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MONODROMY REPRESENTATIONS OF BRAID GROUPS AND YANG-BAXTER EQUATIONS

by Toshitake KOHNO

INTRODUCTION

The purpose of this paper is to give a description of the monodromy of integrable connections over the configuration space arising from classical Yang-Baxter equations. These monodromy representations define a series of linear representations of the braid groups $\theta : B_n \to \mathrm{End}(W^{\otimes n})$ with one parameter, associated to any finite dimensional complex simple Lie algebra $\mathfrak{g}$ and its finite dimensional irreducible representations $\rho : \mathfrak{g} \to \mathrm{End}(W)$. By means of trigonometric solutions of the quantum Yang-Baxter equations due to Jimbo ([10] and [11]), we give an explicit form of of these representations in the case of a non-exceptional simple Lie algebra and its vector representation (Theorem 1.2.8) and in the case of $\mathfrak{sl}(2,\mathbf{C})$ and its arbitrary finite dimensional irreducible representations (Theorem 2.2.4).

Our monodromy representation θ commutes with the diagonal action of the q-analogue of the universal enveloping algebra of $\mathfrak{g}$ in the sense of Jimbo [9], which was discussed as quantum groups by Drinfel'd [7]. In particular, in the case $\mathfrak{g} = \mathfrak{sl}(m,\mathbf{C})$, the representation θ gives Hecke algebra representations of B_n appearing in a recent work of Jones [14].

The study of these monodromy representations is motivated by a recent development of two dimensional conformal field theory initiated by Belavin, Polyakov and Zamolodchikov [5]. The importance of the two dimensional conformal field theory with gauge symmetry was

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pointed out by Knizhnik and Zamolodchikov [18]. They showed that the total differential equations defined by our connections are satisfied by n-point functions in these cases.

Recently Tsuchiya and Kanie [22] developed an operator formalism of two dimensional conformal field theory on $\mathbf{P}^1$ using the Kac-Moody Lie algebra of type $A_1^{(1)}$. It turns out that in the case of the vector representation of $\mathfrak{sl}(2,\mathbf{C})$, the monodromy of n-point functions gives a linear representation of the braid group B_n factoring through the Jones algebra of index $4\cos^2\dfrac{\pi}{\ell+2}$ for a positive integer ℓ (see [13]). In particular this representation is unitarizable. We shall extend this unitarity result to higher representations of $\mathfrak{sl}(2,\mathbf{C})$. A neat description of the monodromy of n-point functions in the case of simple Lie algebras of other types might be pursued from a viewpoint of Brauer's centralizer algebras, which will be discussed in the forthcoming paper.

This paper is organized in the following way. In Sect. 1.1, we explain a process to define an integrable connection associated with a simple Lie algebra and its irreducible representation. We give an explicit description of the monodromy in Sect. 1.2 and 1.3. Sect. 2.1 is devoted to a review of two dimensional conformal field theory due to Tsuchiya and Kanie [22]. We discuss the case of higher representations of $\mathfrak{sl}(2,\mathbf{C})$ in Sect. 2.2 and 2.3.

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The following notations are of frequent use:

B_n : braid group on n strings with generators σ_i, $1 \leqslant i \leqslant n-1$, represented by a braid interchanging strings i and $i+1$ (see [2]).

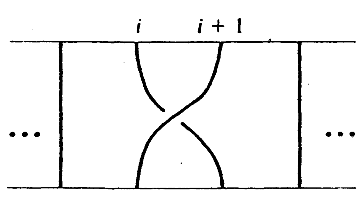

Fig. 1.

P_n : pure braid group on n strings.

$X_n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n;\ \ z_\alpha \neq z_\beta \text{ if } \alpha \neq \beta\}$

$\mathfrak{g}$: a simple finite dimensional complex Lie algebra.
$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form.

$t = \Sigma_\mu I_\mu \otimes I_\mu \in \mathfrak{g} \otimes \mathfrak{g}$.

For a finite dimensional vector space V, we let $\sigma \in \mathrm{End}\,(V \otimes V)$ the transposition defined by $\sigma(x \otimes y) = y \otimes x$. For $X \in \mathrm{End}\,(V \otimes V)$ we put $\bar{X} = \sigma X$.

$\mathbf{C}\{\lambda\}$: ring of the convergent power series.

1. MONODROMY OF INTEGRABLE CONNECTIONS ARISING FROM CLASSICAL YANG-BAXTER EQUATIONS

1.1. Construction of connections.

Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra and let $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. We put

$$t = \Sigma_\mu I_\mu \otimes I_\mu$$

which may also be expressed as

$$t = \frac{1}{2}(\Delta\Omega - \Omega \otimes 1 - 1 \otimes \Omega).$$

Here Ω is the Casimir operator $\Sigma_\mu I_\mu . I_\mu$ in the universal enveloping algebra $U(\mathfrak{g})$ and $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ stands for the comultiplication as a Hopf algebra.

Associated with a simple Lie algebra $\mathfrak{g}$ and its finite dimensional irreducible representations $\rho_\alpha : \mathfrak{g} \to \mathrm{End}\,(W_\alpha)$, $1 \leqslant \alpha \leqslant n$, we consider the total differential equations with a parameter λ

$$(1.1.1) \qquad d\Phi = \Sigma_{1 \leqslant \alpha < \beta \leqslant n} \lambda \Omega_{\alpha\beta}\, d \log (z_\alpha - z_\beta) . \Phi , \qquad \lambda \in \mathbf{C}$$

defined over

$$X_n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n;\ z_\alpha \neq z_\beta \text{ if } \alpha \neq \beta\}.$$

Here $\Omega_{\alpha\beta} \in \mathrm{End}\ (W_1 \otimes \ldots \otimes W_n)$ are defined by

$$\Omega_{\alpha\beta} = \Sigma_\mu \rho_\alpha(I_\mu) \otimes \rho_\beta(I_\mu)$$

where ρ_α stands for the representation ρ_α on the α-th factor acting as the identity on the other factors.

The matrix valued 1-form

$$(1.1.3) \qquad \omega = \Sigma_{1 \leqslant \alpha < \beta \leqslant n} \lambda \Omega_{\alpha\beta} d \log (z_\alpha - z_\beta), \qquad \lambda \in \mathbf{C}$$

is considered to be a connection of the trivial vector bundle over X_n with fiber $W_1 \otimes \cdots \otimes W_n$. The integrability condition for ω

$$d\omega + \omega \wedge \omega = 0$$

is satisfied in our case since we have the following relations among $\Omega_{\alpha\beta}$:

$$(1.1.4) \quad [\Omega_{\alpha\beta}, \Omega_{\alpha\gamma} + \Omega_{\beta\gamma}] = [\Omega_{\alpha\beta} + \Omega_{\alpha\gamma}, \Omega_{\beta\gamma}] = 0 \quad \text{for} \quad \alpha < \beta < \gamma$$
$$[\Omega_{\alpha\beta}, \Omega_{\gamma\delta}] = 0 \quad \text{for distinct } \alpha, \beta, \gamma, \delta.$$

In fact the above relations are derived from the fact that the Casimir operator Ω lies in the center of $U(\mathfrak{g})$. We shall call (1.1.4) the *infinitesimal pure braid relations*. These relations are relevant to the classical Yang-Baxter equation in the following sense.

