ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 37, nº 4 (1987), p. 99-137
<http://www.numdam.org/item?id=AIF_1987__37_4_99_0>

Ann. Inst. Fourier, Grenoble
37, 4 (1987), 99-137.

SUPER BOSON-FERMION CORRESPONDENCE (*)

by V. G. KAC and J. W. van de LEUR

0. Introduction.

Since the pioneering work of Skyrme [20], the boson-fermion correspondence has been playing an increasingly important role in 2-dimensional quantum field theory. More recently, it has become an important ingredient in the work of the Kyoto school on the KP hierarchy of soliton equations [16], [1], [2], [4] (see also [13]).

In the present paper we establish a super boson-fermion correspondence, having in mind its applications to super KP hierarchies.

Let us first recall the discrete counterpart of the Skyrme construction (see e.g. [13]). Consider the *Clifford algebra* on generators ψ_i and ψ_i^*, $i \in \mathbf{Z}$, called *free fermions*, with defining relations :

$$(0.1) \qquad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}, \qquad \psi_i\psi_j + \psi_j\psi_i = 0,$$
$$\psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \qquad i, j \in \mathbf{Z},$$

and its *spin representation* with *vacuum vector* $|0\rangle$ satisfying :

$$(0.2) \quad \psi_i|0\rangle = 0 \quad \text{for} \quad i \leqslant 0, \qquad \psi_i^*|0\rangle = 0 \quad \text{for} \quad i > 0.$$

The besonization procedure consists of introducing *bosons*

$$(0.3) \qquad \alpha_k = \sum_{j \in \mathbf{Z}} :\psi_j\psi_{j+k}^*: \quad , \qquad k \in \mathbf{Z},$$

where : $\psi_i\psi_j^*$: $= \psi_i\psi_j^*$ if $j > 0$ and $= -\psi_j^*\psi_i$ if $j \leqslant 0$. They are

(*) Partially supported by the NSF grant DMS 8508953.

Key-words : Superbosons - Superfermions - Super oscillator algebra - Super vertex operators - SKP hierarchy - Spin module - Oscillator module.

operators on the spin representation space and they form an *oscillator algebra :*

$$\alpha_m\alpha_n - \alpha_n\alpha_m = m\delta_{m,-n}. \tag{0.4}$$

Under the oscillator algebra the spin representation breaks into direct sum of irreducible representations, according to the *charge number* defined by charge $(\psi_i) = -$ charge $(\psi_i^*) = 1$, with vacuum vectors $|m\rangle$, $m \in \mathbf{Z}$, defined by :

$$|m\rangle = \psi_m \dots \psi_1|0\rangle \quad \text{if} \quad m \geqslant 0,$$
$$|m\rangle = \psi_{m+1}^* \dots \psi_0^*|0\rangle \quad \text{if} \quad m < 0.$$

Introducing the generating series (*fields*)

$$\alpha(z) = \sum_{j \in \mathbf{Z}} \alpha_j z^{-j} \quad (z \in \mathbf{C}^\times), \tag{0.5}$$

$$\psi(z) = \sum_{j \in \mathbf{Z}} \psi_j z^j, \qquad \psi^*(z) = \sum_{j \in \mathbf{Z}} \psi_j^* z^{-j} \quad (z \in \mathbf{C}^\times), \tag{0.6}$$

we can rewrite (0.3) in a more compact form :

$$\alpha(z) = \,:\psi(z)\psi^*(z):. \tag{0.5a}$$

The fermionization procedure consists of reconstructing the fermionic fields $\psi(z)$ and $\psi^*(z)$ in terms of the bosonic field $\alpha(z)$. Since the spin representation does not remain irreducible under the oscillator algebra, we need to introduce more operators for that. Let p_m denote the projection operator on the subspace spanned by elements of charge m, and let q denote the operator determined by

$$q|m\rangle = |m+1\rangle, \qquad q\psi_i = \psi_{i+1}q, \qquad q\psi_i^* = \psi_{i+1}^* q. \tag{0.7}$$

Let $p(z) = \sum_{j \in \mathbf{Z}} p_j z^j$ $(= z^{\alpha_0})$; then we have :

$$\begin{aligned} \psi(z) &= p(z)q\Gamma_-(z)\Gamma_+(z), \\ \psi^*(z) &= q^{-1}p(z)^{-1}\Gamma_-(z)^{-1}\Gamma_+(z)^{-1}, \end{aligned} \tag{0.8}$$

where $\Gamma_\pm(z)$ are defined by

$$\Gamma_\pm(z) = \exp - \sum_{n=1}^{\infty} \frac{\alpha_{\pm n}}{\pm n} z^{\mp n} \quad (z \in \mathbf{C}^\times). \tag{0.9}$$

The operators on the right-hand side of (0.8) are called *vertex operators*.

We can proceed now to explain the main construction of the present paper, the super boson-fermion correspondence. Consider the *Clifford superalgebra* on generators ψ_i and ψ_i^*, $i \in \frac{1}{2}\mathbf{Z}$, which may be called *free superfermions*, with defining relations

$$(0.10) \quad \psi_i\psi_j^* + (-1)^{4ij}\psi_j^*\psi_i = \delta_{ij}, \qquad \psi_i\psi_j + (-1)^{4ij}\psi_j\psi_i = 0,$$
$$\psi_i^*\psi_j^* + (-1)^{4ij}\psi_j^*\psi_i^* = 0, \qquad i, j \in \frac{1}{2}\mathbf{Z},$$

and its spin representation with vacuum vector $|0\rangle$ satisfying (0.2). Consider the super fermonic fields

$$(0.11a) \quad \psi_{\bar{0}}(z) = \sum_{j \in \mathbf{Z}} \psi_j z^j, \qquad \psi_{\bar{0}}^*(z) = \sum_{j \in \mathbf{Z}} \psi_j^* z^{-j},$$

$$(0.11b) \quad \psi_{\bar{1}}(z) = \sum_{j \in \frac{1}{2}+\mathbf{Z}} \psi_j z^j, \qquad \psi_{\bar{1}}^*(z) = -\sum_{j \in \frac{1}{2}+\mathbf{Z}} \psi_j^* z^{-j}.$$

The super bosonization procedure consists of introducing the *super bosonic* fields :

$$(0.12a) \quad \lambda(z) := \sum_{n \in \mathbf{Z}} \lambda(n) z^{-n} = :\psi_{\bar{1}}(z)\psi_{\bar{0}}^*(z):,$$

$$(0.12b) \quad \mu(z) := \sum_{n \in \mathbf{Z}} \mu(n) z^{-n} = :\psi_{\bar{1}}(z)\psi_{\bar{1}}^*(z):,$$

$$(0.12c) \quad e(z) := \sum_{n \in \mathbf{Z}} e(n) z^{-n-\frac{1}{2}} = :\psi_{\bar{1}}(z)\psi_{\bar{0}}^*(z):,$$

$$(0.12d) \quad f(z) := \sum_{n \in \mathbf{Z}} f(n) z^{-n+\frac{1}{2}} = :\psi_{\bar{0}}(z)\psi_{\bar{1}}^*(z):,$$

where the normal ordering is defined by

$$(0.13) \quad :\psi_i\psi_j^*: = \psi_i\psi_j^* \ \text{ if } \ j > 0, \quad = -(-1)^{4ij}\psi_j^*\psi_i \ \text{ if } \ j \leqslant 0.$$

The operators $\lambda(n)$, $\mu(n)$, $e(n)$ and $f(n)$ together with the identity form a Lie superalgebra $\tilde{\mathcal{g}}$, $\lambda(n)$ and $\mu(n)$ (resp. $e(n)$ and $f(n)$) being its even (resp. odd) elements. We call $\tilde{\mathcal{g}}$ the *superoscillator algebra.*

A simple but crucial observation is that $\mathcal{g}$ is isomorphic to the affine superalgebra $\tilde{gl}_{1|1}$, associated to the 4-dimensional Lie superalgebra $gl_{1|1}(\mathbf{C})$, defined as follows:

$$\tilde{gl}_{1|1} = gl_{1|1}(\mathbf{C}[t,t^{-1}]) \oplus \mathbf{C}c,$$
$$[x(m), y(n)] = [x,y](m+n) + m\delta_{m,-n}(\mathrm{Str}\, xy)c,$$

where $x, y \in gl_{1|1}(\mathbf{C})$, $x(m)$ stands for $t^m x$, Str stands for the supertrace and c is the central element. Explicitly $\tilde{g}$ can be identified with $\tilde{gl}_{1|1}$ as follows:

$$(0.14)\quad 1 = c, \qquad \lambda(n) = \begin{bmatrix} -t^n & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mu(n) = \begin{bmatrix} 0 & 0 \\ 0 & -t^n \end{bmatrix},$$
$$e(n) = \begin{bmatrix} 0 & t^n \\ 0 & 0 \end{bmatrix}, \qquad f(n) = \begin{bmatrix} 0 & 0 \\ -t^n & 0 \end{bmatrix}.$$

Note that the oscillator algebra is a central extension of the (abelian) Lie algebra $gl_1(\mathbf{C}[t,t^{-1}])$, the loop algebra of $gl_1(\mathbf{C})$. From this point of view, the superoscillator algebra is its natural generalization: it is a central extension of the Lie superalgebra $gl_{1|1}(\mathbf{C}[t,t^{-1}])$, the loop algebra of $gl_{1|1}(\mathbf{C})$.

Introduce the *charge* (resp. *fermionic charge*) by putting charge $(\psi_i) = -$ charge $(\psi_i^*) = 1$ for $i \in \frac{1}{2}\mathbf{Z}$ (resp. fermionic charge $(\psi_i) = -$ fermionic charge $(\psi_i^*) = 1$ for $i \in \mathbf{Z}$, and $= 0$ for $i \in \frac{1}{2} + \mathbf{Z}$). Then, restricted to the superoscillator algebra $\tilde{\mathcal{g}}$, the spin representation breaks into a direct sum of irreducible representations according to the charge number $m \in \mathbf{Z}$, with vacuum vectors $|m\rangle$ defined by

$$(0.15)\quad \begin{aligned} |m\rangle &= \psi_{\frac{1}{2}}^{m}\psi_0^*\,|0\rangle \quad \text{if} \quad m \geqslant 0, \\ |m\rangle &= \psi_{-\frac{1}{2}}^{*\,-m-1}\psi_0^*\,|0\rangle \quad \text{if} \quad m < 0. \end{aligned}$$

This irreducibility is one of the key results of the paper (Theorem 2). It is proved by making use of the super fermionization explained below.

Let P_m denote the projection operator on the subspace spanned by elements of fermionic charge m, let $\mathrm{P}(z) = \sum_{m \in \mathbf{Z}} \mathrm{P}_m z^m (= z^{\lambda(0)})$, and let

Q denote the operator determined by

$$(0.16a)\qquad Q|0\rangle = \psi_1|0\rangle; \qquad Q\psi_i = \psi_{i+1}Q,$$

$$Q\psi_i^* = \psi_{i+1}^*Q \quad \text{if} \quad i \in \mathbf{Z};$$

$$(0.16b)\qquad Q\psi_i = -\psi_iQ, \qquad Q\psi_i^* = -\psi_i^*Q \quad \text{if} \quad i \in \frac{1}{2} + \mathbf{Z}.$$

Then we have (Theorem 1) (*):

$$(0.17a)\qquad \psi_{\bar 0}(z) = P(z)Q\Gamma_-(z)\Gamma_+(z),$$

$$(0.17b)\qquad \psi_{\bar 0}^* = Q^{-1}P(z)^{-1}\Gamma_{-1}(z)^{-1}\Gamma_+(z)^{-1},$$

$$(0.17c)\qquad \psi_{\bar 1}(z) = -P(z)Q\Gamma_-(z)e(z)\Gamma_+(z),$$

$$(0.17d)\qquad \psi_{\bar 1}^*(z) = Q^{-1}P(z)^{-1}\Gamma_-(z)^{-1}f(z)\Gamma_+(z)^{-1},$$

where

$$(0.18)\qquad \Gamma_\pm(z) = \exp - \sum_{n=1}^{\infty} \frac{\lambda(\pm n)}{\pm n} z^{\mp n}.$$

It is natural to call the right-hand side of (0.17b and c) the *super vertex operators*.

