(N; \mathbf{F})|e)^\Gamma \cong \bigoplus_{k=3}^{p+1} (\mathcal{M}_k(\Gamma_1(N); \mathbf{F})|e)$ ([13, Th. 4.2]), where

$\mathcal{M}_k(\Gamma_1(N);\mathbf{F}) = \mathcal{M}_k(\Gamma_1(N);W) \otimes_W \mathbf{F}$. Thus, if $f \in (\mathcal{M}(N;\mathbf{F})|e)^\Gamma$ is in the kernel of δ, f is an element of $\bigoplus_{k=3}^{p+1} S_k(\Gamma_1(N);\mathbf{F})$ and hence is a cusp form. Thus $((\mathrm{Ker}(\delta)|e) \otimes_W \mathbf{F})^\Gamma \subset (S(N;\mathbf{F})|e)^\Gamma$. The other inclusion $(S(N;\mathbf{F})|e)^\Gamma \subset ((\mathrm{Ker}(\delta)|e) \otimes_W \mathbf{F})^\Gamma$ is obvious and thus

$$(*) \qquad (S(N;\mathbf{F})|e)^\Gamma = ((\mathrm{Ker}(\delta)|e) \otimes_W \mathbf{F})^\Gamma.$$

Let X be the Pontryagin dual module of $(\mathrm{Ker}(\delta)|e) \otimes_W K/W$, and write r for $\dim_{\mathbf{F}}(S(N;\mathbf{F})|e)^\Gamma$. Since $S(N;K/W)|e \hookrightarrow (\mathrm{Ker}(\delta)|e) \otimes_W K/W$, we have a natural surjection : $X \twoheadrightarrow \mathbf{h}^o(N;W)$ by [13, Th. 2.2]. By (*), we have a surjection : $\Lambda_K^r \twoheadrightarrow X$. Note that $\mathbf{h}^o(N;W)$ is Λ_K-free of rank r. Thus $X \simeq \mathbf{h}^o(N;W)$ and by duality we have

$$S(N;K/W)|e = (\mathrm{Ker}(\delta)|e) \otimes_W K/W.$$

This implies that $\overline{S}^o(N;W) = \mathrm{Ker}(\delta)|e$. Thus what we have to show is that $S(Np;W)|(1 - T(p^m))$ is dense in $\mathcal{M}(Np;W)|(1 - T(p^m))$. Take $f \in \mathcal{M}(Np;W)|(1-T(p^m))$. Then for any $\gamma \in \Gamma_0(p)$, $a(0,f|\gamma) = 0$. In fact, if $s = \gamma(\infty) \in \mathbf{P}^1(\mathbf{Q})$, then we can choose $\gamma' \in \Gamma_0(p^m)$ such that $\gamma'(\infty)$ and s are $\Gamma_0(Np)$-equivalent. Thus we may suppose that $\gamma \in \Gamma_0(p^m)$. Then as already seen, $g|\gamma \circ T(p^m) = g|T(p^m) \circ \gamma$ for any $g \in \mathcal{M}(Np;W)$, and therefore, we have that

$$a(0,g|\gamma) = a(0,g|\gamma T(p^m)) = a(0,g|T(p^m)\gamma).$$

Thus, if $f = g|(1-T(p^m))$, then $a(0,f|\gamma) = 0$. To construct a series of cusp forms convergent to f, we use an argument of Serre [26, § 3]. Let $r = p^n(p-1)$ and put

$$E_r = 1 - \frac{2r}{B_r}\sum_{n=1}^{\infty}\left(\sum_{0<d|n} d^{r-1}\right) q^n \in \mathcal{M}_r(SL_2(\mathbf{Z});\mathbf{Z}_p)$$

and

$$G_r = E_r - p^r E_r|[p] \in \mathcal{M}_r(\Gamma_0(p);\mathbf{Z}_p).$$

Then $G_r \equiv 1 \bmod p^{n+1}$ and $a(0,G_r|\gamma) = 0$ for all $\gamma \in \Gamma_0(p)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Gamma_0(p)$. Note that $\Gamma_0(p)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Gamma_0(p) = SL_2(\mathbf{Z}) - \Gamma_0(p)$. Then we see easily that $\lim_{r\to\infty} G_r f = f$ and $G_r f \in S(Np;W)$. This finishes the proof.

For each positive integer M, we denote by $\imath_M$ the identity Dirichlet character modulo M; thus

$$\imath_M(n) = \begin{cases} 1 & \text{if} \quad (n,M)=1, \\ 0 & \text{if} \quad (n,M)>1. \end{cases}$$

COROLLARY 2.3. — *For any character χ modulo p^r, we have that $\overline{M}(N;\mathcal{O}_K)|\chi \subset \overline{S}(N;\mathcal{O}_K)$ and $d(\overline{\mathcal{M}}(N;\mathcal{O}_K)) \subset \overline{S}(N;\mathcal{O}_K)$ for $d = q\dfrac{d}{dq}$. Especially, one has that*

$$\overline{\mathcal{M}}(N;\mathcal{O}_K)|\imath_p \subset \overline{S}(N;\mathcal{O}_K).$$

Proof. — One sees easily from definition that $e \circ d = 0$ (e.g. [11, (6.12)]) and $T(p) \circ \chi = 0$. This shows that result by Theorem 2.2.

PROPOSITION 2.4. — *Let $\chi : (\mathbf{Z}/p^r\mathbf{Z})^\times \to \mathcal{O}_K^\times$ be a character. Then we have that $e(f(g|\chi)) = \chi(-1)e(g(f|\chi))$ and $e(fdg) = -e(gdf)$ for $d = q\dfrac{d}{dq}$.*

Proof. — The second assertion follows from the fact that $e \circ d = 0$ and $d(fg) = fdg + gdf$. For any $n \geq 0$, we see that

$$\begin{aligned} a(np^r, f(g|\chi)) &= \sum_{i+j=np^r} \chi(j)a(i,f)a(j,g) \\ &= \chi(-1) \sum_{i+j=np^r} \chi(i)a(i,f)a(j,g) \quad (j \equiv -i \bmod p^r) \\ &= \chi(-1)a(np^r, g(f|\chi)). \end{aligned}$$

Thus we know that

$$((f(g|\chi)) - \chi(-1)(g(f|\chi)))|e = ((f(g|\chi)) - \chi(-1)(g(f|\chi)))|T(p^r) = 0.$$

3. A generalization of p-adic measure theory.

Let X be a product of a finite set and several copies of $\mathbf{Z}_p$. We call this type of topological space a p-adic space. Let $C(X;\mathcal{O}_K)$ be the (compact) p-adic Banach space of all continuous functions : $X \to \mathcal{O}_K$ with the norm : $\| \phi \| = \operatorname{Sup}_{x \in X} |\phi(x)|_p$. Put

$$\mathrm{Meas}(X;\mathcal{O}_K) = \mathrm{Hom}_{\mathcal{O}_K}(C(X;\mathcal{O}_K),\mathcal{O}_K).$$

We may define a norm on $\mathrm{Meas}(X;\mathcal{O}_K)$ by

$$\| \varphi \| = \sup_{\|\varphi\|=1} |\varphi(\phi)|_p \quad (\phi \in C(X;\mathcal{O}_K)).$$

Then $\mathrm{Meas}(X;\mathcal{O}_K)$ becomes a compact p-adic Banach space, and $C(X;\mathcal{O}_K) \cong \mathrm{Hom}_{\mathcal{O}_K}(\mathrm{Meas}(X;\mathcal{O}_K),\mathcal{O}_K)$ naturally, if X is a p-adic group, $\mathrm{Meas}(X;\mathcal{O}_K)$ becomes a Banach algebra under the convolution product. Let us fix a topological generator $\imath(u)$ of Γ with u in $\mathbf{Z}_p$ and identify Λ_K with $\mathcal{O}_K[[X]]$ by the correspondence : $\imath(u) \mapsto 1+X$. Then, it is well known that if μ is an element of $\mathrm{Meas}(\Gamma;\mathcal{O}_K)$, there exists an element F of $\Lambda_K = \mathcal{O}_K[[X]]$ such that $\int_\Gamma x^s d\mu = F(u^s - 1)$ for all $s \in \mathbf{Z}_p$. The map $\mu \mapsto F$ is an isomorphism of Banach spaces between $\mathrm{Meas}(\Gamma;\mathcal{O}_K)$ and Λ_K (e.g. [21, Chap. 4]).

We identify $\mathcal{X}(\mathcal{I};\mathcal{O}_K) = \mathrm{Hom}_{\mathcal{O}_K-\mathrm{alg}}(\mathcal{I},\mathcal{O}_K)$ with a subset of $\mathrm{Spec}(\mathcal{I})$ by $P \mapsto \mathrm{Ker}(P)$; so, $P \in \mathcal{X}(\mathcal{I};\mathcal{O}_K)$ can be considered as an $\mathcal{O}_K$-algebra homomorphism as well as a prime ideal of $\mathcal{I}$. Then $P \in \mathcal{X}(\Lambda_K;\mathcal{O}_K)$ is a prime ideal of height 1 and hence is principal, because Λ_K is a unique factorization domain. Thus we can find a generator F of P. By the Weierstrass preparation theorem (e.g. [21, Chap. 5 Th. 2.2]), we can find a distinguished polynomial $\Phi(X) \in \mathcal{O}_K[X]$ such that $P = (\Phi) = (F)$. Since P is prime, Φ must be irreducible and $\mathcal{O}_K[X]/(\Phi) \cong \mathcal{O}_K$. Thus Φ is of the form : $X - x$ with $x \in \mathcal{O}_K$ and $|x|_p < 1$.

Let $\mathcal{X}(\Lambda_K;\mathcal{O}_K)$ be the set of all $\mathcal{O}_K$-valued points of $\mathrm{Spec}(\Lambda_K)$. Then, we have the isomorphism :

$$\mathcal{X}(\Lambda_K;\mathcal{O}_K) \cong \{x \in \mathcal{O}_K \big| |x|_p < 1\}.$$

Therefore, to give a measure on Γ amounts to give an algebraic function on $\mathcal{X}(\Lambda_K;\mathcal{O}_K)$ induced by its structure sheaf Λ_K. Let $\mathcal{X}_{\mathrm{alg}}(\Lambda_K;\mathcal{O}_K)$ be the subset of all points in $\mathcal{X}(\Lambda_K;\mathcal{O}_K)$ of the form $P_{k,\varepsilon}$ of weight $k \geq 0$ and of finite order character $\varepsilon : \Gamma \to \mathcal{O}_K^\times$.

Let $\mathcal{L}_K$ denote the quotient field of Λ_K, and let $\mathcal{K}$ be a finite extension of $\mathcal{L}_K$. Let $\mathcal{I}$ be the integral closure of Λ_K in $\mathcal{K}$. We say that an ideal P of $\mathcal{I}$ is a *prime divisor* if P is a prime ideal of height 1. An ideal $\mathfrak{a}$ of $\mathcal{I}$ is called a divisor if $\mathfrak{a}$ is an intersection of $\bigcap_P P^{e(P)}$ of finitely many prime divisors. For the general theory of divisors, we refer to [2, Chap. 7].

Lemma 3.1. — *Let $\mathcal{M}/\mathcal{K}$ is a finite extension and $\mathcal{J}$ be the integral closure of $\mathcal{I}$ in $\mathcal{M}$. Then $\mathcal{J}$ is $\mathcal{I}$-reflexive; in particular, $\mathcal{J}$ is Λ_K-free of finite* *rank. Moreover, if there is a prime divisor P of $\mathcal{I}$ such that $\mathcal{I}/P\mathcal{I} \cong \mathcal{O}_K$ and $\mathcal{J}/P\mathcal{J}$ is $\mathcal{I}_K$-flat, then $\mathcal{J}$ is free of finite rank over $\mathcal{I}$.*

Proof. — Put $\tilde{\mathcal{J}} = \bigcap_P \mathcal{J}_P$, where the intersection is taken in the field, $\mathcal{M}$, P runs over all prime divisors of Λ_K and $\mathcal{J}_P$ is the localization of $\mathcal{J}$ at P. Then $\tilde{\mathcal{J}}$ is reflexive and finite over $\mathcal{I}$. Since $\mathcal{J}$ is integrally closed, we know that $\mathcal{J} = \tilde{\mathcal{J}}$. Any reflexive Λ_K-module of finite type is free, since Λ_K is regular of dimension 2. Let P be the prime divisor of $\mathcal{I}$ as in the last assertion. By assumption, $\mathcal{J}/P\mathcal{J}$ is flat over $\mathcal{O}_K \cong \mathcal{I}/P\mathcal{I}$, and thus we can find a basis $\overline{x}_1, \ldots, \overline{x}_r$ of $\mathcal{J}/P\mathcal{J}$ over $\mathcal{O}_K$. Let $x_1, \ldots, x_r \in \mathcal{J}$ be elements such that $x_i \bmod P\mathcal{J} = \overline{x}_i$. By Nakayama's lemma, we have a surjective morphism of $\mathcal{I}$-modules $\varphi : \mathcal{I}^r \to \mathcal{J}$ given by $(a_1, \ldots, a_r) \to a_1x_1 + \ldots + a_rx_r$. From the first assertion, we know that $\mathcal{J}$ is free of the same rank as $\mathcal{I}^r$ over Λ_K. Thus φ must be an isomorphism.

We shall suppose that

(3.1a) *the algebraic closure of $\mathbf{Q}_p$ inside $\mathcal{K}$ coincides with K.*

If (3.1a) is satisfied, we say that $\mathcal{K}$ is defined over K. We put

$$\mathcal{X}(\mathcal{I};\mathcal{O}_K) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{I},\mathcal{O}_K)$$
$$\mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K) = \{P \in \mathcal{X}(\mathcal{I};\mathcal{O}_K) \big| P|_{\Lambda_K} \in \mathcal{X}_{\mathrm{alg}}(\Lambda_K;\mathcal{O}_K)\}.$$

For each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$, we can write $P|_{\Lambda_K} = P_{k,\varepsilon}$ for an integer $k \geq 0$ and a finite order character $\varepsilon : \Gamma \to \mathcal{O}_K^\times$. We write $k = k(P)$ and $\varepsilon = \varepsilon_P$, and we define an integer $r(P) \geq 1$ by $\mathrm{Ker}(\varepsilon_P) = \Gamma_{r(P)} = 1 + p^{r(P)}\mathbf{Z}_p$.

LEMMA 3.2. — *For a given integer n, the number of extensions of K of degree n inside $\overline{\mathbf{Q}}_p$ is finite.*

The proof is easy and is left to the reader.

Write $d(\mathcal{I})(= d(\mathcal{K}))$ for the rank of $\mathcal{I}$ over Λ_K. If we take the composite M of all extensions of K of degree $d(\mathcal{I})!$ and if we write $\mathcal{J} = \mathcal{I} \otimes_{\mathcal{O}_K} \mathcal{O}_M = \mathcal{I} \otimes_{\Lambda_K} \Lambda_M$, then any point $P \in \mathcal{X}(\Lambda_K;\mathcal{O}_K)$ is an image of at least one point of $\mathcal{X}(\mathcal{J};\mathcal{O}_M)$. Thus, by extending scalars, we may assume that

(3.1b) *$\mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ is Zariski dense* (i.e. *has infinitely many points*) *in the space of $\overline{\mathbf{Q}}_p$-valued points of* $\mathrm{Spec}(\mathcal{I})$.

Terminology. — Let $\mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I}$ be the profinite completion of $\mathrm{Meas}(X;\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{I}$. An element of $\mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K} \mathcal{I}$ will be called a generalized measure on $X \times \mathcal{X}(\mathcal{I};\mathcal{O}_K)$.

When $\mathcal{I} = \Lambda_K$, the space $\mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\Lambda_K$ can be naturally identified with the measure space $\mathrm{Meas}(X \times \Gamma;\mathcal{O}_K)$. We may define a p-adic norm on $\mathcal{I}$ by $\| \varphi \| = \| \mathcal{N}_{\mathcal{K}/\mathcal{L}_K}(\varphi) \|$, where $\mathcal{N}_{\mathcal{K}/\mathcal{L}_K} : \mathcal{I} \to \Lambda_K$ is the norm map. Then $\mathcal{I}$ becomes an $\mathcal{O}_K$-Banach algebra under this norm. As a linear topological space, $\mathcal{I}$ is isomorphic to Λ_K^d for $d = d(\mathcal{I})$, and thus as a linear topological space, $\mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I} \cong \mathrm{Meas}(X \times \Gamma;\mathcal{O}_K)^d$.

Let Φ be a generalized measure on $X \times \mathcal{X}(\mathcal{I};\mathcal{O}_K)$. Let $P \in \mathcal{X}(\mathcal{I};\mathcal{O}_K)$ and $\lambda_P : \mathcal{I} \to \mathcal{O}_K$ be the corresponding $\mathcal{O}_K$-algebra homomorphism. Then we have an induced continuous morphism :

$$\mathrm{id} \otimes \lambda_P : \mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I} \to \mathrm{Meas}(X;\mathcal{O}_K) \qquad (\cong \mathrm{Meas}(X;\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{I}/P\mathcal{I}).$$

We shall write $\Phi_P \in \mathrm{Meas}(X;\mathcal{O}_K)$ for the image of Φ under $\mathrm{id.}\otimes\lambda_P$.

LEMMA 3.3. — *Let $\mathcal{X}$ be a subset of $\mathcal{X}(\mathcal{I};\mathcal{O}_K)$, and suppose that $\mathcal{X}$ is Zariski dense in the space of all $\overline{\mathbf{Q}}_p$-valued points of* $\mathrm{Spec}(\mathcal{I})$. *Then Φ is uniquely determined by its values Φ_P at $P \in \mathcal{X}$.*

Proof. — For each $\phi \in C(X;\mathcal{O}_K)$, $\phi : \mathrm{Meas}(X;\mathcal{O}_K) \to \mathcal{O}_K$ defined by $\varphi \to \varphi(\phi)$ ($\varphi \in \mathrm{Meas}(X;\mathcal{O}_K)$) gives a continuous linear form. Then $\phi \otimes \mathrm{id} : \mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I} \to \mathcal{I}$ is a continuous morphism. Note that

$$\lambda_P(\phi \otimes \mathrm{id}(\Phi)) = \phi \otimes \lambda_P(\Phi) = \int_X \phi d\Phi_P.$$

Thus $\phi \otimes \mathrm{id}(\Phi) \in \mathcal{I}$ is determined by its values on $\mathcal{X}$, since $\mathcal{X}$ is Zariski dense. The lemma is obviously true when $\mathcal{I} = \Lambda_K$. If we fix an isomorphism : $\mathcal{I} \cong \Lambda_K^d$, then it induces an isomorphism : $\mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I} \cong \mathrm{Meas}(X \times \Gamma;\mathcal{O}_K)^d$. We consider the commutative diagram :

$$\begin{array}{rcl} \phi \otimes \mathrm{id} : & \mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I} & \to \mathcal{I} \\ & \wr\,\| & \quad \wr\,\| \\ & \mathrm{Meas}(X \times \Gamma;\mathcal{O}_K)^d & \to \Lambda_K^d. \end{array}$$

In the lower line, it is obvious that if $\phi \otimes \mathrm{id}(\Phi) = \phi \otimes \mathrm{id}(\Phi')$ for all $\phi \in C(X;\mathcal{O}_K)$, then $\Phi = \Phi'$. This shows the lemma.

We may extend our theory by replacing $\mathcal{I}$ by $\mathcal{I}\hat{\otimes}_{\mathcal{O}_K}\mathcal{J}$, where $\mathcal{J}$ is the integral closure of Λ_K in another extension $\mathcal{M}/\mathcal{L}_K$ satisfying (3.1a,b). In this case, Lemma 3.3 can be stated as follows :

Let $\mathcal{X}$ and $\mathcal{X}'$ be Zariski dense subsets of $\mathcal{X}(\mathcal{I};\mathcal{O}_K)$ and $\mathcal{X}(\mathcal{J} : \mathcal{O}_K)$ respectively. Then $\Phi \in \mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I}\hat{\otimes}_{\mathcal{O}_K}\mathcal{J}$ is uniquely determined *by its values* $\Phi_{P,Q}$ *at* $(P,Q) \in \mathcal{X} \times \mathcal{X}'$, *where*

$$\Phi_{P,Q} = \mathrm{id} \otimes \lambda_P \otimes \lambda_Q(\Phi).$$

We leave the proof of this generalized version to the reader.

4. Modules of congruences.

In this section, we shall generalize the theory developed in [13, § 3] and [14, § 1 and 10] for modules of congruences over Λ_K to those over an arbitrary normal finite extension $\mathcal{I}$ of Λ_K. Let $\mathcal{K}$ be a finite extension over $\mathcal{L}_K$ and $\mathcal{I}$ be the integral closure of Λ_K in $\mathcal{K}$. We shall suppose that K is sufficiently large so that

(4.1a) $\mathcal{K}$ *is defined over* K;

(4.1b) $\mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ *is Zariski dense in* $\mathrm{Spec}(\mathcal{I})(\overline{\mathbf{Q}}_p)$.

Let N be a positive integer prime to p, and we shall consider the ordinary part $\mathbf{h}^o(N;\mathcal{O}_K)$ of the Hecke algebra defined in § 1 and 2. We shall consider a base change over Λ_K to $\mathcal{I}$; namely, we shall deal with the $\mathcal{I}$-algebra : $\mathbf{h}^o(N;\mathcal{O}_K)\otimes_{\Lambda_K}\mathcal{I}$, which is canonically isomorphic to $\mathbf{h}^o(N;\mathbf{Z}_p)\otimes_{\Lambda}\mathcal{I}$, where $\Lambda = \Lambda_{\mathbf{Q}_p}$.

Let $\lambda : \mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$ be an $\mathcal{I}$-algebra homomorphism. The $\mathcal{O}_K[[Z_N]]$-algebra structure on $\mathbf{h}^o(N;\mathcal{O}_K)$ induces a homomorphism of groups from the subgroup $(\mathbf{Z}/Np\mathbf{Z})^\times$ of Z_N into the unit group of $\mathbf{h}^o(N;\mathcal{O}_K)$. This combined with λ gives a character $\psi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}_K^\times$, which we shall call the character of λ. It is known that ψ is even, i.e. $\psi(-1) = 1$ [14, Cor. 1.6]. By definition for each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$, we have $\mathcal{I}/P\mathcal{I} \cong \mathcal{O}_K$. For $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$, we consider the reduction of λ mod P :

$$\lambda_P : \mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} (\mathcal{I}/P\mathcal{I}) \to \mathcal{I}/P\mathcal{I} \cong \mathcal{O}_K.$$

If the weight $k(P)$ of P is greater than or equal to 2 and if the character ε_P of P has conductor $p^{r(P)}$, we have by Th. 2.1,

$$\mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} (\mathcal{I}/P\mathcal{I}) \cong \mathbf{h}^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P;\mathcal{O}_K),$$

whence an $\mathcal{O}_K$-algebra homomorphism denoted by the same symbol :

$$\lambda_P : \mathbf{h}_{k(P)}(\Phi_{r(P)}, \varepsilon_P;\mathcal{O}_K) \to \mathcal{O}_K.$$

By the duality theorem (Prop. 1.2), we can find a unique normalized eigenform $f_P \in S_{k(P)}(\Gamma_0(Np^{r(P)}), \varepsilon_P\psi\omega^{-k(P)})$ such that $f_P|T(n) =$

$\lambda_P(T(n))f_P$ for all $n \geq 0$. This form will be called the *ordinary form belonging to* λ at $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$. We say that $\mathcal{K}$ is a *splitting* field of $\mathbf{h}^o(N;\mathcal{O}_K)$ if for any minimal prime ideal $\mathfrak{p}$ of $\mathbf{h}^o(N;\mathcal{O}_K), \mathbf{h}^o(N;\mathcal{O}_K)/\mathfrak{p}$ can be embedded isomorphically into $\mathcal{K}$ as Λ_K-algebras.

The following fact is known :

THEOREM 4.1. — *Let* $\lambda : \mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$ *be an* $\mathcal{I}$*-algebra homomorphism. Then the following two statements are equivalent :*

(i) *there exists* $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ *with* $k(P) \geq 2$ *such that* f_P *is primitive of conductor* $Np^{r(P)}$;

(ii) f_P *is primitive for every* P *with* $k(P) \geq 2$ *such that the p-part of* $\varepsilon_P\psi\omega^{-k(P)}$ *is non-trivial.*

Moreover, suppose that the above equivalent conditions are satisfied by λ, *and let* $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ *be a point such that the p-part of* $\varepsilon_P\psi\omega^{-k(P)}$ *is trivial. Then either* f_P *is primitive and* $k(P) = 2$ *or there exists a primitive form* f *in* $S_{k(P)}(\Gamma_0(N), \psi\omega^{-k(P)})$ *such that* $|a(p,f)|_p = 1$ *and* $f_P = f - \alpha f|[p]$, *where* α *is the non-unit p-adic root of the equation :* $X^2 - a(p,f)X + (\psi\omega^{-k(P)})_0(p)p^{k(P)-1} = 0$ *for the primitive character* $(\psi\omega^k)_0$ *associated with* $\psi\omega^{-k}$. *Conversely, suppose that* $\mathcal{K}$ *is a splitting field of* $\mathbf{h}^o(N;\mathcal{O}_K)$. *Then if* $f \in S_k^o(\Phi_r, \varepsilon; \mathcal{O}_K)$ *is primitive or is associated with a primitive form in* $S_k(\Gamma_1(N);\mathcal{O}_K)$ *in the above manner, then* f *belongs to a unique homomorphism :* $\mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$ *satisfying* (i) *and* (ii).

When λ satisfies one of the equivalent conditions (i) and (ii), we say that λ is *primitive.*

Proof. — Since λ induces a Λ_K-algebra homomorphism

$$\check{\lambda} : \mathbf{h}^o(N;\mathcal{O}_K) \to \mathcal{I}$$

and we have, $\check{\lambda} : \mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{L}_K \to \mathcal{K}$, we may suppose that $\mathcal{K}$ is a surjective image of $\check{\lambda}$. Then f_P belongs to $\check{\lambda}$ in the sense of [14, (1.11)] and the theorem follows from [14, Cor. 1.3].