Let us recall that the classical Yang-Baxter equation is a functional equation for a $\mathfrak{g} \otimes \mathfrak{g}$-valued meromorphic function $r(u)$, $u \in \mathbf{C}$, given by

$$(1.1.5) \quad [r_{12}(u-v), r_{13}(u)] + [r_{12}(u-v), r_{23}(v)] + [r_{13}(u), r_{23}(v)] = 0.$$

Here the above triangular equality is considered in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ and r_{ij} signifies the r on the i-th and j-th factors acting as the identity on the other factor. Solutions of the classical Yang-Baxter equation are classified by Belavin and Drinfel'd (see [3] for a precise statement). In particular, they discovered a rational solution $r(u) = t/u$. The infinitesimal pure braid relations are obtained from the fact that t/u satisfies the classical Yang-Baxter equation.

As the monodromy of the connection ω we obtain a linear representation of the pure braid group

$$\theta : P_n \to \mathrm{End}\ (W_1 \otimes \cdots \otimes W_n)$$

depending on the parameter λ. Let us now suppose that the representations ρ_α, $1 \leqslant \alpha \leqslant n$, are the same. In this case the connection ω defined in the above way is invariant by the diagonal action of the symmetric group S_n on $X_n \times (W_1 \otimes \cdots \otimes W_n)$, hence it defines a local system over the quotient space $Y_n = X_n/S_n$. Considering λ as a parameter we obtain a linear representation of the braid group on n strings

$$\theta : B_n \to \mathrm{End}\,(W^{\otimes n}) \otimes \mathbf{C}\{\lambda\}.$$

Here $\mathbf{C}\{\lambda\}$ denotes the ring of the convergent power series. Our main object is to give a description of this monodromy representation.

The total differential equations of the above type appear in the two dimensional conformal field theory with gauge symmetry due to Knizhnik and Zamolodchikov [18]. Although in their situation the parameter λ is given by $(\ell + g)^{-1}$ where ℓ is a positive integer and g is the corresponding dual Coxeter number, we shall deal with the monodromy by considering λ as a parameter.

1.2. Description of the monodromy by means of solutions of quantum Yang-Baxter equations.

Let W be a finite dimensional complex vector space. By the quantum Yang-Baxter equation written in a multiplicative form we mean the following functional equation for a meromorphic function $R(x)$ with values in $\mathrm{End}\,(W \otimes W)$:

$$(1.2.1)\quad R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x).$$

Here the equality is considered in $\mathrm{End}\,(W \otimes W \otimes W)$ and the notation R_{ij} is standard as is explained in the previous section. Let us consider the case where $R(x)$ contains an extra parameter q so that $R(x,q)$ has an expansion around $q = 1$:

$$(1.2.2)\qquad R(x,q) = 1 + (q-1)r(x) + \cdots$$

In this situation we verify that $r(x)$ is a solution of the multiplicative classical Yang-Baxter equation

$$[r_{12}(x), r_{13}(xy)] + [r_{12}(x), r_{23}(y)] + [r_{13}(xy), r_{23}(y)] = 0.$$

We call $r(x)$ the *classical limit* of $R(x,q)$. The following typical solutions of the above classical Yang-Baxter equation was discovered by Belavin and Drinfel'd [3] (see also [10]). Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let Δ be the set of roots of $\mathfrak{g}$. For a root α, we denote by X_α the root vector normalized by $(X_\alpha, X_{-\alpha}) = 1$ with respect to the Cartan-Killing form. Putting $r = \Sigma_{\alpha \in \Delta} \operatorname{sgn} \alpha . X_\alpha \otimes X_{-\alpha}$, we define a $\mathfrak{g} \otimes \mathfrak{g}$-valued function $r(x)$ by

$$r(x) = r - t + \frac{2t}{x - 1} \tag{1.2.3}$$

where t is defined in the previous section. These solutions are called *trigonometric* in the sense that they are rational functions of $x = e^u$.

The quantization problem of the above solutions was treated by Jimbo. In a series of papers [9], [10] and [11], he constructed a matrix $R(x,q)$ whose expansion around $q = 1$ is given by

(1.2.4) $R(x,q) = f(x)\{\mathbf{1} + (q-1)((\rho\otimes\rho)r(x) + \varkappa(x)\mathbf{1}) + \cdots\}$
with some $\mathbf{C}$-valued functions $f(x)$ and $\varkappa(x)$,

for the following simple Lie algebras $\mathfrak{g}$ and their representations $\rho : \mathfrak{g} \to \operatorname{End}(W)$

(1.2.5) $\mathfrak{g}$ is non-exceptional and ρ is the vector representation,
(1.2.6) $\mathfrak{g}$ is $\mathfrak{sl}(2,\mathbf{C})$ and ρ is an arbitrary finite dimensional irreducible representation.

In this section we discuss the case 1.2.5. Our matrices $R(x,q)$ are given by formulae 3.5 and 3.6 in [10] by putting $k = q$. In the formula 1.2.4, $f(x)$ is given by $(x-1)$ if $\mathfrak{g}$ is of type A and by $(x-1)^2$ if $\mathfrak{g}$ is of type B, C or D.

We put $\bar{R} = \sigma R$ where $\sigma \in \operatorname{End}(W \otimes W)$ is the transposition defined by $\sigma(x \otimes y) = y \otimes x$. One of the important properties of the matrix $\bar{R}(x,q)$ is that it commutes with the diagonal action of $U\hat{}(\mathfrak{g})$. Here $U\hat{}(\mathfrak{g})$ denotes the q-analogue of the corresponding Lie algebra $\mathfrak{g}$ due to Jimbo [9], which is also denoted by $U_h(\mathfrak{g})$ with $q = e^h$ by Drinfel'd [7]. Instead of giving the complete definition we recall the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{C})$, which is originally due to Kulish and Reshetikhin (see the references of [7]). We define $U\hat{}(\mathfrak{g})$ to be the $\mathbf{C}$-algebra generated by the symbols $\hat{e}$, $\hat{f}$, q^h and q^{-h} with relations

$$q^{h/2}\hat{e}q^{-h/2} = q\hat{e}, \qquad q^{h/2}\hat{f}q^{-h/2} = q^{-1}\hat{f}, \qquad [\hat{e},\hat{f}] = \frac{q^h - q^{-h}}{q - q^{-1}}.$$

We define the comultiplication $\Delta : U\hat{}(\mathfrak{g}) \to U\hat{}(\mathfrak{g}) \otimes U\hat{}(\mathfrak{g})$ by the algebra homomorphism characterized by

$$\Delta(q^{\pm h/2}) = q^{\pm h/2} \otimes q^{\pm h/2},\ \Delta(X) = X \otimes q^{-h/2} + q^{h/2} \otimes X \ \text{ for } \ X = \hat{e},\ \hat{f}.$$

With respect to the comultiplications Δ and $\bar{\Delta} = \sigma\Delta$, $U\hat{}(\mathfrak{g})$ has a structure of a non-commutative Hopf algebra which is considered to be a deformation of the universal enveloping algebra of $\mathfrak{sl}(2,\mathbf{C})$ (see Drinfel'd [7] and Verdier [23] for a more extensive treatment).