Three immediate applications of the super boson-fermion correspondence are these. First, it is an explicit fermionic construction of all degenerate level 1 irreducible highest weight representations of the Lie superalgebra $\tilde{gl}_{1|1}$, and a multiplicative formula for the q-dimension and a vertex operator construction of some representations of the Lie superalgebra $gl_{\infty|\infty}(\mathbf{C})$. For this we use some results on Verma modules over $\tilde{gl}_{1|1}$, which is a special case of the theory developed in [12], [9], [11].

Second, by comparing two expressions for q-dimensions of representations of $\tilde{gl}_{1|1}$, we derive the following new combinatorial identities. A partition of k into a sum of black and white positive integers such that odd parts of each color are distinct is called a *super bipartition.* Then, for each $m \in \mathbf{Z}_+ = \{0,1,\ldots\}$, the number of super bipartitions of k such that #(white parts) − #(black parts) $\leqslant m$, is equal to the number of super bipartitions of k such that no black part equals

(*) The operators $\mu(n)$ do not appear explicitly in (0.17). However, they do appear implicitly since $[e(n),f(m)] = \lambda(m+n) + \mu(m+n) - m\delta_{m,\,n}c$.

$2m + 1$. G. Andrews provided us with another, analytic proof of this result, but no combinatorial proof is known so far. After the work [5] it has become clear that a whole range of combinatorial identities can be obtained by computing characters of representations of infinite-dimensional Lie algebras in two different ways. The above identity is probably the first where considering a Lie superalgebra representation is essential.

Third, a series of irreducible highest weight modules of the affine Lie superalgebra $\tilde{g}l_{n|n} = gl_{n|n}(\mathbf{C}[t,t^{-1}]) \oplus \mathbf{C}c$ is constructed, and an explicit formula for the q-dimension is found. Unfortunately, these representations are not unitarizable, which is not surprizing since one can show that, in fact, $\tilde{g}l_{m|n}$ has no nontrivial unitarizable highest weight representations. We hope that our construction of the $\tilde{g}l_{n|n}$-modules will give an indication which property in the representation theory of affine Lie superalgebras will take over the role of unitarity.

Now, recall that the KP hierarchy of equations on f in the fermionic picture is [4], [13]:

$$\sum_{i \in \mathbf{Z}} \psi_i f \otimes \psi_i^* f = 0. \tag{0.19}$$

When translated into the the bosonic picture, it gives the celebrated KP hierarchy of PDE's [1], [4], [13]. The importance of this hierarchy of equations stems from the facts that, on the one hand, it is a « universal » system of soliton equations ([16], [2], [4], [13]) and, on the other hand, it characterizes the Jacobians of algebraic curves [17], [18], [19].

A natural « super » analog of (0.19) is

$$\sum_{i \in \frac{1}{2}\mathbf{Z}} (-1)^{2i} \psi_i f \otimes \psi_i^* f = 0, \tag{0.20}$$

which we call the *super* KP *hierarchy*. Unfortunately, we still do not know how to interpret (0.20) in the bosonic picture as a system of « super » soliton equations, or how to relate (0.20) to Jacobians of supercurves. In particular, its relationship to the super KP of Manin-Radul and Kupershmidt [15] (cf. [22]) remains unclear.

We also sketch a similar formalism with the Clifford superalgebra replaced by the Weyl superalgebra.

We would like to thank P. Goddard and D. Olive for discussions while the work was in progress and G. Andrews for correspondence.

1. The Clifford superalgebra and its spin module.

We use the notation and conventions of the superalgebra theory adopted in [6].

The *Clifford superalgebra* $Cl = Cl_{\bar{0}} \oplus Cl_{\bar{1}}$ is defined as the associative superalgebra over $\mathbf{C}$ with a unit element on the generators ψ_j and ψ_j^*, $j \in \frac{1}{2}\mathbf{Z}$, and the following defining relations:

$$\begin{aligned} \psi_i\psi_j^* + (-1)^{4ij}\psi_j^*\psi_i &= \delta_{ij}, \\ \psi_i\psi_j + (-1)^{4ij}\psi_j\psi_i &= 0, \\ \psi_i^*\psi_j^* + (-1)^{4ij}\psi_j^*\psi_i^* &= 0, \end{aligned} \tag{1.1}$$

with the $\mathbf{Z}_2$-gradation given by

$$\psi_j, \psi_j^* \in Cl_{\bar{0}} \quad \text{if} \quad j \in \mathbf{Z}, \quad \psi_j, \psi_j^* \in Cl_{\bar{1}} \quad \text{if} \quad j \in \frac{1}{2} + \mathbf{Z}. \tag{1.2}$$

The superalgebra Cl carries an antilinear anti-involution ω defined by

$$\omega(\psi_j) = (-1)^{2j}\psi_j^*, \quad \omega(\psi_j^*) = (-1)^{2j}\psi_j. \tag{1.3}$$

The *spin module* over Cl is the irreducible Cl-module V which admits a non-zero even vector $|0\rangle$, called the *vacuum vector*, such that

$$\psi_j|0\rangle = 0 \quad \text{for} \quad j \leqslant 0, \quad \psi_j^*|0\rangle = 0 \quad \text{for} \quad j > 0. \tag{1.4}$$

The module V carries a unique Hermitian form $\langle\cdot,\cdot\rangle$ such that the square length of the vacuum vector is 1 and the operators a and $\omega(a)$ are adjoint, $a \in Cl$.

Elements

$$\psi_{i_s}^{*l_s} \dots \psi_{i_1}^{*l_1}\psi_{j_r}^{k_r} \dots \psi_{j_1}^{k_1}|0\rangle \tag{1.5}$$

with

$$j_r > \dots > j_1 > 0 \geqslant i_1 > \dots > i_s,$$

such that $l_t = 1$ (resp. $k_t = 1$) if i_t (resp. j_t) $\in \mathbf{Z}$, form a basis of V. These elements are pairwise orthogonal with respect to $\langle\cdot,\cdot\rangle$ and the square length of the element (1.5) is

$$(-1)^{2\sum_t l_t i_t} k_1 ! \ldots k_r ! l_1 ! \ldots l_s !$$

Thus, the Cl-module V is not unitary, ghosts being elements (1.5) with odd number of ψ_i^* with half-integral i.

2. $gl_{\infty|\infty}$ and $a_{\infty|\infty}$; decomposition of the spin module.

Let $\Psi = \Psi_{\bar{0}} \oplus \Psi_{\bar{1}} = \bigoplus_{j \in \frac{1}{2}\mathbf{Z}} \mathbf{C}\psi_j$ be an (infinite-dimensional) complex $\mathbf{Z}_2$-graded vector space with fixed basic $\{\psi_j\}_{j \in \frac{1}{2}\mathbf{Z}}$ where $\psi_i \in \Psi_{\bar{0}}$ if $i \in \mathbf{Z}$ and $\psi_i \in \Psi_{\bar{1}}$ if $i \in \mathbf{Z} + \frac{1}{2}\cdot$ The map $\sum_{j \in \frac{1}{2}\mathbf{Z}} c_j\psi_j \to (c_j)_{j \in \frac{1}{2}\mathbf{Z}}$ identifies Ψ with the space of column vectors whose coordinates are indexed by $\frac{1}{2}\mathbf{Z}$, all but a finite number of them being 0.

Introduce the infinite complex matrix Lie algebra

$$gl_{\infty|\infty} = gl_{\infty|\infty;\bar{0}} \oplus gl_{\infty|\infty;\bar{1}},$$

where for $\alpha \in \mathbf{Z}_2 = \{\bar{0},\bar{1}\}$:

$$gl_{\infty|\infty;\alpha} = \{(a_{ij})_{i,j \in \frac{1}{2}\mathbf{Z}} | a_{ij} = 0 \quad \text{for} \quad i + j = \alpha + 1 \bmod 2 ;$$
$$\text{all but a finite number of } a_{ij} \text{ are } 0 \text{ for } i + j = \alpha \bmod 2\}.$$

The Lie bracket is defined by

$$[a,b] = ab - (-1)^{\alpha\beta} ba \quad \text{for} \quad a \in gl_{\infty|\infty;\alpha}, b \in gl_{\infty|\infty;\beta}.$$

The Lie superalgebra $gl_{\infty|\infty}$ operates on Ψ via the multiplication of a matrix and a column vector, viz.,

$$E_{ij}(\psi_j) = \psi_i,$$

where E_{ij} denotes the matrix with the (i,j) entry 1 and the rest 0.

Another way to introduce $gl_{\infty|\infty}$ is as a contragredient Lie superalgebra [6] of infinite rank on Chevalley generators

$$e_i = E_{i,i+\frac{1}{2}}, \qquad f_i = E_{i+\frac{1}{2},i},$$

$$h_i = [e_i,f_i] = E_i + E_{i+\frac{1}{2},i+\frac{1}{2}}, \qquad i \in \frac{1}{2}\mathbf{Z}.$$

Its Cartan matrix is

$$\begin{bmatrix} \ddots & & & & & 0 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & & & \ddots \\ 0 & & & & & \end{bmatrix}.$$

Its corresponding Dynkin diagram is the infinite chain

$$\cdots - \otimes - \otimes - \otimes - \cdots .$$

The standard representation of $gl_{\infty|\infty}$ on Ψ can also be defined in terms of the Clifford generators as follows:

$$\text{(2.1)} \qquad \mathrm{E}_{ij} \to (-1)^{2j}\psi_i\psi_j^*,$$

with the action defined by the commutator:

$$\begin{aligned} \mathrm{E}_{ij}\psi_j &= [(-1)^{2j}\psi_i\psi_j^*, \psi_j] \\ &= (-1)^{2j}(\psi_i\psi_j^*\psi_j - (-1)^{4(i+j)j}\psi_j\psi_i\psi_j^*) = \psi_i . \end{aligned}$$

The antilinear anti-involution ω leaves $gl_{\infty|\infty}$, embedded via (2.1) in Cl, invariant and induces its compact anti-involution $a \to {}^t\bar{a}$.

Let V be as in § 1. We define a representation π of $gl_{\infty|\infty}$ on V by

$$\pi(\mathrm{E}_{ij}) = (-1)^{2j}\psi_i\psi_j^*,$$

by which we mean $\pi(\mathrm{E}_{ij})(\psi|0\rangle) = (-1)^{2j}\psi_i\psi_j^*\psi|0\rangle$.

Given $m \in \mathbf{Z}$, define $|m\rangle \in \mathrm{V}$ by

$$|m\rangle = \begin{cases} \psi_{\frac{1}{2}}^{m}|0\rangle & \text{for } m \geqslant 0 \\ \psi_{-\frac{1}{2}}^{*\,-m-1}\psi_0^*|0\rangle & \text{for } m < 0 . \end{cases}$$

Putting $\deg|0\rangle = 0$, $\deg\psi_i = 1$ and $\deg\psi_i^* = -1$ defines the decomposition into a direct sum of vector spaces

$$\mathrm{V} = \bigoplus_{m \in \mathbf{Z}} \mathrm{V}_m ,$$

so that $|m\rangle \in \mathrm{V}_m$. We call m the *charge number*.

If we put $\deg|0\rangle = 0$ and $\deg\psi_i = 1$, $\deg\psi_i^* = -1$ for $i \in \mathbf{Z}$, $\deg\psi_i = \deg\psi_i^* = 0$ for $i \in \mathbf{Z} + \frac{1}{2}$ (resp. $\deg\psi_i = \deg\psi_i^* = 0$ for $i \in \mathbf{Z}$, $\deg\psi_i = 1$, $\deg\psi_i^* = -1$ for $i \in \mathbf{Z} + \frac{1}{2}$), we get other decompositions of V :

$$(2.2) \qquad \mathrm{V} = \bigoplus_{m \in \mathbf{Z}} \mathrm{V}_{(f);m} \ (\text{resp. } \mathrm{V} = \bigoplus_{m \in \mathbf{Z}} \mathrm{V}_{(b);m}).$$

In this case the number m is called the *fermionic* (resp. *bosonic*) *charge number*.