Suppose that λ is primitive. For each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ with $k(P) \geq 2$, we have by Th. 4.1 a K-algebra decomposition :

$$\mathbf{h}_k(\Phi_{r(P)}, \varepsilon_P; K) = K_P \oplus A_P, K_P \cong K \tag{4.2}$$

such that the first projection is induced by λ_P. Let $\mathbf{h}(K_P)$ and $\mathbf{h}(A_P)$ be the projections of $\mathbf{h}_k(\Phi_{r(P)}, \varepsilon_P; \mathcal{O}_K)$ in K_P and A_P, respectively. We shall define the module of congruences for f_P (or λ_P) by

$$C(f_P) = C(\lambda_P) = (\mathbf{h}(K_P) \oplus \mathbf{h}(A_P))/\mathbf{h}_k(\Phi_{r(P)}, \varepsilon_P; \mathcal{O}_K).$$

The above proof of Th. 4.1 combined with [13, Cor. 3.3] and [14, Cor. 1.4] shows

THEOREM 4.2. — *Let $\lambda : \mathbf{h}^o(N, \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$ be a primitive homomorphism of $\mathcal{I}$-algebras. Then λ induces a decomposition of $\mathcal{K}$-algebras :*

$$\mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{K} \cong \mathcal{K} \oplus \mathcal{A}$$

such that the projection of $\mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$ into the first factor coincides with λ. Moreover, let $\mathbf{h}(\mathcal{K})$ and $\mathbf{h}(\mathcal{A})$ be the images of $\mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$ in $\mathcal{K}$ and $\mathcal{A}$, respectively. Then, the diagonal inclusion : $\mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathbf{h}(\mathcal{K}) \oplus \mathbf{h}(\mathcal{A})$ induces an isomorphism for each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_K)$ with $k_P \geq 2$:

$$\mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}_P \cong \mathbf{h}(\mathcal{K})_P \oplus \mathbf{h}(\mathcal{A})_P,$$

where the subscript "P" indicates the localization at P. In particular, $\mathbf{h}(K_P) \cong \mathbf{h}(\mathcal{K})/P\mathbf{h}(\mathcal{K})$ and $A_P = \mathbf{h}(\mathcal{A})_P/P\mathbf{h}(\mathcal{A})_P$.

With the same notation and assumption as in the theorem, put

$$\tilde{\mathbf{h}}(\mathcal{A}) = \bigcap_P \mathbf{h}(\mathcal{A})_P, \tag{4.3a}$$

where the intersection is taken in $\mathcal{A}$ and where P runs over all prime divisors of $\mathcal{I}$,

$$\text{(4.3b)}\quad \mathcal{C}_0(\lambda; \mathcal{I}) = (\mathbf{h}(\mathcal{K}) \oplus \mathbf{h}(\mathcal{A}))/(\mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}),$$
$$\mathcal{C}(\lambda; \mathcal{I}) = (\mathbf{h}(\mathcal{K}) \oplus \tilde{\mathbf{h}}(\mathcal{A}))/(\mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}),$$
$$\text{(4.3c)}\quad \mathcal{N}_s(\lambda; \mathcal{I}) = \tilde{\mathbf{h}}(\mathcal{A})/\mathbf{h}(\mathcal{A}) \cong \mathcal{C}(\lambda; \mathcal{I})/\mathcal{C}_0(\lambda; \mathcal{I}).$$

Then we have the following result (for the definition of the pseudo-nullity, see [2, VII]) :

THEOREM 4.3. — *Let $\mathcal{M}$ be a finite extension of $\mathcal{K}$ and $\mathcal{J}$ be the integral closure of $\mathcal{I}$ in $\mathcal{M}$. Suppose λ to be primitive, and let*

$$\lambda \otimes \mathrm{id} : \mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{J} (= \mathbf{h}^o(N : \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \otimes_{\mathcal{I}} \mathcal{J}) \to \mathcal{J}.$$

Then we have the following assertions :

(4.4a) *$\mathcal{N}_s(\lambda; \mathcal{I})$ is a pseudo-null $\mathcal{I}$-module;*

(4.4b) $\mathcal{C}_0(\lambda \otimes \mathrm{id}; \mathcal{J}) = \mathcal{C}_0(\lambda; \mathcal{I}) \otimes_{\mathcal{I}} \mathcal{J}$;

(4.4c) *$\mathcal{C}_0(\lambda; \mathcal{I}) \cong \mathcal{I}/\mathfrak{a}$ with a non-zero divisor $\mathfrak{a}$ of $\mathcal{I}$.*

Proof. — The assertion (4.4a,b) is obvious from the definition. For simplicity, we shall write $L = \mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$. For (4.4c), we have by definition that $L + \mathbf{h}(\mathcal{K}) = \mathbf{h}(\mathcal{K}) \otimes \mathbf{h}(\mathcal{A})$ and thus

$$\mathcal{C}_0(\lambda;\mathcal{I}) = L + \mathbf{h}(\mathcal{K})/L \cong \mathbf{h}(\mathcal{K})/L \cap \mathbf{h}(\mathcal{K}).$$

Note that $\mathcal{I} \cong \mathbf{h}(\mathcal{K})$ canonically and $\mathfrak{a} = L \cap \mathbf{h}(\mathcal{K}) = L \cap \mathcal{K}$ is reflexive (cf. [2, VII.4 Prop. 6]). Thus $\mathfrak{a}$ is a non-zero divisor.

THEOREM 4.4. — *Let R be a local ring of $\mathbf{h}^o(N;\mathcal{O}_K)$ through which λ factors. Then the following two conditions are equivalent :*

(4.5a) $R \cong \mathrm{Hom}_{\Lambda_K}(R,\Lambda_K)$ *as R-modules;*

(4.5b) $R \otimes_{\Lambda_K} \mathcal{I} \cong \mathrm{Hom}_{\mathcal{I}}(R \otimes_{\Lambda_K} \mathcal{I},\mathcal{I})$ *as $R \otimes_{\Lambda_K} \mathcal{I}$-modules.*

Moreover, if we suppose one of the above equivalent conditions and that λ is primitive, then we have

(4.6a) $\mathcal{N}_s(\lambda;\mathcal{I}) = 0$ *and* $\mathcal{C}_0(\lambda;\mathcal{I}) = \mathcal{C}(\lambda;\mathcal{I})$;

(4.6b) $\mathcal{C}_0(\lambda;\mathcal{I}) \cong \mathcal{I}/H\mathcal{I}$ *for a non-zero element H of $\mathcal{I}$;*

(4.6c) *For each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ with $k(P) \geq 2$, we have*

$$\mathcal{C}_0(\lambda;\mathcal{I}) \otimes_{\mathcal{I}} \mathcal{I}/P\mathcal{I} \cong C(f_P) \textit{ canonically.}$$

Proof. — Note that as $R \otimes_{\Lambda_K} \mathcal{I}$-module,

$$\mathrm{Hom}_{\mathcal{I}}(R \otimes_{\Lambda_K} \mathcal{I},\mathcal{I}) \cong \mathrm{Hom}_{\Lambda_K}(R,\mathrm{Hom}_{\mathcal{I}}(\mathcal{I},\mathcal{I})) = \mathrm{Hom}_{\Lambda_K}(R,\mathcal{I}).$$

Since the Λ_K-module $\mathcal{I}$ is free of finite rank, we see that

$$\mathrm{Hom}_{\Lambda_K}(R,\mathcal{I}) \cong \mathrm{Hom}_{\Lambda_K}(R,\Lambda_K) \otimes_{\Lambda_K} \mathcal{I}.$$

Thus the implication (4.5a) $\Rightarrow$ (4.5b) is obvious; so, we shall suppose (4.5b) and show (4.5a). Since $\mathcal{I}$ is Λ_K-free, we have an isomorphism of R-modules for $d = d(\mathcal{I})$

$$\begin{aligned}\varphi : R^d \otimes_{\Lambda_K} \mathcal{I} &\cong \mathrm{Hom}_{\Lambda_K}(R \otimes_{\Lambda_K} \mathcal{I},\mathcal{I}) \\ &\cong \mathrm{Hom}_{\Lambda_K}(R,\Lambda_K) \otimes_{\Lambda_K} \mathcal{I} \cong \mathrm{Hom}_{\Lambda_K}(R,\Lambda_K)^d.\end{aligned}$$

We write this isomorphism matricially as (φ_{ij}) with homomorphisms of R-modules $\varphi_{ij} : R \to \mathrm{Hom}_{\Lambda_K}(R,\Lambda_K)$. Let $\mathfrak{m}$ be the maximal ideal of R. If $\varphi_{ij}(R) \subset \mathfrak{m}\mathrm{Hom}_{\Lambda_K}(R,\Lambda_K)$ for all i and j, then

$$\varphi(R^d) \subset \mathfrak{m}(\mathrm{Hom}_{\Lambda_K}(R,\Lambda_K)^d).$$

This contradicts to the surjectivity of φ. Thus we can find at least one pair (i,j) such that φ_{ij} is surjective. Since R is Λ_K-free of finite rank by Th. 2.1, we conclude that φ_{ij} is an isomorphism by comparing the rank of R and $\mathrm{Hom}_{\Lambda_K}(R,\Lambda_K)$ over Λ_K; so, $R \cong \mathrm{Hom}_{\Lambda_K}(R,\Lambda_K)$ as R-modules and we have proven the implication (4.5b) $\Rightarrow$ (4.5a). Now, assuming one of (4.5a,b), we shall prove (4.6a). By Th. 4.2, λ induces a decomposition of $\mathcal{K}$-algebras; $R \otimes_{\Lambda_K} \mathcal{K} \cong \mathcal{K} \oplus \mathcal{B}$. Let $R(\mathcal{K})$ and $R(\mathcal{B})$ be the image of R in $\mathcal{K}$ and $\mathcal{B}$. Then, by definition, it is plain that

$$C_0(\lambda;\mathcal{I}) = (R(\mathcal{K}) \oplus R(\mathcal{B}))/(R \otimes_{\Lambda_K} \mathcal{I})$$
$$\text{and } C(\lambda;\mathcal{I}) = (R(\mathcal{K}) \oplus \tilde{R}(\mathcal{B}))/(R \otimes_{\Lambda_K} \mathcal{I}),$$

where $\tilde{R}(\mathcal{B}) = \bigcap_P R(\mathcal{B})_P$ (the intersection is taken over all prime divisors P of $\mathcal{I}$). We have an exact sequence of $R \otimes_{\Lambda_K} \mathcal{I}$-modules : $0 \to \mathrm{Ker}(\Lambda) \to R \otimes_{\Lambda_K} \mathcal{I} \to R(\mathcal{K}) \to 0$. Note that $R(\mathcal{K}) \cong \mathcal{I}$. Thus this sequence is split exact (non-canonically) as that of $\mathcal{I}$-modules and thus $\mathrm{Ker}(\lambda)$ is $\mathcal{I}$-free. By the duality, we have another exact sequence :

$$0 \to \mathrm{Hom}_{\mathcal{I}}(R(\mathcal{K}),\mathcal{I}) \to \mathrm{Hom}_{\mathcal{I}}(R \otimes_{\Lambda_K} \mathcal{I},\mathcal{I}) \to \mathrm{Hom}_{\mathcal{I}}(\mathrm{Ker}(\lambda),\mathcal{I}) \to 0.$$

If we identify $\mathrm{Hom}_{\mathcal{I}}(R \otimes_{\Lambda_K} \mathcal{I},\mathcal{I})$ with $R \otimes_{\Lambda_K} \mathcal{I}$ by (4.5b), we know that $R(\mathcal{B}) \cong \mathrm{Hom}_{\mathcal{I}}(\mathrm{Ker}(\lambda),\mathcal{I})$ (cf. the proof of [15, Lemma 1.6] or [13, Prop. 3.9]). Thus $R(\mathcal{B})$ is $\mathcal{I}$-free and hence coincides with $\tilde{R}(\mathcal{B})$. This shows (4.6a). The assertion (4.6b) follows from [15, Lemma 1.6]. We shall prove (4.6c). We have an exact sequence :

$$0 \to (R \otimes_{\Lambda_K} \mathcal{I}) \to R(\mathcal{K}) \oplus R(\mathcal{B}) \to C_0(\lambda;\mathcal{I}) \to 0.$$

By Th. 4.2, we know that $R \otimes_{\Lambda_K} \mathcal{I}_P = R(\mathcal{K})_P \oplus R(\mathcal{B})_P$ for $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ with $k(P) \geq 2$. Note that $R \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I}$ is a local factor of $\mathbf{h}_{k(P)}(\Phi_{r(P)},\varepsilon_P;\mathcal{O}_K)$ and $R \otimes_{\Lambda_K} \mathcal{I}_P/P\mathcal{I}_P = (R \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I}) \otimes_{\mathcal{O}_K} K$ is a direct factor of $\mathbf{h}_{k(P)}(\Phi_{r(P)},\varepsilon_P;K)$. This shows that the induced sequence :

$$R \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I} \xrightarrow{\alpha}$$
$$(R(\mathcal{K})/PR(\mathcal{K})) \oplus (R(\mathcal{B})/PR(\mathcal{B})) \to C_0(\lambda;\mathcal{I}) \otimes_{\mathcal{I}} \mathcal{I}/P\mathcal{I} \to 0$$

is exact and $\mathrm{Ker}(\alpha)$ is finite. Since $R \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I}$ is $\mathcal{O}_K$-free, α must be injective. The natural projections : $R \otimes_{\Lambda_K} \mathcal{I} \to R(\mathcal{K})$ and $R \otimes_{\Lambda_K} \mathcal{I} \to R(\mathcal{B})$ induce the natural surjections : $R \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I} \to (R(\mathcal{K})/PR(\mathcal{K}))$ and $R \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I} \to (R(\mathcal{B})/PR(\mathcal{B}))$. Since α is the diagonal map of these surjections, we know from the definition of $C(f_P)$ that

$$C(f_P) = \mathrm{Coker}(\alpha) \cong C_0(\lambda;\mathcal{I}) \otimes_{\mathcal{I}} \mathcal{I}/P\mathcal{I} \text{ canonically.}$$

This finishes the proof.

Here let us add a supplement to the result in [14, § 10]. We shall use the same notation as in [14, § 10]. We have defined there a transcendental factor $U_\infty(k,\varepsilon) \in \mathbf{C}^\times$ for the set $\Psi(k,\varepsilon) = \{f_P$ such that $k(P) = k$ and $\varepsilon_P = \varepsilon\}$. We now define the transcendental factor for each f_P. (When $d(\mathcal{I}) = 1$, the set $\Psi(k,\varepsilon)$ consists of a single element f_P and we have nothing to add to [14, § 10] but there are non-trivial examples with $d(\mathcal{I}) > 1$; cf. [12, § 4]). Suppose λ to be primitive. We shall decompose

(4.7) $\quad \mathbf{h}_k(\Phi_r, \varepsilon; K) = K \oplus A$ as an algebra direct sum

according to λ_P for $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_K)$, $r = r(P)$, $k = k(P) \geq 2$ and $\varepsilon = \varepsilon_P$. Let K_0 be the subfield of $\overline{\mathbf{Q}}$ generated by $a(n, f_P)$ for all $n > 0$. Then $\mathbf{h}_k(\Phi_r, \varepsilon; K) = \mathbf{h}_k(\Phi_r, \varepsilon; K_0) \otimes_{K_0} K$ and λ_P induces a morphism of K_0-algebras : $\mathbf{h}_k(\Phi_r, \varepsilon; K_0) \to K_0$. Thus we can decompose

(4.8) $\quad \mathbf{h}_k(\Phi_r, \varepsilon; K_0) = K_0 \oplus A_0$ and $\mathbf{h}_k(\Phi_r, \varepsilon; \mathbf{C}) = \mathbf{C} \oplus (A_0 \otimes_{K_0} \mathbf{C})$.

Let C be the conductor of f_P and put $S(\varepsilon) = S_k(\Phi_r, \varepsilon) + \overline{S_k(\Phi_r, \bar{\varepsilon})}$ as a subspace of the parabolic cohomology group $H^1_P(\Gamma_1(C), L_n(\mathbf{C}))$ $(n = k-2)$ (see [14, (10.4)]). Define $S(f_P) = 1_{K_0}(S(\varepsilon))$ and $S(A) = 1_{A_0}(S(\varepsilon))$, where 1_{K_0} (resp. 1_{A_0}) is the idempotent of the component K_0 (resp. A_0) of the Hecke algebra (4.8). Let $\mathcal{V}$ be the discrete valuation ring $K_0 \cap \mathcal{O}_K$, and write L for the image of $H^1_P(\Gamma_1(C), L_n(\mathcal{V}))$ in $H^1_P(\Gamma_1(C), L_n(\mathbf{C}))$. Put

(4.9) $\quad L_{\varepsilon,P} = L \cap S(f_P),\ L^\varepsilon_P = \pi_\varepsilon(L) \cap S(f_P),$

where $\pi_\varepsilon : H^1_P(\Gamma_1(C), L_n(\mathbf{C})) \to S(\varepsilon)$ be the projection as in [14, (10.6)]. Note that $S(f_P) = L_{\varepsilon,P} \otimes_{\mathcal{V}} \mathbf{C} = L^\varepsilon_P \otimes_{\mathcal{V}} \mathbf{C} = \mathbf{C}f^0_P + \mathbf{C}\overline{(f^0_P)^\rho}$ for $(f^0_P)^\rho(z) = \overline{f^0_P(-\bar{z})}$, where f^0_P denotes the primitive form in $S_k(\Gamma_1(C))$ associated with f_P. Let δ_1, δ_2 (resp. δ'_1, δ'_2) be a basis of $L_{\varepsilon,P}$ (resp. L^ε_P) over V, and define $X, Y \in GL_2(\mathbf{C})$ by the matricial identity :

$$(\delta_1, \delta_2)X = (f^0_P, \overline{(f^0_P)^\rho}), (\delta'_1, \delta'_2)Y = (f^0_P, \overline{(f^0_P)^\rho}).$$

We define

$$(4.10) \qquad u_\infty(\lambda_P) = \det(X\overline{Y})^{\frac{1}{2}} \text{ and}$$
$$U_\infty(\lambda_P) = \pi^{k+1} u_\infty(\lambda_P)/\{(k-1)!CC(\varepsilon\psi\omega^{-k})\varphi(C/C(\varepsilon\psi\omega^{-k}))\},$$

where $C(f_P) = C$ (resp. $C(\varepsilon\psi\omega^{-k})$) is the conductor of f_P (resp. $\varepsilon\psi\omega^{-k}$) and φ is the Euler function.

THEOREM 4.5. — *Suppose one of the equivalent conditions* (4.5a,b) *and λ to be primitive. Suppose also that the p-part of ψ is not equal to ω^2. Let $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ with $k(P) \geq 2$. Let $\mathcal{D}(s, f_P^0)$ be the L-function of f_P^0 defined in* [14, (10.2)] *and $H \in \mathcal{I}$ be a generator of the annihilator of $\mathcal{C}_0(\lambda; \mathcal{I})$. Then we can find a p-adic unit $U_P(\lambda_P) \in \overline{\mathbf{Q}}_P$ such that*

$$H(P) = \mathcal{D}(k(P), f_P^0)/U_\infty(\lambda_P)U_p(\lambda_P). \tag{4.11}$$

For the proof of this fact, see [16]. We will not need this result later in this paper. Here are several remarks about the theorem :

Remark 4.6. — (i) Under the hypothesis of Theorem 4.5, it is known (e.g. [10, Th. 5.1]) that

$$\begin{aligned}(4.12)\quad &\mathcal{D}(k, f_P^0)\\ &= 2^{2k}\pi^{k+1}\Gamma(k)^{-1}\{\delta(C)CC(\varepsilon\psi\omega^{-k})\varphi(C/C(\varepsilon\psi\omega^{-k}))\}^{-1} < f_P^0, f_P^0 >_{\Gamma_1(C)},\end{aligned}$$

where φ is the Euler function, and $\delta(C) = 2$ or 1 according as $C \geq 2$ or not, and $k = k(P)$. By this formula, we know that $\mathcal{D}(k(P), f_P^0)/(2\pi i)^{k(P)+1}u_\infty(\lambda_P)$ is algebraic.

(ii) The Gorenstein condition (4.5a,b) is known to hold in the following cases, where the p-part of ψ is ω^a with $0 \leq a < p-1$:

A. $a = 2$ and $N = 1$ (Mazur [22, Cor. 15.2 and 16.3 of Chap. II]);

B. Let $\pi(\lambda)$ be the Galois representation into $GL_2(\mathcal{K})$ attached to λ defined in [14, Th. 2.1.]. Let $\mathfrak{m}$ (resp. $\mathfrak{p}$) be the maximal ideal of $\mathcal{I}$ (resp. $\mathcal{O}_K$). We shall define the residual representation of $\pi(\lambda)$ mod $\mathfrak{m}$:

$$\tilde{\pi}(\lambda) : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathcal{I}/\mathfrak{m})$$

according to Mazur-Wiles [23, § 10] as follows : Choose $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_K)$ with $k(P) \geq 2$. Then we have Deligne's Galois representation $\pi(f_P)$: $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathcal{O}_K)$ attached to f_P (e.g. [14, § 2]), and the residual representation $\pi(\lambda)$ is the semi-simplification of the composite of $\pi(f_P)$ with the reduction map : $GL_2(\mathcal{O}_K) \to GL_2(\mathcal{O}_K/\mathfrak{p})$. Note that $\tilde{\pi}(\lambda)$ depends only on $\pi(\lambda)$ and is independent of the choice of $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_K)$. Then the Gorenstein condition (4.5a,b) is satisfied by λ if $a \neq 1$, $a \neq 2$ and $\tilde{\pi}(\lambda)$ is irreducible. This is shown by Mazur-Wiles [23, § 10] when $N = 1$ and is generalized to an arbitrary N by Tilouine [33].

C. $a \neq 1$, $a \neq 2$ and λ is with complex multiplication in the sense of [14, Prop. 2.3]. This is a special case of Case B but a simple proof can be found in [15, § 6].

(iii) Let $\hat{\lambda}$ be the restriction of λ to $\mathbf{h}^o(N;\mathcal{O}_K)$. We may assume that $\mathcal{I}$ is the smallest, i.e. the integral closure of Λ_K in the quotient field of the image of $\hat{\lambda}$. If $\mathcal{I} = \Lambda_K$, then $\mathcal{C}(\lambda;\mathcal{I})$ coincides with the module of congruences defined in [13, (3.9b)]. However, if $d(\mathcal{I}) > 1$, the module $\mathcal{C}(\lambda;\mathcal{I})$ and that given in [13, (3.9b)] are generally different. We note here only the existence of a surjective morphism of $\mathcal{C}_0(\lambda;\mathcal{I})$ onto the other. A detailed study of relations among these modules can be found in [16]. Some part of this has already been discussed in [12].

(iv) We briefly discuss here the relation between our transcendental factor $U_\infty(\lambda_P)$ and that defined by Deligne in [6, § 7] in the language of motives. Let g be a primitive form of weight ℓ and of character ξ. Let $M(\lambda_P) = M(f_P^0)$ (resp. $M(g)$) denote the motive attached to the primitive form f_P^0 (resp. g) defined in [6, § 7]. Then $M(\lambda_P)$ is of rank 2 and has coefficient in the field E generated over $\mathbf{Q}$ by $a(n, f_P^0)$ for all n. Let $\mathrm{Ad}(M(\lambda_P))$ be the unique direct factor of rank 3 in $M(\lambda_P) \otimes M(\lambda_P)\check{}$ which is the kernel of the natural morphism :

$$M(\lambda_P) \otimes M(\lambda_P)\check{} \to \mathbf{Z}(0) \otimes_{\mathbf{Z}} \mathbf{r}_E = \mathbf{r}_E(0),$$

where $\mathbf{r}_E$ is the integer ring of E and $M(\lambda_P)\check{}$ is the dual of $M(\lambda_P)$. Then $\mathrm{Ad}(M(\lambda_P))$ can be realized in $H_c^1(Y, F_n(E)) \otimes H^1(Y, F_n(E))$ for the modular curve $Y_{/\mathbf{Q}}$ with $Y(\mathbf{C}) = \Gamma_1(C) \setminus \mathfrak{H}$, where $F_n(E)$ is the locally constant sheaf defined in [16, § 2]. Thus, as the Deligne's periods $c^+(\mathrm{Ad}(M(\lambda_P)) = c^-(\mathrm{Ad}(\lambda_P))$, we may take [6, Prop. 7.7]

$$(4.13)\quad \Omega(P) = \int_{Y'} \delta(f) \wedge \overline{\Theta_n \delta(f)} = (2i)^{k+1}\pi^2 < f_P^0, f_P^0 >_C, (k = k(P)),$$

where $Y' = \Gamma_0(C) \setminus \mathfrak{H}$ and $\delta(f) = (2\pi i) f(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^n dz$ with $\begin{pmatrix} z \\ 1 \end{pmatrix}^n dz$ with $\begin{pmatrix} z \\ 1 \end{pmatrix}^n = {}^t(z^n, z^{n-1}, \ldots, 1)$ for $n = k-2$ and Θ_n is the matrix defined in [27, (8.2.2)] giving the self duality of $F_n(E)$. Noting that the Deligne's periods c^+ and c^- are defined only up to scalar factors in $E^\times$, we hereafter write $a \underset{E}{\sim} b$ (or simply, $a \sim b$) for a, $b \in \mathbf{C}^\times$ if $a = b$ in $\mathbf{C}^\times / E^\times$. Then we see from [31, Th. 1] or [6, Prop. 7.7]

$$(4.14)\quad c^+(\mathrm{Ad}(M(\lambda_P)) \sim c^-(\mathrm{Ad}(M(\lambda_P)) \sim c^+(M(\lambda_P))c^-(M(\lambda_P)) \sim (2\pi i)^{-2}\Omega(P).$$

Similarly to [6, Prop. 7.7], one knows

$$(4.15)\quad c^+(M(\lambda_P) \otimes M(g))(\ell + m) \sim (2\pi i)^{\ell+2m-1}\Omega(P)\overline{G(\xi)} \text{ if } k > \ell.$$

On the other hand, by extending scalar of $M(\lambda_P)$ to K_0 as in (4.8), we know from the definition of $u_\infty(\lambda_P)$ (see also [6, 1.7], [10, Remark 6.4a,b]) that

$$(4.16)\qquad u_\infty(\lambda_P) \underset{K_0}{\sim} (2\pi i)^{-2}\Omega(P) \underset{E}{\sim} c^+(M(\lambda_P))c^-(M(\lambda)).$$

5. Arithmetic measures and the main results.

Let J be a positive integer prime to p. We shall begin with a definition of a class of measures with values in $\overline{\mathcal{M}}(J;\mathcal{O}_K)$. Let X be a p-adic space in the sense of § 3. A measure μ on X with values in $\overline{\mathcal{M}}(J;\mathcal{O}_K)$ is an $\mathcal{O}_K$-linear homomorphism of $C(X;\mathcal{O}_K)$ into $\overline{\mathcal{M}}(J;\mathcal{O}_K)$. If μ is such a measure, μ satisfies automatically the inequality : $|\mu(\phi)|_p \leq \| \phi \| = \sup_x |\phi(x)|_p$ ($\phi \in C(X;\mathcal{O}_K)$). For any ring A, let $LC(X;A)$ denote the space of locally constant functions on X with values in A. When we have an action of Z_J on X, we let $z \in Z_J$ acts on $\phi \in C(X;\mathcal{O}_K)$ by $(\phi|z)(x) = \phi(z \cdot x)$. We write z_p for $z \in Z_J = \mathbf{Z}_p^\times \times (\mathbf{Z}/J\mathbf{Z})^\times$ the projection of z to $\mathbf{Z}_p^\times$.