Let us go back to the situation of the previous section. Associated with a non-exceptional simple Lie algebra $\mathfrak{g}$ and its vector representation, we consider the connection

$$\omega = \Sigma_{1 \leqslant \alpha < \beta \leqslant n} \lambda \Omega_{\alpha\beta} d \log (z_\alpha - z_\beta).$$

As the monodromy of ω we get a one parameter family of linear representation $\theta : B_n \to \operatorname{End}(W^{\otimes n}) \otimes \mathbf{C}\{\lambda\}$. To describe θ we introduce the matrix $T(q)$ by

(1.2.7)
$$T(q) = \lim_{x \to \infty} x^{-d} \bar{R}(x,q)$$

where d is the degree of the corresponding $\bar{R}(x,q)$ with respect to x, which is given by $d = 1$ in the case $\mathfrak{g}$ is of type A and by $d = 2$ in the other cases. We put $\nu = \dfrac{m-1}{2m}$ if $\mathfrak{g} = \mathfrak{sl}(m,\mathbf{C})$ and $\nu = \dfrac{1}{2}$ otherwise. Our main theorem in this section is the following:

THEOREM 1.2.8. — *Let $\mathfrak{g}$ be a non-exceptional complex simple Lie algebra and let $\rho : \mathfrak{g} \to \operatorname{End}(W)$ be its vector representation. As the monodromy of the associated connection*

$$\omega = \Sigma_{1 \leqslant \alpha < \beta \leqslant n} \lambda \Omega_{\alpha\beta} d \log (z_\alpha - z_\beta)$$

we get a linear representation $\theta : B_n \to \operatorname{End}(W^{\otimes n}) \otimes \mathbf{C}\{\lambda\}$ given by

$$\theta(\sigma_i) = q^\nu \{ 1 \otimes \cdots \otimes 1 \otimes T(q) \otimes 1 \otimes \cdots \otimes 1 \}, \qquad 1 \leqslant i \leqslant n-1.$$

Here $q = \exp(-\pi\sqrt{-1}\lambda)$ and $T(q)$ is situated on the i-th and $(i+1)$-st factors. Moreover this representation commutes with the diagonal action of $U\hat{}(\mathfrak{g})$ on $W^{\otimes n}$.

The action of $U\hat{}(\mathfrak{g})$ is defined by the multi-diagonal map in the sense of [9] and [12]. In the case $\mathfrak{g} = \mathfrak{sl}(m,\mathbf{C})$, the monodromy representation obtained above is known as the higher order Pimsner-Popa-Temperley-Lieb representation (see [17]). In fact the matrix $T(q)$ is given by

$$(1.2.9)\quad T(q)=\Sigma E_{\alpha\alpha}\otimes E_{\alpha\alpha}+q\Sigma_{\alpha\neq\beta}E_{\alpha\beta}\otimes E_{\beta\alpha}+(1-q^2)\Sigma_{\alpha<\beta}E_{\alpha\alpha}\otimes E_{\beta\beta}$$

where $E_{\alpha\beta}$ signify $m \times m$ matrix units. In this case the matrix $T(q)$ defines a linear representation of the braid group factoring through the Iwahori's Hecke algebra of the symmetric group.

1.3. Proof of Theorem 1.2.8.

Let us start with an integrable connection ω over X_n of the form $\omega = \Sigma_{1\leqslant\alpha<\beta\leqslant n}M_{\alpha\beta}d\log(z_\alpha - z_\beta)$, $M_{\alpha\beta}\in\mathfrak{gl}(m,\mathbf{C})$. The monodromy of ω is expressed by an infinite sum using Chen's iterated integrals [6].

$$(1.3.1)\qquad \theta(\gamma) = \mathbf{1} + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots$$

for $\gamma\in P_n$. Here we have used the following standard notation for the Chen's iterated integrals.

Let X be a smooth manifold and let ω_i, $1 \leqslant i \leqslant n$, be matrix valued 1-forms on X. For a path $\gamma:[0,1]\to X$, we define the iterated integral $\int_\gamma \omega_1\omega_2 \ldots \omega_n$ by

$$\int_\Delta A_1(t_1)A_2(t_2)\ldots A_n(t_n)dt_1dt_2\ldots dt_n$$

where $\gamma^*\omega_i = A_i(t_i)dt_i$ and $\Delta = \{(t_1,\ldots,t_n);\ 0\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant 1\}$.

Let $\mathbf{C}\langle\langle X_{\alpha\beta}\rangle\rangle$ denote the ring of non-commutative formal power series with indeterminates $X_{\alpha\beta}$, $1\leqslant\alpha<\beta\leqslant n$, and let J be its two sided ideal generated by the following infinitesimal pure braid relations among $X_{\alpha\beta}$:

$$(1.3.2)\qquad \begin{array}{c}[X_{\alpha\beta},\ X_{\alpha\gamma}+X_{\beta\gamma}],\quad [X_{\alpha\beta}+X_{\alpha\gamma},\ X_{\beta\gamma}],\quad \alpha<\beta<\gamma \\ [X_{\alpha\beta},X_{\gamma\delta}]\ \text{ for distinct } \alpha,\beta,\gamma,\delta.\end{array}$$

We denote by A the quotient algebra $\mathbf{C}\langle\langle X_{\alpha\beta}\rangle\rangle/J$. As a universal expression of 1.3.1, we obtain a homomorphism $\tilde{\theta}: P_n \to A$ defined by

$$\tilde{\theta}(\gamma) = 1 + \int_\gamma \tilde{\omega} + \int_\gamma \tilde{\omega}\tilde{\omega} + \cdots \text{ with}$$

$$\tilde{\omega} = \Sigma_{1 \leqslant \alpha < \beta \leqslant n} X_{\alpha\beta} \otimes d \log (z_\alpha - z_\beta).$$

Let $C[P_n]\hat{}$ denote the completion of the group ring $C[P_n]$ with respect to the powers of the augmentation ideal and let $j: P_n \to C[P_n]\hat{}$ denote the natural homomorphism. We have the following assertions:

Proposition 1.3.3. — *(i) We have an isomorphism of complete Hopf algebras* $C[P_n]\hat{} \xrightarrow{\sim} A$ *such that the following diagram is commutative.*

$$\begin{array}{ccc} & \overset{j}{\nearrow} & C[P_n]\hat{} \\ P_n & & \downarrow \\ & \underset{\tilde{\theta}}{\searrow} & A \end{array}$$

(ii) The universal expression of the monodromy $\tilde{\theta}: P_n \to A$ *is injective.*

The assertion (i) has been discussed by several authors in a more general situation (see [1], [8] and [16]). The primitive part of A is the Malcev Lie algebra of P_n, which is the dual of the Sullivan's 1-minimal model of X_n (see [21], [19] and [16]). The assertion (ii) is proved in [17] by the induction with respect to n by using the fibration $\pi: X_{n+1} \to X_n$. The essential points are that the monodromy of the fibration π is trivial on the homology and that the natural homomorphism j is injective in the case of free groups. By using the assertion (ii) we have shown in [17] the following theorem:

Theorem 1.3.4 ([17]). — *Let* $\gamma_{\alpha\beta}$, $1 \leqslant \alpha < \beta \leqslant n$, *be a system of generators of* P_n *given by*

$$(1.3.5) \qquad \gamma_{\alpha\beta} = \sigma_\alpha \sigma_{\alpha+1} \ldots \sigma_{\beta-1} \sigma_\beta^2 \sigma_{\beta-1}^{-1} \ldots \sigma_\alpha^{-1}.$$

If $\theta: P_n \to GL(m,C)$ *is a linear representation such that* $\|\theta(\gamma_{\alpha\beta}) - \mathbf{1}\|$ *is sufficiently small for each* $1 \leqslant \alpha < \beta \leqslant n$, *then there exist constant matrices* $M_{\alpha\beta}$, $1 \leqslant \alpha < \beta \leqslant n$, *close to* 0, *satisfying the infinitesimal pure braid relations, such that the monodromy of the connection* $\omega = \Sigma_{1 \leqslant \alpha < \beta \leqslant n} M_{\alpha\beta}\, d \log (z_\alpha - z_\alpha)$ *is equivalent to* θ.