The subspaces V_m are invariant and irreducible with respect to $gl_{\infty|\infty}$ (this is not the case for $\mathrm{V}_{(f);m}$ or $\mathrm{V}_{(b);m}$). We denote this representation by π_m. Its highest weight vector is $|m\rangle$, in the sense that

$$\pi_m(\mathrm{E}_{ij})|m\rangle = 0 \quad \text{for} \quad i < j$$

and

$$\pi_m(\mathrm{E}_{ii})|m\rangle = \lambda_i^{(m)}|m\rangle, \quad i \in \frac{1}{2}\mathbf{Z}.$$

Using the relations in the Clifford superalgebra, we calculate its corresponding highest weight $(\lambda_i^{(m)})_{i \in \frac{1}{2}\mathbf{Z}}$:

$$\begin{array}{cccccccccc} & 0 & 0 & 0 & m+1 & m & 0 & 0 & & \\ \cdots & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \cdots & \text{for } m \geqslant 0. \\ & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{3}{2} & & \end{array}$$

$$\begin{array}{cccccccccc} & 0 & m+1 & m & 0 & 0 & 0 & 0 & & \\ \cdots & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \cdots & \text{for } m < 0. \\ & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{3}{2} & & \end{array}$$

Note that the operators $\pi_m(a)$ and $\pi_m({}^t\bar{a})$ are adjoint with respect to the Hermitian form $\langle\cdot,\cdot\rangle$ on V_m. However, π_m is not a unitary representation (see § 1). Moreover, one can prove that no highest weight representation of $gl_{\infty|\infty}$, except the trivial one, is unitary.

We introduce now a Lie superalgebra $\bar{a}_{\infty|\infty}$ containing $gl_{\infty|\infty}$:

$$\bar{a}_{\infty|\infty} = \{(a_{ij})_{i,j \in \frac{1}{2}\mathbf{Z}} \mid \text{for each } k \text{ the number of non-zero } a_{ij} \text{ with } j \leqslant k \text{ and } i \geqslant k \text{ is finite}\}.$$

It acts on a completion $\bar{\Psi}$ of the space Ψ, where

$$\bar{\Psi} = \left\{ \sum_{j \in \frac{1}{2}\mathbf{Z}} c_j \Psi_j | c_j = 0 \text{ for } j \gg 0 \right\}.$$

However, if we try to extend the representation π_m to the Lie superalgebra $\bar{a}_{\infty|\infty}$, we encounter an « anomaly », e.g., in

$$\pi_m(\operatorname{diag}(\lambda_i)_{i\in\frac{1}{2}\mathbf{Z}})|m\rangle = (m\lambda_{\frac{1}{2}} + \sum_{i\leqslant 0}(-1)^{2i}\lambda_i)|m\rangle \quad \text{if} \quad m \geqslant 0,$$

the right-hand side is in general a divergent series. To remove this anomaly, we change the representation π_m as follows. Put:

(2.3*a*) $\quad \hat{\pi}_m(\mathrm{E}_{ii}) = \pi_m(\mathrm{E}_{ii}) - (-1)^{2i}\mathrm{I} \quad$ for $\quad i \leqslant 0$.

(2.3*b*) $\quad \hat{\pi}_m(\mathrm{E}_{ij}) = \pi_m(\mathrm{E}_{ij})$ if $i \neq j$ or $i = j > 0$.

Note that (2.3*a*) simply means that diagonal matrices kill the vacuum $|0\rangle$.

Extending $\hat{\pi}_m$ by linearity, we get a projective representation of the Lie superalgebra $\bar{a}_{\infty|\infty}$. Equivalently, introduce the central extension $a_{\infty|\infty} = \bar{a}_{\infty|\infty} \oplus \mathbf{C}c$ with center $\mathbf{C}c$ and bracket

$$[a,b] = ab - (-1)^{\alpha\beta}ba + \mathrm{C}(a,b)c,$$

for

$$a \in \bar{a}_{\infty|\infty;\alpha}, \qquad b \in \bar{a}_{\infty|\infty;\beta},$$

where the cocycle C is defined by:

(2.4*a*) $\quad \mathrm{C}(\mathrm{E}_{ij},\mathrm{E}_{ji}) = -(-1)^{2(i+j)}\mathrm{C}(\mathrm{E}_{ji},\mathrm{E}_{ij}) = (-1)^{2i}$ if $i \leqslant 0 < j$,

(2.4*b*) $\quad \mathrm{C}(\mathrm{E}_{ij},\mathrm{E}_{kl}) = 0$ in all other cases.

Then, extending $\hat{\pi}_m$ to $a_{\infty|\infty}$ by $\hat{\pi}_m(c) = \mathrm{I}$, we obtain a linear representation of the Lie superalgebra $a_{\infty|\infty}$ on the space V_m, which we again denote by $\hat{\pi}_m$.

Introduce the *principal gradation* $gl_{\infty|\infty} = \bigoplus_{j\in\mathbf{Z}} g_j$ by putting $\deg \mathrm{E}_{ij} = 2(j-i)$, so that $[g_i, g_j] = g_{i+j}$. By putting $\deg c = 0$, we extend this gradation to $a_{\infty|\infty} = \bigoplus_{j\in\mathbf{Z}} a_j$. This principal gradation on the Lie superalgebra induces a *principal gradation*

(2.5*a*) $$\mathrm{V}_m = \bigoplus_{k\in\mathbf{Z}} \mathrm{V}_m^{(k)},$$

which is consistent with the gradation on $a_{\infty|\infty}$, i.e.,

$$\hat{\pi}_m(a_j)\mathrm{V}_m^{(k)} \subseteq \mathrm{V}_m^{(k-j)} \qquad \text{for} \qquad a_j \in a_j.$$

More explicity, the space $\mathrm{V}_m^{(k)}$ is the linear span of all elements of V_m of the form

(2.5*b*) $$\hat{\pi}_m(\mathrm{E}_{i_1 j_1}) \ldots \hat{\pi}_m(\mathrm{E}_{i_r j_r})|m\rangle$$

with $(i_1+i_2+\cdots+i_r) - (j_1+j_2+\cdots+j_r) = \frac{1}{2}k$.

Note that in (2.5*b*) we could assume $j_t < i_t$ for all $1 \leqslant t \leqslant r$ (by the PBW theorem for superalgebras); it follows that $V_m^{(k)} = 0$ for $k < 0$, and $V_m^{(0)} = \mathbf{C}|m\rangle$.

We can rewrite (2.5*b*) in terms of the Clifford generators: $V_m^{(k)}$ is the linear span of all elements

$$\psi_{i_s}^{*l_s} \ldots \psi_{i_1}^{*l_1} \psi_{j_r}^{k_r} \ldots \psi_{j_1}^{k_1} |m\rangle \tag{2.6}$$

with $(j_1k_1+\cdots+j_rk_r) - (i_1l_1+\cdots+i_sl_s) = \frac{1}{2}k$.

Now notice that

$$k_t \,(\text{resp. } l_t) = 1 \quad \text{if} \quad j_t \,(\text{resp. } i_t) \in \mathbf{Z}; \tag{2.7a}$$

furthermore we can order the i_t 's and j_t 's in such a way that

$$i_{t+1} < i_t \qquad \text{and} \qquad j_{t+1} > j_t. \tag{2.7b}$$

With these and the following conditions (2.7*c*,*d*), the elements of the form (2.6) form a basis of $V_m^{(k)}$. Assume $m \geqslant 0$, so that $|m\rangle = \psi_{\frac{1}{2}}^m|0\rangle$. Then we get the following extra conditions for the i_t 's and j_t 's:

$$i_t \leqslant \frac{1}{2}, \qquad j_t > 0, \tag{2.7c}$$

(2.7*d*) $i_t = \frac{1}{2}$ *occurs at most* m times, and if it occurs then $i_1 \neq \frac{1}{2}$.

If $m < 0$, so that $|m\rangle = \psi^{*\,-m-1}_{-\frac{1}{2}}\psi_0^*|0\rangle$, then the extra conditions are:

$$i_t < 0, \qquad j_t \geqslant -\frac{1}{2}. \tag{2.7c'}$$

(2.7*d'*) $j_t = -\frac{1}{2}$ occurs at most $-m-1$ times, and if it occurs then $j_1 \neq -\frac{1}{2}$.

From all these conditions we conclude that $\dim V_m^{(k)} < \infty$ for all k. Let $p_m(k) = \dim V_m^{(k)}$; we may write the formal power series.

$$\dim_q V_m = \sum_{k \geqslant 0} p_m(k) q^k , \tag{2.8}$$

called the *q-dimension* (or the *partition function*) of V_m.

For $m > 0$, $p_m(k)$ is equal to the number of partitions of k into integers, where each integer has black or white color, with the following conditions (cf. (2.7)) :

(2.9)
(1) # whites = # blacks,
(2) even parts of each color are distinct,
(3) whites > 0, blacks $\geqslant -1$,
(4) -1 occurs at most m times,
(5) if -1 occurs, then white 1 does not occur.

Now adding 1 to all blacks and subtracting 1 off all whites, $p_m(k)$ stays the same, but the conditions (2.9) become the following ones :

(2.9′)
(1) # whites − # blacks $\leqslant m$,
(2) odd parts of each color are distinct,
(3) all parts are positive integers.

If $m < 0$ then one can verify that

$$p_m(k) = p_{-m-1}(k) \quad \text{for} \quad k \geqslant 0 . \tag{2.10}$$

In § 7 we shall give another formula for $p_m(k)$, which will give an explicit expression for $\dim_q V_m$.

3. The principal subalgebra of $a_{\infty|\infty}$.

We shall construct a subsuperalgebra $\tilde{g}$ of $a_{\infty|\infty}$ for which the modules V_m considered as $\tilde{g}$-moduleswill turn out to remain irreducible.

Put

$$\lambda(n) = \sum_{k \in Z} E_{k,k+n}, \qquad \mu(n) = \sum_{k \in Z+\frac{1}{2}} E_{k,k+n},$$

$$e(n) = \sum_{k \in Z} E_{k-\frac{1}{2},k+n}, \qquad f(n) = \sum_{k \in Z} E_{k,k+n-\frac{1}{2}},$$

for $n \in \mathbf{Z}$. Then these elements of $a_{\infty|\infty}$ together with the central element c form a basis of a sub(super)algebra $\tilde{g}$ of $a_{\infty|\infty}$. One easily computes its commutation relations:

$$\begin{aligned}
&[\lambda(n), e(m)] = -e(m+n), \qquad [\lambda(n), f(n)] = f(m+n),\\
&[\mu(n), e(m)] = e(m+n), \qquad [\mu(n), f(n)] = -f(m+n),\\
&[\lambda(n), \lambda(m)] = n\delta_{m,-n}c, \qquad [\mu(n), \mu(m)] = -n\delta_{m,-n}c,\\
&[\lambda(n), \mu(m)] = 0, \qquad [e(n), f(m)] = \lambda(m+n) + \mu(m+n) - n\delta_{m,-n}c.
\end{aligned}$$

We will call this algebra $\tilde{g}$ the *principal subalgebra* of $a_{\infty|\infty}$. Note that $\tilde{g} = \{(a_{ij}) \oplus \lambda c \in a_{\infty|\infty} | a_{ij} = a_{i+n,j+n} \text{ for } n \in \mathbf{Z}\}$, and that the principal gradation of $\lambda(n)$, $\mu(n)$, $e(n)$, $f(n)$ is $2n$, $2n$, $2n+1$, $2n-1$, respectively.

We have:

(3.1*a*) $\hat{\pi}_m(e(n))|m\rangle = 0, \quad \hat{\pi}_m(f(n+1))|m\rangle = 0$ for $n \geqslant 0$,

(3.1*b*) $\hat{\pi}_m(\lambda(n))|m\rangle = 0, \quad \hat{\pi}_m(\mu(n))|m\rangle = 0$ for $n > 0$,

(3.1*c*) $\hat{\pi}_m(c) = I, \quad \hat{\pi}_m(\lambda(0))|m\rangle = 0$,

(3.1*d*) $\hat{\pi}_m(\mu(0))|m\rangle = m|m\rangle$ if $m \geqslant 0$, $= (m+1)|m\rangle$ if $m < 0$.