Terminology. — We say that a measure $\mu : C(X;\mathcal{O}_K) \to \overline{\mathcal{M}}(J;\mathcal{O}_K)$ is *arithmetic* if the following three conditions are satisfied :

(5.1a) *There exists a positive integer ℓ such that for every $\phi \in LC(X;\overline{\mathbf{Q}}) \cap C(X;\mathcal{O}_K)$,*

$$\mu(\phi) \in \mathcal{M}_\ell(Jp^\infty;\overline{\mathbf{Q}}).$$

(This integer ℓ will be called the weight of μ).

(5.1b) *There are a continuous action : $Z_J \times X \to X$ and a finite order character $\xi : Z_J \to \mathcal{O}_K^\times$ such that $\mu(\phi)|z = z_p^\ell \xi(z)\mu(\phi|z)$ for every $\phi \in C(X;\mathcal{O}_K)$, where ℓ is the weight of μ.*

(5.1c) *There exists a continuous function $\nu : X \to \mathcal{O}_K$ such that $(\nu|z)(x) = z_p^2\nu(x)$ for $z \in Z_J$ and for each $0 < r \in \mathbf{Z}$,*

$$d^r(\mu(\phi)) = \mu(\nu^r\phi) \quad \text{for } d = q\frac{d}{dq}\cdot$$

We say a measure $\mu : C(X;\mathcal{O}_K) \to \overline{\mathcal{M}}(J;\mathcal{O}_K)$ *cuspidal* if μ has values in $\overline{S}(J;\mathcal{O}_K)$.

After stating the main result, we shall discuss several examples of arithmetic measures : Let $\mathcal{K}$ be a finite extension of $\mathcal{L}_K$ (the quotient field of Λ_K), and let $\mathcal{I}$ be the integral closure of Λ_K in $\mathcal{K}$. We shall suppose that (5.2) *$\mathcal{K}$ is defined over K* (cf. (3.1a)).

Replacing K by a finite extension if necessary, we may assume this condition without losing much generality. We fix throughout this section a primitive homomorphism $\mathcal{I}$-algebra $\lambda : \mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$ with character $\psi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}_K^\times$. We write ψ_p (resp. ψ') for the restriction of ψ to $(\mathbf{Z}/p\mathbf{Z})^\times$ (resp. $(\mathbf{Z}/N\mathbf{Z})^\times$) and ψ_P for $\varepsilon_P\psi_p\omega^{-k}$ for each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ of weight k. For $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ with weight $k \geq 2$, let $f_P \in S_k(\Gamma_0(Np^r), \psi_P\psi')$ be the ordinary form belonging to λ at P. Let f_P^0 denote the primitive form associated with f_P, and we write $Np^{r_0(P)}$ for the conductor of f_P^0. By Theorem 4.1, if ψ_P is non-trivial, then $f_P = f_P^0$ and $r(P) = r_0(P)$. If ψ_P is trivial and $k(P) > 2$, then $f_P \neq f_P^0$ and $r(P) = 1$ but $r_0(P) = 0$. When ψ_P is trivial and $k(P) = 2$, both the cases occur, and we then have that $r_0(P) = 1$ or 0 according as $f = f_P^0$ or not (the special case when $r_0(P) = 1$ with trivial ψ_P corresponds to the case where the automorphic representation of f_P^0 is special at p; otherwise, it is always principal at p). For each normalized eigenform f of conductor C and of weight k, taking the primitive form f_0 associated with f, we shall define a root number $W(f)$ by

$$f_0|_k \begin{pmatrix} 0 & -1 \\ C & 0 \end{pmatrix} = W(f) f_0^\rho.$$

This number $W(f)$ gives the constant term of the functional equation of

$$L(s,f) = \sum_{n=1}^{\infty} a(n, f_0) n^{-s}$$

and also appears in the constant term of the functional equation of the (primitive) Rankin product $D(s,f,g)$ (e.g. [11, § 9]). Thus $W(f)$ can be canonically factorized into a product of local factors. Let $W_p(f)$ be the p-factor of $W(f)$ and write $W(f) = W_p(f)W'(f)$. The explicit form of $W_p(f_P)$ can be given in the language of Jacquet-Langlands theory of automorphic representations (cf. [17]). To explain this, let π be an automorphic representation of $Gl_i(\mathbf{A})$ for $i = 1$ or 2, where $\mathbf{A}$ denotes the ring of adeles of $\mathbf{Q}$. (When $i = 1$, π is nothing but a Hecke character of the idele group $\mathbf{A}^\times$). We factorize π into the tensor product of local representations $\bigotimes_q \pi_q$ over all the places q of $\mathbf{Q}$. Let e denote the standard additive character of $\mathbf{A}/\mathbf{Q}$ (which coincides with the character $: x \mapsto \exp(-2\pi i x)$ on the infinite part $\mathbf{R}$ of $\mathbf{A}$). Then the constant term $\varepsilon(\pi, e)$

of the functional equation can be factorized as $\varepsilon(\pi, e) = \prod_q \varepsilon(\pi_q, e_q)$ ($e_q = e|_{\mathbf{Q}_q}$). Let $C(\pi) = \prod_q C(\pi_q)$ be the conductor of π (see Weil [34, VII] when $i = 1$, and Casselman [4] when $i = 2$). The conductor $C(\pi_q)$ is a power of the prime q. When $i = 1$, the restriction of π_q to $\mathbf{Z}_q^\times$ and that of π to $\hat{\mathbf{Z}}^\times = \prod_q \mathbf{Z}_q^\times$ are induced by a primitive Dirichlet character modulo $C(\pi_q)$ and $C(\pi)$, respectively. Using the same symbol π and π_q for the corresponding Dirichlet characters, we define the Gauss sum by

$$G(\pi) = \sum_{u \bmod C(\pi)} \pi(u) \exp(2\pi i u/C(\pi))$$

$$\text{and } G(\pi_q) = \sum_{u \bmod C(\pi_q)} \pi_q(u) \exp(2\pi i u/C(\pi_q)).$$

Then it is well known (e.g. [34, VII.7]) that

(5.3) If $i = 1$, then

$$\varepsilon(\pi_p, e_p) = \begin{cases} \pi_p(C(\pi_p))G(\overline{\pi}_p)/|\pi_p(C(\pi_p))G(\overline{\pi}_p)| & \text{if } C(\pi_p) > 1, \\ 1 & \text{if } C(\pi_p) = 1. \end{cases}$$

Now we consider the case of $i = 2$ and suppose that π is the automorphic representation attached to f_0. Then we have that

$$W(f) = \prod_q \varepsilon(\pi_q, e_q), W_p(f) = \varepsilon(\pi_p, e_p) \text{ and } W'(f) = \prod_{q \neq p} \varepsilon(\pi_q, e_q).$$

It is known (e.g. [4, Remark, p. 306] or [5, § 4, 5, and 6]) that

(5.4a) *$W'(f) = W(f)$ if $C = C(f)$ is prime to p,*

(5.4b) *$|W'(f)|_p = 1$; i.e., $W'(f)$ is a p-adic unit in $\overline{\mathbf{Q}}_p$.*

(5.4c) *Write $C = C_pC'$ for a p-power C_p and C' prime to p. Then*

$$W'(f|\chi) = \chi(C')W'(f)$$

for each Dirichlet character χ modulo a power of p.

When π_p is super cuspidal, the nature of $W_p(f)$ is purely non-abelian, and thus one cannot express it by the Gauss sums. However, when π_p is not super cuspidal, $W_p(f)$ has an expression in terms of the Gauss sums, and we shall give here its explicit form. Since we have greater interest in the case where $f = f_P|\chi$ for a point $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ and since in this case, π_p is either principal or special, we do not lose much generality by this restriction. We thus assume that π_p is either a principal series representation $\pi(\alpha,\alpha')$ or a special representation $\sigma(\alpha,\alpha')$ for quasi characters α, α' of $\mathbf{Q}_p^\times$. Then one knows from [7, Remark 4.25 and Th. 6.15] that

(5.5a)

$$W_p(f)=\begin{cases}\varepsilon(\alpha,e_p)\varepsilon(\alpha',e_p) & \textit{if either } \pi_p=\sigma(\alpha,\alpha') \textit{ with ramified } \alpha \\ & \textit{or } \pi_p=\pi(\alpha,\alpha'),\\ -\alpha(p)/|\alpha(p)| & \text{if } \pi_p=\sigma(\alpha,\alpha') \textit{ with unramified } \alpha.\end{cases}$$

We say that f is p-minimal, if f has a minimal conductor in the class of twists $f|\chi$ by characters χ modulo powers of p (i.e., $C(f|\chi)\geq C(f)$ for all finite order characters $\chi:\mathbf{Z}_p^\times\to\overline{\mathbf{Q}}^\times$). If π_p is not super cuspidal, then f is p-minimal if and only if either $a(p,f_0)\neq 0$ or $p\not| C(f)$. In this case, $\pi_p=\overline{\sigma}(\alpha,\alpha')$ or $\pi_p=\pi(\alpha,\alpha')$, and α is unramified [14, Lemma 10.1]. Suppose hereafter that π_p is not super cuspidal and f is p-minimal. Let ξ be the character of f_0. If $\pi_p=\pi(\alpha,\alpha')$, then we may assume that α is unramified and thus $\alpha(p)$ is one of the non-zero root of the quadratic polynomial : $X^2-a(p,f_0)X+\xi(p)p^{k-1}, \alpha\alpha'(p)=\xi'(p)p^{k-1}(\xi'=\xi|_{(\mathbf{Z}/C'\mathbf{Z})^\times}), G(\alpha)=1$ and $\alpha'=\overline{\xi}_p$ on $\mathbf{Z}_p^\times$. If $\pi_p=\sigma(\alpha,\alpha')$, then ξ_p is trivial and $\alpha(p)=a(p,f_0)$. Thus we have by (5.5a)

(5.5b)

$$W_p(f)=\begin{cases}1 & \text{if } C_p=1,\\ -a(p,f_0)/p^{(k-2)/2} & \text{if } \pi_p=\sigma(\alpha,\alpha'),\\ (\xi'(p)a(p,f_0)^\rho p^{-k/2})^r G(\xi_p) & \text{if } C_p=p^r\end{cases}$$

with $r>0$ and $\pi_p=\pi(\alpha,\alpha')$.

If we take the convention that $G(\xi_p)=1$ when ξ_p is trivial, and also if we take the convention that $(\xi'(p)a(p,f_0)p^{-k/2})^0=1$, the first case of (5.5b) can be considered as a special case of the last formula in (5.5b). Let ξ be a Dirichlet character of conductor p^γ, and write the conductor of $\xi_p\chi i$ as $p^{\gamma'}$. Then, similarly as above, we have, if $\gamma>0(\gamma'\geq 0)$,

(5.5c) $W_p(f|\chi)=$

$$\begin{cases}(\alpha(p)/p^{k/2})^\gamma(\xi'(p(p)\overline{\alpha(p)}/p^{k/2})^{\gamma'}G(\chi)G(\xi_p\chi) & \text{if } \pi_p=\pi(\alpha,\alpha')\\ G(\chi)^2\xi'(p)^\gamma p^{-\gamma} & \text{if } \pi_p=\sigma(\alpha,\alpha').\end{cases}$$

When $\pi_p=\pi(\alpha,\alpha')$, we can also write

$$W_p(f|\chi)=(p^{k-1)/2}/\overline{\alpha(p)})^\gamma(p^{(k-1)/2}/\alpha(p)\overline{\xi}'(p))^{\gamma'}p^{-(\gamma+\gamma')/2}G(\chi)G(\xi_p\chi).$$

When $f=f_P$ for some $P\in\mathcal{X}_{\mathrm{alg}}(\mathcal{I})$, we always have that $\alpha(p)=a(p,f_P)$, and by (5.5b,c), we know the exact form of $W_p(f_P|\chi)$.

We now define an Euler p-factor of $\mathcal{D}(s, f_P^0)$ as in Th. 4.5 for $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$:

(5.6) $S(P) = S(\lambda_P)$

$$= \begin{cases} -1 & \text{if } \psi_P = \mathrm{id} \text{ and } f_P = f_P^0, \\ \left(\dfrac{\psi'(p)a(p, f_P^\rho)^2}{p^{k(P)}}\right)^{r_0(P)} & \\ \qquad \times \left(1 - \dfrac{\psi'\psi_P(p)p^{k(P)-1}}{a(p, f_P)^2}\right)\left(1 - \dfrac{\psi'\psi_P(p)p^{k(P)-2}}{a(p, f_P)^2}\right), & \end{cases}$$

if either ψ_P is non-trivial or $f_P \neq f_P^0$.

Note that the condition that $\psi_P = \mathrm{id}$ and $f_P = f_P^0$ is equivalent to saying that π_p is special and $k(P) = 2$ (for $f = f_P$). Here we follow the convention that $\psi_P(p) = 1$ if ψ_P is trivial, and $\psi_P(p) = 0$ if ψ_P is non-trivial. We take

$$\Omega(P) = \Omega(\lambda_P) = (2i)^{k(P)+1}\pi^2 < f_P^o, f_P^o >_C \tag{5.7}$$

in (4.13) as a transcendental factor of the Rankin product for f_P, where C is the conductor of f_P. As shown by Shimura [30][31] (see also § 6 in the text), we know that if g belongs to $\mathcal{M}_\ell(Jp^\infty; \overline{\mathbf{Q}})$, then for integers m with $0 \leq m < k - \ell$,

$$\frac{\mathcal{D}_{JNp}(\ell + m, f_P, g)}{(2\pi i)^{\ell+2m-1}\Omega(P)} \textit{ is an algebraic number.}$$

We shall choose and fix an element $H \in \mathcal{I}$ ($H \neq 0$) which annihilates the module of congruence $\mathcal{C}_0(\lambda; \mathcal{I})$ defined in (4.3). This is possible because of (4.4c) in Th. 4.3. When the annihilator of $\mathcal{C}_0(\lambda; \mathcal{I})$ is principal, we shall choose $H \in \mathcal{I}$ so that *H generates the annihilator of* $\mathcal{C}_0(\lambda; \mathcal{I})$. This happens in the following situations : Let $\mathcal{R}$ be the local ring of $\mathbf{h}^o(N; \mathcal{O}_K)$ through which λ factors. The annihilator of $\mathcal{C}_0(\lambda; \mathcal{I})$ is principal when one of the following conditions are satisfied :

(i) $\mathcal{R} \cong \mathrm{Hom}_{\Lambda_K}(\mathcal{R}, \Lambda_K)$ *as $\mathcal{R}$-modules* (cf. Th. 4.4 (4.6b));

(ii) $\mathcal{I}$ *is a unique factorization domain;* for example, when $\mathcal{I} = \Lambda_K$ (cf. Th. 4.3).

Thus, only when the annihilator of $\mathcal{C}_0(\lambda; \mathcal{I})$ is principal, an intrinsic choice of H up to unit factors in $\mathcal{I}$ is possible. Let $\mu : C(X; \mathcal{O}_K) \to \overline{\mathcal{M}}(J; \mathcal{O}_K)$ be an *arithmetic* measure. We write ℓ (resp. $\xi : Z_J \to \mathcal{O}_K^\times$) for the weight (resp. the character) of μ as in (5.1a,b). Suppose that

$$\mu(\phi)|T(p) = 0 \textit{ for all } \phi \in C(X; \mathcal{O}_K). \tag{5.8}$$

Under (5.8), μ automatically becomes *cuspidal* by Th. 2.2. For each finite order character $\chi : Z_J \to \overline{\mathbf{Q}}^\times$, put

$$LC(X,\chi;\overline{\mathbf{Q}}) = \{\phi \in LC(X;\overline{\mathbf{Q}}) \mid (\phi|z)(x) = \chi(z)\phi(x) \text{ for all } z \in Z_J\}.$$

Now we are ready to state our main result :

THEOREM 5.1. — *Let the notation and the assumption be as above. Then there exists a unique generalized measure (in the sense of § 3) $\Phi = \Phi^{\mu,\lambda} \in \mathrm{Meas}(X;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I}$ with the following interpolation property : for each pair $(P,m) \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}) \times \mathbf{Z}$ with $0 \le m < k(P) - \ell$ and for each $\phi \in LC(X,\chi;\overline{\mathbf{Q}})$, we have the following evaluation of the measure Φ_P at P :*

$$S(P)H(P)^{-1}\int_W \phi\nu^m d\Phi_P$$
$$= tG(\psi_P)^{-1}W'(f_P)^{-1}p^{\beta(\ell+2m)/2}a(p,f_P)^{-\beta}\frac{\mathcal{D}_{JN_P}(\ell+m,f_P,\mu(\phi)|_\ell\tau_\beta)}{(2\pi i)^{\ell+2m-1}\Omega(P)},$$
$$t = t(P,\ell,m) = [N,J]N^{-k(P)/2}J^{\ell/2+m}\Gamma(\ell+m)\Gamma(m+1),$$

where β is a positive integer such that $\mu(\phi) \in \mathcal{M}_\ell(\Gamma_1(Jp^\beta)), \tau_\beta = \begin{pmatrix} 0 & -1 \\ Jp^\beta & 0 \end{pmatrix}$ and $[N,J]$ is the least common multiple of N and J.

For the validity of the above evaluation formula for $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$, we of course have to assume that $H(P) \neq 0$, but by Th. 4.2, we can always choose H so that $H(P) \neq 0$. Since the right-hand side of the formula is independent of the choice of H, the measure Φ divided by H is intrinsically determined. The uniqueness of the measure Φ follows from Lemma 3.3. The existence will be proven in § 9. Now we shall give several examples of arithmetic measures and for each of them, we write again the version of Theorem 5.1 :

Example a. Theta measures ([11, § 2]). — Let V be a vector space over $\mathbf{Q}$ of even dimension 2κ, and let $n : V \to \mathbf{Q}$ be a positive definite quadratic form. We shall write $S(x,y) = n(x+y) - n(x) - n(y)$ for the corresponding inner product. Take a lattice I in V so that $n(I) \subset \mathbf{Z}$, and write I^* for the dual lattice of I and Δ for the discriminant of I; i.e., we put

$$I^* = \{x \in V \mid S(x,I) \subset \mathbf{Z}\}, \Delta = [I^* : I],$$
$$\mathcal{W} = \{x \in I^* \mid n(x) \in \mathbf{Z}\}, W = \varprojlim_r \mathcal{W}/p^r I$$

and
$$W^\times = \{w \in W \mid n(x) \in \mathbf{Z}_p^\times\}.$$

Let M be the level of I, i.e. the smallest positive integer such that $Mn(I^*) \subset \mathbf{Z}$, and write $M = Jp^\delta$ with $(J,p) = 1$ and $0 \leq \delta \in \mathbf{Z}$. Then as seen in [11, § 2], we have the theta measure on W whose values belong to $\overline{\mathcal{M}}(J;\mathcal{O}_K)$ defined by

$$\theta(\phi) = \sum_{w \in \mathcal{W}} \phi(w) q^{n(w)}$$

where ϕ is in $C(W;\mathcal{O}_K)$. Let $\eta : V \to \overline{\mathbf{Q}}$ be a spherical function of degree α in the sense of [11, § 1], and we denote by the same symbol the natural extension of η to a function on W to $\overline{\mathbf{Q}}_p$ by continuity. (Since the spherical function always has values in a finite extension of $\mathbf{Q}$ on V, to define its continuous prolongation η on W, we do not have to take the p-adic completion of $\overline{\mathbf{Q}}_p$). We suppose that η has values in $\mathcal{O}_K$ on W. Then we shall also consider the measure $\eta\theta$ defined by

$$\eta\theta(\phi) = \theta(\eta\phi) = \sum_{w \in \mathcal{W}} \eta\phi(w) q^{n(w)} \in \overline{\mathcal{M}}(J;\mathcal{O}_K).$$

By definition, the restriction of θ to $W^\times$ satisfies (5.8) and is thus cuspidal. The natural action of $\mathbf{Z}$ on $\mathcal{W}$ via the multiplication of integers extends by continuity to an action of Z_J on W and $W^\times$. By the classical transformation formula of theta series (e.g. [11, Prop. 1.1]), we can verify that $\eta\theta$ satisfy the conditions (5.1a,b) for $\ell = \kappa + \alpha$, and the character ξ is given by

$$\xi(m) = \left(\frac{(-1)^\kappa \Delta}{m}\right) \quad (m \in \mathbf{Z},\ (m, Jp) = 1)$$

for the quadratic residue symbol $\left(\frac{a}{b}\right)$. The condition (5.1c) is obviously satisfied by the function $n : W^\times \to \mathbf{Z}_p$. By the construction of $\Phi^{\mu,\lambda}$, which will be done in § 9, it is obvious that

$$\int_{W^\times} \phi n^m d\Phi_P^{\eta\theta,\lambda} = \int_{W^\times} \phi\eta n^m d\Phi_P^{\theta,\lambda}.$$

Thus, we obtain from Th. 5.1 the following result which is in appearance a little stronger than Th. 5.1 :

THEOREM 5.1a. — *Let the notation be as above. Then there exists a unique generalized measure $\Phi^\theta = \Phi\theta,\lambda \in$ Meas $(W^\times;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I}$ with the following interpolation property : for each quadruple (P,m,ν,ϕ) consisting of $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}), m \in \mathbf{Z}$ with $0 \leq m < k(P) - \kappa - \alpha$ and $\phi \in LC(W^\times,\chi;\overline{\mathbf{Q}})$,* *we have that*

$$S(P)H(P)^{-1}\int_{W^\times}\eta\phi n^m d\Phi_P^\theta$$
$$= t(P,\kappa+\alpha,m)W'(f_P)^{-1}G(\psi_P)^{-1}p^{\beta(\kappa+\alpha+2m)/2}a(p,f_P)^{-\beta}$$
$$\times\frac{\mathcal{D}_{JNp}(\kappa+\alpha+m,f_P,\theta(\eta\phi)|_{\kappa+\alpha}\tau_\beta)}{(2\pi i)^{\kappa+\alpha+2m-1}\Omega(P)}$$

where β is a positive integer such that $\theta(\eta\phi)\in\mathcal{M}_{\kappa+\alpha}(\Gamma_1(Jp^\beta))$.

Here are some remarks about the theorem, which is a generalization of [11, Th. 2.1] :

(i) If we write the level M of θ as $M=Jp^\delta$ with $\delta\geq 0$ and J prime to p and if ϕ factors through $\mathcal{W}/p^\gamma I$, then β as in the theorem can be given by $\delta+2\gamma$ (e.g. [11, Prop. 1.1]).

(ii) If K is sufficiently large, any homogeneous polynomial function F of degree d on V can be written as a finite sum of functions of the form ηn^r with $d=\alpha+2r$. Thus if $k(P)>d+\kappa$, by Th. 5.1a, we can evaluate the integral :

$$\int_{W^\times}\phi F d\Phi_P^\theta.$$

(iii) When $\dim(V)=2$ (i.e. $\kappa=1$), n is essentially a norm form of an imaginary quadratic field. In this special case, a detailed study of the measure Φ_P^θ for a fixed P with $k(P)=2$ is done by B. Perrin-Riou [25], which includes with other things a p-adic interpolation of the Hasse-Weil zeta function of the abelian variety attached to f_P ([16, Th. 7.14]) over the imaginary quadratic field associated with n.

Example b. The measure attached to modular forms ([11, § 8]). — Fix a p-adic modular form $g\in\overline{\mathcal{M}}(J;\mathcal{O}_K)$. Then we can define a measure $\mu_g : C(\mathbf{Z}_p;\mathcal{O}_K)\to\overline{\mathcal{M}}(J;\mathcal{O}_K)$ by

$$\mu_g(\phi)=\sum_{n=0}^{\infty}\phi(n)a(n,g)q^n\in\overline{\mathcal{M}}(J;\mathcal{O}_K)\quad\text{(cf. [11, Prop. 8.1]).}$$

If $g\in\overline{S}(J;\mathcal{O}_K)$, then μ_g is cuspidal. Even if g may not be in $\overline{S}(J;\mathcal{O}_K)$, its restriction to $\mathbf{Z}_p^\times$ is cuspidal and satisfies (5.8). If we take the linear form $\nu:\mathbf{Z}_p\to\mathcal{O}_K$ given by $\nu(w)=w$ and let Z_J act on $\mathbf{Z}_p$ by $z\cdot w=z_p^2 w$, then μ_g satisfies (5.1c). Further suppose that

g is a classical modular form in $\mathcal{M}_\ell(\Gamma_0(Jp^\delta),\xi;\overline{\mathbf{Q}})$.

Then μ_g satisfies (5.1a,b) for ℓ and $\xi : Z_J \to \mathcal{O}_K^\times$ ([11, Prop. 8.1]). Thus we can specialize Th. 5.1 to the measure μ_g. Since it is a routine work to derive the formulation of the general theorem for μ_g from Th. 5.1, we shall make explicit the p-adic L-function attached to λ and μ_g in the case where g is a normalized eigenform in $S_\ell(\Gamma_0(Jp^\delta), \xi)$. We consider the Rankin product

$$\mathcal{D}(s, f_P, g) = \mathcal{D}_C(s, f_P^0, g_0), \tag{5.9}$$

where C is the least common multiple of $C(g)$ and $C(f_P)$ and where g_0 is the primitive form associated with g.

Let $M(g)$ (resp. $M(\lambda_P)$) be the motive associated with g_0 (resp. f_P^0 for $P \in \mathcal{X}_{\rm alg}(\mathcal{I})$). Then the primitive Rankin product $D(s, f_P, g)$ is the L-function attached to the motive $M(\lambda_P) \otimes M(g)$ ([6, § 1]). By the recent solution of the local Langlands conjecture for $GL(2)$, we know that

$$D(s, f_P, g) = L(s, \pi \times \pi'),$$

where π and π' are the automorphic representations associated with f_P^0 and g_0, respectively, and $L(s, \pi \times \pi')$ is the L-function attached to $\pi \times \pi'$ defined by Jacquet [18]. The Euler factors of $L(s, \pi \times \pi')$ are completely determined by Gelbart and Jacquet [18] and [8,§ 1]. By their results, $\mathcal{D}(s, f_P, g)$ coincides with $D(s, f_P, g)$ if there is no finite place q where π_q is super cuspidal and π'_q is equivalent to the contragredient representation of π_q up to the twists by unramified characters. Let Σ_P be the set of primes where $\mathcal{D}(s, f_P, g)$ and $D(s, f_P, g)$ have different Euler factors. Then Σ_P is conjectured to be independent of P but this assertion is still an open question in general. The following facts are known ([16, § 7]) :

(5.10a) *If f_P^0 has no supercuspidal prime at least for one P (i.e. $\Sigma_P = \phi$), then $\Sigma_P = \phi$ for all $P \in \mathcal{X}_{\rm alg}(\mathcal{I})$ (with $k(P) \geq 2$),*

(5.10b) *Σ_P is independent of P except for finitely many P in $\mathcal{X}_{\rm alg}(\mathcal{I})$.*

Thus in the case of (5.10a), we know the identity $\mathcal{D}(s, f_P, g) = D(s, f_P, g)$ for all P. Anyway, by this difficulty, we are forced to consider $\mathcal{D}(s, f_P, g)$ instead of the primitive $D(s, f_P, g)$.