To deduce Theorem 1.3.4 from Propositon 1.3.3 we used an argument due to Hain [8].

Now let us go back to the situation of Theorem 1.2.8.

LEMMA 1.3.6. — *We put* $\lambda = -(\pi\sqrt{-1})^{-1}\log q$, $-\pi \leqslant \mathrm{Im}\ \log q < \pi$. *The matrix* $T(q)^2$ *has an expansion with respect to* λ *of the form*

$$T(q)^2 = \mathbf{1} + 2\pi\sqrt{-1}\,\lambda\{(\rho\otimes\rho)(t) - 2\nu.\mathbf{1}\} + \mathcal{O}(\lambda^2).$$

Here ρ *is the vector representation as in Sect.* 1.2.

Proof of Lemma 1.3.6. — Let us recall that $T(q)$ is defined as the leading coefficient of the matrix $\bar{R}(x,q)$ with respect to x. By means of the expansion 1.2.4 and the definition of $r(x)$ (see 1.2.3), we have

$$T'(1) = \sigma.\{(\rho\otimes\rho)(r-t) + 2\nu.\mathbf{1}\}. \tag{1.3.7}$$

Here we have used $2\nu = \lim_{x\to\infty} \varkappa(x)$, which is verified by a direct computation. Let us now observe that $T(1)$ is equal to the transposition σ. By using

$$\begin{aligned} \sigma.(\rho\otimes\rho)(t).\sigma &= (\rho\otimes\rho)(t) \\ \sigma.(\rho\otimes\rho)(r).\sigma &= -(\rho\otimes\rho)(r) \end{aligned} \tag{1.3.8}$$

we obtain the formula

$$T(1)T'(1) + T'(1)T(1) = -2(\rho\otimes\rho)(t) + 4\nu.\mathbf{1}.$$

Our Lemma follows immediately.

It follows from the definition of the Yang-Baxter equation 1.2.1 that the matrix $\bar{R}(x,q)$ satisfies

$$\bar{R}_{12}(x)\bar{R}_{23}(xy)\bar{R}_{12}(y) = \bar{R}_{23}(y)\bar{R}_{12}(xy)\bar{R}_{23}(x). \tag{1.3.9}$$

This shows that the correspondence

$$\sigma_i \to 1\otimes\cdots\otimes T(q)\otimes\cdots\otimes 1 \tag{1.3.10}$$

appearing in the statement of Theorem 1.2.8 actually defines a linear representation of the braid group. In the following we denote this representation by φ.

If $|\lambda|$ is sufficiently small, then we may apply Theorem 1.3.4. Hence in this situation we have a matrix $M(\lambda)\in \mathrm{End}(W\otimes W)$ close to 0 and analytic with respect to λ, so that the monodromy of the connection $\Sigma_{1\leqslant\alpha<\beta\leqslant n} M_{\alpha\beta}(\lambda)\, d\log(z_\alpha - z_\beta)$ expressed by the iterated integrals 1.3.1 is equal to φ restricted to P_n.

Let $M(\lambda) = Z_1\lambda + Z_2\lambda^2 + \cdots$ be an expansion of $M(\lambda)$ around $\lambda = 0$. By means of the expression of the monodromy using iterated integrals and Lemma 1.3.6 we have

$$Z_1 = (\rho\otimes\rho)(t) - 2\nu.\mathbf{1}.$$

In the following, we denote the above matrix by Ω'.

LEMMA 1.3.11. — *If $|\lambda|$ is sufficiently small, there exists a matrix $P(\lambda) \in \mathrm{End}(W^{\otimes n})$ with $\lim_{\lambda\to 0} P(\lambda) = \mathbf{1}$ such that*

$$P(\lambda)^{-1}M_{\alpha\beta}(\lambda)P(\lambda) = \lambda\Omega'_{\alpha\beta}.$$

Proof of Lemma 1.3.11. — Let $H_{\alpha\beta}$ denote the hyperplane in $\mathbf{C}^n$ defined by $z_\alpha = z_\beta$. Let $\mu : X \to \mathbf{C}^n$ be a blowing up with exceptional divisors E_k, $3 \leqslant k \leqslant n$, such that $\mu(E_k) = \bigcap_{1\leqslant\alpha<\beta\leqslant k} H_{\alpha\beta}$. We denote by E_2 the proper transform of H_{12}. Then the residue of the connection $\mu^*\omega$ along the divisor E_k is expressed as $\Sigma_{1\leqslant\alpha<\beta\leqslant k}M_{\alpha\beta}(\lambda)$. Let us observe that a normal loop around E_k is given by $\gamma_k = (\sigma_1\ldots\sigma_{k-1})^k$ which lies in the center of B_k. For a generic value $\lambda\in\mathbf{C}$, the matrix $\varphi(\gamma_k)$ is diagonalizable, which implies that the residue $\Sigma_{1\leqslant\alpha<\beta\leqslant k}M_{\alpha\beta}(\lambda)$ is diagonalizable. Moreover, by means of the infinitesimal pure braid relations for $M_{\alpha\beta}(\lambda)$ we conclude that the residues $\Sigma_{1\leqslant\alpha<\beta\leqslant k}M_{\alpha\beta}(\lambda)$, $k = 2, 3, \ldots$ are diagonalized simultaneously. We have a matrix $Q(\lambda) = Q_0 + Q_1\lambda + Q_2\lambda^2 + \cdots$ such that for $2 \leqslant k \leqslant n$

$$Q(\lambda)^{-1}(\Sigma_{1\leqslant\alpha<\beta\leqslant k}M_{\alpha\beta}(\lambda))Q(\lambda) \tag{1.3.12}$$

is diagonal. It can be shown by using the explicit form of $T(q)$ that the eigenvalues of $\varphi(\gamma_k)$ is of the form q^m with some integer m. This implies that the matrix 1.3.12 is linear with respect to λ. Hence it is written as $Q_0^{-1}(\Sigma_{1\leqslant\alpha<\beta\leqslant k}\lambda\Omega'_{\alpha\beta})Q_0$. Putting $P(\lambda) = Q(\lambda).Q_0^{-1}$, we obtain a desired matrix. This proves Lemma.