We shall give another construction of the subalgebra $\tilde{g}$. Let $\mathscr{L} = \mathbf{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in the indeterminate t. Let

$$gl_{1|1}(\mathscr{L}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathscr{L}\right\},$$

be the Lie superalgebra with the $\mathbf{Z}_2$-gradation

$$gl_{1|1;\bar{0}}(\mathscr{L}) = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right\} \quad \text{and} \quad gl_{1|1;\bar{1}}(\mathscr{L}) = \left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right\},$$

and the usual Lie superbracket:

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]^0 = \begin{pmatrix} a\alpha - \alpha a + b\gamma + \beta c & b\delta - \beta d + a\beta - \alpha b \\ c\alpha - \gamma a + d\gamma - \delta c & d\delta - \delta d + c\beta + \gamma b \end{pmatrix}.$$

The subalgebra $\tilde{g}$ is an extension by a 1-dimensional center $\mathbf{C}c$ of $gl_{1|1}(\mathscr{L})$: $\tilde{gl}_{1|1} = gl_{1|1}(\mathscr{L}) \oplus \mathbf{C}c$. The Lie bracket is given by

$$[x + \lambda c, y + \mu c] = [x, y]^0 + \operatorname{Res}_{t=0} \operatorname{Str}\left(\frac{dx}{dt} y\right) c$$

for $x, y \in gl_{1|1}(\mathscr{L})$. Here Str means the *supertrace*; for an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in gl_{1|1}(\mathscr{L})$ this is defined by $\operatorname{Str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a - d$.

The two constructions of $\tilde{g}$ are identified as follows :

$$e(n)=\begin{bmatrix}0 & t^n\\ 0 & 0\end{bmatrix},\quad f(n)=\begin{bmatrix}0 & 0\\ -t^n & 0\end{bmatrix},$$

$$\mu(n)=\begin{bmatrix}0 & 0\\ 0 & -t^n\end{bmatrix},\quad \lambda(n)=\begin{bmatrix}-t^n & 0\\ 0 & 0\end{bmatrix},\quad 1=c.$$

We put $\lambda = \lambda(0)$ and $\mu = \mu(0)$ for short.

In order to perform calculations, it is more convenient to enlarge the algebra $\tilde{gl}_{1|1}$ by adding an even derivation d which operates on $gl_{1|1}(\mathscr{L})$ as $t\dfrac{d}{dt}$ and kills c. Denote this superalgebra by $\hat{gl}_{1|1}$; then

$$\hat{gl}_{1|1} = gl_{1|1}(\mathscr{L}) \oplus \mathbf{C}c \oplus \mathbf{C}d,$$

and

$$[d,x] = t\frac{dx}{dt} \quad \text{for} \quad x \in gl_{1|1}(\mathscr{L}), \qquad [d,c] = 0.$$

4. The super boson-fermion correspondence.

In § 3 we have described the elements $\lambda(n)$, $\mu(n)$, $e(n)$ and $f(n)$ of the principal subalgebra in terms of $E_{i,j} \in a_{\infty|\infty}$. Using (2.1) and (2.3) we can describe these elements in terms of ψ_i, $\psi_i^* \in Cl$. In order to do this it will be convenient to introduce a *normal ordering* of the quadratic expressions $\psi_i\psi_j^*$, which we define to be

$$:\psi_i\psi_j^*: = \begin{cases} \psi_i\psi_j^* & \text{for} \quad j > 0, \\ -(-1)^{4ij}\psi_j^*\psi_i & \text{for} \quad j \leqslant 0. \end{cases}$$

Then we have,

$$\lambda(n) = \sum_{k \in \mathbf{Z}} :\psi_k\psi_{k+n}^*:, \qquad \mu(n) = -\sum_{k \in \mathbf{Z}+\frac{1}{2}} :\psi_k\psi_{k+n}^*:,$$

$$e(n) = \sum_{k \in \mathbf{Z}} :\psi_{k-\frac{1}{2}}\psi_{k+n}^*:, \qquad f(n) = -\sum_{k \in \mathbf{Z}} :\psi_k\psi_{k+n-\frac{1}{2}}^*:.$$

The converse of the above description, i.e., describing the ψ_i and ψ_i^* in terms of the $\lambda(n)$, $\mu(n)$, $e(n)$ and $f(n)$, is also possible. This will give us a super boson-fermion correspondence. We shall give this description in the rest of this section.

Let $\hat{V}_m$ denote the formal completion of V_m (see e.g. [10, § 1.5]), and put

$$\hat{V} = \prod_{m \in Z} \hat{V}_m .$$

We introduce generating series of the ψ_i and ψ_i^*, which are operators that map V into $\hat{V}$ ($z \in \mathbf{C}^\times$):

$$\psi_{\bar{0}}(z) = \sum_{i \in Z} \psi_i z^i, \qquad \psi_{\bar{0}}^*(z) = \sum_{i \in Z} \psi_i^* z^{-i}$$

$$\psi_{\bar{1}}(z) = \sum_{i \in Z+\frac{1}{2}} \psi_i z^i, \qquad \psi_{\bar{1}}^*(z) = -\sum_{i \in Z+\frac{1}{2}} \psi_i^* z^{-i}.$$

LEMMA 4.1. — *The following commutation relations hold :*

$$[\lambda(n), \psi_{\bar{0}}(z)] = z^n \psi_{\bar{0}}(z), \qquad [\lambda(n), \psi_{\bar{0}}^*(z)] = -z^n \psi_{\bar{0}}^*(z),$$

$$[\mu(n), \psi_{\bar{0}}(z)] = 0, \qquad [\mu(n), \psi_{\bar{0}}^*(z)] = 0,$$

$$[\mu(n), \psi_{\bar{1}}(z)] = z^n \psi_{\bar{1}}(z), \qquad [\mu(n), \psi_{\bar{1}}^*(z)] = -z^n \psi_{\bar{1}}^*(z),$$

$$[\lambda(n), \psi_{\bar{1}}(z)] = 0, \qquad [\lambda(n), \psi_{\bar{1}}^*(z)] = 0,$$

$$[e(n), \psi_{\bar{0}}(z)] = z^{n+\frac{1}{2}} \psi_{\bar{1}}(z), \qquad [e(n), \psi_{\bar{0}}^*(z)] = 0,$$

$$[f(n), \psi_{\bar{0}}(z)] = 0, \qquad [f(n), \psi_{\bar{0}}^*(z)] = -z^{n-\frac{1}{2}} \psi_{\bar{1}}^*(z),$$

$$[f(n), \psi_{\bar{1}}(z)] = z^{n-\frac{1}{2}} \psi_{\bar{0}}(z), \qquad [f(n), \psi_{\bar{1}}^*(z)] = 0,$$

$$[e(n), \psi_{\bar{1}}(z)] = 0, \qquad [e(n), \psi_{\bar{1}}^*(z)] = z^{n+\frac{1}{2}} \psi_{\bar{0}}^*(z).$$

Proof. — Is straightforward.

In order to describe the generating series of the ψ_i and ψ_i^* explicitly, we need the fermionic charge decomposition of V defined by (2.2).

Let P_m be the projection on $V_{(f);m}$, and let

$$P(z) = \sum_{m \in Z} P_m z^m, \qquad P(z)^{-1} = \sum_{m \in Z} P_m z^{-m}$$

be the corresponding generating series.

We also need another operator Q mapping $V_{(f);m}$ into $V_{(f);m+1}$, which is uniquely defined by the following properties:

$$Q|0\rangle = \psi_1|0\rangle, \tag{4.1a}$$

$$(4.1b) \qquad Q\psi_i = \psi_{i+1}Q, \qquad Q\psi_i^* = \psi_{i+1}^*Q \qquad \text{for} \qquad i \in \mathbf{Z},$$

$$(4.1c) \qquad Q\psi_i = -\psi_i Q, \qquad Q\psi_i^* = -\psi_i^* Q \qquad \text{for} \qquad i \in \mathbf{Z} + \frac{1}{2}.$$

It is straightforward to verify the following

LEMMA 4.2. – *One has for* $n \in \mathbf{Z}$, $z \in \mathbf{C}^\times$:

$$\begin{array}{ll}
QP(z) = z^{-1}P(z)Q, & QP(z)^{-1} = zP(z)^{-1}Q; \\
\lambda(n)P(z) = P(z)\lambda(n), & \lambda(n)P(z)^{-1} = P(z)^{-1}\lambda(n); \\
\mu(n)P(z) = P(z)\mu(n), & \mu(n)P(z)^{-1} = P(z)^{-1}\mu(n); \\
e(n)P(z) = zP(z)e(n), & e(n)P(z)^{-1} = z^{-1}P(z)^{-1}e(n); \\
f(n)P(z) = z^{-1}P(z)f(n), & f(n)P(z)^{-1} = zP(z)^{-1}f(n); \\
Qe(n) = -e(n+1)Q, & Qf(n) = -f(n-1)Q; \\
Q\lambda(0) = (\lambda(0)-I)Q, & Q\lambda(n) = \lambda(n)Q \quad \text{for} \quad n \neq 0; \\
Q\mu(n) = \mu(n)Q. &
\end{array}$$

□

Define for $z \in \mathbf{C}^\times$:

$$\Gamma_-(z) = \exp\left(\sum_{n>0} \frac{\lambda(-n)}{n} z^n\right)$$

and

$$\Gamma_+(z) = \exp\left(-\sum_{n>0} \frac{\lambda(n)}{n} z^{-n}\right).$$

Their product, i.e., $\Gamma(z) = \Gamma_-(z)\Gamma_+(z)$ is the well-known *vertex operator*, which appears in string theory and goes back to Skyrme [20]. Note that $\Gamma(z)$ maps V into $\hat{V}$.

Then (see e.g. [13] or the introduction) we have:

$$(4.2) \qquad \begin{aligned} \psi_{\bar{0}}(z) &= P(z)Q\Gamma_-(z)\Gamma_+(z) \\ \psi_{\bar{0}}^*(z) &= Q^{-1}P(z)^{-1}\Gamma_-(z)^{-1}\Gamma_+(z)^{-1}. \end{aligned}$$

Using Lemma 4.1 we deduce:

$$(4.3) \qquad \psi_{\bar{1}}(z) = [e(0), \psi_{\bar{0}}(z)]z^{-\frac{1}{2}}, \quad \psi_{\bar{1}}^*(z) = -[f(0), \psi_{\bar{0}}^*(z)]z^{\frac{1}{2}}.$$

Moreover, it is possible to get a somewhat nicer description of these operators. For this we need the following lemma:

LEMMA 4.3. — *As equalities of formal power series in z we have :*

$$\begin{aligned}
\Gamma_+(z)e(m)\Gamma_+(z)^{-1} &= \sum_{i=0}^{\infty} e(m+i)z^{-i},\\
\Gamma_+(z)f(m)\Gamma_+(z)^{-1} &= f(m) - f(m+1)z^{-1},\\
\Gamma_+(z)^{-1}e(m)\Gamma_+(z) &= e(m) - e(m+1)z^{-1},\\
\Gamma_+(z)^{-1}f(m)\Gamma_+(z) &= \sum_{i=0}^{\infty} f(m+i)z^{-i},\\
\Gamma_-(z)e(m)\Gamma_-(z)^{-1} &= e(m) - e(m-1)z,\\
\Gamma_-(z)f(m)\Gamma_-(z)^{-1} &= \sum_{i=0}^{\infty} f(m-i)z^{i},\\
\Gamma_-(z)^{-1}e(m)\Gamma_-(z) &= \sum_{i=0}^{\infty} e(m-i)z^{i},\\
\Gamma_-(z)^{-1}f(m)\Gamma_-(z) &= f(m) - f(m-1)z.
\end{aligned}$$

Proof. — The exponentials and logarithms in the proof are to be understood by means of their formal power series expansions. Completely formally we have the following equality:

$$\exp q(z)a \exp(-q(z)) = \exp(ad q(z))(a),$$

here $q(z)$ is some formal power series in z. We will prove the lemma for the formulas which contain $\Gamma_+(z)$ (the proof of other formulas is similar).

Define the polynomials $q_n(\vec{x})$ as follows:

$$\exp \sum_{n>0} x_n \frac{z^{-n}}{n} = \sum_n q_n(\vec{x}) z^{-n},$$

where $\vec{x} = (x_1, x_2, \ldots)$.

Let $y(m) = t^m \otimes y$ for short, where $y = e$ or $f \in g\ell_{1|1}(\mathbf{C})$, and $t^m \in \mathscr{L} = \mathbf{C}[t, t^{-1}]$. We have:

$$[\lambda(n), y(m)] = (-1)^{\delta} y(m+n),$$

where $\delta = 0$ for $y = f$ and $\delta = 1$ for $y = e$.

