

ANNALES DE L'INSTITUT FOURIER

HANSPETER KRAFT

CLAUDIO PROCESI

Graded morphisms of G -modules

Annales de l'institut Fourier, tome 37, n° 4 (1987), p. 161-166

http://www.numdam.org/item?id=AIF_1987__37_4_161_0

© Annales de l'institut Fourier, 1987, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

GRADED MORPHISMS OF G-MODULES

by H. KRAFT and C. PROCESI

1. Introduction.

During the 1987 meeting in honor of J. K. Koszul, Steve Halperin explained to us the following conjecture (motivated by the study of the spectral sequence associated to a homogeneous space).

1.1. CONJECTURE. — *If f_1, f_2, \dots, f_n is a regular sequence in the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_n]$, the connected component of the automorphism group of the (finite dimensional) algebra $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$ is solvable.*

In this paper we prove a weak form of this (Corollary 4.3) which implies the conjecture at least when the f_i 's are homogeneous (Remark 4.4).

2. Preliminaries.

Our base field is \mathbb{C} , the field of complex numbers, or any other algebraically closed field of characteristic zero.

2.1. DEFINITION. — *A morphism $\varphi : V \rightarrow W$ between finite dimensional vector spaces V and W is called graded if there is a basis of W such that the components of φ are all homogeneous polynomials.*

Let us denote by $\mathcal{O}(V)$, $\mathcal{O}(W)$ the ring of regular functions on V and W . These \mathbb{C} -algebras are naturally graded by degree: $\mathcal{O}(V) = \bigoplus_i \mathcal{O}(V)_i$. A subspace $S \subset \mathcal{O}(V)$ is called *graded* if $S = \bigoplus_i S \cap \mathcal{O}(V)_i$.

Key-words: Automorphism group of an algebra - G-module - Equivariant graded morphism - Regular sequence.

If $\varphi: V \rightarrow W$ is a morphism and $\varphi^*: \mathcal{O}(W) \rightarrow \mathcal{O}(V)$ the corresponding comorphism we have the following equivalence:

φ is graded $\Leftrightarrow \varphi^*(W^*)$ is a graded subspace of $\mathcal{O}(V)$.

2.2. LEMMA. — *For any graded morphism $\varphi: V \rightarrow W$ there is a unique decomposition $W = \bigoplus_{v \geq 0} W_v$ and homogeneous morphisms $\varphi_v: V \rightarrow W_v$ of degree v such that*

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots): V \rightarrow W_0 \oplus W_1 \oplus W_2 \oplus \dots$$

(This is clear from the definitions.)

2.3. Remark. — Let G be an algebraic group. Assume that V and W are G -modules and that $\varphi: V \rightarrow W$ is graded and G -equivariant. Then in the notations of lemma 2.2 all W_v are submodules and all components φ_v are G -equivariant.

2.4. Remark. — If $\varphi: V \rightarrow W$ is graded and dominant with $\varphi^{-1}(0) = \{0\}$, then φ is a finite surjective morphism. In fact given a finitely generated graded algebra $A = \bigoplus_{i \geq 0} A_i$ with $A_0 = \mathbb{C}$ and a graded subspace $S \subset A$ such that the radical $\text{rad}(S)$ of the ideal generated by S is the homogeneous maximal ideal $\bigoplus_{i > 0} A_i$ of A , then A is a finitely generated module over the subalgebra $\mathbb{C}[S]$ generated by S (see [1, II.4.3 Satz 8]).

3. The Main Theorem.

3.1. THEOREM. — *Let G be a connected reductive algebraic group and let V, W be two G -modules. Assume that V and W do not contain 1-dimensional submodules. Then any graded G -equivariant dominant morphism with finite fibres is a linear isomorphism.*

We first prove this for $G = \text{SL}_2$ and then reduce to this situation.

For any \mathbb{C}^* -module V we have the weight decomposition

$$V = \bigoplus_j V_j, \quad V_j := \{v \in V \mid t(v) = t^j \cdot v\}.$$

We say that V has *only positive weights* if $V = \bigoplus_{j > 0} V_j$.

3.2. LEMMA. — *Let V, W be two C^* -modules with only positive weights, and let $\varphi: V \rightarrow W$ be a C^* -equivariant graded morphism with finite fibres. For all $k \geq 0$ we have*

$$\varphi^{-1}\left(\bigoplus_{j \leq k} W_j\right) \subseteq \bigoplus_{j \leq k} V_j,$$

and the inclusion is strict for at least one k in case φ is not linear.