Since $\mu_{g|\chi} = \chi\mu_g$ for each finite order character $\chi : \mathbf{Z}_p^\times \to \overline{\mathbf{Q}}^\times$, we may assume that g is p-minimal without losing much generality. As seen, for example, in [14,Lemma 10.1], a primitive form f is p-minimal and is not super cuspidal at p if and only if either $a(p, f) \neq 0$ or the conductor of f is prime to p. Especially, f_P^0 is p-minimal and is not super cuspidal at p. Let h be the primitive form associated with $g|\overline{\xi}_p$ (or equivalently, it is the primitive form associated with $g^\rho|\xi'$ for the restriction ξ' of ξ to $(\mathbf{Z}/J\mathbf{Z})^\times$). We define for $f = g, g^\rho, h$ and f_P^0 the algebraic numbers $\alpha(f)$ and $\alpha'(f)$ by

$$(5.11) \qquad [(1-\alpha(f)p^{-s})(1-\alpha'(f)p^{-s})]^{-1} = \sum_{n=0}^{\infty} a(p^n, f)p^{-ns}.$$

We assume that $\alpha(f_P) = \alpha(f_P^0) = a(p, f_P)$, which is a p-adic unit by definition. We further suppose that g is primitive of conductor Jp^δ and p-minimal. If π_p' is not super cuspidal, at least one of $\alpha(g)$ and $\alpha'(g)$ is non-zero. Thus we suppose in this case that $\alpha(g) \neq 0$ and $\alpha(g^\rho) = \alpha(g)^\rho$. Define $\alpha(h^\rho)$ and $\alpha'(h^\rho)$ by $\alpha'(h^\rho) = \alpha'(g)^\rho$ and $\alpha(h^\rho)\alpha(g^\rho) = p^{\ell-1}\overline{\xi}'(p)$ (we use these symbols only when π_p' is not super cuspidal; i.e., $\alpha(g) \neq 0$). These numbers $\alpha(h^\rho)$ and $\alpha'(h^\rho)$ coincides with the numbers defined by (5.11) for $f = h^\rho$ except when π_p' is special. Now let us define an Euler p-factor supposing that π_p' is not super cuspidal : Write the conductors of χ and $\xi_p\chi$ as p^γ and $p^{\gamma'}$, respectively. Then we put

(5.12a) $E_1(s) =$

$$\begin{cases} \left(\dfrac{p^{s-1}}{\alpha(g^\rho)\alpha(f_P^0)}\right)^\gamma \left(\dfrac{p^{s-1}}{\alpha(h^\rho)\alpha(f_P^0)}\right)^{\gamma'} & \text{if } \pi_p' \text{ is principal or } \gamma > 0 \\ -\left(\dfrac{p^{s-1}}{\alpha(g^\rho)\alpha(f_P^0)}\right) & \text{if } \pi_p' \text{ is special and } \gamma = 0, \end{cases}$$

(5.12b) $E_2(s) =$

$$\begin{cases} \left(1 - \dfrac{\chi(p)p^{s-1}}{\alpha(g^\rho)\alpha(f_P^0)}\right)\left(1 - \dfrac{\xi_p\chi(p)p^{s-1}}{\alpha(h^\rho)\alpha(f_P^0)}\right) & \text{if } \pi_p' \text{ is principal or } \gamma > 0 \\ \left(1 - \dfrac{p^{s-1}}{\alpha(h^\rho)\alpha(f_P^0)}\right) & \text{if } \pi_p' \text{ is special and } \gamma = 0, \end{cases}$$

(5.12c) $E_3(s) = (1 - \overline{\chi}(p)\alpha'(f_P^0)\alpha(g^\rho)p^{-s})(1 - (\overline{\xi_p\chi})(p)\alpha'(f_P^0)\alpha'(h^\rho)p^{-s}),$

where we have taken the convention that $\chi(p) = \overline{\chi}(p) = 0$ or 1 according as $\gamma > 0$ or $\gamma = 0$ (this convention also applies to $\xi_p\chi$). We further put

$$E(s) = E_1(s)E_2(s)E_3(s).$$

The following lemma is due to (a suggestion made by) B. Perrin-Riou :

LEMMA 5.2. — *Let the notation be as above. Suppose that g is primitive of conductor Jp^δ and p-minimal. Let χ be a Dirichlet character of conductor p^γ. We regard χ as a character of $\mathbf{Z}_p^\times$ and extend it to a function on $\mathbf{Z}_p$ by putting 0 outside $\mathbf{Z}_p^\times$ (thus $a(n, g|\chi) = 0$ if p divides n even if*

χ is trivial). Let β be the smallest exponent so that $g|\chi \in S_\ell(\Gamma_0(Jp^\beta),\xi)$, and put

$$D(\ell+m) = p^{\beta(\ell+2m)/2}a(p,f_P)^{-\beta}\mathcal{D}_{JNp}(\ell+m,f_P,(g|\chi)|_\ell\tau_\beta).$$

Then we have

(i) *Suppose that π'_p is not super cuspidal. Then we have*

$$D(\ell+m) = W'(g)G(\chi)G(\xi_p\chi)\chi(J)E(\ell+m)\mathcal{D}(\ell+m,f_P,g^\rho|\overline{\chi}).$$

Moreover β is given as follows : Let $\imath_p$ denote the trivial character mod p. *Then*

$$\beta = \begin{cases} 2 & \text{if } \chi = \imath_p \text{ and } \delta = 0, \\ \delta+1 & \text{if } \delta > 0 \text{ and either } \chi = \imath_p \text{ or } \chi = \overline{\xi}_p, \\ \gamma+\gamma' & \text{otherwise.} \end{cases}$$

(ii) *Suppose that π'_p is super cuspidal. Then we have*

$$D(\ell+m) = \chi(J)W'(g)W_p(g|\chi)p^{\beta(\ell+2m)/2}a(p,f_P)^{-\beta}\mathcal{D}(\ell+m,f_P,g^\rho|\overline{\chi}),$$

where

$$\beta = \begin{cases} \max(\delta,2\gamma) & \text{if } \delta \text{ is even,} \\ 2\gamma & \text{if } \delta \text{ is odd and } 2\gamma > \delta+1, \\ \delta & \text{if } \delta \text{ is odd and } 2\gamma \le \delta+1. \end{cases}$$

Proof. — (i) Almost by definition (cf. [30, Lemma 1]), we have that if g_0 is the primitive form associated with $g|\chi$, then

$$\mathcal{D}_{JNp}(s,f_P,g_0) = E_3(s)\mathcal{D}(s,f_P,g^\rho|\overline{\chi}).$$

Thus we shall express $D(\ell+m)$ by $\mathcal{D}_{JNp}(\ell+m,f_P,g_0^\rho)$. By the p-minimality of g, if $\pi'_p = \pi(\alpha,\alpha')$ for quasi characters α and α' of $\mathbf{Q}_p^\times$, then one of α and α', say α, is unramified. If $\pi'_p = \sigma(\alpha,\alpha')$, then both α and α' are unramified. Note that $\pi(\alpha,\alpha')\otimes\chi = \pi(\alpha\chi,\alpha'\chi), \sigma(\alpha,\alpha')\otimes\chi = \sigma(\alpha\chi,\alpha'\chi)$ and $\alpha'|_{\mathbf{Z}_p^\times} = \xi_p$. Thus, if both $\alpha\chi$ and $\alpha'\chi$ are ramified (i.e. $\gamma > 0, \gamma' > 0$), then $g|\chi$ is primitive of conductor $JC(\chi)C(\xi_p\chi)$, and thus in this case, $\beta = \gamma+\gamma'$ and

$$(g|\chi)|\tau_\beta = W(g|\chi)g^\rho|\overline{\chi}.$$

Since $W(g|\chi) = W'(g)\chi(J)W_p(g|\chi)$ by (5.4c), the desired formula follows from (5.5c). When either $\chi = \imath_p$ or $\overline{\xi}_p$, then an explicit computation shows that

$$g|\chi = \begin{cases} g_0 - a(p,g_0)g_0|[p] & \text{if } \delta > 0 \\ & \text{and either } \chi = \imath_p \text{ or } \overline{\xi}_p, \\ g - a(p,g)g|[p] + \xi'(p)p^{\ell-1}g|[p^2] & \text{if } \delta = 0 \text{ and } \chi = \imath_p, \end{cases}$$

where g_0 is the primitive form associated with $g|\chi$. This shows the value of β as in (i). Now, applying τ_β to this formula, we have

$(g|\chi)|\tau_\beta =$

$$\begin{cases} -a(p,g_0)p^{-\ell/2}W(g|\chi)(g_0^\rho - a(p,g_0)^{-1}p^\ell g_0^\rho|[p]) \\ \qquad \text{if } \delta > 0 \text{ and either } \chi = \imath_p \text{ or } \overline{\xi}_p, \\ p^{-1}\xi'(p)W(g)(g^\rho - \overline{\xi}'(p)a(p,g)pg^\rho|[p] + \overline{\xi}'(p)p^{\ell+1}g^\rho|[p^2]) \\ \qquad \text{if } \delta = 0 \text{ and } \chi = \imath_p. \end{cases}$$

We now suppose that $\delta > 0, \chi = \imath_p$ and π'_p is principal. Then we see that $\delta = \gamma'$ and

$$D(\ell+m) = -p^{\gamma'\ell/2+m(\gamma'+1)}\alpha(f_P^0)^{-\gamma'-1}\alpha(g)W(g) \\ \times(1-(\alpha(f_P^0)/\alpha(g))p^{-m})\mathcal{D}_{JN_p}(\ell+m,f_P,g^\rho).$$

Note that

$$(1-(\alpha(f_P^0)/\alpha(g))p^{-m}) = -\alpha(f_P^0)\alpha(g^\rho)p^{1-m-\ell}\Big(1-\frac{p^{m+\ell-1}}{\alpha(f_P^0)\alpha(g^\rho)}\Big).$$

This combined with (5.5c) (or (5.5b)) shows the result. The case where $\delta > 0$ but either π'_p is special or $\chi = \overline{\xi}_p$ can be treated similarly; so, we next suppose that $\delta = 0$ and $\chi = \imath_p$. Then $D(\ell+m)$ is equal to

$$p^{\ell+2m-1}\alpha(f_P^0)^{-2}\xi'(p)W(g) \\ \times(1-\overline{\xi}'(p)a(p,g)p^{1-\ell-m}+\overline{\xi}'(p)\alpha(f_P^0)^2p^{1-\ell-2m})\mathcal{D}_{JN_p}(\ell+m,f_p,g^\rho).$$

Since $\alpha(g^\rho)\alpha(h^\rho) = \overline{\xi}'(p)p^{\ell-1} = \alpha(g^\rho)\alpha'(g^\rho)$ and $\overline{\xi}'(p)a(p,g) = \alpha(g^\rho) + \alpha(h^\rho)$, we know that

$$\begin{aligned}(1-&\overline{\xi}'(p)a(p,g)\alpha(f_P^0)p^{1-\ell-m}+\overline{\xi}'(p)\alpha(f_P^0)^2p^{1-\ell-2m}) \\ &= (1-\alpha(g^\rho)\alpha(f_P^0)p^{1-\ell-m})(1-\alpha(h^\rho)\alpha(f_P^0)p^{1-\ell-m}) \\ &= \overline{\xi}'(p)p^{1-\ell-2m}\alpha(f_P^0)^2E_2(\ell+m).\end{aligned}$$

This finishes the proof of the assertion (i). The value of β as in the second assertion can be found in Carayol [3, p. 208 (g) and 8.1]. The set of supercuspidal representations is stable under the twist by quasi characters of $\mathbf{Q}_p^\times$, and hence if π'_p is supercuspidal, then $g|\chi$ is always primitive. In this case $E_3(s)$ is reduced to 1 and hence the assertion (ii) is obvious from (5.4c).

THEOREM 5.1b. — *Fix an integer a with $0 \le a < p-1$. Let g be a primitive form of conductor Jp^δ, of character ξ and of weight ℓ. Suppose that g is p-minimal and let $\pi' = \otimes \pi'_q$ be the automorphic representation of $GL_2(\mathbf{A})$ attached to g. Then there exists a unique element $D = D_g$ in the quotient field of $\mathcal{I}\hat{\otimes}_{\mathcal{O}_K}\Lambda_K$ such that*

(i) *If $H \in \mathcal{I}$ annihilates the module of congruence $C_0(\lambda;\mathcal{I})$, then $HD \in \mathcal{I}\hat{\otimes}_{\mathcal{O}_K}\Lambda_K$,*

(ii) *For each point $(P,R) \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}) \times \mathcal{X}_{\mathrm{alg}}(\Lambda_K)$ such that $0 \le k(R) < k(P) - \ell$, $D(P,r)$ is finite and its value is given by*

$$D(P,R) = tW'(f_P)^{-1}W'(g)G(\psi_P)^{-1}\overline{G(\xi)}\varepsilon_R\omega^{a-k(R)}(J)S(P)^{-1}$$

$$\times \begin{cases} G(\varepsilon_R\omega^{a-k(R)})G(\varepsilon_R\omega^{a-k(R)}\xi_p)E(\ell+k(R)) \\ \quad \mathcal{D}(\ell+k(R), f_P, g^\rho|\varepsilon_R^{-1}\omega^{k(R)-a})/\Omega(P,g,R) \\ \qquad\qquad\qquad \textit{if } \pi'_p \textit{ is not super cuspidal,} \\ W_p(g|\varepsilon_R\omega^{a-k(R)})p^{\beta(R)(\ell+2k(R))/2} \\ \quad \alpha(f_P^0)^{-\beta(R)}\mathcal{D}(\ell+k(R), f_P, g^\rho|\varepsilon_R^{-1}\omega^{k(R)-a})/\Omega(P,g,R) \\ \qquad\qquad\qquad \textit{if } \pi'_p \textit{ is super cuspidal,} \end{cases}$$

where $t = t(P,\ell,k(R))$ as in Th. 5.1, *$\beta(R)$ is equal to β for $\chi = \varepsilon_R\omega^{a-k(R)}$ as in Lemma* 5.2, *and*

$$\Omega(P,g,R) = (2\pi i)^{\ell+2k(R)-1}\Omega(P)\overline{G(\xi)} \text{ (see 4.15)).}$$

This result follows directly from Lemma 5.2 and Th. 5.1.

Example c. Eisenstein measures. — Fix an integer $b > 1$ prime to Jp. Let $\zeta^b : C(Z_J;\mathcal{O}_K) \to \mathcal{O}_K$ denote the well known measure corresponding to the Kubota-Leopoldt p-adic L-function; namely, for each finite order character $\chi : Z_J \to \overline{\mathbf{Q}}^\times$ and for each positive integer m, it satisfies (e.g. [12, Chap. 4])

$$\int_{Z_J} \chi(z)z_p^{m-1}d\zeta^b = (1-b^m\chi(b))L(1-m,\chi).$$

We shall define several measures $E^b : C(Z_J;\mathcal{O}_K) \to \overline{\mathcal{M}}(J;\mathcal{O}_K), G_m$, $E : C(Z_J;\mathcal{O}_K) \to \overline{S}(J;\mathcal{O}_K)$ for each $m > 1$ and $\mathcal{E} : C(\mathbf{Z}_p \times Z_J;\mathcal{O}_K) \to \overline{S}(J;\mathcal{O}_K)$ by the following formulae :

$$2\int_{Z_J}\phi(z)dE^b = \int_{Z_J}\phi d\zeta^b + \sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n \\ (d,Jp)=1}} \mathrm{sgn}(d)(\phi(d) - b\phi(bd))\Big)q^n,$$

$$2\int_{Z_J} \phi(z)dE = \sum_{\substack{n=1\\(n,p)=1}}^{\infty} \Big(\sum_{\substack{d|n\\(d,Jp)=1}} \mathrm{sgn}(d)\phi(d)\Big) q^n,$$

$$2\int_{Z_J} (\phi(z)dG_m = \sum_{n=1}^{\infty} \Big(\sum_{\substack{d|n\\(n/d,Jp)=1}} \mathrm{sgn}(d)d^{m-1}\phi(n/d)\Big) q^n$$

for $\phi \in C(Z_J; \mathcal{O}_K)$, and

$$2\int_{\mathbf{Z}_p\times Z_J} \phi(w,z)d\mathcal{E} = \sum_{\substack{n=1\\(n,p)=1}}^{\infty} \Big(\sum_{\substack{d|n\\(d,Jp)=1}} \mathrm{sgn}(d)\phi(n,d)\Big) q^n$$

for $\phi \in C(\mathbf{Z}_p \times Z_J; \mathcal{O}_K)$.

The measure $2E^b$ has been employed in [11, § 6] and we have the following formulae by definition :

(5.13a) $$\Big(\int_{Z_J} \phi(z)dE^b\Big)|\imath_p = \int_{Z_J} \big(\phi(z) - b\phi(bz)\big)dE,$$

(5.13b) $$G_m(\phi)|\imath_p = d^{m-1}\Big(\int_{Z_J} z_p^{1-m}\phi(z)dE\Big) \text{ for } d = q\frac{d}{dq},$$

(5.13c) $$\int_{\mathbf{Z}_p\times Z_J} w^m\phi(z)d\mathcal{E} = d^m\Big(\int_{Z_J}\phi dE\Big) \text{ for each } 0 \le m \in \mathbf{Z}.$$

The existence $\mathcal{E}$ and E follows from that of E^b (by (5.13a,c)), which is verified in [11, § 6]. The existence of G_m will be shown in the next section. Anyway, the existence of all these measures follows from a general result of Katz [20, VI]. Note that $\mathcal{E}$ and E satisfy (5.8) and hence are cuspidal. We can verify directly that $G_m(\phi)|e = 0$ for all $\phi \in C(Z_J; \mathcal{O}_K)$ and thus, G_m is also cuspidal. For each pair of finite order characters $\chi, \eta : Z_J \to \overline{\mathbf{Q}}^\times$ and for each positive integer k, define

$$E_k(\chi,\eta) = \sum_{\substack{n=1\\(n,p)=1}}^{\infty} \Big(\sum_{0<d|n} \eta(n/d)\chi(d)d^{k-1}\Big)q^n \in \overline{\mathbf{Q}}[[q]].$$

If $\eta : \mathbf{Z}_p^\times \to \overline{\mathbf{Q}}$ and $\chi(-1) = (-1)^k$, then we have

(5.14) $$\int_{\mathbf{Z}_p\times Z_J} \eta(w)\chi(z)z_p^{k-1}d\mathcal{E} = E_k(\chi\eta,\eta) \in \mathcal{M}_k(Jp^\infty; \overline{\mathbf{Q}}).$$

When one considers the series $E_k(\chi,\eta)$ for the primitive trivial character η, the constant term of the form $L(1-k,\chi)/2$ or the term of the form

$c(z-\overline{z})^{-1}$ with $c \in \mathbf{C}$ may appear. However, after applying the twisting operator for the character $\imath_p$, we see easily that these terms disappear. Then (5.14) follows from [14, Lemma 5.2] (see Lemma 5.3 below). Even if $\chi(-1) \neq (-1)^k$, the formula (5.14) remains true but it simply vanishes; i.e. $E_k(\chi\eta,\eta) = 0$. This shows that $\mathcal{E}$ is arithmetic of weight 1 with the trivial character $\imath_{Jp}$, where the action of Z_J on $X = \mathbf{Z}_p^\times \times Z_J$ is given by $z(w,z') = (z_p^2 w, zz')$ and the function $\nu : \mathbf{Z}_p^\times \times Z_J \to \mathbf{Z}_p$ is given by $(w,z) \mapsto w$. Now we shall apply Th. 5.1 to the measure $\mathcal{E}$. To formulate our result in a final form, we prepare a lemma.

LEMMA 5.3. — *Let χ and η be primitive Dirichlet characters modulo u and v, respectively. Define a function on the Poincaré upper half plane by*

$$E_k^0(\chi,\eta)(z) = \delta_1(u)L(0,\eta) + \delta(v)L(1-k,\chi) + \delta_2(u,v)\frac{i}{2\pi(z-\overline{z})} + \sum_{n=1}^{\infty}\sum_{0<d|n}\chi(d)\eta(n/d)d^{k-1}q^n,$$

where

$$\delta_1(u) = \begin{cases} 2^{-1} & \text{if } k = u = 1, \\ 0 & \text{otherwise,}\end{cases}$$

$$\delta_2(u,v) = \begin{cases} 2^{-1} & \text{if } k = 2 \text{ and } u = v = 1, \\ 0 & \text{otherwise,}\end{cases}$$

$$\delta(v) = \begin{cases} 2^{-1} & \text{if } v = 1, \\ 0 & \text{otherwise.}\end{cases}$$

Suppose that $\eta\chi(-1) = (-1)^k$. Then $E_k^0(\chi,\eta)|_k\gamma = \chi\eta(\gamma)E_k^0(\chi,\eta)$ for all $\gamma \in \Gamma_0(uv)$ and for $\tau = \begin{pmatrix} 0 & -1 \\ uv & 0 \end{pmatrix}$, we have

$$E_k^0(\chi,\eta)|_k\tau = (uv^{-1})^{k/2}\eta(-1)G(\eta)/G(\overline{\chi})E_k^0(\overline{\eta},\overline{\chi}).$$

(On the space of Eisenstein series, the action of τ is no longer unitary and hence the root number $(uv^{-1})^{k/2}\eta(-1)G(\eta)/G(\overline{\chi})$ may not be of absolute value 1).

Proof. — Since the above fact is well known, we here only give a sketch of a proof in the (absolute convergent) case : $k > 2$. The case where $k = 1$ and 2 can be treated similarly (see § 6 in the text). According to Hecke, we define a convergent Eisenstein series by

$$G_k(z;a,b) = \sum_{\substack{(c,d)\equiv(a,b) \bmod uv \\ (c,d)\neq 0}} (cs+d)^{-k}.$$

Then, as shown by Hecke [9], we have the following Fourier expansion of this series :

$$G_k(z;a,b) = \delta(a)\zeta(k,b) + c_k\Big(\sum_{\substack{mn>0 \\ m\equiv a \bmod uv}} k^{k-1}\mathrm{sgn}(n)e\Big(\frac{bn+mnz}{uv}\Big)\Big),$$

where $\zeta(k,b) = \sum_{\substack{n\equiv b \bmod uv \\ n\neq 0}} n^{-k}$,

$$\delta(a) = \begin{cases} 1 & \text{if } a\equiv 0 \bmod uv, \text{ and } c_k = \dfrac{(-2\pi i)^k}{(uv)^k\Gamma(k)}, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the sum

$$E'_k(\chi,\eta) = \sum_{a=1}^{v}\sum_{b=1}^{uv}\eta(a)\overline{\chi}(b)G_k(z;au,b).$$

Then the coefficient of $e\Big(\dfrac{nz}{uv}\Big)$ $(n\neq 0)$ of $E'_k(\chi,\eta)$ is equal to

$$c_k\cdot\sum_{a=1}^{v}\eta(a)\sum_{\substack{d\mid n \\ n/d\equiv au \bmod uv}} d^{k-1}\mathrm{sgn}(d)\sum_{b=1}^{uv}\overline{\chi}(b)e\Big(\frac{bd}{uv}\Big).$$

Taking the following well known formula into account :

$$\sum_{b=1}^{uv}\overline{\chi}(b)e\Big(\frac{bd}{uv}\Big) = \begin{cases} 0 & \text{if } v\nmid d, \\ vG(\overline{\chi})\chi(d/v) & \text{if } v\mid d, \end{cases}$$

we see easily that

$$E'_k(\chi,\eta) = c_k v^k G(\overline{\chi})E^o_k(\chi,\eta).$$

This shows the first assertion. To prove the second assertion, we note that

$$E'_k(\chi,\eta)(z) = \sum_{(c,d)\in u\mathbf{Z}\times\mathbf{Z}}\eta(c/u)\overline{\chi}(d)(cz+d)^{-k}.$$

Thus we have that

$$E'_k(\chi,\eta)|\tau(z) = (uv)^{k/2}\sum_{(c,d)\in u\mathbf{Z}\times\mathbf{Z}}\eta(c/u)\overline{\chi}(d)(uvdz-c)^{-k}.$$

By substituting (c,d) for $(dv,-c/u)$, we know that

$$E'_k(\chi,\eta)|\tau = (vu^{-1})^{k/2}\eta(-1)E'_k(\overline{\eta},\overline{\chi}).$$

This shows the last assertion.