Let $\varepsilon = 0$ or 1, $|t| < |z| < |t|^{-1}$ and $\vec{\lambda} = (\lambda(1), \lambda(2), \ldots)$, $\vec{1} = (1,1,\ldots)$. We have:

$$\begin{aligned}
&\exp\left((-1)^{\varepsilon} \sum_{n>0} \operatorname{ad} \lambda(n) \frac{z^{-n}}{n}\right) y(m) \\
&\quad = \sum_{n} q_n((-1)^{\varepsilon} \operatorname{ad} \vec{\lambda}) y(m) z^{-n} \\
&\quad = \sum_{n} q_n((-1)^{\varepsilon+\delta}\vec{1}) y(m+n) z^{-n} \\
&\quad = t^m \sum_{n} q_n((-1)^{\varepsilon+\delta}\vec{1}) \left(\frac{z}{t}\right)^{-n} \otimes y \\
&\quad = t^m \left(\exp\left((-1)^{\varepsilon+\delta} \Sigma \frac{1}{n}\left(\frac{t}{z}\right)^n\right)\right) \otimes y \\
&\quad = t^m \left(\exp\left((-1)^{\varepsilon+\delta} \log\left(\frac{1}{1-t/z}\right)\right)\right) \otimes y \\
&\quad = t^m \sum_{i=0}^{\infty} \left(\frac{t}{z}\right)^i \otimes y = \sum_{i=0}^{\infty} y(m+i) z^{-i} \qquad \text{if } \varepsilon + \delta \text{ is even}, \\
&\quad = t^m \left(1 - \frac{t}{z}\right) \otimes y = y(m) - y(m+1) z^{-1} \quad \text{if } \varepsilon + \delta \text{ is odd}. \quad \square
\end{aligned}$$

Using Lemmas 4.2 and 4.3 we get:

$$\begin{aligned}
\psi_{\bar{1}}(z) &= [e(0), \psi_{\bar{0}}(z)] z^{-\frac{1}{2}} \\
&= (e(0) P(z) Q \Gamma_-(z) \Gamma_+(z) - P(z) Q \Gamma_-(z) \Gamma_+(z) e(0)) z^{-\frac{1}{2}} \\
&= (-P(z) Q (e(-1) \Gamma_-(z) \Gamma_+(z) z - P(z) Q \Gamma_-(z) \Gamma_+(z) e(0)) z^{-\frac{1}{2}} \\
&= (-P(z) Q \Gamma_-(z) \sum_{i=0}^{\infty} e(-1-i) z^{i+1} \Gamma_+(z) \\
&\qquad\qquad - P(z) Q \Gamma_-(z) \sum_{i=0}^{\infty} e(i) z^{-i} \Gamma_+(z)) z^{-\frac{1}{2}} \\
&= - P(z) Q \Gamma_-(z) \sum_{i \in \mathbf{Z}} e(i) z^{-i-\frac{1}{2}} \Gamma_+(z).
\end{aligned}$$

We can make a similar computation for $\psi_{\bar{1}}^*(z)$. Putting

$$e(z) = \sum_{i \in \mathbf{Z}} e(i) z^{-i-\frac{1}{2}} \qquad \text{and} \qquad f(z) = \sum_{i \in \mathbf{Z}} f(i) z^{-i+\frac{1}{2}},$$

we obtain the central result of this paper.

THEOREM 1 (The super boson-fermion correspondence). —

$$\begin{aligned}\psi_{\bar 0}(z) &= \mathrm{P}(z)\mathrm{Q}\Gamma_-(z)\Gamma_+(z)\\ \psi^*_{\bar 0}(z) &= \mathrm{Q}^{-1}\mathrm{P}(z)^{-1}\Gamma_-(z)^{-1}\Gamma_+(z)^{-1}\\ \psi_{\bar 1}(z) &= -\ \mathrm{P}(z)\mathrm{Q}\Gamma_-(z)e(z)\Gamma_+(z)\\ \psi^*_{\bar 1}(z) &= \mathrm{Q}^{-1}\mathrm{P}(z)^{-1}\Gamma_-(z)^{-1}f(z)\Gamma_+(z)^{-1}. \qquad \square\end{aligned}$$

Using this theorem, we express the generating series of $\pi_m(\mathrm{E}_{ij})$, $i,j\in\frac{1}{2}\mathbf{Z}$, in terms of the $\lambda(n)$, $\mu(n)$, $e(n)$ and $f(n)$. This generating series is given by

$$\pi_m\Big(\sum_{i,j\in\frac12\mathrm{Z}}\mathrm{E}_{ij}y^iz^{-j}\Big)=\psi_{\bar 0}(y)\psi^*_{\bar 0}(z)+\psi_{\bar 0}(y)\psi^*_{\bar 1}(z)+\psi_{\bar 1}(y)\psi^*_{\bar 0}(z)+\psi_{\bar 1}(y)\psi^*_{\bar 1}(z)$$

where $y,z\in\mathbf{C}^\times$. Using (2.3), Lemma 4.2 and Lemma 4.3, we get the following result:

PROPOSITION 4.4. — *For* $|y|>|z|$ *we have*

$$(a)\quad \pi_m\Big(\sum_{i,j\in\frac12\mathrm{Z}}\mathrm{E}_{ij}y^iz^{-j}\Big)=\Big(\frac{y}{z}\Big)^m\Gamma_-(y,z)\Big[\frac{1}{1-z/y}+f(z)+\Big(\frac{y}{z}\Big)e(y)+\Big(\frac{y}{z}-1\Big)e(y)f(z)\Big]\Gamma_+(y,z),$$

$$(b)\quad \hat\pi_m\Big(\sum_{i,j\in\frac12\mathrm{Z}}\mathrm{E}_{ij}y^iz^{-j}\Big)=\Big(\frac{y}{z}\Big)^m\Gamma_-(y,z)\Big[\frac{1}{1-z/y}+f(z)+\Big(\frac{y}{z}\Big)e(y)+\Big(\frac{y}{z}-1\Big)e(y)f(z)\Big]\Gamma_+(y,z)+\Big(\frac{y}{z}\Big)^m\Big(1-\Big(\frac{z}{y}\Big)^{\frac12}\Big)\frac{1}{1-z/y}$$

where

$$\Gamma_-(y,z)=\exp\Big(\sum_{n>0}\frac{\lambda(-n)}{n}(y^n-z^n)\Big)$$

and

$$\Gamma_+(y,z)=\exp\Big(-\sum_{n>0}\frac{\lambda(-n)}{n}(y^{-n}-z^{-n})\Big). \qquad \square$$

In order to calculate $\pi_m(\mathrm{E}_{ij})$ or $\hat\pi_m(\mathrm{E}_{ij})$, one can develop $\Big(1-\frac{z}{y}\Big)^{-1}=\sum_{k\geqslant 0}\Big(\frac{z}{y}\Big)^k$, $\Gamma_-(y,z)$, $\Gamma_+(y,z)$, $e(y)$ and $f(z)$ in formal power series in y and z and then collect all the coefficients of $y^i z^{-j}$. Then one obtains a complicated operator of infinite order and in infinite many variables.

From Proposition 4.4 we can immediately deduce the second key result of this paper.

THEOREM 2. – *The presentation $\hat{\pi}_m$ of $a_{\infty|\infty}$ on V_m remains irreducible when restricted to the principal subalgebra $\tilde{\mathscr{g}}$.*

Proof. – By Proposition 4.4, a $\tilde{\mathscr{g}}$-invariant subspace of V_m is $a_{\infty|\infty}$-invariant. □

In § 5 we shall develop a representation theory of $\hat{gl}_{1|1}$, in order to prove in § 6 a formula for the q-dimension of the representation $\hat{\pi}_m$ of $a_{\infty|\infty}$ on V_m.

5. Structure of Verma modules over $\hat{gl}_{1|1}$.

In this section we develop a representation theory of the Lie superalgebra $\hat{\mathscr{g}} = \hat{gl}_{1|1}$ constructed in § 3.

Let $\hat{\mathscr{h}} = \mathbf{C}\lambda \oplus \mathbf{C}\mu \oplus \mathbf{C}c \oplus \mathbf{C}d$ be the *Cartan subalgebra* of $\hat{\mathscr{g}}$. Introduce a $\mathbf{C}$-valued bilinear for $(\,,)$ on $\hat{\mathscr{g}}$ by:

$$(x+\alpha c+\beta d, y+\gamma c+\delta d) = \mathrm{Res}_0(t^{-1}\mathrm{Str}(xy)) + \alpha\delta + \beta\gamma$$

for $\quad x, y \in gl_{1|1}(\mathscr{L}), \quad \alpha, \beta, \gamma, \delta \in \mathbf{C}.$

One easily verifies that this bilinear form is supersymmetric, non-degenerate and invariant. Moreover, its restriction to $\hat{\mathscr{h}}$ is non-degenerate.

The Cartan subalgebra $\hat{\mathscr{h}}$ is diagonalizable in $\hat{\mathscr{g}}$, so that we have the *root space decomposition :*

$$\hat{\mathscr{g}} = \bigoplus_{\gamma \in \hat{\mathscr{h}}^*} \hat{\mathscr{g}}_\gamma, \qquad \text{where} \qquad \hat{\mathscr{g}}_\gamma = \{x \in \hat{\mathscr{g}} \,|\, [h,x] = \gamma(h)x \text{ for all } h \in \hat{\mathscr{h}}\}.$$

We call $0 \neq \gamma \in \hat{\mathscr{h}}^*$ a *root* if $\hat{\mathscr{g}}_\gamma \neq 0$. The set of all roots is denoted by Δ, which we call the *root system.* Let α and δ be the roots corresponding to the root spaces

$$\hat{\mathscr{g}}_\alpha = \mathbf{C}e(0), \quad \hat{\mathscr{g}}_\delta = \mathbf{C}\lambda(1) \oplus \mathbf{C}\mu(1)$$

so that we have

$$- \alpha(\lambda) = \alpha(\mu) = 1 , \qquad \alpha(c) = \alpha(d) = 0 ;$$
$$\delta(\lambda) = \delta(\mu) = \delta(c) = 0 , \qquad \delta(d) = 1 .$$

Then we can express Δ in terms of α and δ :

$$\Delta = \{k\delta \pm \alpha \mid k \in \mathbf{Z}\} \cup \{k\delta \mid k \in \mathbf{Z}\backslash 0\} .$$

Let $\Delta_{\bar{0}} = \{\gamma \in \Delta \mid \hat{\mathscr{g}}_\gamma \subset \hat{\mathscr{g}}_{\bar{0}}\}$ and $\Delta_{\bar{1}} = \{\gamma \in \Delta \mid \hat{\mathscr{g}}_\gamma \subset \hat{\mathscr{g}}_{\bar{1}}\}$, be the sets of *even* and *odd roots*, respectively. Then Δ is the disjoint union of

$$\Delta_{\bar{0}} = \{k\delta \mid k \in \mathbf{Z}\backslash 0\} \qquad \text{and} \qquad \Delta_{\bar{1}} = \{k\delta \pm \alpha \mid k \in \mathbf{Z}\} .$$

Since the restriction of $(,)$ to $\hat{\mathscr{h}}$ is non-degenerate, we can identify $\hat{\mathscr{h}}$ with $\hat{\mathscr{h}}^*$, obtaining for $\Lambda \in \hat{\mathscr{h}}^*$:

$$(5.1a) \qquad (\Lambda,\alpha) = - \Lambda(\lambda+\mu) , \qquad (\Lambda,\delta) = \Lambda(c) ,$$

so that

$$(5.1b) \qquad (\beta, \gamma) = 0 \qquad \text{for} \qquad \beta , \gamma \in \Delta .$$

Let $\{\alpha_0 = \delta - \alpha, \alpha_1 = \alpha\}$, be the set of *simple roots*. We set $L = \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$ and $L_+ = \mathbf{Z}_+\alpha_0 + \mathbf{Z}_+\alpha_1$. The lattice L is called the *root lattice*. For $\gamma = k\alpha_0 + l\alpha_1 \in L$ the number $ht(\gamma) = k + l$ is called the *height* of γ. For any $\gamma \in \Delta$, either $\gamma \in L_+$ or $- \gamma \in L_+$; γ is called a *positive root* in the first case. We denote the set of positive roots by Δ_+. Then Δ is a disjoint union of Δ_+ and $- \Delta_+$.