The proof of Theorem 1.2.8 is completed in the following way. We put $\omega' = \Sigma\lambda\Omega'_{\alpha\beta}d\log(z_\alpha - z_\beta)$. By Lemma 1.3.11 the expression

$$\mathbf{1} + \int_\gamma \omega' + \int_\gamma \omega'\omega' + \cdots \tag{1.3.13}$$

is equal to $P(\lambda)^{-1}\varphi(\lambda)P(\lambda)$ if $|\lambda|$ is sufficiently small. We observe that $P(\lambda)$ is analytically continued to a meromorphic function of λ on the whole complex plane. Since the expression 1.3.13 is an entire function of λ we conclude by an analytic continuation that 1.3.13 is expressed as $P(\lambda)^{-1}\varphi(\lambda)P(\lambda)$ in $\mathrm{End}(W^{\otimes n}) \otimes C\{\lambda\}$. Thus we have shown the statement of Theorem 1.2.8 on the pure braid group P_n. To extend this to the full braid group B_n it suffices to observe that both $\theta(\sigma_i)$ and $\varphi(\sigma_i)$ are the transposition of the i-th factor and $(i+1)$-st factors if $\lambda = 0$ and that they are holomorphic with respect to λ. This shows the first assertion of Theorem 1.2.8. The second assertion is derived from the fact that $\bar{R}(x,q)$ commutes with the diagonal action of $\hat{U}(\mathfrak{g})$. This completes the proof of Theorem 1.2.8.

(1.3.14) *Remark.* — For a complex number $\lambda \in C$, the above proof implies that the correspondence described in Theorem 1.2.8 holds true if $\varphi(\gamma_k)$, $2 \leqslant k \leqslant n$, are diagonalizable. This condition is satisfied if φ is completely reducible.

2. MONODROMY OF n-POINT FUNCTIONS IN TWO DIMENSIONAL CONFORMAL FIELD THEORY

2.1. Review of $A_1^{(1)}$ model due to Tsuchiya and Kanie.

In this section we recall briefly the operator formalism of the two dimensional conformal field theory on $\mathbf{P}^1$ with gauge symmetry of type $A_1^{(1)}$ following a recent work of Tsuchiya and Kanie [22].

Integrable highest weight modules. — Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{C})$ and let $\hat{\mathfrak{g}}$ be the affine Lie algebra of type $A_1^{(1)}$ which is defined by the canonical central extension of the loop algebra $\mathfrak{g} \otimes \mathbf{C}[t,t^{-1}]$ (see [15]). Putting $\mathfrak{M}_\pm = \Sigma_{n\geqslant 1}\mathfrak{g} \otimes t^{\pm n}$, $\hat{\mathfrak{g}}$ is decomposed into

$$\hat{\mathfrak{g}} = \mathfrak{M}_+ \oplus \mathfrak{g} \oplus \mathbf{C}c \oplus \mathfrak{M}_-$$

where c is the central element. For a positive integer ℓ and a half integer j such that $0 \leqslant j \leqslant \ell/2$ it is known by Kac [15] that there exists a unique irreducible left $\hat{\mathfrak{g}}$-module $\mathscr{H}_j(\ell)$ with a non zero vector $|\ell,j\rangle$ such that

$$(2.1.1) \qquad \mathfrak{M}_+|\ell,j\rangle = E|\ell,j\rangle = 0, \; H|\ell,j\rangle = 2j|\ell,j\rangle, \quad c|\ell,j\rangle = \ell|\ell,j\rangle.$$

In the same way, we have a unique irreducible right $\hat{\mathfrak{g}}$-module $\mathcal{H}_j^\dagger(\ell)$ with $\langle j,\ell|$ such that

$$\langle j,\ell|\mathfrak{N}_- = \langle j,\ell|\mathrm{F} = 0,\ \langle j,\ell|\mathrm{H} = 2j\langle j,\ell|, \quad \langle j,\ell|c = \ell\langle j,\ell|. \tag{2.1.2}$$

Here H, E and F stand for the usual Chevalley basis of $\mathfrak{g}$. In the following we fix ℓ and we write $\mathcal{H}_j$ instead of $\mathcal{H}_j(\ell)$. There exists a unique bilinear form $\mathcal{H}_j^\dagger \times \mathcal{H}_j \to \mathbf{C}$ such that $\langle j,\ell|\ell,j\rangle = 1$ and $\langle ua|v\rangle = \langle u|av\rangle$ for any $a \in \hat{\mathfrak{g}}$, $u \in \mathcal{H}_j^\dagger$ and $v \in \mathcal{H}_j$.

Operation of the Virasoro Lie algebra. – For $\mathrm{X} \in \mathfrak{g}$, we put $\mathrm{X}[n] = \mathrm{X} \otimes t^n$ and $\mathrm{X}(z) = \Sigma_{n \in \mathbf{Z}}\mathrm{X}[n]z^{-n-1}$ with $z \in \mathbf{C}\backslash\{0\}$. The *Segal-Sugawara form* $\mathrm{T}(z)$ is defined to be

$$\mathrm{T}(z) = \frac{1}{2(2+\ell)}\{\Sigma_\mu :\ \mathrm{I}_\mu(z)\mathrm{I}_\mu(z):\}. \tag{2.1.3}$$

Here $\{\mathrm{I}_\mu\}$ denotes an orthonormal basis of $\mathfrak{g}$ and $::$ stands for the usual normal order product defined by

$$:\mathrm{X}[m]\mathrm{Y}[n]: = \begin{cases} \mathrm{X}[m]\mathrm{Y}[n]) & \text{if } m < n \\ \frac{1}{2}\{\mathrm{X}[m]\mathrm{Y}[n] + \mathrm{Y}[n]\mathrm{X}[m]\} & \text{if } m = n \\ \mathrm{Y}[n]\mathrm{X}[m] & \text{if } m > n. \end{cases}$$

We define L_m, $m \in \mathbf{Z}$ as the coefficients of the expansion

$$\mathrm{T}(z) = \Sigma_{m \in \mathbf{Z}}\mathrm{L}_m z^{-m-2}. \tag{2.1.4}$$

We may also express L_m as

$$\mathrm{L}_m = \frac{1}{2(2+\ell)}\Sigma_{k \in \mathbf{Z}}\Sigma_\mu\ :\ \mathrm{I}_\mu(-k)\mathrm{I}_\mu(m+k)\ : \tag{2.1.5}$$

These L_m, $m \in \mathbf{Z}$, satisfy the fundamental relations of the *Virasoro Lie algebra*:

$$[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c'. \tag{2.1.6}$$

Here $c' = \dfrac{3\ell}{\ell + 2}\, id$, which we shall call the *central charge*. With respect to the operation of L_0, $\mathcal{H}_j$ is decomposed into finite dimensional subspaces

$$\mathcal{H}_j = \bigoplus_{d \geqslant 0} \mathcal{H}_{j,d} \tag{2.1.7}$$

where $\mathscr{H}_{j,d}$ is the eigenspace with the eigenvalue $\dfrac{j^2+j}{\ell+2}+d$. In particular, $\mathscr{H}_{j,0}$ is identified with the spin j representation of $\mathfrak{g}$, which is denoted by V_j.