We fix a primitive character χ modulo J and consider, for each pair of finite order characters (ξ,η) of $\mathbf{Z}_p^\times$, the Eisenstein series

$$E_\ell(\chi\xi\eta,\eta)=\int_{\mathbf{Z}_p^\times\times Z_J}\eta(w)\chi\xi(z)z_p^{\ell-1}d\mathcal{E}.$$

Write $C(\xi\eta)=p^{\gamma'}$ and $C(\eta)=p^{\gamma}$. Then by Lemma 5.3, the root number $W_\ell(\chi\xi\eta,\eta)=W(E_\ell(\chi\xi\eta,\eta))$ is given by

$$\begin{aligned}W_\ell(\chi\xi\eta,\eta)&=(Jp^{\gamma'-\gamma})^{\ell/2}\eta(-1)G(\eta)/G(\overline{\chi\xi\eta})\\&=J^{\ell/2-1}(-1)^\ell\xi\eta(J)\chi(p^{\gamma'})p^{\ell(\gamma'-\gamma)/2-\gamma'}G(\chi)G(\eta)G(\xi\eta).\end{aligned}$$

We now define, for each primitive character θ and for $P\in\mathcal{X}_{\mathrm{alg}}(\mathcal{I})(k(P)\geq 2)$, the primitive L-function of f_p by

$$L(s,f_P,\theta)=\sum_{n=1}^{\infty}\theta(n)a(n,f_P^0)n^{-s}.$$

Let θ_p be the p-part of θ and decompose $\theta=\theta'\theta_p$ and write $C(\theta_p)=p^{\nu}$. We define $\alpha(f_P^0)$ and $\alpha'(f_P^0)$ by (5.11) for $f=f_P^0$ and assume that $\alpha(f_P^0)=a(p,f_P)$. We define an Euler p-factor by

$$E_\theta(s)=\Big(\frac{p^{s-1}}{\alpha(f_P^0)\overline{\theta}'(p)}\Big)^{\nu}\Big(1-\frac{\theta(p)p^{s-1}}{\alpha(f_P^0)}\Big)(1-\alpha'(f_P^0)\overline{\theta}(p)p^{-s}).$$

Then, by virtue of Lemma 5.3, taking $E_\ell^0(\chi\xi\eta,\eta)$ as g (and $E_\ell^0(\overline{\eta},\overline{\chi\xi\eta})$ as g^ρ) in the proof of Lemma 5.2, and regarding g as if it were a cusp form whose automorphic representation at p is the principal series representation $\pi((\xi\eta)_p,\eta_p)$, we obtain

COROLLARY 5.4. — *Let the notation be as above, and suppose that χ is primitive of conductor J. Let β be the smallest exponent of p so that*

$$E_\ell(\chi\xi\eta,\eta)\in\mathcal{M}_\ell(\Gamma_0(Jp^\beta),\chi\xi\eta^2).$$

Then, we have

$$\begin{aligned}&p^{\beta(\ell+2m)/2}a(p,f_P)^{-\beta}\mathcal{D}_{JNp}(\ell+m,f_P,E_\ell(\chi\xi\eta,\eta)|\tau_\beta)\\&\quad=(-1)^\ell J^{\ell/2-1}\xi\eta(J)G(\chi)G(\eta)G(\xi\eta)\\&\qquad\times E_{\chi\xi\eta}(\ell+m)L(\ell+m,f_P^0,\overline{\chi\xi\eta})E_\eta(m+1)L(m+1,f_P^o,\overline{\eta}).\end{aligned}$$

Let $\Phi = \Phi^{\mathcal{E},\lambda}$ be the generalized measure obtained by applying Th. 5.1 to the Eisenstein measure $\mathcal{E}$ on $\mathbf{Z}_p^\times \times Z_J$. We fix a primitive character χ of conductor J (i.e. $\chi : Z_J \to \overline{\mathbf{Q}}$). Then we derive another measure $\Psi^{\chi,\lambda}$ on $\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times$ out of Φ by putting

$$\int_{\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times} \phi(w,z)d\Psi_P^{\chi,\lambda} = \int_{\mathbf{Z}_p^\times \times Z_J} \phi(w,z_p)\chi(z)d\Phi_P \quad (\phi \in C(\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times;\mathcal{O}_K))$$

for all $P \in \mathcal{X}(\mathcal{I})$. Then we obtain easily from Th. 5.1 and Cor. 5.4 the following result :

THEOREM 5.1c. — *Let χ be a primitive Dirichlet character of conductor J. Then there exists a unique generalized measure $\Psi = \Psi^{\chi,\lambda}$ in* $\mathrm{Meas}(\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times;\mathcal{O}_K)\hat{\otimes}\mathcal{I}$ *with the following interpolation property : For each pentad $(P,\xi\eta,\ell,m)$ consisting of finite order characters $\xi,\eta : \mathbf{Z}_p^\times \to \overline{\mathbf{Q}}^\times$, $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ and non-negative integers ℓ, m with $0 \le m < k(P) - \ell$, we have*

$$S(P)H(P)^{-1}\int_{\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times} \eta(w)w^m\xi(z)z^{\ell-1}d\Psi_P$$
$$= t'\xi\eta(J)\Big(\frac{G(\chi)G(\eta)G(\xi\eta)}{W'(f_P)G(\psi_P)}\Big)E_{\chi\xi\eta}(\ell+m)L(\ell+m,f_P^0,\overline{\chi\xi\eta})$$
$$\times E_\eta(m+1)L(m+1,f_P^0,\overline{\eta})/(2\pi i)^{\ell+2m-1}\Omega(P),$$

where $t' = (-1)^\ell[N,J]N^{-k(P)/2}J^{m+\ell-1}\Gamma(\ell+m)\Gamma(m+1)$.

Note that $(2\pi i)^{\ell+2m-1}\Omega(P)G(\chi)^{-1}G(\eta)^{-1}G(\xi\eta)^{-1} \sim \Omega(P,g,R)$ as in Th. 5.1b for $R = P_{m,\eta}$ and $g = E_\ell^0(\chi\xi\eta,\eta)$.

Example d. The measure associated with homomorphisms of Hecke algebras. — We now want to interpolate the p-adic L-functions $D_g(P,R)$ obtained in Example b considering even g as a variable moving along an irreducible component of the Hecke algebra. Thus the result here include Th. 5.1b as a special case if g is ordinary. Let $\lambda' : \mathbf{h}^o(J;\mathcal{O}_K) \to \Lambda_K$ be a homomorphism of Λ_K-algebras and $\xi : (\mathbf{Z}/Jp\mathbf{Z})^\times \to \mathcal{O}_K^\times$ be its character. We fix a topological generator $u \in \Gamma$ and identify Λ_K with $\mathcal{O}_K[[X]]$ by $\imath(u) \mapsto 1+X$. Put $A(n;X) = \lambda'(T(n)) \in \mathcal{O}_K[[X]]$. Then we shall define a measure $\lambda'_0 : C(\mathbf{Z}_p^\times \times \Gamma;\mathcal{O}_K) \to \overline{S}(J;\mathcal{O}_K)$ by

$$\int_{\mathbf{Z}_p^\times \times \Gamma} \phi(w)\varepsilon(\gamma)\gamma^s d\lambda'_0 = \sum_{n=1}^{\infty} \phi(n)A(n;\varepsilon(u)u^s-1)q^n$$

for $\phi \in C(\mathbf{Z}_p^\times;\mathcal{O}_K)$, $s \in \mathbf{Z}_p$ and for each finite order character $\varepsilon : \Gamma \to \mathcal{O}_K^\times$. We let Z_J act on $\mathbf{Z}_p^\times \times \Gamma$ by $z(w,\gamma) = (z_p^2 w, < z_p > \gamma)$, where $< z_p > \in \Gamma$ is defined by $< z_p > = \omega(z_p)^{-1} z_p$. If we write $f_{k,\varepsilon}$ for f_P with $P = P_{k,\varepsilon}$, then we see easily that

$$\int_{\mathbf{Z}_p^\times \times \Gamma} \phi(w)\varepsilon(\gamma)\gamma^k d\lambda_0' = \int_{\mathbf{Z}_p^\times} \phi(w) d\mu_{f_{k,\varepsilon}}$$

for the measure $\mu_{f_{k,\varepsilon}}$ as in Example b and thus λ_0' satisfies (5.8). Let $\mathcal{M}$ be a finite extension of $\mathcal{L}_K$ defined over K and $\mathcal{J}$ be the integral closure of Λ_K in $\mathcal{M}$. We can generalize the above construction for a more general homomorphism $\lambda' : \mathbf{h}^o(J;\mathcal{O}_K) \to \mathcal{J}$ of Λ_K-algebra. Put $\mathcal{J}^* = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{J};\mathcal{O}_K)$. Then, by the duality in Th. 1.3, we have an $\mathcal{O}_K$-linear map

$$\lambda'^* : \mathcal{J}^* \to \overline{S}^o(J;\mathcal{O}_K), \tag{5.15}$$

which is the adjoint of λ'. We shall now extend λ'^* to $C(\mathbf{Z}_p;\mathcal{O}_K) \hat{\otimes}_{\mathcal{O}_K} \mathcal{J}^*$, which will be denoted by λ_0' : for each $\phi \in C(\mathbf{Z}_p^\times;\mathcal{O}_K)$ and $j \in \mathcal{J}^*$, we define

$$\lambda_0'(\phi \otimes j) = \int_{\mathbf{Z}_p^\times} \phi d\mu_g \quad \text{for } g = \lambda'^*(j) \in \overline{S}^o(J;\mathcal{O}_K),$$

where μ_g is the measure as in Example b. Then, by continuity, we have an $\mathcal{O}_K$-linear form

$$\lambda_0' : C(\mathbf{Z}_p^x;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K} J^* = (\mathrm{Meas}(\mathbf{Z}_p;O_K)\hat{\otimes}_{\mathcal{O}_J}\mathcal{J})^* \to \overline{S}(J;\mathcal{O}_K).$$

If we let Z_J act on $\mathbf{Z}_p$ via $w \mapsto z_p^2 w$ and if we define $\lambda_0'|z$ for $z \in Z_J$ by $(\lambda_0'|z)(\phi \otimes j) = \lambda_0'((\phi|z) \otimes j| < z_p >)$, then we have

$$\lambda_0'|z = \xi(z)\lambda_0' \quad \text{for any } z \in Z_J, \tag{5.16}$$

where ξ is the character of λ'.

The linear form λ_0' is not exactly a measure but a generalized measure with values in $\overline{S}(J;\mathcal{O}_K)$ in the sense of § 3. Then (5.16) is the formula corresponding to (5.1b); thus, λ_0' is of weight 0 with an abuse of language. For each $Q \in \mathcal{X}_{\mathrm{alg}}(\mathcal{J})$ with $k(Q) \geq 2$, we denote by g_Q the ordinary form belonging to λ' at Q in the sense of [14, Cor. 1.5]. We fix an integer a with $0 \leq a < p-1$, and for $(P,Q,R) \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}) \times \mathcal{X}_{\mathrm{alg}}(\mathcal{J}) \times \mathcal{X}_{\mathrm{alg}}(\Lambda_K)$, we denote by $E(P,Q,R)$ the Euler factor $E(k(Q)+k(R))$ in (5.12a,b,c) for $g = g_Q^0$ and $\chi = \omega^{a-k(R)}\varepsilon_R$.

THEOREM 5.1d. — *Let $\lambda : \mathbf{h}^o(N;\mathcal{O}_K)\otimes_{\Lambda_K}\mathcal{I}\to\mathcal{I}$ be a primitive $\mathcal{I}$-algebra homomorphism and let $\lambda' : \mathbf{h}^o(J;\mathcal{O}_K)\to\mathcal{J}$ be another primitive Λ_K-algebra homomorphism. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are defined over K. Then there exists a unique element D in the quotient field of $\mathcal{I}\hat{\otimes}\mathcal{J}\hat{\otimes}\Lambda_K$ such that* (i) *if $H\in\mathcal{I}$ annihilates $C_0(\lambda;\mathcal{I})$, then $H(P)D(P,Q,R)$ is integral; i.e., $HD\in\mathcal{I}\hat{\otimes}\mathcal{J}\hat{\otimes}\Lambda_K$ and* (ii) *for each point (P,Q,R) in $\mathcal{X}_{\mathrm{alg}}(\mathcal{I})\times\mathcal{X}_{\mathrm{alg}}(\mathcal{J})\times\mathcal{X}_{\mathrm{alg}}(\Lambda_K)$ with $0\le k(R)+k(Q)<k(P)$ and $k(P)>k(Q)\ge 2$,*

$$D(P,Q,R)=cwS(P)^{-1}E(P,Q,R)\\ \times\mathcal{D}(k(Q)+k(R),f_P,g_Q^\rho|\varepsilon_R^{-1}\omega^{k(R)-a}/\Omega(P,Q,R),$$

where $\Omega(P,Q,R)=(2\pi i)^{k(Q)+2k(R)-1}\Omega(P)\overline{G(\xi_Q\xi')}$ as in (4.15),

$$c=c(P,Q,R)=[N,J]\\ \times N^{-k(P)/2}J^{(k(Q)+2k(R))/2}\Gamma(k(Q)+k(R))\Gamma(k(R)+1),$$
$$w=w(P,Q,R)=W'(g_Q)W'(f_P)^{-1}G(\psi_P)^{-1}\\ \times\overline{G(\xi_Q\xi')}G(\varepsilon_R\omega^{a-k(R)})G(\varepsilon_R\omega^{a-k(R)}\xi_Q)\varepsilon_R\omega^{a-k(R)}(J).$$

In Th. 5.1d, we have interpolated the values of the function $E(s)\mathcal{D}(s,f_P,g_Q^\rho|\chi)$. When λ' is the restriction of λ to $\mathbf{h}^o(N;\mathcal{O}_K)$, the added Euler factor $E(s)$ has a trivial zero. In fact, in $E(s)$, we have the following factors : with the notation of (5.12b), $E_2(s)=E_2'(s)E_2''(s)$ and

$$E_2''(s)=\begin{cases}\left(1-\dfrac{\chi(p)p^{s-1}}{\alpha(g_Q^\rho)\alpha(f_P^0)}\right) & \text{if } \pi_p' \text{ is principal or } \gamma>0,\\[2ex] \left(1-\dfrac{p^{s-1}}{\alpha(h^\rho)\alpha(f_P^0)}\right) & \text{if } \pi_p' \text{ is special and } \gamma=0.\end{cases}$$

In either case as above, we can express $E_2''(s)$ uniformly as :

$$E_2''(s)=\left(1-\chi(p)\frac{\alpha(g_Q)}{\alpha(f_P^0)}p^{s-k(Q)}\right).$$

Since $\alpha(g_Q)/\alpha(f_P^0)$ is always a p-adic unit, if $k(R)>0$, $E_2''(k(Q)+k(R))$ does not vanish for all $(P,Q)\in\mathcal{X}(\mathcal{I})\times\mathcal{X}(\mathcal{I})$. However, taking $R=P_0=P_{0,\imath_p}$ and varying (P,Q) on $\mathcal{X}(\mathcal{I})\times\mathcal{X}(\mathcal{I})$, we know that

$$E''(P,Q)=E_2''(k(Q))=1-\alpha(g_Q)/\alpha(f_P^0)=1-a(p,f_Q)/a(p,f_P)$$

has a trivial zero at the diagonal divisor $\Delta=\{(P,P)\mid P\in\mathcal{X}(\mathcal{I})\}$ on $\mathcal{X}(\mathcal{I})^2$. Note that as a function of (P,Q) on $\mathcal{X}(\mathcal{I})^2$, $E''(P,Q)$ is an element of $\mathcal{I}\otimes\mathcal{I}$. Thus we may ask whether $D(P,Q,P)/E''(P,Q)$ has a pole at

Δ or not. An answer to this question can be given as follows : we fix a topological generator $\imath(u) \in \Gamma$ and identify $\Lambda_K \hat{\otimes}_{\mathcal{O}_K} \Lambda_K$ with $\mathcal{O}_K[[X,Y]]$ by regarding X (resp. Y) as a function on $\mathbf{Z}_p \times \mathbf{Z}_p$ given by $(s,t) \to u^s - 1$ (resp. $(s,t) \to u^t - 1$). We regard $\mathcal{O}_K[[X,Y]]$ naturally as a subalgebra of $\mathcal{I}\hat{\otimes}\mathcal{I}$.

THEOREM 5.1d$'$. — *Let the notation be as in Th.* 5.1d. *Suppose that* λ' *is the restriction of* λ *to* $\mathbf{h}^o(N;\mathcal{O}_K)$. *Write* $D(P,Q)$ *for the p-adic L-function :*

$$\mathcal{X}(\mathcal{I})^2 \ni (P,Q) \mapsto D(P,Q,P_0) \in \overline{\mathbf{Q}}_p$$

as in Th. 5.1d *for* $a = 0$, *and put* $D'(P,Q) = D(P,Q)/E''(P,Q)$ *as an element of the quotient field of* $\mathcal{I}\hat{\otimes}\mathcal{I}$. *Then*

(i) *If* $H \in \mathcal{I}$ *annihilates the module of congruence* $C_0(\lambda;\mathcal{I})$, *then* $(X-Y)HD' \in \mathcal{I}\hat{\otimes}\mathcal{I}$,

(ii) $(X(P)-Y(Q))D'(P,Q)|_{P=Q} = (1+Y(P))(\log(u))\varphi([N,J]p)/[N,J]p$ *for all non-critical* $P \in \mathcal{X}(\mathcal{I})$.

(We say that P is critical if P lies on the support of $C_0(\lambda;\mathcal{I})$ in $\mathcal{X}(\mathcal{I})$.)

By this theorem, D' has a non-trivial simple pole at Δ and interpolates the values $E'(P,Q,P_0)\mathcal{D}(kQ,f_P,f_Q^\rho)/\Omega(P,Q,P_0)$, where

$$E'(P,Q,P_0) = E(P,Q,P_0)/E''(P,Q).$$

Theorems I and II in § 0 are a special case of Theorems 5.1d and d'. We shall give a proof of these theorems in § 9.

6. Real analytic Eisenstein series and the holomorphic projection.

Here we shall review the explicit Fourier expansions of (group theoretic) Eisenstein series of integral weight by following Shimura's method which has been applied to those of half integral weight [28]. We shall do this here because this explicit Fourier expansion gives one of keys for the proof of Th. 5.1 and it is hard to find adequate references. We shall give only the outline of the proofs. Main references are [28] and [32].

In this section, we always write, for $v \in \mathbf{C}$ and $s \in \mathbf{C}$, $v^s = \exp(s\log v)$, where we shall define the logarithm by $\log v = \log|v| + i\theta$ with $-\pi < \theta \leq \pi$. We also write $e(x)$ for $\exp(2\pi\sqrt{-1}x)$ for $x \in \mathbf{C}$. Let N be a positive integer (which may be divisible by p in this section), and let ψ be a Dirichlet character modulo N. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, put $\psi(d) = \psi(d)$ and $j(\gamma, z) = (cz + d)$ for $z \in \mathfrak{H}$. Put $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} | m \in \mathbf{Z}\}$. If $\psi(-1) = (-1)^\lambda$ for $\lambda \in \mathbf{Z}$, the function $\gamma \longmapsto \psi(\gamma) j(\gamma, z)^{-\lambda} |j(\gamma, z)|^{-2s}$ depends only on the left coset of γ mod Γ_∞. Assuming that $\psi(-1) = (-1)^\lambda$, put

$$L_N(s, \psi) = \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \psi(n) n^{-s},$$

$$E^*_{\lambda,N}(z, s; \psi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \psi(\gamma) j(\gamma, z)^{-\lambda} |j(\gamma, z)|^{-2s} \tag{6.1}$$

$$G^*_{\lambda,N}(z, s; \psi) = E^*_{\lambda,N}(z, s; \psi)|_{\lambda,s} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where we define $f|_{\lambda,s}\gamma$ for each function $f : \mathfrak{H} \to \mathbf{C}$ by

$$(f|_{\lambda,s}\gamma)(z) = f(\gamma(z)) j(\gamma, z)^{-\lambda} |j(\gamma, z)|^{-2s}.$$

Then these series are absolutely convergent if $\mathrm{Re}(s) > 1 - \frac{\lambda}{2}$, and we can express $G^*_{\lambda,N}$ as follows, when $\mathrm{Re}(s) > 1 - \frac{\lambda}{2}$,

$$G^*_{\lambda,N}(z, s; \psi) = N^{-\lambda-2s} \sum_{d=1}^{\infty} \psi(d) d^{-\lambda-2s} \sum_{\substack{r=0 \\ (r,d)=1}}^{d-1} S\Big(\frac{Z}{N} + \frac{r}{d}; \lambda + s, s\Big), \tag{6.2}$$

where $S(z; \alpha, \beta) = \sum_{m=-\infty}^{\infty} (z + m)^{-\alpha} (\overline{z} + m)^{-\beta}$ for $\alpha, \beta \in \mathbf{C}$, which is absolutely convergent if $\mathrm{Re}(\alpha + \beta) > 1$. By the Poisson summation formula, we have

$$S(z; \alpha, \beta) = \sum_{m=-\infty}^{\infty} e(mx) \xi(y, m; \alpha, \beta) \quad \text{for } z = x + iy \in \mathfrak{H}, \tag{6.3}$$

where ξ is a function on $\mathbf{R}_+ \times \mathbf{R} \times \mathbf{C}^2 (\mathbf{R}_+ = \{x \in \mathbf{R} | x > 0\})$ which is given by an absolutely convergent integral

$$\xi(y, t; \alpha, \beta) = \int_{-\infty}^{\infty} e(-t \cdot x)(x + iy)^{-\alpha} (x - iy)^{-\beta} dx \text{ if } Re(\alpha + \beta) > 1.$$

Put $\mathfrak{H}' = \{z \in \mathbf{C} | \mathrm{Re}(z) > 0\}$ and

$$\omega(z;\alpha,\beta) = \Gamma(\beta)^{-1} z^{\beta} \int_0^{\infty} e^{-zx}(x+1)^{\alpha-1}x^{\beta-1}dx,$$

which is absolutely and uniformly convergent if $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(\beta) > 0$. This function has a holomorphic continuation on $\mathfrak{H}' \times \mathbf{C}^2$ and has a functional equation $\omega(z;1-\beta,1-\alpha) = \omega(z;\alpha,\beta)$ (cf. [28, § 2] and [32, Th. 3.1]). Furthermore, we have the explicit formulae ([32, Prop. 3.2]) :

(6.4a) For $0 < n \in \mathbf{Z}$, $\omega(z,n+1,\beta)$

$$= \sum_{k=0}^{n} \binom{n}{k} \beta(\beta+1)\dots(\beta+k-1)z^{-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\beta+k)}{\Gamma(\beta)} z^{-k}.$$

(6.4b) For $0 < n \in \mathbf{Z}$, $\omega(z;\alpha,-n)$

$$= \sum_{k=0}^{n} \binom{n}{k} (1-\alpha)(2-\alpha)\dots(k-\alpha)z^{-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(k-\alpha+1)}{\Gamma(1-\alpha)} z^{-k}.$$

(6.4c) $\omega(z;\alpha,0) = \omega(z;1,\beta) = 1$.

We can express $\xi(y,t;\alpha,\beta)$ by $\omega(z;\alpha,\beta)$ as follows :

$$\xi(y,t;\alpha,\beta) = \begin{cases} (\sqrt{-1})^{\beta-\alpha}(2\pi)^{\alpha}\Gamma(\alpha)^{-1}(2y)^{-\beta}t^{\alpha-1}e^{-2\pi yt}\omega(4\pi yt;\alpha,\beta) & \text{if } t > 0, \\ (\sqrt{-1})^{\beta-\alpha}(2\pi)^{\beta}\Gamma(\beta)^{-1}(2y)^{-\alpha}|t|^{\beta-1}e^{-2\pi y|t|} \\ \qquad \times \omega(4\pi y|t|;\beta,\alpha) & \text{if } t < 0, \\ (\sqrt{-1})^{\beta-\alpha}(2\pi)^{\alpha+\beta}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1} \\ \qquad \times \Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta} & \text{if } t = 0. \end{cases} \tag{6.5}$$

Thus the function ξ also has an analytic continuation, and the series (6.3) always converges at the point where ξ has no singularity. In exactly the same manner as in [28, § 3] (actually, it is much simpler), we have

THEOREM 6.1. — *We have the following explicit Fourier expansion :*

$$\begin{aligned}(2y)^s(\sqrt{-1})^{\lambda}L_N(\lambda+2s,\psi)G^*_{\lambda,N}(z,s;\psi) \\ = \frac{2\pi}{N}(2y)^{1-\lambda-s}\Gamma(s)^{-1}\Gamma(\lambda+s)^{-1}\Gamma(\lambda+2s-1)L_N(\lambda+2sq-1,\psi) \\ + \left(\frac{2\pi}{N}\right)^{\lambda+s}\Gamma(\lambda+s)^{-1}\sum_{m=1}^{\infty}\sum_{0<d|m}^{\infty} \\ \psi(d)d^{-\lambda-2s+1}m^{\lambda+s-1}e\left(\frac{mz}{N}\right)\omega\left(\frac{4\pi my}{N};\lambda+s,s\right)\end{aligned}$$

$$+\left(\frac{2\pi}{N}\right)^{s}\Gamma(s)^{-1}(2y)^{-\lambda}\sum_{m=1}^{\infty}\sum_{0<d|m}^{\infty}$$
$$\psi(d)d^{-\lambda-2s+1}e\left(\frac{-m\overline{z}}{N}\right)\omega\left(\frac{4\pi my}{N};s,\lambda+s\right).$$

Moreover, this series converges absolutely and uniformly on any compact subset in $\mathbf{C}$ *outside the small circles of singularities of the constant term.*

By evaluating, this series for $\lambda>0$ at $s=0$, we have

COROLLARY 6.2. — *We have for each* $\lambda>0$ *that*

$$\left(\frac{N}{-2\pi\sqrt{-1}}\right)^{\lambda}\Gamma(\lambda)L_N(\lambda,\psi)G^{*}_{\lambda,N}(Nz,0;\psi)$$
$$=\left(\frac{N^{\frac{1}{2}}}{-2\pi\sqrt{-1}}\right)^{\lambda}\Gamma(\lambda)L_N(\lambda,\psi)(E^{*}_{\lambda,N}(z,0;\psi))|_{\lambda}\tau_N$$
$$=\delta_{\lambda,1}\cdot\frac{1}{2}L(0,\psi)+\delta_2(\psi)\frac{1}{8\pi y}\prod_{\ell|N}\left(1-\frac{1}{\ell}\right)$$
$$+\sum_{n=1}^{\infty}\Big(\sum_{0<d|n}\left(\frac{n}{d}\right)^{\lambda-1}\psi(d)\Big)e(nz),$$

where $\delta_{\lambda,1}$ *is the Kronecker symbol,* $\tau_N=\begin{pmatrix}0&-1\\N&0\end{pmatrix}$ *and*

$$\delta_2(\psi)=\begin{cases}1 & \text{if } \lambda=2 \text{ and } \psi=\imath_N\\ 0 & \text{otherwise.}\end{cases}$$

Moreover $g=L_N(\lambda,\psi)G^{*}_{\lambda,N}(Nz,0;\psi)$ *satisfies* $g|_{\lambda}\gamma=\psi(\gamma)g$ *for any* $\gamma\in\Gamma_0(N)$*; especially,* $g\in\mathcal{M}_{\lambda}(\Gamma_0(N),\psi)$ *if* $\lambda\neq 2$ *or* $\psi\neq\imath_N$.

By evaluating, the series in Th. 6.1 for $\lambda>0$ at $s=1-\lambda$, we have

COROLLARY 6.3. — *We have for each* $\lambda>0$ *that*

$$\pi^{-1}N2^{-\lambda}(\sqrt{-1})^{\lambda}L_N(2-\lambda,\psi)(Ny)^{1-\lambda}G^{*}_{\lambda,N}(Nz,1-\lambda;\psi)$$
$$=\pi^{-1}2^{-\lambda}N^{1-\frac{\lambda}{2}}(\sqrt{-1})^{\lambda}L_N(2-\lambda,\psi)(y^{1-\lambda}E^{*}_{\lambda,N}(z,1-\lambda;\psi))|_{\lambda}\tau_N$$
$$=\frac{1}{2}L_N(1-\lambda,\psi)+\sum_{n=1}^{\infty}\sum_{0<d|m}\psi(d)d^{\lambda-1}e(nz).$$

This function gives an element of $\mathcal{M}_{\lambda}(\Gamma_0(N),\psi)$.