We consider the subalgebras $\mathscr{n}_- = \bigoplus_{\gamma \in \Delta_+} \hat{\mathscr{g}}_{-\gamma}$ and $\mathscr{n}_+ = \bigoplus_{\gamma \in \Delta_+} \hat{\mathscr{g}}_\gamma$. This gives us the *triangular decomposition* :

$$(5.2) \qquad \hat{\mathscr{g}} = \mathscr{n}_- \oplus \hat{\mathscr{h}} \oplus \mathscr{n}_+ .$$

The antilinear anti-involution ω extends to $a_{\infty|\infty}$ by :

$$\omega(a) = {}^t\bar{a} \qquad \text{for} \qquad a \in \alpha_{\infty|\infty} , \ \omega(c) = c .$$

The principal subalgebra $\tilde{\mathscr{g}}$ is invariant under ω ; we have :

$$\omega(e(k)) = f(-k) , \qquad \omega(f(k)) = e(-k) ,$$
$$\omega(\lambda(k)) = \lambda(-k) , \qquad \omega(\mu(k)) = \mu(-k) , \qquad \omega(c) = c ;$$

this extends to the whole $\hat{\mathscr{g}}$ by $\omega(d) = d$.

$U(\mathscr{g})$ will always denote the universal enveloping superalgebra of the Lie superalgebra $\mathscr{g}$. From (5.2), we obtain

$$U(\hat{\mathscr{g}}) = U(n_-) \otimes_C U(\hat{h}) \otimes_C U(n_+).$$

The root space decomposition of $\hat{\mathscr{g}}$ induces an L-gradation on $U(\hat{\mathscr{g}})$:

$$U(\hat{\mathscr{g}}) = \bigoplus_{\beta \in L} U(\hat{\mathscr{g}})_\beta.$$

Now consider the left ideal $\mathscr{I}(\Lambda$, in $U(\hat{\mathscr{g}})$, generated by n_+ and the elements $h - \Lambda(h)$, $h \in \hat{h}$. We set

$$M(\Lambda) = U(\hat{\mathscr{g}})/\mathscr{I}(\Lambda).$$

The left multiplication on $U(\hat{\mathscr{g}})$ induces a structure of $U(\hat{\mathscr{g}})$-module on $M(\Lambda)$. We denote the image of 1 in $M(\Lambda)$ by v_Λ. We call $M(\Lambda)$ the *Verma module* with highest weight Λ.

PROPOSITION 5.1.

(1) $n_+ \cdot v_\Lambda = 0$; $h \cdot v_\Lambda = \Lambda(h) v_\Lambda$ *for* $h \in \hat{h}$.

(2) $M(\Lambda)$ *is a free* $U(n_-)$*-module of rank* 1 *with generator* v_Λ.

(3) $M(\Lambda) = U(n_-) v_\Lambda$.

(4) *The elements*

$$\ldots \mu(-2)^{n_2} \lambda(-2)^{m_2} f(-2)^{l_2} e(-2)^{k_2} \mu(-1)^{n_1} \lambda(-1)^{m_1} f(-1)^{l_1} e(-1)^{k_1} f(0)^{l_0} v_\Lambda$$

with n_i, $m_i \in \mathbf{Z}_+$ *and* k_i, $l_i = 0$ *or* 1, *and only a finite number of* k_i, l_i, m_i, n_i *non-zero, form a basis of* $M(\Lambda)$.

(5) $M(\Lambda)$ *contains a unique proper maximal* $\hat{\mathscr{g}}$*-submodule* $I(\Lambda)$, *so that*

$$L(\Lambda) = M(\Lambda)/I(\Lambda)$$

is an irreducible $\hat{\mathscr{g}}$*-module.*

Proof. — Is standard. □

The L-gradation of $U(n_-)$ induces a weight space decomposition $M(\Lambda) = \bigoplus_{\beta \in L_+} M(\Lambda)_{\Lambda - \beta}$, where

$$M(\Lambda)_{\Lambda-\beta} = U(n_-)_{-\beta}(v_\Lambda) = \{v \in M(\Lambda) \mid h(v) = (\Lambda - \beta)(h),\ h \in \hat{h}\}.$$

Let $P(\eta)$, $\eta \in L$, denote the number of partitions of η into a sum of positive roots ($k\delta$ taken with multiplicity 2 and the rest with multiplicity 1), odd roots appearing at most ones (by definition, $P(0)=1$). By Proposition 5.1 (4) we have:

$$(5.3) \qquad P(\eta) = \dim M(\Lambda)_{\Lambda-\eta}.$$

A quotient $V(\Lambda)$ of $M(\Lambda)$ is called a highest weight $\hat{\mathscr{g}}$-module. It has the induced weight space decomposition:

$$V(\Lambda) = \bigoplus_{\beta \in L_+} V(\Lambda)_{\Lambda-\beta}.$$

As usual, we define its character by:

$$\mathrm{ch}\ V(\Lambda) = \sum_{\beta \in L_+} (\dim V(\Lambda)_{\Lambda-\beta}) e^{\Lambda-\beta}.$$

Putting $V(\Lambda)_{[j]} = \bigoplus_{\beta \in L_+, ht(\beta)=j} V(\Lambda)_{\Lambda-\beta}$, we obtain the *principal gradation* of $V(\Lambda)$. Define the q-dimension of $V(\Lambda)$ by:

$$\dim_q V(\Lambda) = \sum_{j \in \mathbb{Z}_+} \dim V(\Lambda)_{[j]} q^j.$$

Then, defining the principal specialization F by $F(e^{-\beta}) = q^{ht(\beta)}$, $\beta \in L_+$, we have

$$(5.4) \qquad \dim_q V(\Lambda) = F(e^{-\lambda} \mathrm{ch}\ V(\Lambda)).$$

In particular, we have by Proposition 5.1 (4) (cf. (5.3)):

$$(5.5) \quad \mathrm{ch}\ M(\Lambda) = e^{\Lambda} \sum_{\beta \in L_+} P(\beta) e^{-\beta}$$

$$= e^{\Lambda} \prod_{k \geqslant 1} (1-e^{-k\delta})^{-2}(1+e^{-k\delta+\alpha})(1+e^{-(k-1)\delta-\alpha}),$$

$$(5.6) \qquad \dim_q M(\Lambda) = \prod_{k \geqslant 1} (1+q^{2k-1})^2/(1-q^{2k})^2.$$

We can introduce a Hermitian form F on $M(\Lambda)$, called the *contravariant form*, which is uniquely defined by the properties:

$$F(v_\Lambda, v_\Lambda) = 1 \quad \text{and} \quad F(g(v), w) = F(v, \omega(g)(w)),\ v,\ w \in M(\Lambda),\ g \in \hat{\mathscr{g}}.$$

We have the following properties:

$$F(M(\Lambda)_{-\eta}, M(\Lambda)_{-\theta}) = 0 \quad \text{if} \quad \eta \neq \theta\,; \qquad \operatorname{Ker} F = I(\Lambda)\,.$$

We set $F_\eta = F|M_{-\eta}$. The function $(\det F_\eta)(\Lambda)$ is a polynomial in $\Lambda \in \hat{h}^*$, which is independent of the choice of the basis in $M(\Lambda)_{-\eta}$ up to a positive constant factor. Clearly $M(\Lambda)$ is irreducible if and only if $\det F_\eta \neq 0$ for all $\eta \in L_+ \backslash 0$.

For contragredient Lie superalgebras there exists a formula for $\det F_\eta$. One can find the correct version in [11] (its proof is the same as in [12]). This formula also holds for $\hat{g}$. Recall that in this case $(\gamma,\gamma) = 0$ for all $\gamma \in \Delta_+$ (see (5.1*b*), so that for $\rho \in \hat{h}^*$ such that $(\rho,\alpha_i) = \frac{1}{2}(\alpha_i,\alpha_i)$, we have $(\rho,\gamma) = 0$ for all $\gamma \in \Delta$. Hence we get from the general formula in [11] (here and further, $N = \{1, 2, \ldots\}$):

PROPOSITION 5.2. — *For the representation* $M(\Lambda)$ *of* $\hat{g}$ *one has*

$$(\det F_\eta)(\Lambda) = \prod_{\gamma \in \Delta_{\bar{0}+}} \prod_{k \in N} (\Lambda,\gamma)^{P(\eta - k\gamma)} \prod_{\gamma \in \Delta_{\bar{1}+}} (\Lambda,\gamma)^{P_\gamma(\eta - \alpha)},$$

where P_γ *denotes the number of partitions not involving the root* γ. *In particular* $M(\Lambda)$ *is irreducible iff* $(\Lambda,\gamma) \neq 0$ *for all* $\gamma \in \Delta_+$. □

Following [3], we construct the Jantzen filtration

$$M(\Lambda) = M \supset M^1 \supset M^2 \supset \cdots.$$

This filtration satisfies the following conditions:

(5.7*a*) $\qquad M(\Lambda)_{\Lambda-\eta} \cap M^i = 0 \qquad$ for some i.

(5.8*b*) M^1 is the kernel of F and M/M^1 is an irreducible $\hat{g}$-module isomorphic to $L(\Lambda)$.

Now using Proposition 5.2, we can prove as in [3, 12]:

(5.7*c*)
$$\sum_{i \geq 1} \operatorname{ch} M^i = \sum_{\substack{\gamma \in \Delta_{\bar{0}+} \\ (\Lambda,\gamma)=0}} e^\Lambda \sum_{n \in N} \sum_{\eta \in L_+} P(\eta - n\gamma) e^{-\eta} + \sum_{\substack{\gamma \in \Delta_{\bar{1}+} \\ (\Lambda,\gamma)=0}} e^\Lambda \sum_{\eta \in L_+} P_\gamma(\eta - \gamma) e^{-\eta}$$
$$= \sum_{\substack{\gamma \in \Delta_{\bar{0}+} \\ (\Lambda,\gamma)=0}} \sum_{n \in N} \operatorname{ch} M(\Lambda - n\gamma) + \sum_{\substack{\gamma \in \Delta_{\bar{1}+} \\ (\Lambda,\gamma)=0}} \operatorname{ch} M_\gamma(\Lambda - \gamma),$$

where $M_\gamma(\Lambda-\gamma)$ is a submodule of $M(\Lambda)$ generated by a vector of weight $\Lambda - \gamma$.

PROPOSITION 5.3. — *Let* $\Lambda \in \hat{h}^*$ *be such that* $\Lambda(c) \neq 0$. *Then the* $\hat{\mathcal{g}}$*-module* $M(\Lambda)$ *is irreducible except for the following three cases :*

(a) $(\Lambda,\alpha) = 0$,

(b) $(\Lambda,\alpha) = k\Lambda(c)$ *for a positive integer* k,

(c) $(\Lambda,\alpha) = -k\Lambda(c)$ *for a positive integer* k.

In these cases, $M(\Lambda)$ *contains a unique irreducible submodule* $L(\Lambda-\beta)$, *where* $\beta = \alpha$, $k\delta - \alpha$ *and* $k\delta + \alpha$ *for cases* (a), (b) *and* (c) *respectively, and we have :*

$$\mathrm{ch}\, L(\Lambda) = (1+e^{-\beta})^{-1}\, \mathrm{ch}\, M(\Lambda). \tag{5.8}$$

Proof. — Follows from the general description of irreducible subquotients of Verma modules given in [9] (which is deduced from (5.7c)). The fact that the irreducible subquotient $L(\Lambda-\beta)$ occurs with multiplicity 1 (this is true in the general setup of [9] for any odd β, actually) is clear from (5.7c). □

Remark 5.1. — Replacing $\hat{\mathcal{g}}$ by $\tilde{\mathcal{g}}$ and $\hat{\mathcal{g}}$-modules by graded $\tilde{\mathcal{g}}$-modules in this section does not change the claims of Proposition 5.3, since, apart from Cv_Λ, the only other vectors killed by n_+ are in $M(\Lambda)_{\Lambda-\beta}$ (where β is as in Proposition 5.3).