Definition of primary fields. — We are interested in operators on the space $\mathscr{H} = \bigoplus_{j=0}^{\ell/2} \mathscr{H}_j$. The basic operators are so called *primary fields.* A primary field of spin j is defined to be a bilinear form $\phi(u,z)$: $\mathscr{H}^\dagger \times \mathscr{H} \to \mathbf{C}$ parametrized by $u \in V_j$ and $z \in \mathbf{C}\backslash\{0\}$ in such a way that

(i) $\phi(u,z)$ is linear with respect to u

(ii) $\langle v|\phi(u,z)|w\rangle$ is a multivalued holomorphic function of z for any $v \in \mathscr{H}^\dagger$ and $w \in \mathscr{H}$,

satisfying the following conditions:

(2.1.8) $[X \otimes t^m,\ \phi(u,z)] = z^m\phi(Xu,z)$ (*gauge condition*)

(2.1.9) $$[L_m, \phi(u,z)] = z^m\left\{z\frac{\partial}{\partial z} + (m+1)\Delta_j\right\}\phi(u,z)$$

where $\Delta_j = \dfrac{j^2+j}{\ell+2}$, which we shall call the *conformal dimension* of ϕ.

Existence of vertex operators. — Given a primary field of spin j, we associate to the triple $v = (j_1,j,j_2)$ the (j_1,j_2) component of $\phi(u,z)$ with respect to the decomposition 2.1.7, which we denote by $\phi_v(u,z)$. This operator is called a *vertex operator of type* v. We have a Laurent series expansion $\phi_v(u,z) = \Sigma_{n\in\mathbf{Z}}\phi_n(u)z^{-n-\Delta}$ with $\Delta = \Delta_j + \Delta_{j_1} - \Delta_{j_2}$ ([22] Prop. 2.1.). This gives a $\mathfrak{g}$ invariant trilinear form φ: $V_{j_1}^\dagger \otimes V_{j_2} \otimes V_{j_3} \to \mathbf{C}$ defined by $\varphi(u,v,w) = \langle u|\phi_0(v)|w\rangle$, which we shall call the *initial form.*

THEOREM 2.1.10 ([22] Th. 2.2.). — (i) *A non trivial vertex operator of type v exists if and only if the following conditions are satisfied:*

(2.1.11) $|j_1-j_2| \leqslant j \leqslant j_1+j_2$, $j_1+j+j_2 \in \mathbf{Z}$ (*Clebsch-Gordan condition*)

(2.1.12) $$j_1+j+j_2 \leqslant \ell$$

(ii) *Under the above conditions, a vertex operator of type v is unique up to scalar and is determined by its initial form.*

Differential equation of n-point functions. – For an operator A on $\mathcal{H}$, we denote by $\langle A \rangle$ its *vacuum expectation* defined by $\langle \mathrm{vac}|A|\mathrm{vac}\rangle = \langle 0,\ell|A|\ell,0\rangle$. Our purpose is to give a description of *n-point functions* $\langle \phi_1(u_1,z_1)\dots\phi_n(u_n,z_n)\rangle$ for primary fields ϕ_i. A main tool to deduce differential equations satisfied by n-point functions is the following *operator product expansions*

$$(2.1.13)\quad X(\zeta)\phi(u,z) = \frac{1}{\zeta - z}\phi(Xu,z) + (\text{regular terms})$$

$$(2.1.14)\quad T(\zeta)\phi(u,z) = \left(\frac{\Delta_j}{(\zeta-z)^2} + \frac{1}{\zeta - z}\frac{\partial}{\partial z}\right)\phi(u,z) + (\text{regular terms})$$

for a primary field ϕ of spin j. Here the meaning of the compositions of operators is justified by the use of the decomposition 2.1.7 (see [22] for a precise definition). Following [18], we define the operation of $\hat{\mathfrak{g}}$ on vertex operators by

$$(2.1.15)\quad [X[m]\phi](u,z) = \frac{1}{2\pi\sqrt{-1}}\int_C d\zeta(\zeta-z)^m X(\zeta)\phi(u,z)$$

$$(2.1.16)\quad [L_m\phi](u,z) = \frac{1}{2\pi\sqrt{-1}}\int_C d\zeta(\zeta-z)^{m+1}T(\zeta)\phi(u,z)$$

for a positively oriented small contour C around z. Combining with the operator product expansions, we obtain

$$(2.1.17)\quad \begin{aligned}&[X[0]\phi](u,z) = \phi(Xu,z),\\ &[X[m]\phi](u,z) = 0 \quad \text{for} \quad m > 0\end{aligned}$$

$$(2.1.18)\quad [L_{-1}\phi](u,z) = \frac{\partial}{\partial z}\phi(u,z), \qquad [L_0\phi](u,z) = \Delta_j\phi(u,z),$$

$$[L_m\phi](u,z) = 0 \quad \text{for} \quad m > 0.$$

Starting from a primary field ϕ of spin j, we get new operators by the iterations of the operations of $X[m]$ and L_m, $m \leqslant 0$, of type 2.1.15 and 16. They are classified into the *levels* by the eigenvalues of the operator L_0, e.g., $L_{-n_1}L_{-n_n}\dots L_{-n_k}\phi$ has an eigenvalue $\Sigma_{j=1}^k n_k + \Delta_j$ with respect to the operation of L_0. This is the whole spectrum of our operators. From the operator product expansions, we deduce the following *local Ward identities :*

$$(2.1.19)\quad \langle X(\zeta)\phi_1(z_1)\dots\phi_n(z_n)\rangle = \Sigma_{\alpha=1}^{n}\frac{1}{\zeta - z_\alpha}\langle \phi_1(z_1)\dots[X[0]\phi_\alpha](z_\alpha)\dots\phi_n(z_n)\rangle$$

$$(2.1.20)\quad \langle T(\zeta)\phi_1(z_1)\dots\phi_n(z_n)\rangle = \Sigma_{\alpha=1}^{n}\left(\frac{\Delta_{j\alpha}}{(\zeta - z_\alpha)^2} + \frac{1}{\zeta - z_\alpha}\frac{\partial}{\partial z_\alpha}\right)\langle \phi_1(z_1)\dots\phi_n(z_n)\rangle.$$

Here ϕ_α is supposed to be a primary field of spin j_α.

THEOREM 2.1.21 (Knizhnik and Zamolodchikov [18]). – *The n-point function* $\Phi = \langle\phi_1(z_1)\dots\phi_n(z_n)\rangle$ *satisfies the total differential equation*

$$d\Phi = \Sigma_{1\leqslant\alpha<\beta\leqslant n}\frac{1}{\ell+2}\Omega_{\alpha\beta}d\log(z_\alpha - z_\beta).\Phi.$$

Here ϕ_α *is a primary field of spin* j_α *and* $\Omega_{\alpha\beta}\in \mathrm{End}(V_{j_1}\otimes\cdots\otimes V_{j_n})$ *is determined by* 1.1.2 *via spin* j_α *representations of* $\mathfrak{sl}(2,\mathbf{C})$.