Now we shall introduce Shimura's differential operator :

$$\delta_\ell = \frac{1}{2\pi i}\Big(\frac{\ell}{2iy} + \frac{d}{dz}\Big),\ \delta_\ell^m = \delta_{\ell+2m-2}\ldots\delta_{\ell+2}\delta_\ell \text{ for } \ell, m \in \mathbf{Z}.$$

For $\gamma \in GL_2(\mathbf{R})$ with $\det(\gamma) > 0$, it satisfies the formula :

$$\delta_\ell^m(f|_\ell\gamma) = (\delta_\ell^m f)|_{\ell+2m}\gamma.$$

A relation between differential operators δ and $d = q\dfrac{d}{dq} = \dfrac{1}{2\pi i}\dfrac{d}{dz}$ is given by

$$(6.6)\quad \delta_m^r = \sum_{t=0}^{r}\binom{r}{t}\frac{\Gamma(m+r)}{\Gamma(m+r-t)}(-4\pi y)^{-t}d^{r-t}$$
$$= \sum_{t=0}^{r}\binom{r}{t}\frac{\Gamma(t-m-r+1)}{\Gamma(1-m-r)}(4\pi y)^{-t}d^{r-t}.$$

We may evaluate the series of Th. 6.1 at $s = -r$ or $s = 1-\lambda+r$ with $0 \le r < \dfrac{\lambda}{2}$. Then, by applying (6.4a,b) and (6.6), we know that for each $0 \le r < \dfrac{\lambda}{2}$:

$$(6.7a)\quad y^{-r}E^*_{\lambda+2r,N}(z,-r;\psi)$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\lambda+r)}(-4\pi)^r\delta_\lambda^r E^*_{\lambda,N}(z,0;\psi)\ ([30,\ (2.9)]),$$
$$(6.7b)\quad y^{1-\lambda-r}E^*_{\lambda+2r,N}(z,1-\lambda-r;\psi)$$
$$= \frac{(-4\pi)^r}{\Gamma(r+1)}\delta_\lambda^r(y^{1-\lambda}E^*_{\lambda,N}(z,1-\lambda;\psi)).$$

This can be also verified directly by using the series expression (6.1) (cf. [29, (2.4)]).

Let L be a positive integer prime to p. We shall consider the Eisenstein measures on Z_L introduced in Example c in § 5 : for each $\chi : (\mathbf{Z}/Lp^\beta\mathbf{Z})^\times \to \overline{\mathbf{Q}}^\times$, we have

$$(6.8a)\quad \int_{Z_L}\chi(z)dG_m = -\delta_2(\chi)\frac{1}{8\pi y}\prod_{\ell|Lp}\Big(1-\frac{1}{\ell}\Big)$$
$$+(2\pi)^{-m}(Lp^\beta)^{\frac{m}{2}}(\sqrt{-1})^m\Gamma(m)L_{Lp}(m,\chi)E^*_{m,Lp}(z,0;\chi)|_m\tau_\beta,$$

$$(6.8b)\quad (1-\chi(b)b^m)^{-1}\int_{Z_L}\chi(z)z_p^{m-1}dE^b$$

$$= \pi^{-1}2^{-m}L^{1-\frac{m}{2}}p^{\beta(1-\frac{m}{2})}(\sqrt{-1})^m L_{Lp}(2-m,\chi) \\ \times(y^{1-m}E^*_{m,Lp}(z,1-m;\chi))|_m\tau_\beta,$$

where $\tau_\beta = \begin{pmatrix} 0 & -1 \\ Lp^\beta & 0 \end{pmatrix}$. Note that the above formulae are valid for any β such that the finite order character $\chi : Z_L \to \overline{\mathbf{Q}}^\times$ factors through $(\mathbf{Z}/Lp^\beta\mathbf{Z})^\times$. We put

(6.9)

$$E_{m,Lp}(\chi) = (1-\chi(b)b^m)^{-1}\int_{Z_L}\chi(z)z_p^{m-1}dE^b$$
$$= \frac{1}{2}L_{Lp}(1-m,\chi) + \sum_{n=1}^{\infty}\sum_{0<d|n}\chi(d)d^{m-1}q^n \quad \text{for } m \geq 1,$$
$$G_{m,Lp}(\chi) = \int_{Z_L}\chi(z)dG_m = \sum_{n=1}^{\infty}\sum_{0<d|n}\chi(d)\left(\frac{n}{d}\right)^{m-1}q^n$$
$$G'_{m,Lp}(\chi) = \delta_2(\chi)\frac{\varphi(Lp)}{8\pi Lpy} + G_{m,Lp}(\chi) \quad \text{for } m \geq 2.$$

These are modular forms in $\mathcal{M}_m(\Gamma_0(Lp^\beta);\overline{\mathbf{Q}})$ except for $G_{2,Lp}(\imath_{Lp})$.

For each integer $m \geq 0$ and for each subalgebra A of $\mathbf{C}$, let $\mathcal{A}^m(A)$ denote the space of functions f on $\mathfrak{H}$ with the Fourier expansion of the following form :

$$f = \sum_{n=0}^{\infty}a(n,y)e\left(\frac{n}{M}z\right) \quad \text{for some } 0 < M \in \mathbf{Z},$$

where $a(n,y)$ is a polynomial in $(4\pi y)^{-1}$ with coefficients in A of degree less than or equal to m. For each congruence subgroup Δ of $SL_2(\mathbf{Z})$ and its finite order character $\psi : \Delta \to A$, we define a subspace $\mathcal{A}_k^m(\Delta; A)$ (resp. $\mathcal{A}_k^m(\Delta,\psi,A)$) of $\mathcal{A}^m(A)$ consisting of functions f in $\mathcal{A}^m(A)$ such that

(i) $f|_k\gamma \in \mathcal{A}^m(\mathbf{C})$ for all $\gamma \in SL_2(\mathbf{Z})$;

(ii) $f|_k\gamma = f$ for all $\gamma \in \Delta$ (resp. $f|_k\gamma = \psi(\gamma)f$ for all $\gamma \in \Delta$).

The following fact is shown in [30, Lemma 7] :

LEMMA 6.4. — *Let ψ be a Dirichlet character modulo N, and suppose that A is a $\mathbf{Q}$-algebra. Then for each $f \in \mathcal{A}_k^m(\Gamma_0(N),\psi;A)$ (resp.*

$f \in \mathcal{A}_k^m(\Gamma_1(N); A)$), if $k > 2m$, we can express

$$f = \sum_{r=0}^{m} \delta_{k-2r}^r f_r \text{ with } f_r \in \mathcal{M}_{k-2r}(\Gamma_0(N), \psi; A)$$

$$(\text{resp. } \mathcal{M}_{k-2r}(\Gamma_1(N); A)).$$

These modular forms are uniquely determined by f and are cusp forms if f is a cusp form.

We write $\mathcal{H}(f)$ for f_0 in the above formula. This gives a map, if $k > 2m$, $\mathcal{H} : \mathcal{A}_k^m(\Gamma_1(N); \overline{\mathbf{Q}}) \to \mathcal{M}_k(\Gamma_1(N); \overline{\mathbf{Q}})$, which will be called the *holomorphic projection*. For each $f = \sum_{r=0}^{m} (4\pi y)^{-r} f^{(r)} \in \mathcal{A}^m(\mathbf{C})$ with $f^{(i)}$ holomorphic, we put $c(f) = f^{(0)}$, which will be called the *constant term* of f. Then c defines a map $c : \mathcal{A}_k^m(\Gamma_1(N); A) \to A[[q]]$.

LEMMA 6.5.

(i) (Shimura) *If f is a cusp form in $S_k(\Gamma_0(N), \psi)$, then $< f, g >_N = < f, \mathcal{H}(g) >_N$ for all $g \in \mathcal{A}_k^m(\Gamma_0(N), \psi; \mathbf{C})$ with $k > 2m$.*

(ii) *If $k > 2m$ and $\ell > 2n$, then $\mathcal{H}(g\delta_k^r h) = (-1)^r \mathcal{H}(h\delta_\ell^r g)$ for all $h \in \mathcal{A}_k^m(\Gamma_1(N), \mathbf{C})$ and $g \in \mathcal{A}_\ell^n(\Gamma_1(N); \mathbf{C})$.*

(iii) *Suppose that N is prime to p and $f \in \mathcal{A}_k^m(\Gamma_1(Np^\beta); K_0)$ for a finite extension $K_0/\mathbf{Q}$. Let K be the closure of K_0 in Ω. If $k > 2m$, then the formal q-expansion $c(f)$ is in fact an element of $\overline{\mathcal{M}}(N; K)$.*

(iv) *Let $g \in \mathcal{A}_\ell^n(\Gamma_1(Np^\beta); K_0)$ and $h \in \mathcal{M}_k(Np^\beta); K_0)$, and let e be the projection to the ordinary part on $\overline{\mathcal{M}}(N; K)$. If $\ell > 2n$, then $e(\mathcal{H}(g\delta_k^r h)) = e(c(g) d^r h)$ in $\overline{\mathcal{M}}(N; K)$ for $d = q\dfrac{d}{dq}$.*

Proof. — The assertion (i) is given in [30, Lemma 6]. The second assertion follows from the argument which proves [11, Lemma 5.3]. If we write $f \in \mathcal{A}_k^m(\Gamma_1(Np^\beta); K_0)$ as $f = \sum_{r=0}^{m} \delta_{k-2r}^r f_r$ for $f_r \in \mathcal{M}_{k-2r}(\Gamma_1(Np^\beta); K_0)$, then we have $c(f) = \sum_{r=0}^{m} d^r f_r$ by comparing the constant term of the both sides. This shows the assertion (iii). To see (iv), we write

$$g\delta_k^r h = \sum_{j=0}^{t} \delta_{s-2j}^j f_j \quad \text{for} \quad t = n + r \quad \text{and} \quad s = k + \ell + 2r.$$

Then by the formula (6.6), we have that

$$c(g\delta_k^r h) = \mathcal{H}(g\delta_k^r h) + \sum_{j=1}^{t} d^j f_j = c(g)d^r h.$$

Note that $e \circ d = 0$ (cf. [11, (6.13)]). This shows the assertion (iv).

The following fact is a modification of a result of Shimura [30, Th. 2] and [31, p. 217] :

THEOREM 6.6. — *Let* $h \in S_k(\Gamma_0(Lp^\beta), \psi)$ *and* $g \in \mathcal{M}_\ell(\Gamma_0(Lp^\beta), \xi)$. *Then if* $0 \leq m\frac{1}{2}(k-\ell)$,

$$\mathcal{D}_{Lp}(\ell+m, h, g)$$
$$= t < h^\rho|_k\tau_{Lp^\beta}, \mathcal{H}((g|_\ell\tau_{Lp^\beta})\delta_{k-\ell-2m}^m(E_{k-\ell-2m,Lp}(\xi\psi))) >_{Lp^\beta}$$

and if $\frac{1}{2}(k-\ell) \leq m < k-\ell$,

$$\mathcal{D}_{Lp}(\ell+m, h, g)$$
$$= t < h^\rho|_k\tau_{Lp^\beta}, \mathcal{H}((g|_\ell\tau_{Lp^\beta})\delta_{\ell-k+2m+2}^{k-\ell-m-1}(G'_{\ell-k+2m+2,Lp}(\xi\psi))) >_{Lp^\beta},$$

where $t = 2^{k+j}\pi^{j+1}(Lp^\beta)^{\frac{1}{2}(k-j-2)}(\sqrt{-1})^{\ell-k}(\Gamma(m+1)\Gamma(j-m))^{-1}$ *for* $j = \ell + 2m$.

Proof. — We shall give a proof of the case when $\frac{1}{2}(k-\ell) \leq r < k-\ell$, since the other case can be treated similarly. By the Rankin-Selberg convolution, we have (cf. [30, (2.4)]) that

$$(4\pi)^{-s}\Gamma(s)\mathcal{D}_{Lp}(s,h,g) = L_{Lp}(2s+2-k-\ell, \xi\psi)$$
$$< h^\rho, gE_{k-\ell,Lp}^*(z, s+1-k; \xi\psi)y^{s+1-k} >_{Lp^\beta}$$

and thus,

$$\mathcal{D}_{Lp}(\ell+m, h, g) =< h^\rho, g \cdot E >_{Lp^\beta},$$

where $E = C_0 L_{Lp}(\ell-k+2+2m, \xi\psi)E_{k-\ell,Lp}^*(z, \ell+m+1-k; \xi\psi)y^{\ell-k+m+1}$ with a constant $C_0 \neq 0$. Write $r = k-\ell-m-1$ and $\lambda = k-\ell-2r$. Then $E|\tau_{Lp^\beta} = C_1\delta_\lambda^r G'_{\lambda,Lp}(\xi\psi)$ for a non-zero constant C_1 by Cor. 6.3 and (6.7a). We see easily that

$$< h^\rho, gE >_{Lp^\beta} =< h^\rho|\tau, (g|\tau)(E|\tau) >_{Lp^\beta} \text{ for } \tau = \tau_{Lp^\beta}.$$

This shows the assertion.

Th. 6.6 combined with Lemma 6.4 shows the following result of Shimura [30] and [31] :

Corollary 6.7. — *Suppose that h is primitive of conductor C. Then the number*

$$\frac{\mathcal{D}_{Lp}(\ell+m,h,g)}{\pi^{\ell+2m+1} < h,h >_C}$$

is algebraic for each integer m with $0 \le m < k-\ell$.

7. Duality theorems.

We shall study here the $\mathcal{O}_K$-dual space of $\mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$ and generalize Th. 1.3. This duality theorem is crucial to the construction of the convoluted measures which will be constructed in the next section.

Write C_K for $C(\Gamma;\mathcal{O}_K)$ for simplicity. As explained in the beginning of § 3, we have a duality :

$$\text{(7.1)} \qquad \mathrm{Hom}_{\mathcal{O}_K}(C_K,\mathcal{O}_K) \simeq \Lambda_K,\ \mathrm{Hom}_{\mathcal{O}_K}(\Lambda_J,\mathcal{O}_K) \simeq C_K.$$

We shall fix a topological generator u of Γ. Let $\mathcal{A}$ be a reduced algebra finite flat over Λ_K. Let M be a compact $\mathcal{A}$-module and put $M^* = \mathrm{Hom}_{\mathcal{O}_K}(M,\mathcal{O}_K)$. Then we can define a pairing

$$\text{(7.2)} \qquad <,>: M \times M^* \to \mathcal{A}^*$$

by $< m,m^* > (a) = m^*(a\cdot m)$ for $m \in M,\ m^* \in M^*$ and $a \in \mathcal{A}$.

Proposition 7.1. — *Suppose that there exists a projective system $\{M_i,\rho_{ij}\}_{i,j\in N}$ of $\mathcal{A}$-modules such that*

(i) *$M \simeq \varprojlim_i M_i$ as $\mathcal{A}$-module;*

(ii) *ρ_{ij} is surjective for all $i \ge j$;*

(iii) *The $\mathcal{O}_K$-module M_i is free of finite rank for each i.*

Then we have that $M^* \simeq \varprojlim_m ((\varinjlim_i M_i^*) \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^m\mathcal{O}_K)$ and $(M^*)^* \simeq M$ as $\mathcal{A}$-module. Moreover the pairing (7.2) induces isomorphisms : $\mathrm{Hom}_{\mathcal{A}}(M,\mathcal{A}^*) \simeq M^*$ and $\mathrm{Hom}_{\mathcal{A}}(M^*,\mathcal{A}^*) \simeq M$.

This fact may be well known but it is important for the sequel; so, we shall give a proof. If M is finite flat over Λ_K, the conditions (i), (ii) and (iii) can be verified easily, and they are also satisfied by $\mathbf{h}(N;\mathcal{O}_K)$ and $\mathrm{Meas}(X;\mathcal{O}_K)$ for a p-adic space X as in § 5 (for $\mathcal{A}=\mathcal{O}_K[[Z_J]]$ or Λ_K).

Proof. — Since $\mathrm{Hom}_{\mathcal{A}}(M,\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{A},\mathcal{O}_K))\simeq \mathrm{Hom}_{\mathcal{O}_K}(M\otimes_{\mathcal{A}}\mathcal{A},\mathcal{O}_K)$ $\simeq M^*$ naturally, the last assertion follows from the first. By the assumption (ii), the adjoint map $\rho_{ij}^*: M_i^*\to M_j^*$ is injective for each pair $j>i$. Put $E=\varinjlim_i M_i^*$. Then we see from [1, II.6.6] that

$$M=\varprojlim_i M_i=\varprojlim_i \mathrm{Hom}_{\mathcal{O}_K}(M_i^*,\mathcal{O}_K)=\mathrm{Hom}_{\mathcal{O}_K}(E,\mathcal{O}_K).$$

On the other hand, for each pair $m>n$, we have a commutative diagram of natural maps :

$$\begin{array}{ccccccc}
0 & \longrightarrow & p^{m-1}E/p^mE & \longrightarrow & E/p^mE & \xrightarrow{p} & E/p^mE \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & p^{n-1}E/p^nE & \longrightarrow & E/p^nE & \xrightarrow{p} & E/p^nE.
\end{array}$$

Obviously, $\varprojlim_m p^{m-1}E/p^mE=0$ and thus $\overline{E}=\varprojlim_m E/p^mE$ is without p-torsion. There is a natural map :

$$\mathrm{Hom}_{\mathcal{O}_K}(\overline{E},\mathcal{O}_K)\longrightarrow \mathrm{Hom}_{\mathcal{O}_K}(E,\mathcal{O}_K),$$

which is bijective because E is dense in $\overline{E}$ and every $\mathcal{O}_K$-linear form is uniformly continuous. Thus we know that

$$M=\mathrm{Hom}_{\mathcal{O}_K}(\overline{E},\mathcal{O}_K)=\overline{E}^*.$$

By the assumption (iii), M_i/p^mM_i and $M_i^*/p^mM_i^*$ are mutually Pontryagin dual modules. Thus we know that

$$\begin{aligned}E/p^mE &\simeq \mathrm{Hom}_{\mathbf{Z}_p}(\varprojlim_i M_i/p^mM_i,\mathbf{Z}/p^m\mathbf{Z})\\ &\simeq \mathrm{Hom}_{\mathbf{Z}_p}(M/p^mM,\mathbf{Z}/p^m\mathbf{Z})\simeq \mathrm{Hom}_{\mathbf{Z}_p}(M,\mathbf{Z}/p^m\mathbf{Z}).\end{aligned}$$

Therefore we see that

$$E=\varprojlim_m E/p^mE=\varprojlim_m \mathrm{Hom}_{\mathbf{Z}_p}(M,\mathbf{Z}/p^m\mathbf{Z})=\mathrm{Hom}_{\mathbf{Z}_p}(M,\mathbf{Z}_p)$$

([1, II.6.3.Prop.5]).

Note that $M^*\simeq \mathrm{Hom}_{\mathcal{O}_K}(M,\mathcal{O}_K)\simeq \mathrm{Hom}_{\mathcal{O}_K}(M,\mathrm{Hom}_{\mathbf{Z}_p}(\mathcal{O}_K,\mathbf{Z}_p))\simeq$ $\mathrm{Hom}_{\mathbf{Z}_p}(M,\mathbf{Z}_p)$. This shows that $\overline{E}=M^*$. Note that

$$(M^*)^*=\mathrm{Hom}_{\mathcal{O}_K}(M^*,\mathcal{O}_K)\simeq \mathrm{Hom}_{\mathcal{O}_K}(\overline{E},\mathcal{O}_K)\simeq \mathrm{Hom}_{\mathcal{O}_K}(E,\mathcal{O}_K)\simeq M.$$

This finishes the proof.

COROLLARY 7.2. — *Let M and E be compact $\mathcal{A}$-modules. Suppose that M satisfies the condition of Proposition 7.1 and $E \simeq \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}^*)$ as $\mathcal{A}$-module. Then if we denote by $<,>$ the pairing : $M \times E \to \mathcal{A}^*$ which gives the above isomorphism, then the pairing*

$$<,>_0 : M \times E \to \mathcal{O}_K$$

defined by $< m, e >_0 = < m, e > (1)$ gives an isomorphism : $E \simeq M^*$.

This is a direct consequence of Proposition 7.1.

PROPOSITION 7.3. — *Let M be a compact $\mathcal{A}$-module, and let $\mathfrak{a}$ be an ideal of $\mathcal{A}$. Put $M^*[\mathfrak{a}] = \{m^* \in M^* | \alpha \cdot m^* = 0 \text{ for all } \alpha \in \mathfrak{a}\}$. Then we have a natural isomorphism : $(M/\mathfrak{a}M)^* \simeq M^*[\mathfrak{a}]$.*

Proof. — Since $\mathcal{A}$ is noetherian, we may choose a finitely many generators $\alpha_1, \ldots, \alpha_r$ of $\mathfrak{a}$. Then by definition, we have that $M/\mathfrak{a}M = M/\alpha_1 M + \ldots + \alpha_r M$ and $M^*[\mathfrak{a}] = \bigcap_i M^*[\alpha_i]$. We consider an exact sequence :

$$\begin{array}{ccccc} M^r & \longrightarrow & M & \longrightarrow & M/\mathfrak{a}M \longrightarrow 0, \\ \Psi & & \Psi & & \\ (m_1, \ldots, m_r) & \mapsto & \alpha_1 m_1 + \ldots + \alpha_r m_r & & \end{array}$$

which yields another :

$$\begin{array}{ccccc} 0 \longrightarrow (M/\mathfrak{a}M)^* & \longrightarrow & M^* & \longrightarrow & (M^*)^r \\ & & \Psi & & \Psi \\ & & m & \longmapsto & \bigoplus_{i=1}^{r} \alpha_i m. \end{array}$$

This shows the result.

Now we shall return to the situation of § 5 : Let $\mathcal{K}$ be a finite extension of $\mathcal{L}_K$ defined over K and $\mathcal{I}$ be the integral closure of Λ_K in $\mathcal{K}$. Put $\mathcal{I}^* = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{I}, \mathcal{O}_K)$ and $\hat{\mathcal{I}} = \mathrm{Hom}_{\Lambda_K}(\mathcal{I}, \Lambda_K)$. They are naturally $\mathcal{I}$-modules. By Lemma 3.1 and (7.1), we know that as Λ_K-modules, $\mathcal{I}^* \simeq C_K^d, \hat{\mathcal{I}} \simeq \Lambda_K^d$ for $d = d(\mathcal{I}) = [\mathcal{K} : \mathcal{L}_K]$. Let N be a positive integer prime to the fixed prime $p \geq 5$, and write

$$<,> : \mathbf{h}(N; \mathcal{O}_K) \times \overline{S}(N; \mathcal{O}_K) \to \Lambda_K^* = C_K$$

for the pairing defined by $< h, f > (\gamma) = a(1, f|h\gamma)(\gamma \in \Gamma)$. Then, Th. 1.3 combined with Prop 7.1 shows that

$$\text{(7.3)} \qquad \mathbf{h}(N;\mathcal{O}_K) \simeq \mathrm{Hom}_{\Lambda_K}(\overline{S}(N;\mathcal{O}_K), C_K),$$
$$\overline{S}(N;\mathcal{O}_K) \simeq \mathrm{Hom}_{\Lambda_K}(\mathbf{h}(N;\mathcal{O}_K), C_K)$$

as Λ_K-modules under $<,>$.

THEOREM 7.4. — *Let M be a compact Λ_K-module satisfying the conditions of Prop. 7.1 for $\mathcal{A} = \Lambda_K$. Then we have a canonical isomorphism of $\mathcal{I}$-modules : $(M \otimes_{\Lambda_K} \mathcal{I})^* \simeq M^* \otimes_{\Lambda_K} \hat{\mathcal{I}}$. In particular, we have a canonical isomorphism of modules over $\mathbf{h}(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$:*

$$\mathrm{Hom}_{\mathcal{O}_K}(\overline{S}(N;\mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}}, \mathcal{O}_K) \simeq \mathbf{h}(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I},$$

which is given by the pairing

$$(,) : (\mathbf{h}(N;\mathcal{O}_K) \otimes_{\Lambda_N} \mathcal{I}) \times (\overline{S}(N;\mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}}) \to \mathcal{O}_K$$

defined by $(h \otimes i, f \otimes \varphi) = < h, f > (\varphi(i))$ for $i \in \mathcal{I}, h \in \mathbf{h}(N;\mathcal{O}_K)$ and $f \in \overline{S}(N;\mathcal{O}_K)$.

(The tensor product $h(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$ and $\overline{S}(N;\mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}}$ are automatically complete under the p-adic topology, since the Λ_K-modules $\mathcal{I}$ and $\hat{\mathcal{I}}$ are free of finite rank (Lemma 3.1).).

Proof. — We simply write Λ for Λ_K. The result follows from the following formal calculation and Cor. 7.2 :

$$\begin{array}{ll}
\mathrm{Hom}_{\mathcal{I}}(M \otimes_\Lambda \mathcal{I}, \mathcal{I}^*) & \\
\simeq \mathrm{Hom}_{\mathcal{I}}(M \otimes_\Lambda \mathcal{I}, \mathrm{Hom}_\Lambda(\mathcal{I}, \Lambda^*)) & \text{(Prop. 7.1)} \\
\simeq \mathrm{Hom}_\Lambda(M \otimes_\Lambda \mathcal{I} \otimes_{\mathcal{I}} \mathcal{I}, \Lambda^*) & \text{[1, II.4.1]} \\
\simeq \mathrm{Hom}_\Lambda(M \otimes_\Lambda \mathcal{I}, \Lambda^* \otimes_\Lambda \Lambda) & \\
\simeq \mathrm{Hom}_\Lambda(M, \Lambda^*) \otimes_\Lambda \mathrm{Hom}_\Lambda(\mathcal{I}, \Lambda) & \text{(Lemma 3.1, [1, II.4.4])} \\
\simeq M^* \otimes_\Lambda \hat{\mathcal{I}} & \text{(Prop. 7.1),}
\end{array}$$

since $M \otimes_\Lambda \mathcal{I}$ again satisfies the condition of Prop. 7.1 for $\mathcal{A} = \mathcal{I}$. The fact that the isomorphism given for $\overline{S}(N;\mathcal{O}_K)$ is a morphism of modules over $\mathbf{h}(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$ follows from the explicit form of the pairing given in the theorem.