6. The q-dimension formula for the $a_{\infty|\infty}$-module V_m.

By Theorem 2 we already know that V_m remains irreducible if we restrict to the principal subalgebra $\tilde{\mathcal{g}}$. So V_m is an irreducible highest weight module $L(\Lambda_m)$ over $\tilde{\mathcal{g}}$ with highest weight vector $|m\rangle$ (see (3.1)) and the highest weight defined by :

$$\Lambda_m(c) = (\Lambda_m,\delta) = 1\ ; \quad (\Lambda_m,\alpha) = -m \text{ if } m \geqslant 0, -(m+1) \text{ if } m<0.$$

Define β by :

$$\beta = m\delta + \alpha \quad \text{if} \quad m \geqslant 0, \qquad \beta = \alpha \quad \text{if} \quad m = -1,$$
$$\beta = -(m+1)\delta - \alpha \quad \text{if} \quad m < -1.$$

Then (5.8) gives us

$$\mathrm{ch}\, L(\Lambda_m) = (1+e^{-\beta})^{-1}\, \mathrm{ch}\, M(\Lambda_m). \tag{6.1}$$

Hence, using (5.6) and the fact that the principal gradation of the $a_{\infty|\infty}$-module V_m is consistent with that of the $\tilde{\mathfrak{g}}$-module $V_m = L(\Lambda_m)$ we obtain :

$$(6.2)\quad \dim^q V^m = \begin{cases} (1+q^{2m+1})^{-1} \prod\limits_{k \geq 1} (1-q^{2k})^{-2}(1+q^{2k-1})^2 & \text{if} \quad m \geq 0 \\ (1+q^{-2m-1})^{-1} \prod\limits_{k \geq 1} (1-q^{2k})^{-2}(1+q^{2k-1})^2 & \text{if} \quad m < 0. \end{cases}$$

As a corollary, we obtain an interesting combinatorial identity. We shall call a partition of k into a sum of positive integers of white and black color a *super bipartition* if odd parts of each color are distinct.

COROLLARY 6.1. — *Fix $m \geq 0$. Then the number of super bipartitions of k such that the number of white parts minus the number of black parts is less than or equal to m, is equal to the number of super bipartitions of k such that no black part equals $2m+1$.*

7. The operators S and T.

Introduce the following two operators on $V \otimes V$ which are adjoint to each other :

$$S = \sum_{i \in \frac{1}{2}\mathbb{Z}} (-1)^{2i} \psi_i \otimes \psi_i^* \qquad \text{and} \qquad T = \sum_{i \in \frac{1}{2}\mathbb{Z}} \psi_i^* \otimes \psi_i .$$

Using the defining relations of Cl it is straightforward to verify the following lemma :

LEMMA 7.1.

(1) *S and T commute with the action of $\pi \otimes \pi$ $(gl_{\infty|\infty})$.*

(2) $S(|m\rangle \otimes |n\rangle) = 0$ *iff* $m \geq 0$ *and* $n \leq 0$
$T(|m\rangle \otimes |n\rangle) = 0$ *iff* $m \leq 0$ *and* $n \geq 0$.

In the Lie algebras case, i.e., gl_∞, a similar operator S leads to the definition of the KP and MKP hierarchies. (See [13].) In that case one can construct Hirota bilinear equations, which are certain « bilinear » partial differential equations. We had hoped that also in the super case, i.e., $gl_{\infty|\infty}$, this operator S would lead to some hierarchy of differential equations with commuting and anticommuting variables. Generalizing the definition of the KP hierarchy, a natural definition of the super KP hierarchy is then given by the following equation

$$(7.1) \qquad S(F \otimes F) = 0 \qquad \text{for} \qquad F \in V_0 .$$

Let $F = F_{\bar{0}} + F_{\bar{1}}$ be the decomposition in even and odd elements of V_0, then (7.1) is equivalent to

$$(7.2) \qquad \sum_{i \in \frac{1}{2}\mathbb{Z}} \psi_i(F_{\bar{0}}) \otimes \psi_i^*(F_{\bar{0}} + F_{\bar{1}}) + (-1)^{2i} \psi_i(F_{\bar{1}}) \otimes \psi_i^*(F_{\bar{0}} + F_{\bar{1}}) = 0,$$

and this again is equivalent to the following bilinear identity

$$(7.3) \qquad \oint (\psi_{\bar{0}}(z)(F_{\bar{0}} + F_{\bar{1}}) \otimes \psi_{\bar{0}}^*(z)(F_{\bar{0}} + F_{\bar{1}}) + \psi_{\bar{1}}(z)(F_{\bar{0}} + F_{\bar{1}}) \otimes \psi_{\bar{1}}^*(z)(F_{\bar{0}} + F_{\bar{1}}) \frac{dz}{z} = 0 .$$

Unfortunately, it is not clear to us how one can develop this in order to get « super » Hirota bilinear equations of the Super KP-hierarchy.

In [15], Manin and Radul use even and odd pseudodifferential operators to define the Super KP (SKP)-hierarchy. The time evolution in SKP is defined with respect to a non-abelian Lie superalgebra θ of flows θ_i, $i \geqslant 1$, $\deg \theta_i \equiv i \bmod 2$ with the commutation relations

$$[\theta_{2i}, \theta_{2j}] = [\theta_{2i}, \theta_{2j-1}] = 0, \qquad [\theta_{2i-1}, \theta_{2j-1}] = 2\theta_{2i+2j-2} .$$

This Lie superalgebra θ can be found as a subalgebra in the principal subalgebra $\tilde{\mathscr{g}}$. There are two choices for θ, viz,

(1) $\theta_{2j} = \lambda(j) + \mu(j), \qquad \theta_{2j-1} = e(j) + f(j) \qquad j \geqslant 1$, or

(2) $\theta_{2j} = \lambda(j) + \mu(j), \qquad \theta_{2j-1} = i(e(j) - f(j)) \quad j \geqslant 1$.

This would suggest that there is a relation between the SKP hierarchy of Manin and Radul and the representation theory of $g\ell_{\infty|\infty}$ developed in this paper. Until now it is unclear, however, how these two theories are related.

8. The Weyl superalgebra, its oscillator module and the boson-fermion correspondence.

In this section we review the theory developed in § 1-§ 7 but now for the Weyl superalgebra instead of Clifford superalgebra.

The *Weyl superalgebra* $W = W_{\bar{0}} \oplus W_{\bar{1}}$ is defined as the associative superalgebra over $\mathbf{C}$ with a unit element on the generators ψ_j and ψ_j^*, $j \in \frac{1}{2}\mathbf{Z}$, and the following defining relations:

$$
\begin{aligned}
&\psi_i\psi_j^* - (-1)^{4ij}\psi_j^*\psi_i = \delta_{ij},\\
(8.1)\qquad &\psi_i\psi_j - (-1)^{4ij}\psi_j\psi_i = 0,\\
&\psi_i^*\psi_j^* - (-1)^{4ij}\psi_j^*\psi_i^* = 0,
\end{aligned}
$$

with the $\mathbf{Z}_2$-grading given by

$$(8.2)\qquad \psi_j, \psi_j^* \in W_{\bar{0}} \quad \text{if} \quad j \in \mathbf{Z}, \qquad \psi_j, \psi_j^* \in W_{\bar{1}} \quad \text{if} \quad j \in \frac{1}{2} + \mathbf{Z}.$$

The antilinear anti-involution ω on W is defined by

$$(8.3)\qquad \omega(\psi_j) = -(-1)^{2j}\psi_j^*, \qquad \omega(\psi_j^*) = -(-1)^{2j}\psi_j.$$

The *oscillator module* over W is the irreducible W-module U which admits a non-zero even vacuum vector $|0\rangle$, such that (1.4) holds. The Hermitian form $\langle \bullet, \bullet \rangle$ on U is defined in the same way as on the Cl-module V.

Elements

$$(8.4)\qquad \psi_{i_s}^{*l_s} \dots \psi_{i_1}^{*l_1} \psi_{j_r}^{k_r} \dots \psi_{j_1}^{k_1} |0\rangle$$

with

$$j_1 > \dots > j_r > 0 \geqslant i_1 > \dots > i_s,$$

such that $l_t = 1$ (resp. $k_t = 1$) if i_t (resp. j_t) $\in \frac{1}{2} + \mathbf{Z}$ form a basis of U. These elements are pairwise orthogonal with respect to $\langle \cdot , \cdot \rangle$ and the square length of the element (8.4) is

$$(-1)^{2\sum_t l_t\left(i_t-\frac{1}{2}\right)} k_1 ! \ldots k_r ! \, l_1 ! \ldots l_s ! .$$

Let ψ and $gl_{\infty|\infty}$ be as in § 2, we define the standard representation of $gl_{\infty|\infty}$ on Ψ in terms of the Weyl generators as follows :

$$\text{(8.5)} \qquad E_{ij} \to -(-1)^{2j} \psi_i \psi_j^* ,$$

the action is again defined by the (super) commutator. The map (8.5) also defines a representation ρ of $g\ell_{\infty|\infty}$ on U. Given $n \in \mathbf{Z}$, define

$$|n\rangle = \begin{cases} \psi_1^{n-1} \psi_{\frac{1}{2}} |0\rangle & \text{for} \quad n > 0, \\ \psi_0^{*-n} |0\rangle & \text{for} \quad n \leqslant 0. \end{cases}$$

Putting $\deg |0\rangle = 0$, $\deg \psi_i = 1$ and $\deg \psi_i^* = -1$, we get the following decomposition into irreducible $gl_{\infty|\infty}$-modules U_n with highest weight vectors $|n\rangle$:

$$U = \bigoplus_{n \in \mathbf{Z}} U_n .$$

We denote by ρ_n the representation of $gl_{\infty|\infty}$ on the submodule U_n. We calculate the corresponding highest weights $(\lambda_i^{(n)})_{i \in \frac{1}{2}\mathbf{Z}}$:

$$\begin{array}{ccccccccccc} & 0 & & 0 & & 0 & & 0 & & n & & n-1 & & 0 & & \\ \ldots & \otimes & — & \otimes & — & \otimes & — & \otimes & — & \otimes & — & \otimes & — & \otimes & \ldots & \text{for } n > 0. \\ & -\frac{3}{2} & & -1 & & -\frac{1}{2} & & 0 & & \frac{1}{2} & & 1 & & \frac{3}{2} & & \end{array}$$

$$\begin{array}{ccccccccccc} & 0 & & 0 & & n & & n-1 & & 0 & & 0 & & 0 & & \\ \ldots & \otimes & — & \otimes & — & \otimes & — & \otimes & — & \otimes & — & \otimes & — & \otimes & \ldots & \text{for } n \leqslant 0. \\ & -\frac{3}{2} & & -1 & & -\frac{1}{2} & & 0 & & \frac{1}{2} & & 1 & & \frac{3}{2} & & \end{array}$$

Comparing the highest weights of U_n with the highest weights of V_m, we conclude

$$\text{(8.6)} \qquad \dim_q U_n = \dim_q V_{n-1} .$$

As in the case of Cl and V, we can also define the fermionic and bosonic charge decompositions. Put $\deg |0\rangle = 0$ and $\deg \psi_i = 1$, $\deg \psi_i^* = -1$ for $i \in \mathbf{Z} + \frac{1}{2}$, $\deg \psi_i = \deg \psi_i^* = 0$ for $i \in \mathbf{Z}$

(resp. $\deg \psi_i = 1$, $\deg \psi_i^* = -1$ for $i \in \mathbf{Z}$, $\deg \psi_i = \deg \psi_i^* = 0$ for $i \in \mathbf{Z} + \frac{1}{2}$), obtaining the *fermionic* (resp. *bosonic*) *charge decomposition* of U :

$$(8.7) \qquad \mathrm{U} = \bigoplus_{n \in \mathbf{Z}} \mathrm{U}_{(f);n} \, (\text{resp. } \mathrm{U} = \bigoplus_{n \in \mathbf{Z}} \mathrm{R}\; \mathrm{U}_{(b);n}).$$

Again the subspaces $\mathrm{U}_{(f);n}$ and $\mathrm{U}_{(b);n}$ are not invariant with respect to $gl_{\infty|\infty}$.

In order to avoid « anomalies » in $\bar{a}_{\infty|\infty}$, we change the representation ρ. Define $\hat{\rho}$ as follows :

$$(8.8a) \qquad \hat{\rho}(\mathrm{E}_{ii}) = \rho(\mathrm{E}_{ii}) + (-1)^{2i}\mathrm{I} \qquad \text{for} \qquad i < 0$$

$$(8.8b) \qquad \hat{\rho}(\mathrm{E}_{ij}) = \rho(\mathrm{E}_{ij}) \quad \text{if} \quad i \neq j \qquad \text{or} \qquad i = j > 0\,.$$

Then the corresponding cocycle is

$$\mathrm{C}^-(\mathrm{E}_{ij}, \mathrm{E}_{kl}) = \mathrm{C}(\mathrm{E}_{-k+\frac{1}{2},\, -l+\frac{1}{2}}, \mathrm{E}_{-i+\frac{1}{2},\, -j+\frac{1}{2}}),$$

where C is defined by (2.4).