Proof. – Let $\phi(u,z)$ be a primary field. By the expression of L_0 given in 2.1.5 and the identities 2.1.17 and 18 we have

$$(\ell+2)\frac{\partial}{\partial z}\phi(u,z) = [\Sigma_\mu I_\mu[-1]I_\mu[0]\phi](u,z).$$

The RHS turns out to be the constant term of the operator product expansion of $\Sigma_\mu I_\mu(\zeta)\phi[I_\mu u,z]$. This implies that

$$(\ell+2)\frac{\partial}{\partial z}\phi(u,z) = \lim_{\zeta\to z}[\Sigma_\mu I_\mu(\zeta)\phi[I_\mu u,z] - \frac{1}{\zeta - z}\phi[\Omega u,z]]$$

where Ω denotes the Casimir operator. Combining with the local Ward identity 2.1.19 we have

$$(\ell+2)\frac{\partial}{\partial z_\alpha}\Phi = \Sigma_{\beta\neq\alpha}\frac{\Omega_{\alpha\beta}}{z_\alpha - z_\beta}\Phi$$

which proves our Theorem.

2.2. Monodromy associated with higher representations of $\mathfrak{sl}(2,\mathbf{C})$.

For a half integer $j \geqslant 0$, we denote by V_j the irreducible left $\mathfrak{sl}(2,\mathbf{C})$ module of spin j, which is an irreducible representation of dimension $2j+1$. We now proceed to discuss the monodromy representation $\theta : B_n \to \operatorname{End}(V_j^{\otimes n})$ of the connection associated with the spin j representation of $\mathfrak{sl}(2,\mathbf{C})$ in the sense of Sect. 1.1. For this purpose we first recall a « fusion » process for solutions of Yang-Baxter equations due to Jimbo [11]. Let us start with the matrix $T(q)$ given in 1.2.9 with $m = 2$. We put

$$\bar{R}(x,q) = xq^{-1}T(q) - x^{-1}qT(q)^{-1}.$$

The matrix $R(x,q) = \sigma\bar{R}(x,q)$ is a solution of the quantum Yang-Baxter equation. We have an expansion of the form

$$(2.2.1) \qquad R(x,q) = (x-x^{-1})\{1+r(x)(q-1) + \cdots\}$$

with its classical limit $r(x)$. We put

$$R_k(x,q) = R_{k,2m}(x,q)R_{k,2m-1}(xq,q) \ldots R_{k,m+1}(xq^{m-1},q)$$

which is considered to be an element of $\operatorname{End}(V^{\otimes m} \otimes V^{\otimes m})$. Here $R_{\alpha,\beta}$ stands for the matrix R acting on the α-th and β-th factors and $V = \mathbf{C}^2$. We now define $R^{(m)}(x,q)$ as

$$R^{(m)}(x,q) = R_1(x,q)R_2(xq,q) \ldots R_m(xq^{m-1},q).$$

Let us regard V as a $U\hat{\ }(\mathfrak{sl}(2,\mathbf{C}))$ module and we denote by $V_j\hat{\ }$ the irreducible $U\hat{\ }(\mathfrak{sl}(2,\mathbf{C}))$ module of spin j considered as a subspace of $V^{\otimes 2j}$. This is denoted by L_{2j} in [9] Sect. 3. The matrix $R^{(m)}(x,q)$ defined above determines an endomorphism of $V_j\hat{\ } \otimes V_j\hat{\ }$ with $j = m/2$. Let us define the matrix $T^{(m)}(q)$ by

$$(2.2.2) \qquad T^{(m)}(q) = \lim_{x\to\infty} x^{-m^2}\, \overline{R}^{(m)}(x,q).$$

This matrix is also expressed explicitly as

$$(2.2.3) \quad T^{(m)}(q) = q^{-m^3}(T_mT_{m-1} \ldots T_1)(T_{m+1}T_m \ldots T_2) \ldots \ldots (T_{2m-1}T_{2m-2} \ldots T_m)$$

where T_i denotes the matrix $T(q)$ on the i-th and $(i+1)$-st factors.

THEOREM 2.2.4. — *As the monodromy of the connection associated with the spin $j = m/2$ representation of* $\mathfrak{sl}(2,\mathbf{C})$, *we get a one parameter family of linear representations* $\theta : \mathrm{B}_n \to \mathrm{End}\,(\mathrm{W}^{\otimes n}) \otimes \mathbf{C}\{\lambda\}$ *with* $\mathrm{W} = \hat{\mathrm{V}_j}$ *defined by*

$$\theta(\sigma_i) = q^{-1/4}\{\mathbf{1} \otimes \cdots \otimes \mathrm{T}^{(m)}(q) \otimes \cdots \otimes \mathbf{1}\}, \qquad 1 \leqslant i \leqslant n-1,$$

where $q = \exp\,(-\pi\sqrt{-1}\lambda)$ *and* $\mathrm{T}^{(m)}(q)$ *is on the i-th and* $(i+1)$*-st factors.*

Let $\iota : \mathrm{B}_n \to \mathrm{B}_{mn}$ be a homomorphism defined by

$$\text{(2.2.5)} \quad \iota(\sigma_i) = (\sigma_{\alpha+m}\sigma_{\alpha+m-1}\cdots\sigma_{\alpha+1}).(\sigma_{\alpha+m+1}\sigma_{\alpha+m}\cdots\sigma_{\alpha+2})\cdots \\ \cdots(\sigma_{\alpha+2m-1}\alpha_{\alpha+2m-2}\cdots\sigma_{\alpha+m})$$

with $\alpha = (i-1)m$. This « parallel » embedding is illustrated in the following picture:

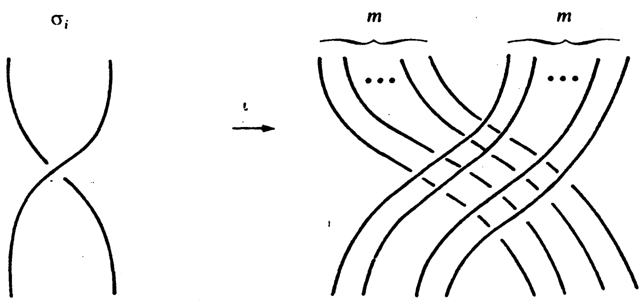

Fig. 2.

By means of this homomorphism our monodromy representation θ is also expressed in the following manner:

COROLLARY 2.2.6. — *Let* $\varphi : \mathrm{B}_{mn} \to \mathrm{End}\,(\mathrm{V}^{\otimes mn}) \otimes \mathbf{C}\{\lambda\}$ *be the Pimsner-Popa-Temperley-Lieb representation defined by* $\varphi(\sigma_i) = \mathbf{1} \otimes \cdots \otimes \mathrm{T}(q) \otimes \cdots \otimes \mathbf{1}$ *(see 1.2.9.). Then the composition* $\varphi \circ \iota$: $\mathrm{B}_n \to \mathrm{End}\,(\mathrm{V}^{\otimes mn}) \otimes \mathbf{C}\{\lambda\}$ *leaves invariant the subplace* $(\mathrm{V}^{\otimes n})$ *and the monodromy representation* θ *is given by*

$$\theta(\sigma_i) = q^{-m^3-\frac{1}{4}}\varphi \circ \iota(\sigma_i), \qquad 1 \leqslant i \leqslant n-1.$$

It turns out that our monodromy representation is the same as that studied by Murakami [20] up to a scalar representation.