Proof. — By lemma 2.2 and remark 2.3 we have $\varphi = \sum_{v \geq 1} \varphi_v$ where $\varphi_v: V \rightarrow W_v$ is homogeneous of degree v and C^* -equivariant. Let $v = \sum_{j=1}^k v_j \in \bigoplus_{j > 0} V_j = V$ with $v_k \neq 0$. Then

$$\lim_{\lambda \rightarrow 0} \lambda^k \cdot t_\lambda^{-1}(v) = v_k.$$

(Here t_λ denotes the action of C^* .) Since φ_v is homogeneous of degree v and C^* -equivariant we obtain

$$(1) \quad \lim_{\lambda \rightarrow 0} \lambda^{vk} \cdot t_\lambda^{-1}(\varphi_v(v)) = \varphi_v(v_k).$$

This implies that $\varphi_v(v) \in \bigoplus_{j \leq vk} W_j$ for all v , proving the first claim.

If φ is not linear, i.e. $\varphi \neq \varphi_1$, then there is a $v > 1$, an index k and an element $v \in V_k$ such that $\varphi_v(v) \neq 0$. But $\varphi_v(v) \in W_{vk}$ by (1) and so $v \notin \varphi^{-1}\left(\sum_{j \leq k} W_j\right)$. □

3.3. COROLLARY. — *Under the assumptions of lemma 3.2 suppose that φ is surjective. Put $\lambda_j := \dim V_j$ and $\mu_j := \dim W_j$. Then for all $k \geq 1$ we have*

$$(2) \quad \lambda_1 + \lambda_2 + \dots + \lambda_k \geq \mu_1 + \mu_2 + \dots + \mu_k.$$

If φ is not linear the inequality is strict for at least one k .

(This is clear.)

3.4. PROPOSITION. — *Let V, W be two SL_2 -modules containing no fixed lines. Let $\varphi: V \rightarrow W$ be a graded SL_2 -equivariant morphism, which is dominant and has finite fibres. Then φ is a linear isomorphism.*

Proof. – Consider the maximal unipotent subgroup

$$U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{SL}_2$$

and the maximal torus

$$T := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^* \right\} \simeq \mathbb{C}^*.$$

By assumption φ is finite and surjective (Remark 2.4), and $\varphi^{-1}(W^U) = V^U$. Hence the induced morphism

$$\varphi|_{V^U} : V^U \rightarrow W^U$$

is graded, T -equivariant, finite and surjective too. Furthermore all weights λ_j of V^U and μ_j of W^U are positive. It follows from (2) that

$$\lambda_k + \lambda_{k+1} + \dots \leq \mu_k + \mu_{k+1} + \dots$$

for all k , because $\sum_j \lambda_j = \dim V^U = \dim W^U = \sum_j \mu_j$. From this we get

$$\begin{aligned} \dim V &= 2\lambda_1 + 3\lambda_2 + \dots + (n+1)\lambda_n \\ &\leq 2\mu_1 + 3\mu_2 + \dots + (n+1)\mu_n = \dim W \end{aligned}$$

for all n which are big enough. (Remember that an irreducible SL_2 -module of highest weight j is of dimension $j+1$). If φ is not linear this inequality is strict (Corollary 3.3), contradicting the fact that φ is finite and surjective. \square

3.5. Proof of the Theorem. – Assume that $\varphi : V \rightarrow W$ is not linear, i.e. there is a $v_0 > 1$ such that the component $\varphi_{v_0} : V \rightarrow W_{v_0}$ is non-zero. Then there is a homomorphism $\mathrm{SL}_2 \rightarrow G$ and a non-trivial irreducible SL_2 -submodule $M \subseteq V$ such that $\varphi_j|_M \neq 0$. (In fact the intersection of the fixed point sets $V^{(\mathrm{SL}_2)}$ for all homomorphisms $\iota : \mathrm{SL}_2 \rightarrow G$ is zero.) Now consider the G -stable decompositions $V = V^{\mathrm{SL}_2} \oplus V'$ and $W = W^{\mathrm{SL}_2} \oplus W'$ and the following morphism :

$$\varphi' : V' \hookrightarrow V \xrightarrow{\varphi} W \xrightarrow{\mathrm{pr}} W'.$$

Since V' and W' are sums of isotypic components the morphism φ' is again graded. Furthermore $\varphi^{-1}(W^{\mathrm{SL}_2}) = V^{\mathrm{SL}_2}$, hence $\varphi^{-1}(0) = V^{\mathrm{SL}_2} \cap V' = \{0\}$. This implies that $\varphi' : V' \rightarrow W'$ is dominant

with finite fibres and satisfies therefore the assumptions of proposition 3.4. As a consequence φ' is linear. Since $\varphi|_{V'} : V' \rightarrow W$ is graded too we have $\varphi_v|_{V'} = 0$ for all $v > 1$. This contradicts the facts that $M \subseteq V'$ and $\varphi_{v_0}|_M \neq 0$ (see the construction above).