COROLLARY 7.5. — *Let M be a compact Λ_K-module satisfying the condition of Prop. 7.1. For each $P \in \mathcal{X}(\mathcal{I};\mathcal{O}_K)$, we have a canonical* *isomorphism of $\mathcal{I}$-modules :* $(M^* \otimes_{\Lambda_K} \hat{\mathcal{I}})[P] \simeq M^*[P \cap \Lambda_K]$. *In particular, if* $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_K)$ *and* $k(P) \geq 2$ *then*

$$(\overline{S}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}})[P] \simeq S^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P; \mathcal{O}_K) \text{ canonically}$$

as $\mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$*-modules.*

Proof. — From Th. 7.4, we know that $(M \otimes_{\Lambda_K} \mathcal{I})^* \simeq M^* \otimes_{\Lambda_K} \hat{\mathcal{I}}$. By Prop. 7.3, we have that $(M \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I})^* = (M \otimes_{\Lambda_K} \mathcal{I} \otimes_{\mathcal{I}} \mathcal{I}/P\mathcal{I})^* \simeq (M^* \otimes_{\Lambda_K} \mathcal{I})[P]$. On the other hand, we know that $M \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I} \simeq M \otimes_{\Lambda_K} \Lambda_K/(P \cap \Lambda_K)$, since $\mathcal{I}/P\mathcal{I} \simeq \Lambda_K/(P \cap \Lambda_K)(\simeq \mathcal{O}_K)$ naturally as Λ_K-modules. Again by Prop. 7.3, we have that $(M \otimes_{\Lambda_K} (\Lambda_K/(P \cap \Lambda_K)))^* \simeq M^*[P \cap \Lambda_K]$, which proves the first assertion. Then Th. 2.1, Prop. 1.2 and Prop. 7.3 show the last assertion since $P \cap \Lambda_K = P_{k(P),\varepsilon_P}$ by definition.

Remark 7.6. — For each $P \in \mathcal{X}(\mathcal{I}; \mathcal{O}_K)$, we have a unique $\mathcal{O}_K$-algebra isomorphism $i_P : \mathcal{I}/P\mathcal{I} \simeq \mathcal{O}_K$ by definition. We thus have a canonical element $i_P \in (\mathcal{I}/P\mathcal{I})^*$. The identification of $(\mathcal{I}/P\mathcal{I})^*$ with $\mathcal{O}_K$ as in the proof of Cor. 7.5 is explicitly given by : $\mathcal{O}_K \ni a \mapsto a \cdot i_P \in (\mathcal{I}/P\mathcal{I})^*$. For each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_K)$ with $k(P) \geq 2$, if we write the natural algebra homomorphism for $k = k(P), r = r(P)$ and $\varepsilon = \varepsilon_P$ as

$$\pi_{k,\varepsilon} : \mathbf{h}^o(N; \mathcal{O}_K) \to \mathbf{h}^o_k(\Phi_r, \varepsilon; \mathcal{O}_K) = \mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \Lambda_K/P_{k,\varepsilon}\Lambda_K$$

and the natural inclusion as

$$\imath_{k,\varepsilon} : S^o_k(\Phi_r, \varepsilon; \mathcal{O}_K) \to \overline{S}^o(N; \mathcal{O}_K),$$

then we have that

$$\begin{aligned} (7.4) \qquad (h \otimes i, \imath_{k,\varepsilon}(f) \otimes b i_P) &= < \pi_{k,\varepsilon}(h) \otimes (i \bmod P), bf > \\ &= b(i \bmod P) a(1, f | \pi_{k,\varepsilon}(h)) \end{aligned}$$

for $h \in \mathbf{h}^o(N; \mathcal{O}_K)$, $i \in \mathcal{I}$, $b \in \mathcal{O}_K$ and $f \in S^o_k(\Phi_r, \varepsilon; \mathcal{O}_K)$, where the pairing $(,)$ is as in Th. 7.4 and $<,>$ is as in Prop. 1.2.

COROLLARY 7.7. — *Let* M *be a compact* Λ_L*-module satisfying the conditions of Prop. 7.1 for* $\mathcal{A} = \Lambda_K$. *Let* $\varphi : M^* \to \overline{S}^o(N; \mathcal{O}_K)$ *be a* Λ_K*-linear map. Put* $\hat{\varphi} = \varphi \otimes \mathrm{id} : M^* \otimes_{\Lambda_K} \hat{\mathcal{I}} \to \overline{S}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}}$. *Then for each* $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_K)$ *with* $k(P) \geq 2$, *we have the following commutative diagram :*

$$\begin{array}{ccc} (M^* \otimes_{\Lambda_K} \hat{\mathcal{I}})[P] & \xrightarrow{\varphi} & (\overline{S}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}})[P] \\ \wr\| & & \wr\| \\ M^*[P_{k(P),\varepsilon_P}] & \xrightarrow{\varphi} & S^o_k(\Phi_{r(P)}, \varepsilon_P; \mathcal{O}_K), \end{array}$$

where the vertical isomorphisms are those of Cor. 7.5.

This is a direct consequence of Cor. 7.5.

Now we shall define a key linear form :

$$\ell = \ell_\lambda : \overline{S}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}} \to \mathcal{O}_K$$

for each primitive homomorphism $\lambda : \mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$. Let $\mathcal{C}_0(\lambda;\mathcal{I})$ be the module of congruences as in (4.3), and let $H \in \mathcal{I}(H \neq 0)$ be the element chosen in § 5 before Th. 5.1 which annihilates $\mathcal{C}_0(\lambda;\mathcal{I})$. Since λ is primitive, λ induces a decomposition of $\mathcal{K}$-algebra : $\mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{K} = \mathcal{K} \oplus \mathcal{A}$ as in Th. 4.2. Let $1_\mathcal{K}$ be the idempotent corresponding to the first factor. Then, by the very definition in (4.3), we know that $H \cdot 1_\mathcal{K} \in \mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$. Then, by Th. 7.4, we shall define ℓ by

$$(7.5) \qquad \ell(g) = \ell_\lambda(g) = (H \cdot 1_\mathcal{K}, g) \quad \text{for} \quad g \in \overline{S}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}}.$$

PROPOSITION 7.8. — *Let $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ with $k(P) \geq 2$. Let $\lambda : \mathbf{h}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$ be a primitive homomorphism of $\mathcal{I}$-algebras and $\lambda_P : \mathbf{h}^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P;\mathcal{O}_K) \to \mathcal{O}_K$ be the induced $\mathcal{O}_K$-algebra homomorphism. Let $\mathbf{h}^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P;K) = K_P \oplus A_P$ be the decomposition of K-algebra induced by λ_P as in (4.2) and 1_P be the idempotent of the factor $K_P(\simeq K)$. Then the restriction of ℓ_λ to the subspace $(\overline{S}^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}})[P](\simeq S^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P;\mathcal{O}_K))$ is given by $\ell_\lambda(g) = < H(P) \cdot 1_P, g > = H(P)a(1, g|1_P)$ for $g \in S^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P;\mathcal{O}_K)$. Especially $H(P) \cdot 1_P \in \mathbf{h}^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P;\mathcal{O}_K)$.*

This is obvious from Th. 4.2, (7.4) and the definition (7.5). If the Gorenstein condition (4.5a) is satisfied, Th. 4.4 (4.6c) asserts that $H(P)$ gives the exact denominator of 1_P in $\mathbf{h}^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P;K)$ relative to $\mathbf{h}^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P;\mathcal{O}_K)$.

We define a linear form for $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ with $k(P) \geq 2$

$$\ell_P = \ell_{f_P} : S^o_{k(P)}(\Phi_{r(P)}, \varepsilon_P;K) \to K \text{ by } \ell_P(g) = a(1, g|1_P)$$

with the notation of Prop. 7.8. This linear form is studied in [11, § 4], and we have a formula for $g \in S_{k(P)}(\Gamma_0(Np^n), \varepsilon_P\psi\omega^{-k};\overline{\mathbf{Q}})$ with each $n \geq r(P)$.

$$(7.6) \qquad \ell_P(g|e) = a(p, f_P)^{r(P)-n} p^{(n-r(P))(k-1)} \frac{< h_P|[p^{n-r(P)}], g >_{Np^n}}{< h_P, f_P >_{Np^{r(P)}}},$$

where $h_P = f_P^\rho|_k \begin{pmatrix} 0 & -1 \\ Np^r & 0 \end{pmatrix}$ and $[p^\alpha]$ is the operator defined in § 1.III ($g|e$ is known to belong to $S_{k(P)}(\Gamma_0(Np^{r(P)}), \varepsilon_P\psi\omega^{-k}; \overline{\mathbf{Q}})$ for any $n \geq r(P)$ [11, Prop. 4.1], and thus $\ell_P(g|e)$ is well defined).

8. Convoluted measures.

In this section, we shall develop a general theory of p-adic convolution of a measure over the group Z_L and a generalized measure as in § 3. Let J be a positive integer prime to p and put

$$Z_J = \varprojlim_r (\mathbf{Z}/Jp^r\mathbf{Z})^\times = \Gamma \times (\mathbf{Z}/Jp\mathbf{Z})^\times \text{ for } \Gamma \cong 1 + p\mathbf{Z}_p \subset \mathbf{Z}_p^\times.$$

Let $\mathcal{A}_J = \mathcal{O}_K[[Z_J]]$ be the continuous group algebra of Z_J. Take a positive integer L which is a multiple of J and is prime to p. Let M (resp. U and V) be a compact $\mathcal{A}_J$-module (resp. compact $\mathcal{A}_L$-modules) satisfying the conditions of Prop. 7.1 for $\mathcal{A} = \mathcal{A}_J$ (resp. $\mathcal{A} = \mathcal{A}_L$). For our later application, we take as M either Meas $(X; \mathcal{O}_K)$ for a p-adic space X, $\mathcal{I}$ as in § 5 or $\mathcal{I}\hat{\otimes}_{\mathcal{O}_K}\mathcal{J}$ as in Th. 5.1d. We shall fix a continuous character $\alpha : Z_L \to \mathcal{O}_K^\times$ and define the action of Z_L on $C(Z_L; \mathcal{O}_K)$ (twisted by α) by

$$(\phi|_\alpha z)(x) = \alpha(z)\phi(zx) \quad \text{for} \quad \phi \in C(Z_L; \mathcal{O}_K).$$

Thus we allow the twisted action of Z_L on $C(Z_L; \mathcal{O}_K)$ which may differ from the usual one. Let $E : C(Z_L; \mathcal{O}_K) \to U^* = \mathrm{Hom}_{\mathcal{O}_K}(U, \mathcal{O}_K)$ and $\varphi : M^* \to V^*$ be $\mathcal{A}_L$-linear maps (for the twisted action on $C(Z_L; \mathcal{O}_K)$). Here we consider M^* as an $\mathcal{A}_L$-module via the natural projection : $\mathcal{A}_L \to \mathcal{A}_J$. We suppose that *there is an* $\mathcal{O}_K$*-linear map* $\mathbf{m} : U^* \otimes_{\mathcal{O}_K} V^* \to \overline{S}(L; \mathcal{O}_K)$ *such that*

$$\mathbf{m}(u \otimes v)|a = \mathbf{m}((u|a) \otimes (v|a)) \quad \text{for} \quad a \in \mathcal{A}_K.$$

As an example of U^* and V^*, we may take

$$U^* = \overline{\mathcal{M}}(L; \mathcal{O}_K) \quad \text{and} \quad V^* = \overline{S}(L; \mathcal{O}_K).$$

Note that $\overline{M}(L; \mathcal{O}_K)$ is a topological ring with the product induced by $\mathcal{O}_K[[q]]$ (which is the usual product of modular forms : $(f, g) \mapsto f \cdot g$), $\overline{S}(L; \mathcal{O}_K)$ is an ideal of $\overline{\mathcal{M}}(L; \mathcal{O}_K)$ and the multiplication of $\overline{\mathcal{M}}(L; \mathcal{O}_K)$ satisfies the condition of $\mathbf{m}$ as above. In applications in this paper, we always work under this choice of U^* and V^*. However, if one considers also modular forms of half integral weight, there is another example of U^* and V^* different from those treated in this paper. The case of half integral weight will be studied in detail in our subsequent paper. Thus, this application in mind, we shall treat the topic a bit more in general than what is necessary here.

We shall naturally extend φ to $\varphi \otimes \mathrm{id} : M^* \otimes_{\mathcal{O}_K} U^* \to V^* \otimes_{\mathcal{O}_K} U^*$ and define

$$\tilde{\varphi} : M^* \hat{\otimes}_{\mathcal{O}_K} U^* \to \overline{S}(L; \mathcal{O}_K) \text{ by } \tilde{\varphi} = \mathbf{m} \circ (\varphi \otimes \mathrm{id}), \tag{8.1}$$

where $M^* \hat{\otimes} U^*$ is the p-adic completion $\varprojlim (M^* \otimes U^*)/p^j(M^* \otimes U^*)$ which may not equal to its profinite completion. We say that a function $\phi : M \to U^*$ is continuous if it is continuous under the p-adic topology on U^* and the topology of the profinite group M. Thus if ϕ is $\mathcal{O}_K$-linear, the ϕ is continuous if and only if there exists $i > 0$ for any $j > 0$ such that ϕ mod $p^j : M/p^jM \to U^*/p^jU^*$ factors through M_i/p^jM_i, where $M = \varprojlim M_i$ as in Proposition 7.1. Then, if $U = \varprojlim U_k$, the image of ϕ mod p^j is actually contained in $U_k^*/p^jU_k^*$ for some k. We denote by $\mathrm{Hom}_c(M, U^*)$ the space of all continuous $\mathcal{O}_K$-linear maps. Then we have

$$\begin{aligned}\mathrm{Hom}_c(M, U^*) &= \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathcal{O}_K}(M_i/p^jM_i, U^*/p^jU^*) \\ &= \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathcal{O}_K}(M_i/p^jM_i \otimes_{\mathcal{O}_K} U, \mathcal{O}_K/p^j\mathcal{O}_K) = (M \hat{\otimes}_{\mathcal{O}_K} U)^*,\end{aligned}$$

where $M \hat{\otimes}_{\mathcal{O}_K} U$ is the profinite completion of $M \otimes_{\mathcal{O}_K} U$.

LEMMA 8.1. — $M^* \hat{\otimes}_{\mathcal{O}_K} U^* = \varprojlim (M^* \otimes U^*)/p^j(M^* \otimes U^*)$

$$\cong \mathrm{Hom}_c(M, U^*) \cong (M \hat{\otimes}_{\mathcal{O}_K} U)^*.$$

Proof. – We have a natural map : $M^* \otimes_{\mathcal{O}_K} U^* \to \mathrm{Hom}_c(M, U^*)$ given by $\phi \otimes u \mapsto (m \mapsto \phi(m)u)$. By definition, $M \hat{\otimes}_{\mathcal{O}_K} U = \varprojlim_{ij} M_i \otimes U_j$ satisfies the assumption of Proposition 7.1 and thus

$$(M \hat{\otimes}_{\mathcal{O}_K} U)^* \cong \varprojlim_m (\varinjlim M_i^* \otimes U_j^*) \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^m\mathcal{O}_K = M^* \hat{\otimes}_{\mathcal{O}_K} U^*,$$

which proves the assertion.

Hereafter by Lemma 8.1, we shall identify $M^* \hat{\otimes}_{\mathcal{O}_K} U^*$ with $\mathrm{Hom}_c(M, U^*)$. We let Z_L act on $C(M \times Z_L; \mathcal{O}_K)$ by

$$(F|z)(m, x) = F(z^{-1}m, zx) \quad \text{for} \quad F \in C(M \times Z_L; \mathcal{O}_K).$$

We then define a function $E_*(F) : M \to U^*$ for each $F \in C(M \times Z_L; \mathcal{O}_K)$ by $E_*(F)(m) = \int_{Z_L} (F|z)(m,1)dE(z)$ for $m \in M$. We equip $C(M \times Z_L; \mathcal{O}_K)$ with the uniform norm : $\|F\| = \sup_{m,z} |F(m,z)|_p$. Since M and Z_L are compact, this space becomes a compact Banach space. The compactness of $M \times Z_L$ assures the uniform continuity of F. Thus for any $\varepsilon > 0$, we find a neighbourhood H of 0 in M such that if $m - m' \in H$, the norm of the function : $Z_L \ni z \mapsto (F|z)(m,1) - (F|z)(m',1)$ in $C(Z_L, \mathcal{O}_K)$ is smaller than ε. This shows that $E_*(F) \in C(M, U^*)$. For any $F \in M^* \hat{\otimes}_{\mathcal{O}_K} C(Z_L; \mathcal{O}_K)$, by regarding F as a function on $M \times Z_L$, we know that $E_*(F) \in \mathrm{Hom}_{\mathcal{O}_K}(M, U^*) = M^* \hat{\otimes}_{\mathcal{O}_K} U^*$. Thus we have a $\mathcal{O}_K$-linear map :

$$E_* : M^* \hat{\otimes}_{\mathcal{O}_K} C(Z_L; \mathcal{O}_K) \to \mathrm{Hom}_c(M, U^*) \simeq M^* \hat{\otimes}_{\mathcal{O}_K} U^*.$$

We shall define

$$E * \varphi : M^* \hat{\otimes}_{\mathcal{O}_K} C(Z_L; \mathcal{O}_K) \to \overline{S}(L; \mathcal{O}_K) \tag{8.2}$$

by $(E * \varphi)(F) = \tilde{\varphi}(E * (F)) \in \overline{S}(L; \mathcal{O}_K)$ for $\tilde{\varphi}$ as in (8.1).

We shall define another action of Z_L on $C(M \times Z_L; \mathcal{O}_K)$ by

$$(\phi\|_\alpha z)(m,x) = \alpha(z)\phi(m, zx). \tag{8.3}$$

PROPOSITION 8.2. — *The action* (8.3) *of* Z_L *preserves the subspace* $M^* \hat{\otimes}_{\mathcal{O}_K} C(Z_L; \mathcal{O}_K)$ *of* $C(M \times Z_L; \mathcal{O}_K)$, *and the convoluted measure* $E * \varphi$: $M^* \hat{\otimes}_{\mathcal{O}_K} C(Z_L; \mathcal{O}_K) \to \overline{S}(L; \mathcal{O}_K)$ *gives a morphism of* $\mathcal{A}_L$*-modules under the action* (8.3).

Proof. — The first assertion is obvious; so, we shall prove the second one. There is a natural projection map : $Z_L \to Z_J$, and we write $z_J \in Z_J$ for the projected image of $z \in Z_L$. We shall let Z_L act on $M^* \otimes_{\mathcal{O}_K} U^*$ diagonally by $(m \otimes u)|z = (m|z_J) \otimes (u|z)$.

We also define an action of Z_L on $C(M, U^*)$ by

$$(\phi|z)(m) = (\phi(z_J \cdot m))|z,$$

where the last z acts on the value $\phi(z_J \cdot m) \in U^*$ through the action of Z_L on U^*. The natural inclusion : $M^* \hat{\otimes}_{\mathcal{O}_K} U^* \to C(M, U^*)$ is compatible under the action of Z_L. Then by definition, $\tilde{\varphi} : M^* \hat{\otimes}_{\mathcal{O}_K} U^* \to \overline{S}(L; \mathcal{O}_K)$ becomes $\mathcal{A}_L$-equivariant. On the other hand, if we let $z \in Z_L$ act on

$E_*(F) \in M^*\hat{\otimes}_{\mathcal{O}_K} U^*$ through the diagonal action defined as above, we have that

$$(E_*(F)|z)(m) = \Big(\int_{Z_L} (F|x)(zm,1)dE(x)\Big)|z,$$

where in the right-hand side, $z \in Z_L$ acts on $\Big(\int_{Z_L} (F|x)(zm,1) dE(x)\Big) \in U^*$ via its action on U^* for fixed m. Since E is $\mathcal{A}_L$-linear for the twisted action by α, we see that

$$\begin{aligned}(\int_{Z_L} (F|x)(zm,1)dE(x))|z &= \int_{Z_L} \alpha(z)(F|zx)(zm,1)dE(x)\\ &= \int_{Z_L} \alpha(z)F(x^{-1}m, zx)dE(x)\\ &= \int_{Z_L} ((F\|_\alpha z)|x)(m,1)dE(x) \quad \text{(cf. (8.3))}.\end{aligned}$$

This shows that $E_*(F)|z = E_*(F\|_\alpha z)$ and the $\mathcal{A}_L$-equivariance of $E * \varphi = \tilde{\varphi} \circ E_*$ by definition, since $\tilde{\varphi}$ is $\mathcal{A}_L$-linear.

LEMMA 8.3. — *Let $\mathfrak{a}$ be an ideal of $\mathcal{A}_L$, and let $z \in Z_L$ act on $M^*\hat{\otimes}_{\mathcal{O}_K} C(Z_L;\mathcal{O}_K)$ by $(m \otimes \phi)\|_\alpha z = m \otimes (\phi|_\alpha z)$. Then, if $\mathcal{A}_L/\mathfrak{a}$ is free of finite rank over $\mathcal{O}_K$ or Λ_K, we have a natural isomorphism : $C(Z_L;\mathcal{O}_K)[\mathfrak{a}]\hat{\otimes}_{\mathcal{O}_K} M^* \simeq (M^*\hat{\otimes}_{\mathcal{O}_K} C(Z_L;\mathcal{O}_K))[\mathfrak{a}]$.*

Proof. — We shall prove the lemma only when $\mathcal{A}_L/\mathfrak{a}$ is free of finite rank over $\mathcal{O}_K$, since the other case can be dealt with similarly. We have an exact sequence of $\mathcal{O}_K$-modules : $0 \to \mathfrak{a} \to \mathcal{A}_L \to \mathcal{A}_L/\mathfrak{a} \to 0$. Since $\mathcal{A}_L/\mathfrak{a}$ is $\mathcal{O}_K$-free, we have a commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{A}_L/\mathfrak{a})^* & \longrightarrow & \mathcal{A}_L^* & \longrightarrow & \mathfrak{a}^* & \longrightarrow & 0\\ & & \wr\| & & \wr\| & & \| & & \\ 0 & \longrightarrow & C(Z_L;\mathcal{O}_K)[\mathfrak{a}] & \longrightarrow & C(Z_L;\mathcal{O}_K) & \longrightarrow & \mathfrak{a}^* & \longrightarrow & 0,\end{array}$$

where the horizontal rows are split exact sequences. Thus we have an (split) exact sequence :

$$\begin{aligned}0 \to C(Z_L;\mathcal{O}_K)[\mathfrak{a}] \otimes_{\mathcal{O}_K} M^* &\longrightarrow C(Z_L;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K} M^*\\ &\longrightarrow \mathfrak{a}^*\hat{\otimes}_{\mathcal{O}_K} M^* \longrightarrow 0.\end{aligned}$$

The finiteness of rank ${}_{\mathcal{O}}\mathcal{A}_L/\mathfrak{a}$ shows that $(C(Z_L;\mathcal{O}_K)[\mathfrak{a}] \otimes_{\mathcal{O}_K} M^*)^* \simeq M\hat{\otimes}_{\mathcal{O}_K}\mathcal{A}_L/\mathfrak{a} \simeq ((C(Z_L;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K} M^*)[\mathfrak{a}])^*$. This shows the result.

COROLLARY 8.4. — *Let $\psi : Z_L \to \mathcal{O}_K^\times$ be a continuous character and $\psi_a : \mathcal{A}_L \to \mathcal{O}_K$ be the corresponding $\mathcal{O}_K$-algebra homomorphism. Put $P_\psi = \mathrm{Ker}(\psi_a)$. Then $E * \varphi$ induces an $\mathcal{O}_K$-linear form :*

$$M \simeq C(Z_L; \mathcal{O}_K)[P_\psi] \otimes_{\mathcal{O}_K} M^* \longrightarrow \overline{S}(L; \mathcal{O}_K)[P_\psi]$$

which is explicitly given by

$$(E * \varphi)(\phi) = \tilde{\varphi}(m \mapsto E(z \mapsto \psi\alpha^{-1}(z)\phi(z^{-1}m))) \ (m \in M, z \in Z_L).$$

Proof. — The identification : $\mathcal{O}_K \simeq C(Z_L; \mathcal{O}_K)[P_\psi]$ is induced by the correspondence : $1 \mapsto \psi\alpha^{-1}$, since we have let Z_L act on $C(Z_L; \mathcal{O}_K)$ by the twisted action by α. Then we identify M^* with

$$C(Z_L; \mathcal{O}_K)[P_\psi] \otimes_{\mathcal{O}_K} M^* = (C(Z_L; \mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K} M^*)[P_\psi] \text{ by}$$
$$\phi \mapsto ((z, m) \mapsto \psi\alpha^{-1}(z)\phi(m)).$$

Then the assertion follows from the definition of $E * \varphi$.

We shall now decompose canonically $Z_L = \Gamma \times (\mathbf{Z}/Lp\mathbf{Z})^\times$. For each character $\psi : (\mathbf{Z}/Lp\mathbf{Z})^\times \to \mathcal{O}_K^\times$, we put

$$\begin{aligned} C(Z_L; \mathcal{O}_K)[\psi] &= \{\phi \in C(Z_L; \mathcal{O}_K) | \phi(\zeta z) = \psi(\zeta)\phi(z) \\ &\qquad\qquad \text{for } \zeta \in (\mathbf{Z}/Lp\mathbf{Z})^\times\} \\ &= \{\phi \in C(Z_L; \mathcal{O}_K) \big| \phi|_\alpha \zeta = \alpha\psi(\zeta)\phi \\ &\qquad\qquad \text{for } \zeta \in (\mathbf{Z}/Lp\mathbf{Z})^\times\}. \end{aligned}$$

If we denote by $\psi_a : \mathcal{O}_K[(\mathbf{Z}/Lp\mathbf{Z})^\times] \to \mathcal{O}_K$ the $\mathcal{O}_K$-algebra homomorphism induced by ψ, we have an extension

$$\mathrm{id} \otimes \psi_a : \mathcal{A}_L \simeq \Lambda_K \otimes_{\mathcal{O}_K} \mathcal{O}_K[(\mathbf{Z}/Lp\mathbf{Z})^\times] \to \Lambda_K$$

which is surjective. We write $\breve{\alpha}$ for the restriction of α to $(\mathbf{Z}/Lp\mathbf{Z})^\times$. Then by applying Lemma 8.3 to $\mathfrak{a} = \mathrm{Ker}(\mathrm{id} \otimes (\breve{\alpha}\psi)_a)$, we have a canonical isomorphism : $C(Z_L; \mathcal{O}_K)[\psi]\hat{\otimes}_{\mathcal{O}_K} M^* \simeq (C(Z_L; \mathcal{O}_K) \otimes_{\mathcal{O}_K} M^*)[\mathfrak{a}]$. Thus, we know that $E * \varphi$ induces

$$E * \varphi : M^* \hat{\otimes}_{\mathcal{O}_K} (C(Z_L; \mathcal{O}_K)[\breve{\alpha}^{-1}\psi]) \longrightarrow \overline{S}(L; \mathcal{O}_K)[\psi],$$

with the notation of § 1.V.