Proof of Theorem 2.2.4. – We put $m = 2j$. It follows from the fact that $R^{(m)}(x,q)$ is a solution of the Yang-Baxter equation ([11] Th. 2) that the correspondence in the statement of Theorem 2.2.4 actually defines a linear representation of B_n. Let ρ denote the spin j representation of $\mathfrak{sl}(2,\mathbf{C})$. By using the classical limit $r(x)$ of $R(x,q)$, we have an expansion

$$(2.2.6)\quad R^{(m)}(x,q) = (x-x^{-1})^{m^2}\{\mathbf{1}+(\rho\otimes\rho)(r(x))(q-1)+\cdots\}.$$

By the definition of $T^{(m)}(q)$ and the above formula we have

$$\frac{d}{dq}\bar{T}^{(m)}(q) = (\rho\otimes\rho)\left(r-t-\frac{1}{2}\mathbf{1}\right).$$

Here r and t are defined in Sect. 1.2. As a consequence we have

$$\frac{d}{dq}T^{(m)}(q)^2|_{q=1} = (\rho\otimes\rho)(-2t-\mathbf{1}).$$

This implies that $T^{(m)}(q)^2$ has an expansion

$$\mathbf{1} + 2\pi\sqrt{-1}\lambda\left\{(\rho\otimes\rho)(t) + \frac{1}{2}\mathbf{1}\right\} + \mathcal{O}(\lambda^2)$$

with $\lambda = -(\pi\sqrt{-1})^{-1}\log q$, $-\pi \leqslant \operatorname{Im}\log q < \pi$. Let us observe that the eigenvalues of $\varphi(\gamma_k)$, $2 \leqslant k \leqslant n-1$, are of the form q^α with some integer α. Hence the same argument as in the proof of Theorem 1.2.8 can be applied to our Theorem.

2.3. Unitarity of the monodromy of n-point functions.

Let us now apply the fusion process introduced in the previous section to a description of the monodromy of n-point functions when ϕ_α, $1 \leqslant \alpha \leqslant n$, are vertex operators of spin j. For a pair of half integers (j,t), we denote by $\Gamma^j_{n,t}$ the set defined by

$$\Gamma^j_{n,t} = \{(p_0,p_1,\ldots,p_n)\,;\ p_i\in\frac{1}{2}\mathbf{Z}_{\geqslant 0} \text{ such that } p_0 = 0,\ p_n = t \text{ and each triple } v_i = (p_{i-1},j,p_i) \text{ satisfies the conditions 2.1.11 and 12}\}.$$

We fix a positive integer ℓ. To each element of $\Gamma^j_{n,t}$ we associate the composition of vertex operators of type v_i, $1 \leqslant i \leqslant n$. This defines the n-point function

$$\phi_{v_1 \dots v_n}(z_1, \dots, z_n) = \langle \mathrm{vac} | \phi_{v_1}(z_1) \dots \phi_{v_n}(z_n) | v \rangle$$

for $v \in \mathrm{V}_t$. It is shown in [22] that this is a holomorphic function in the region $|z_1| > \cdots > |z_n|$ and is analytically continued to a multi-valued holomorphic function on X_n. Moreover, they showed that the monodromy of the n-point functions associated with $\Gamma^j_{n,t}$ defines a linear representation of the braid group B_n, which we denote by $\theta : \mathrm{B}_n \to \mathrm{End}\,(\mathrm{W}^j_{n,t})$. Our main object is to describe this representation.

Let us remark that the above composition of vertex operators is illustrated by the lattice obtained from the decomposition of $\mathrm{V}_j \otimes \cdots \otimes \mathrm{V}_j$ into simple $\mathfrak{sl}(2,\mathbf{C})$ modules. Here are some examples.

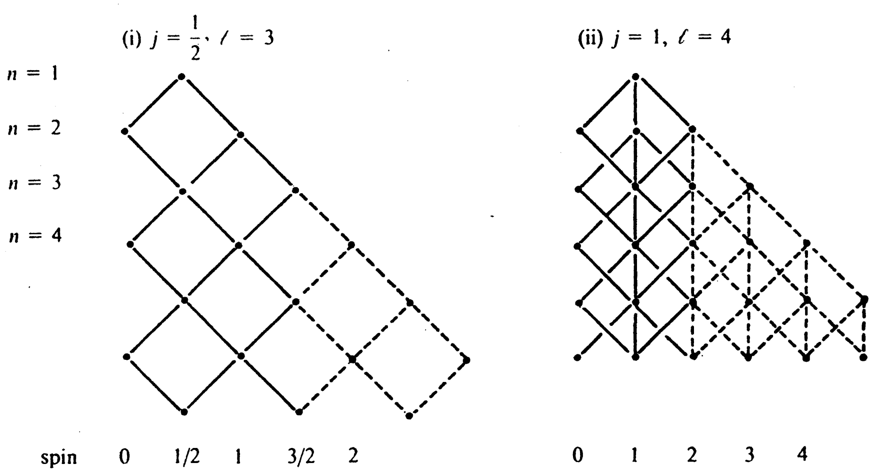

Fig. 3.

We denote by $\langle \alpha, \beta \rangle$ the atom corresponding to $n = \alpha$ and spin β. The composition of vertex operators defined by $(p_0, \dots, p_n) \in \Gamma^j_{n,t}$ is represented by the path connecting $\langle 1, p_1 \rangle, \dots, \langle n-1, p_{n-1} \rangle$. By means of an explicit computation of the 4-point functions Tsuchiya and Kanie showed that in the case $j = 1/2$ the monodromy $\theta : \mathrm{B}_n \to \mathrm{End}\,(\mathrm{W}^{1/2}_{n,t})$ factors through the Jones algebra with index $\tau^{-1} = 4\cos^2 \dfrac{\pi}{\ell + 2}$ (see [13]) and is equivalent to an irreducible unitarizable representation of B_n obtained by Wenzl [24]. Here we may identify the lattice illustrated in Fig. 3 (i) to the Bratteli diagram of the corresponding Jones algebra (see [22] Th. 5.2). Our result is as follows :

THEOREM 2.3.1. — *For any positive half integer* j, *the monodromy of* n*-point functions* $\theta : B_n \to \mathrm{End}\,(W^j_{n,t})$ *is unitarizable.*

Outline of Proof. — Let us first recall the differential equation satisfied by the n-point functions (Th. 2.1.21). Let $\iota : B_n \to B_{2jn}$ be the homomorphism defined by 2.2.5 with $m = 2j$. Let $\theta_0 : B_{2jn} \to \mathrm{End}\,(W^{1/2}_{2jn,t})$ be the monodromy of $2jn$-point functions with spin $1/2$. It follows from [22] Th. 5.2 that θ_0 is unitarizable. In particular, the matrices

$$(\theta_0 \circ \iota)(\sigma_1 \ldots \sigma_{k-1})^k, \qquad 1 \leqslant k \leqslant n,$$

are diagonalizable. Hence we may apply an argument of the proof of Theorem 2.2.5 and Corollary 2.2.6 to our situation (see also Remark 1.3.14). This implies that the monodromy representation $\theta : B_n \to \mathrm{End}\,(W^j_{n,t})$ is equivalent to a subrepresentation of the representation given by the correspondence

$$\sigma_i \to q^{\mu}\theta_0 \circ \iota(\sigma_i)$$

with some constant μ. Combining with the fact that θ_0 is unitarizable we obtain our Theorem.

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