4. Some Consequences.

We add some corollaries of the theorem. Let G be a connected reductive group. For every G -module V we have the canonical G -stable decomposition $V = V^\circ \oplus V'$ where V° is the sum of all 1-dimensional representations (i.e. $V^\circ = V^{(G,G)}$) and V' the sum of all others. The proof of the theorem above easily generalizes to obtain the following result :

4.1. THEOREM. — *Let $\varphi : V \rightarrow W$ be a graded G -equivariant dominant morphism with finite fibres. Then φ induces a linear isomorphism*

$$\varphi|_{V'} : V' \xrightarrow{\sim} W'$$

4.2. COROLLARY. — *Let $\mathcal{O}(V)$ be the ring of regular functions on a G -module V , and let f_1, \dots, f_n be a regular sequence of homogenous elements of $\mathcal{O}(V)$ such that the linear span $\langle f_1, \dots, f_n \rangle$ is G -stable. Then $\langle f_1, \dots, f_n \rangle$ contains all non-trivial representations of (G,G) in $\mathcal{O}(V)_1$, the linear part of $\mathcal{O}(V)$.*

Proof. — The regular sequence f_1, \dots, f_n defines a G -equivariant finite morphism $\varphi : V \rightarrow W$, $W := \langle f_1, \dots, f_n \rangle^*$. By the theorem above the restriction $\varphi'|_{V'} : V' \rightarrow W'$ is a linear isomorphism which means that every non-trivial (G,G) -submodule of $\langle f_1, \dots, f_n \rangle$ is contained in the linear part $\mathcal{O}(V_1)$ of $\mathcal{O}(V)$. □

4.3. Recall that a finite dimensional \mathbb{C} -algebra is called a *complete intersection* if it is of the form $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$ with a regular sequence f_1, \dots, f_n .

COROLLARY. — *Let A be a finite dimensional local \mathbb{C} -algebra with maximal ideal \mathfrak{m} and let $\text{gr}_{\mathfrak{m}}A$ be the associated graded algebra (with respect to the \mathfrak{m} -adic filtration). If $\text{gr}_{\mathfrak{m}}A$ is a complete intersection then the connected component of the automorphism group of A is solvable.*

Proof. — Let G and \bar{G} be the connected components of the automorphism groups of A and of $\text{gr}_m A$ respectively. Since the m -adic filtration of A is G -stable we have a canonical homomorphism $\rho: G \rightarrow \bar{G}$. It is easy to see that $\ker \rho$ is unipotent, so it remains to show that \bar{G} is solvable.

Assume that \bar{G} is not solvable. Then \bar{G} contains a (non-trivial) semisimple subgroup H . By assumption we have an isomorphism

$$\text{gr}_m A \simeq \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$$

with a regular sequence f_1, \dots, f_n where all f_i are homogeneous of degree ≥ 2 . Clearly the action of \bar{G} on $\text{gr}_m A$ is induced from a (faithful) linear representation on $\mathbb{C}[x_1, \dots, x_n]_1 \subset \mathbb{C}[x_1, \dots, x_n]$. Hence it follows from corollary 4.2 that $\langle f_1, \dots, f_n \rangle$ contains all non-trivial H -submodules of $\mathbb{C}[x_1, \dots, x_n]_1$, contradicting the fact that all f_i have degree ≥ 2 . \square

4.4. Remark. — The corollary above implies that conjecture 1.1 is true in case all f_i are homogeneous, i.e. if the algebra

$$A = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$$

is finite dimensional and graded with all x_i of degree 1.

4.5. Remark. — Another formulation of our result is the following: Let V be a representation of a connected algebraic group G and $Z \subset V$ a G -stable graded subscheme, which is a complete intersection supported in $\{0\}$. Then (G, G) acts trivially on Z .

BIBLIOGRAPHY

- [1] H. KRAFT, *Geometrische Methoden in der Invariantentheorie*, Aspekte der Mathematik D1, Vieweg-Verlag, 1985.

H. KRAFT,
Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 Basel.

C. PROCESI,
Istituto Matematico
Guido Castelnuovo
Università di Roma
I-00100 Roma.