Let $\lambda : \mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$ be a primitive $\mathcal{I}$-algebra homomorphism, and let $\psi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}_K^\times$ be the character of λ. Let

$\ell = \ell_\lambda : \overline{S}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}} \to \mathcal{O}_K$ be the linear form defined in (7.5). Suppose that

(8.4) *L is divisible by N and J.*

By extending each function $\phi \in C(\Gamma; \mathcal{O}_K)$ to Z_L by

$$\hat{\phi}(z) = \breve{\alpha}^{-1}\psi(z_0)\phi(< z_p >),$$

where z_0 (resp. $< z_p >$) is the projection of $z \in Z_L$ in $(\mathbf{Z}/Lp\mathbf{Z})^\times$ (resp. Γ), we have an isomorphism : $C(\Gamma; \mathcal{O}_K) \simeq C(Z_L; \mathcal{O}_K)[\breve{\alpha}^{-1}\psi]$, which is the adjoint of $\mathrm{id} \otimes (\breve{\alpha}^{-1}\psi)_a$. We shall identify the spaces as above in this manner. We shall consider the $\mathcal{O}_K$-linear map :

$$\Psi : M^*\hat{\otimes}_{\mathcal{O}_K} C(\Gamma; \mathcal{O}_K) \longrightarrow \overline{S}^o(N; \mathcal{O}_K)$$

defined by the composite

$$\begin{aligned} \Psi : M^*\hat{\otimes}_{\mathcal{O}_K} C(\Gamma; \mathcal{O}_K) &\simeq M^*\hat{\otimes}_{\mathcal{O}_K} C(Z_L; \mathcal{O}_K)[\breve{\alpha}^{-1}\psi] \\ &\xrightarrow{E*\varphi} \overline{S}(L; \mathcal{O}_K)[\psi] \xrightarrow{T_{L/N}} \overline{S}(N; \mathcal{O}_K)[\psi] \xrightarrow{e} \overline{S}^o(N; \mathcal{O}_K)[\psi] \end{aligned}$$

(i.e. $\Psi = e \circ T_{L/N} \circ (E * \varphi)$), where $T_{L/N}$ is the operator defined in § 1.VI and e is the projection to the ordinary part given in §2. Note that $C(\Gamma; \mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}} \simeq \mathcal{I}^*$ by Prop. 7.1 through the correspondence : $\phi \otimes \hat{i} \mapsto (i \mapsto \phi(\hat{i}(i)))$. Thus we know that

$$M^*\hat{\otimes}_{\mathcal{O}_K} C(\Gamma; \mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}} \simeq M^*\hat{\otimes}_{\mathcal{O}_K} \mathcal{I}^* \simeq (M\hat{\otimes}_{\mathcal{O}_K}\mathcal{I})^* \text{ by Lemma 8.1.}$$

DEFINITION. — We shall define a generalized measure $E *_\lambda \varphi \in M\hat{\otimes}_{\mathcal{O}_K}\mathcal{I}$ by the composite

$$(8.5) \quad \begin{aligned} E *_\lambda \varphi = \ell_\lambda \circ (\Psi \otimes \mathrm{id}) : (M\hat{\otimes}_{\mathcal{O}_K}\mathcal{I})^* &\simeq M^*\hat{\otimes}_{\mathcal{O}_K} C(\Gamma; \mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}} \\ &\xrightarrow{\Psi\otimes\mathrm{id}} \overline{S}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}} \xrightarrow{\ell_\lambda} \mathcal{O}_K. \end{aligned}$$

In the first part [11, p. 189], we used the trace operator instead of $T_{L/N}$ to define $E *_\lambda \varphi$ or more precisely Ψ. This change of operators gives much improvement in the result, and the utility of $T_{L/N}$ was found by B. Perrin-Riou [25]. Note that for each $P \in \mathcal{X}(\mathcal{I}; \mathcal{O}_K)$,

$$(M\hat{\otimes}_{\mathcal{O}_K}\mathcal{I}) \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I} \simeq M \otimes_{\mathcal{O}_K} \mathcal{I}/P\mathcal{I} \simeq M.$$

Thus we may consider the image $(E *_\lambda \varphi)_P$ of $E *_\lambda \varphi$ in M according to the above map.

THEOREM 8.5. — *For each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_K)$ with $k_P \geq 2$ and $\phi \in M^*$, we have $(E *_\lambda \varphi)_P(\phi) = H(P)\ell_P \circ T_{L/N} \circ e(\tilde{\varphi}(m \mapsto E_*(z. \mapsto \alpha^{-1}\varepsilon\psi\omega^{-k}(z)z_p^k\phi(z^{-1}m))))$, where $z_p \in \mathbf{Z}_p^\times$ is the projection of $z \in Z_L$, $k = k(P)$, $\varepsilon = \varepsilon_P$ and ℓ_P is as in (7.6).*

Proof. — One can naturally identify

$$\begin{aligned} &((M\hat{\otimes}_{\mathcal{O}_K}\mathcal{I}) \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I})^* &&\simeq (M \otimes_{\mathcal{O}_K} \mathcal{I}/P\mathcal{I})^* \simeq M^* \\ \text{and}\quad &((M\hat{\otimes}_{\mathcal{O}_K}\mathcal{I}) \otimes_{\Lambda_K} \mathcal{I}/P\mathcal{I})^* &&\simeq (M^*\hat{\otimes}_{\mathcal{O}_K}C(\Gamma;\mathcal{O}_K) \otimes_{\Lambda_K} \hat{\mathcal{I}})[P] \\ &&&\qquad\qquad\qquad\qquad \text{(Prop. 7.3)} \\ &&&\simeq M^*\hat{\otimes}_{\mathcal{O}_K}(C(\Gamma;\mathcal{O}_K)[P_{k,\varepsilon}]) \\ &&&\qquad\qquad\qquad\qquad \text{(Cor. 7.5)} \\ &&&\simeq M^* \otimes_{\mathcal{O}_K} (C(Z_L;\mathcal{O}_K)[P_\xi]) \\ &&&\qquad\qquad \text{for } \xi = \varepsilon\psi\omega^{-k}, \end{aligned}$$

where $P_\xi \subset \mathcal{A}_L$ is the ideal as in Cor. 8.4. Then, we may regard $(E *_\lambda \varphi)_P$ as the restriction of $E_\lambda^*\varphi$ to $M^*\hat{\otimes}_{\mathcal{O}_K}C(Z_L;\mathcal{O}_K)[P_\xi]$ (for $\xi = \varepsilon\psi\omega^{-k}$), which is canonically isomorphic to M^*. Then by applying Cor. 8.4 to ξ, we obtain the result from Prop. 7.8.

9. Proof of Theorems 5.1, 5.1d and 5.1d'.

Proof of Theorem 5.1. — We shall use the same notation as in Th. 5.1. Especially, we denote by $\mu : C(X;\mathcal{O}_K) \to \overline{S}(J;\mathcal{O}_K)$ the given arithmetic measure of weight ℓ and of character ξ and by $\lambda : \mathbf{h}^o(N;\mathcal{O}_K)\otimes_{\Lambda_K}\mathcal{I} \to \mathcal{I}$ the primitive $\mathcal{I}$-algebra homomorphism. Let L be the least common multiple of J and N, and let $E : C(Z_L : \mathcal{O}_K) \to \overline{S}(L;\mathcal{O}_K)$ be the Eisenstein measure defined in § 5, Example C, which satisfies

$$2\int_{Z_L} \phi(z)dE = \sum_{\substack{n=1 \\ p|n}}^{\infty} \Big(\sum_{\substack{d|n \\ (d,Lp)=1}} sgn(d)\phi(d)\Big)q^n \in \mathcal{O}_K[[q]].$$

We shall define an arithmetic measure $\mu^L : C(X;\mathcal{O}_k) \to \overline{S}(L,\mathcal{O}_K)$ out of the given μ by $\mu^L(\phi) = \pi(\phi)|[L/J](\phi \in C(X;\mathcal{O}_K))$, where $[L/J] : \overline{S}(J;\mathcal{O}_K) \to \overline{S}(L;\mathcal{O}_K)$ is the operator defined in § 1.III. Then one can easily verify that μ^L is arithmetic of weight ℓ and of character ξ (which factors through Z_J). We shall let Z_L act on $C(Z_L;\mathcal{O}_K)$ by

$$(9.1a)\qquad (\phi\|z)(x) = \xi(z)z_p^\ell\phi(z \cdot x) \quad (\phi \in C(X;\mathcal{O}_K))$$

and on $C(Z_L; \mathcal{O}_K)$ by

$$(9.1b) \qquad (\phi|_\alpha z)(w) = z_p\phi(zw) \qquad (\phi \in C(Z_L; \mathcal{O}_K)),$$

where $x \mapsto z \cdot x (x \in X)$ is the action of $z \in Z_L$ on X as in (5.1b). The action of Z_L on $C(Z_L; \mathcal{O}_K)$ is the twisted action by the character $\alpha : z \mapsto z_p$ considered in § 8. As seen in Example c in § 5, E is of weight 1. Thus, under the action (9.1a,b), the measures $E : C(Z_L; \mathcal{O}_K) \to \overline{S}(L; \mathcal{O}_K)$ and $\mu^L : C(X; \mathcal{O}_K) \to \overline{S}(L; \mathcal{O}_K)$ become $\mathcal{A}_L$-linear. We write M for this $\mathcal{A}_L$-module $C(X; \mathcal{O}_K)$ and apply the theory in § 8 to μ^L and E. We take $\overline{S}(L; \mathcal{O}_K)$ as U^* and V^* in § 8 and the usual multiplication of $\overline{S}(L; \mathcal{O}_K)$ is taken as $\mathbf{m}$. Put $\Phi = E *_\lambda \mu^L \in \mathrm{Meas}(X; \mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I}$. We shall show that this generalized measure satisfies the requirement of the theorem. Let $\chi : Z_J \to \overline{\mathbf{Q}}^\times$ be a finite order character, and let $\phi \in LC(X, \chi; \overline{\mathbf{Q}})$. Suppose that $\mu(\phi) \in \mathcal{M}_\ell(\Gamma_1(Jp^\beta))$. Then, by (5.1b), we know that

$$(9.2) \qquad \mu(\phi) \in \mathcal{M}_\ell(\Gamma_0(Jp^\beta), \xi\chi) \quad \text{and} \quad \mu^L(\phi) \in \mathcal{M}_\ell(\Gamma_0(Lp^\beta), \xi\chi).$$

Since $\mathcal{K} \cap \overline{\mathbf{Q}}_p = K$, for each finite extension K'/K, $\mathcal{K}' = \mathcal{K} \otimes_K K'$ is still a field. If $\mathcal{I}'$ denotes the integral closure of $\Lambda_{K'}$ in $\mathcal{K}'$, we have a unique scalar extension of $\lambda : \lambda' = \lambda \otimes \mathrm{id} : \mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}' = \mathbf{h}^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \otimes_{\mathcal{I}} \mathcal{I}' \to \mathcal{I}'$. By construction, one sees that $\Phi' = E *_{\lambda'} \mu^L$ is the natural image of $\Phi = E *_\lambda \mu^L$ under the scalar extension map : $\mathrm{Meas}(X; \mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{I} \to \mathrm{Meas}(X; \mathcal{O}_{K'})\hat{\otimes}_{\mathcal{O}_{K'}}\mathcal{I}'$. Thus to prove the theorem, replacing λ, $\mathcal{I}$ and $\mathcal{O}_K$ by its extension λ', $\mathcal{I}'$ and $\mathcal{O}_{K'}$ if necessary, we may assume that the condition (3.1b) holds for $\mathcal{I}$ and $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_K)$.

Write ε for ε_P, k for $k(P)$ and r for $r(P)$ for $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_K)$ with $k(P) \geq 2$. Then we have by Th. 8.5 that

$$\int_X \phi\nu^m d\Phi_P$$
$$= H(P)\ell_P \circ T_{L/N} \circ e\Big(\int_X \int_{Z_L} \varepsilon\psi\omega^{-k}(z)z_p^{k-1}(\phi\nu^m\|z^{-1})(x)dE(z)d\mu^L(x)\Big).$$

Note that by (9.1a) and (5.1c)

$$(9.3) \quad e\Big(\int_X \int_{Z_L} \varepsilon\psi\omega^{-k}(z)z_p^{k-1}((\phi\nu^m)\|z^{-1})(x)dE(z)d\mu^L(x)\Big)$$
$$= e\Big(\int_{Z_L} \eta(z)z_p^{k-j-1}dE \cdot \int_X \phi\nu^m d\mu^L\Big),$$

where we write η for $\varepsilon\psi\omega^{-k}\xi^{-1}\chi^{-1}$ and j for $\ell+2m$. Also note that

$$\text{(9.4a)}\quad \int_{Z_L} \eta(z) z_p^{k-j-1} dE = E_{k-j,Lp}(\eta)|\imath_p \qquad \text{if } k > j (\Leftrightarrow 0 \le m < \frac{1}{2}(k-\ell)),$$

$$\text{(9.4b)}\quad d^{1+j-k}\Big(\int_{Z_L} \eta(z) z_p^{k-j-1} dE\Big) = G_{2+j-k,Lp}(\eta)|\imath_p \qquad \text{if } j \ge k (\Leftrightarrow k-\ell \le 2m),$$

where d is the differential operator $q\dfrac{d}{dq}$ defined in § 1.VIII and $\imath_p$ is the twisting operator in § 1.VII for the trivial character modulo p. We have to be careful about the fact : $\int_X \phi\nu^m d\mu^L = (L/J)^{-m} d^m(\mu^L(\phi))$ (cf. (5.1c)) and also we shall use the fact that $e(hdf) = -e(fdh)$ and $e(h(f|\imath_p)) = e((h|\imath_p)f)$ (Prop. 2.4). Then we apply to (9.3) the above facts and obtain the formula :

$$e\Big(\int_{Z_L} \eta(z) z_p^{k-j-1} dE \cdot \int_X \phi\nu^m d\mu^L\Big) = \begin{cases} (-J/L)^m e((\mu^L(\phi|\imath_p) d^m(E_{k-j,Lp}(\eta))) & \text{if } 0 \le m < \frac{1}{2}(k-\ell), \\ (-J/L)^m e((\mu^L(\phi)|\imath_p) d^{k-\ell-m-1}(G_{2+j-k,Lp}(\eta))) & \text{if } \frac{1}{2}(k-\ell) \le m < k-\ell. \end{cases}$$

Note that $f|\imath_p = f - (f|T(p))|[p]$. Thus by the assumption (5.8) that $\mu(\phi)|T(p) = 0$ for all $\phi \in C(X;\mathcal{O}_K)$, we know that $\mu^L(\phi)|\imath_p = \mu^L(\phi)$. Since $\mu^L(\phi) \in \mathcal{M}_\ell(\Gamma_0(Lp^\gamma), \xi\chi; \overline{\mathbf{Q}})$ for any $\gamma \ge \beta$, we may suppose that η is a character of $(\mathbf{Z}/Lp^\beta\mathbf{Z})^\times$. Then by Lemma 6.5 (ii) and (iv), we have with the notation of (6.9) that

$$e\Big(\int_{Z_L} \eta(z) z_p^{k-j-1} dE \cdot \int_X \phi\nu^m d\mu^L\Big) = \begin{cases} (-J/L)^m e\circ\mathcal{H}(\mu^L(\phi)\delta_{k-j}^m(E_{k-j,Lp}(\eta))) & \text{if } 0 \le m < \frac{1}{2}(k-\ell), \\ (-J/L)^m e\circ\mathcal{H}(\mu^L(\phi)\delta_{2+j-k}^{k-\ell-m-1}(G'_{2+j-k,Lp}(\eta))) & \text{if } \frac{1}{2}(k-\ell) \le m < k-\ell. \end{cases}$$

We write simply g for either $\mathcal{H}(\mu^L(\phi)\delta^m_{k-j}(E_{k-j,Lp}(\eta)))$ or $\mathcal{H}(\mu^L(\phi)$ $\delta^{k-\ell-m-1}_{2+j-k}G'_{2+j-k,Lp}(\eta)))$ according as the above condition on m, where $\mathcal{H}$ is the holomorphic projection defined in § 6. Then, by (7.6), we have that

$$\int_X \phi\nu^m d\Phi_P$$
$$= H(P)a(p,f_P)^{r-\beta}p(\beta-r)(k-1)(-J/L)^m \frac{< h_P|[p^{\beta-r}], g|T_{L/N} >_{Np^\beta}}{< h_P, f_P >_{Np^r}}.$$

By (1.7) combined with [27, (3.4.5)], we have that

$$< h_P|[p^{\beta-r}], g|T_{L/N} >_{Np^\beta} = (L/N)^k < h_P|[Lp^{\beta-r}/N], g >_{Lp^\beta}$$
$$= (L/N)^{\ell/2}p^{\ell(r-\beta)/2} < f_P^\rho|\tau_{Lp^\beta}, g >_{Lp^\beta},$$

since $h_P|[Lp^{\beta-r}/N] = (Lp^{\beta-r}/N)^{-k/2}f_P^\rho|\tau_{Lp^\beta}$ for $\tau_{Lp^\beta} = \begin{pmatrix} 0 & -1 \\ Lp^\beta & 0 \end{pmatrix}$. Note that

$$\mu^L(\phi) = ((-1)^k(L/J)^{-k/2}(\mu(\phi)|_\ell\tau_{Jp^\beta})|_\ell\tau_{Lp^\beta}.$$

Then the assertion follows from Th. 6.6 and the following formula :

$$(9.5)\quad \frac{< h_P, f_P >_{Np^r}}{< f_P^0, f_P^0 >_{Np^{r_0}}}$$
$$= \begin{cases} (-1)^k W(f_P) & \text{if } f_P = f_P^0, \\ (-1)^k W(f_P) f^{(2-k)/2} a(p,f_P) & \\ \quad \times \left(1 - \dfrac{\psi_P\psi'(p)p^{k-1}}{a(p,f_P)^2}\right)\left(1 - \dfrac{\psi_P\psi'(p)p^{k-2}}{a(p,f_P)^2}\right) & \text{if } f_P \neq f_P^0. \end{cases}$$

This formula (9.5) is given in [25, Lemma 27] when $k(P) = 2$ and the general case : $k(P) \geq 2$ follows from the same computation there or the formula [30, (3.2)].

Proof of Theorem 5.1d. — Let L be the least common multiple of N and J. Put $\Phi = E *_\lambda (\lambda'_0)^L$ with the notation of the above proof of Th. 5.1, where $(\lambda'_0)^L : C(\mathbf{Z}_p^\times; \mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{J}^* \to \overline{S}(L;\mathcal{O}_K)$ is defined by $(\lambda'_0)^L = [L/J] \circ \lambda'_0$. For each $Q \in \mathcal{X}_{\rm alg}(\mathcal{J};\mathcal{O}_K)$ with $k(Q) \geq 2$, we have a natural map :

$$C(\mathbf{Z}_p^\times;\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{I}^* \xrightarrow{\sim} C(\mathbf{Z}_p^\times;\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{J}^*[Q] \otimes_{\mathcal{O}_K} \mathcal{I}^*$$
$$\longrightarrow C(\mathbf{Z}_p^\times;\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{J}^* \otimes_{\mathcal{O}_K} \mathcal{I}^* \xrightarrow{\Phi} \mathcal{O}_K.$$

By definition, this is nothing but the convoluted measure $E *_\lambda (\mu_{g_Q})^L$ for the measure μ_{g_Q} associated with g_Q as in Example b in § 5. Then we apply Th. 5.1b, which has already been deduced from Th. 5.1, and obtain the result.

Proof of Theorem 5.1d′. — We shall deduce Th. 5.1d′ from the following result :

THEOREM 9.1. — *Let the notation and the assumption be as in Th. 5.1d. Then, for each integer $b > 1$ prime to JNp, there exists a unique generalized Iwasawa function Φ^b in the quotient field of $\mathcal{I}\hat{\otimes}\mathcal{J}$ such that*

(i) *If $H \in \mathcal{I}$ annihilates $\mathcal{C}_0(\lambda;\mathcal{I})$, then $H(P)\Phi^b(P,Q) \in \mathcal{I}\hat{\otimes}\mathcal{J}$,*

(ii) *for each $(P,Q) \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}) \times \mathcal{X}_{\mathrm{alg}}(\mathcal{J})$ with $k(P) \geq k(Q) \geq 2$, we have*

$$\Phi^b(P,Q) = \begin{cases} cwS(P)^{-1}(1-\psi\xi^{-1}\varepsilon_P\varepsilon_Q^{-1}(b) < b >^{k(P)-k(Q)}) \\ \quad \times E'(P,Q)\mathcal{D}(k(Q), f_P, g_Q)/\Omega(P,Q,P_0) \quad \text{if } k(P) > k(Q), \\ 0 \quad \text{if } k(P) = k(Q), \varepsilon_P = \varepsilon_Q \text{ and } \psi = \xi \\ \text{but either } P \neq Q \text{ or } \lambda \text{ does not factors through } \lambda', \\ -(\varphi(Lp)/Lp)(\log(< b >)) \text{ if } P = Q \text{ and } \lambda' = \lambda|_{\mathbf{h}^o(N;\mathcal{O}_K)}, \end{cases}$$

where $c = c(P,Q,P_0)$, $w = w(P,Q,P_0)$ as in Th. 5.1d and $E'(P,Q) = E(P,Q,P_0)/E_2''(P,Q)$ for $E_2''(P,Q)$ defined in Th. 5.1d' and $< b > = b\omega(b)^{-1} \in \mathbf{Z}_p^x$.

Proof. — Put $\Psi^b = E^b * (\lambda'^*)^L \in \mathcal{I}\hat{\otimes}\mathcal{J}$ for $(\lambda'^*)^L = [L/J] \circ \lambda'^*$ and $\Phi^b(P,Q) = H(P)^{-1}\Psi^b(P,Q)$, where $E^b : C(Z_L;\mathcal{O}_K) \to \overline{\mathcal{M}}(J;\mathcal{O}_K)$ is the Eisenstein measure defined in § 5 in Example c. By Th. 8.5 combined with the argument which proves Th. 5.1, we know that

$$\Psi^b(P,Q) = H(P)\ell_P \circ T_{L/N} \circ e\Big(\int_{Z_L} \eta(z)z_p^{k-\ell-1}dE^b \cdot (g_Q|[L/J])\Big),$$

where $\eta = \varepsilon_P\varepsilon_Q^{-1}\psi\xi^{-1}\omega^{\ell-k}$ for $k = k(P)$ and $\ell = k(Q)$. By the well known formula :

$$\int_{Z_L} z_p^{-1}dE^b = -(\varphi(Lp)/Lp)\log(< b >) \in \mathbf{Z}_p,$$

we know that

$\int_{Z_L} \eta(z)z_p^{k-\ell-1}dE^b$

$$= \begin{cases} (1-\eta(b) < b >^{k-\ell})E_{k-\ell,Lp}(\eta) & \text{if } k > \ell, \\ -(\varphi(Lp)/Lp)\log(< b >) & \text{if } k = \ell \text{ and } \eta = \mathrm{id}. \end{cases}$$

When $k > \ell$, the same calculation as in the proof of Theorems 5.1 and 5.1d yields the expression of $\Phi^b(P,Q)$ by the special values of $\mathcal{D}(s, f_P, g_Q|\tau_\beta)$. Then in a similar manner to the proof of Lemma 5.2, we obtain the desired result. Here note that g_Q has non-trivial coefficient in q^p and hence we have the Euler factor $E'(P,Q)$ instead of $E(P,Q,P_0)$. We now suppose that $k = \ell$ and $\eta = \mathrm{id}$. Then, the similar computation as in the proof of Th. 5.1 shows that

$$\Psi^b(P,Q) = -(\varphi(Lp)/Lp)(\log(< b >))H(P)\ell_P(T_{L/N} \circ e(g_Q|[L/J]).$$

If λ does not factors through λ', then by [14, Cor 1.3], we can find an integer n prime to Lp such that $a(n, g_Q) \neq a(n, f_P)$. Note that $\ell_P(h|T(n)) = a(n, f_P)\ell_P(h)$ for any h. On the other hand, for $h = T_{L/N} \circ e(g_Q|[L/J])$, we know that $h|T(n) = a(n, g_Q)h$. This show that $\ell_P(T_{L/N} \circ e(g_Q|[L/J])) = 0$. If λ' is the restriction of λ to $\mathbf{h}^o(N; \mathcal{O}_K)$ $(J = N)$ but if $P \neq Q$, the same argument shows that $\ell_P(h) = 0$ for $h = e(f_Q)$. If $P = Q$, then $L = N = J$ and $\Psi^b(P,Q) = -(\varphi(Lp)/Lp)(\log(< b >))H(P)$, since $\ell_P(f_P) = 1$. This finishes the proof.

Assuming $\lambda' = \lambda|_{\mathbf{h}^o(N;\mathcal{O}_K)}$, we shall now prove Th. 5.1d'. We now eliminate the dependence on b of the function Φ^b defined in Th. 9.1. We choose b so that $< b >$ gives the topological generator u of Γ which we already fixed in § 5. Consider the power series $F(X,Y) \in \mathcal{O}_K[[X,Y]]$ such that $F(\varepsilon(u)u^s - 1, \varepsilon'(u)u^t - 1) = 1 - \varepsilon\varepsilon'^{-1}(u)u^{s-t}$ for each $s, t \in \mathbf{Z}_p$ and finite order characters $\varepsilon, \varepsilon' : \Gamma \to \mathcal{O}_K^\times$. We may show the existence of such a power series as follows : We identify $\mathbf{Z}_p^2$ with Γ^2 by $(s,t) \mapsto (u^s, u^t)$. Define functions S, T on $\mathbf{Z}_p^2$ by $S(s,t) = u^{s+t} - 1, T(s,t) = u^{s-t} - 1$.

Then, we know that $F(X,Y) = -T = 1 - (X + 1)/(Y + 1) \in \mathcal{O}_K[[X,Y]]$. Note that $X - Y = T(1 + Y)$ and $1 + Y$ is a unit in $\Lambda_L = \mathcal{O}_K[[Y]]$. By Th. 9.1, the function $D'(P,Q)$ as in Th. 5.1d' has the expression $D' = \Phi^b/F$ in the quotient field of $\mathcal{I}\hat{\otimes}\mathcal{I}$. (In fact, the set of points $(P,Q) \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})^2$ with $k(P) > k(Q)$ is dense in $\mathcal{X}(\mathcal{I})^2$.) Thus by Th. 9.1, we know that for non-critical P

$$\begin{aligned}((X - Y)D'(P,Q))|_{P=Q} &= T(Y + 1)\Phi^b(P,Q)/F|_{P=Q} \\ &= -(1 + Y(P))\Phi^b(P,P) = (1 + Y(P))(\varphi(Lp)/Lp)(\log(u)),\end{aligned}$$

since $(X - Y)/F|_{P=Q} = -T(1 + Y)/T|_{P=Q} = -(1 + Y(P))$. This finishes the proof.

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