Then extending $\hat{\rho}$ to $a_{\infty|\infty} = \bar{a}_{\infty|\infty} \oplus \mathbf{C}c$ with cocycle C^-, by $\hat{\rho}(c) = \mathrm{I}$, we obtain a linear representation of the Lie algebra $a_{\infty|\infty}$ on the space U_n, which we denote by $\hat{\rho}_n$. One can find a similar construction of $a_{\infty|\infty}$ in [22]. •

The elements $\mu(m)$, $\lambda(m)$, $e(m)$, $f(m)$ and c of § 3 form a basis of the principal subalgebra $\tilde{g} \subset a_{\infty|\infty}$. Moreover, now they have the following commutation relations :

$$\begin{aligned}
&[\lambda(n), e(m)] = -e(m+n), && [\lambda(n), f(m)] = f(m+n),\\
&[\mu(n), e(m)] = e(m+n), && [\mu(n), f(m)] = -f(m+n),\\
&[\lambda(n), \lambda(m)] = -n\delta_{n,-m}c, && [\mu(n), \mu(m)] = n\delta_{n,-m}c,\\
&[\lambda(n), \mu(m)] = 0, && [e(n), f(m)] = \lambda(m+n) + \mu(m+n) + n\delta_{m,-n}c\,.
\end{aligned}$$

So we have the following identification with $\tilde{gl}_{1|1}$:

$$e(n) = \begin{bmatrix} 0 & t^n \\ 0 & 0 \end{bmatrix}, \qquad f(n) = \begin{bmatrix} 0 & 0 \\ t^n & 0 \end{bmatrix},$$

$$\lambda(n) = \begin{bmatrix} 0 & 0 \\ 0 & t^n \end{bmatrix}, \qquad \mu(n) = \begin{bmatrix} t^n & 0 \\ 0 & 0 \end{bmatrix}, \qquad 1 = c\,.$$

If we replace $\hat{\pi}_m$ by $\hat{\rho}_{m+1}$ the relations (3.1) still hold, hence, when restricted to $\tilde{\mathcal{g}}$, we have

$$\hat{\rho}_{m+1} \simeq \hat{\pi}_m . \tag{8.9}$$

This together with (8.6) proves Theorem 2 in the Weyl superalgebra case:

THEOREM 2'. — *The representation $\hat{\rho}_n$ of $a_{\infty|\infty}$ on U_n remains irreducible when restricted to the principal subalgebra $\tilde{\mathcal{g}}$.*

Now we describe the boson-fermion correspondence for W.

We introduce a *normal ordering*:

$$:\psi_i\psi_j^*: = \begin{cases} \psi_i\psi_j^* & \text{for} \quad j > 0 \\ (-1)^{4ij}\psi_j^*\psi_i & \text{for} \quad j \leqslant 0 . \end{cases}$$

Then $\hat{\rho}(\mathrm{E}_{ij}) = (-1)^{2j+1} : \psi_i\psi_j^* :$,

$$\hat{\rho}(\lambda(n)) = -\sum_{k \in \mathbf{Z}} :\psi_k\psi_{k+n}^*:, \qquad \hat{\rho}(\mu(n)) = \sum_{k \in \mathbf{Z}+\frac{1}{2}} :\psi_k\psi_{k+n}^*:,$$

$$\hat{\rho}(e(n)) = -\sum_{k \in \mathbf{Z}} :\psi_{k-\frac{1}{2}}\psi_{k+n}^*:, \qquad \hat{\rho}(f(n)) = \sum_{n \in \mathbf{Z}} :\psi_k\psi_{k+n-\frac{1}{2}}:.$$

Let $\hat{\mathrm{U}}$ be the formal completion of U, and let $\psi_{\bar{0}}(z)$ and $\psi_{\bar{1}}(z)$ be the same generating series as in § 4. The series $\psi_{\bar{0}}^*(z)$ and $\psi_{\bar{1}}^*(z)$ are slightly different:

$$\psi_{\bar{0}}^*(z) = \sum_{i \in \mathbf{Z}} -\psi_i z^{-i}; \qquad \psi_{\bar{1}}^*(z) = \sum_{i \in \mathbf{Z}+\frac{1}{2}} \psi_i^* z^{-i}.$$

These are operators that map U into $\hat{\mathrm{U}}$. Then exactly the same commutation relations hold as in Lemma 4.1.

Let P_n be the projection on $\mathrm{U}_{(f);n}$ (see (9.7)) and let

$$\mathrm{P}(z) = \Sigma \mathrm{P}_n z^{n-\frac{1}{2}} \quad \text{and} \quad \mathrm{P}(z)^{-1} = \Sigma \mathrm{P}_n z^{-n+\frac{1}{2}}$$

be the corresponding generating series of the projections. Define also

Q; now it maps $U_{(f)\,n}$ into $U_{(f)\,n+1}$; and this operator is uniquely defined by the following relations:

$$Q\,|0\rangle = \psi_{\frac{1}{2}}|0\rangle,$$

$$Q\psi_i = \psi_{i+1}Q, \quad Q\psi_i^* = \psi_{i+1}^*Q \qquad \text{for} \qquad i \in \mathbf{Z} + \frac{1}{2},$$

$$Q\psi_i = \psi_i Q, \qquad Q\psi_i^* = \psi_i^* Q \qquad \text{for} \qquad i \in \mathbf{Z}.$$

It is easy to verify the following relations:

LEMMA 8.1. For $n \in \mathbf{Z}$ we have:

$$\begin{array}{llll}
QP(z) = z^{-1}P(z)Q; & QP(z)^{-1} = zP(z)^{-1}Q; \\
\lambda(n)P(z) = P(z)\lambda(n); & \lambda(n)P(z)^{-1} = P(z)^{-1}\lambda(n); \\
\mu(n)P(z) = P(z)\mu(n); & \mu(n)P(z)^{-1} = P(z)^{-1}e(n); \\
e(n)P(z) = z^{-1}P(z)e(n); & e(n)P(z)^{-1} = zP(z)^{-1}e(n); \\
f(n)P(z) = zP(z)f(n); & f(n)P(z)^{-1} = z^{-1}P(z)^{-1}f(n); \\
Qe(n) = e(n-1)Q; & Qf(n) = f(n+1)Q; \\
Q\lambda(n) = \lambda(n)Q; & \\
Q\mu(0) = (\mu(0)-I)Q; & Q\mu(n) = \mu(n)Q \quad \text{for} \quad n \neq 0.
\end{array}$$

□

Define

$$\Gamma_-(z) = \exp\left(\sum_{n>0} \frac{\mu(-n)}{n} z^n\right), \qquad \Gamma_+(z) = \exp\left(-\sum_{n>0} \frac{\mu(-n)}{n} z^n\right),$$

then the following relations hold:

LEMMA 8.2.

$$\Gamma_+(z)e(m)\Gamma_+(z)^{-1} = e(m) - e(m+1)z^{-1},$$

$$\Gamma_+(z)f(m)\Gamma_+(z)^{-1} = \sum_{i=0}^{\infty} f(m+i)z^{-i},$$

$$\Gamma_+(z)^{-1}e(m)\Gamma_+(z) = \sum_{i=0}^{\infty} e(m+i)z^{-i},$$

$$\Gamma_+(z)^{-1}f(m)\Gamma_+(z) = f(m) - f(m+1)z^{-1},$$

$$\Gamma_-(z)e(m)\Gamma_-(z)^{-1} = \sum_{i=0}^{\infty} e(m-i)z^{i},$$

$$\Gamma_-(z)f(m)\Gamma_-(z)^{-1} = f(m) - f(m-1)z,$$

$$\Gamma_-(z)^{-1}e(m)\Gamma_-(z) = e(m) - e(m-1)z,$$

$$\Gamma_-(z)^{-1}f(m)\Gamma_-(z) = \sum_{i=0}^{\infty} f(m-i)z^{i}.$$

□

Now as before, we have:

$$\psi_{\bar{1}}(z) = P(z)Q\Gamma_-(z)\Gamma_+(z), \qquad \psi^*_{\bar{1}}(z) = Q^{-1}P(z)^{-1}\Gamma_-(z)^{-1}\Gamma_+(z)^{-1}.$$

Using Lemma 4.1 we find

$$\psi_{\bar{0}}(z) = [f(0), \psi_{\bar{1}}(z)]z^{\frac{1}{2}} \quad \text{and} \quad \psi^*_{\bar{0}}(z) = [e(0), \psi_{\bar{1}}(z)]z^{-\frac{1}{2}}.$$

Applying Lemma 8.2 we find the analog of Theorem 1 in the Weyl superalgebra case. Let $e(z)$ and $f(z)$ be as in § 4. Then we have:

THEOREM 1′ (The super boson-fermion correspondence for W).

$$\begin{aligned}
\psi_{\bar{0}}(z) &= P(z)Q\Gamma_-(z)f(z)\Gamma_+(z),\\
\psi^*_{\bar{0}}(z) &= Q^{-1}P(z)^{-1}\Gamma_-(z)^{-1}e(z)\Gamma_+(z)^{-1},\\
\psi_{\bar{1}}(z) &= P(z)Q\Gamma_-(z)\Gamma_+(z),\\
\psi^*_{\bar{1}}(z) &= Q^{-1}P(z)^{-1}\Gamma_-(z)^{-1}\Gamma_+(z)^{-1}.
\end{aligned}$$

PROPOSITION 8.3. – *For* $|y| > |z|$ *we have*

$$(a)\quad \rho_m\Big(\sum_{i,j\in\frac{1}{2}\mathbf{Z}} E_{ij}y^iz^{-j}\Big) = \Big(\frac{y}{z}\Big)^{m-1}\Gamma_-(y,z)\Big[\frac{1}{1-z/y} + e(z) + \Big(\frac{y}{z}\Big)f(y) + \Big(\frac{y}{z}-1\Big)f(y)e(z)\Big]\Gamma_+(y,z),$$

$$(b)\quad \hat{\rho}_m\Big(\sum_{i,j\in\frac{1}{2}\mathbf{Z}} E_{ij}y^iz^{-j}\Big) = \Big(\frac{y}{z}\Big)^{m-1}\Gamma_-(y,z)\Big[\frac{1}{1-z/y} + e(z) + \Big(\frac{y}{z}\Big)f(y) + \Big(\frac{y}{z}-1\Big)f(y)e(z)\Big]\Gamma_+(y,z) + \Big(\frac{y}{z}\Big)^m\Big(1-\frac{z}{y}\Big)^{\frac{1}{2}}\Big)\frac{1}{1-z/y},$$

where

$$\Gamma_-(y,z) = \exp\Big(\sum_{n>0}\frac{\mu(-n)}{n}(y^n - z^n)\Big)$$

and

$$\Gamma_+(y,z) = \exp\Big(-\sum_{n>0}\frac{\mu(n)}{n}(y^{-n} - z^{-n})\Big). \qquad \square$$

Finally, we introduce the operators S and T in the Weyl superalgebra case :

$$S = -\sum_{i\in\frac{1}{2}\mathbf{Z}} (-1)^{2i}\psi_i \otimes \psi_i^* \qquad \text{and} \qquad T = \sum_{i\in\frac{1}{2}\mathbf{Z}} \psi_i^* \otimes \psi_i .$$

Lemma 7.1 also holds in this case for the representation ρ instead of π. Hence the super KP hierarchy can be defined in a similar way :

$$S(F \otimes F) = 0 \qquad \text{for} \qquad F \in U_0 .$$

9. Reduction to $\tilde{gl}_{k|k}$.

In this section we return to the description of $gl_{\infty|\infty}$ and $a_{\infty|\infty}$ and their highest weight module V_m in terms of the Clifford generators.

Let $\mathscr{L} = \mathbf{C}[t,t^{-1}]$ be the algebra of Laurent polynomials in t, previously introduced in § 3. Fix two positive integers k and l. Denote by $\varphi_1, \varphi_2, \ldots, \varphi_{k+l}$ the standard basis of $\mathbf{C}^{k+l}$. We identify the vector space $\mathscr{L}^{k+l}$ over $\mathbf{C}$ with the space Ψ (see § 2) by

$$(9.1)\qquad \begin{aligned} \psi_{ki+j} &= t^{-i}\varphi_j &&\text{if } i\in\mathbf{Z}, && j = 1,\ldots,k,\\ \psi_{li+j+\frac{1}{2}} &= t^{-i}\varphi_{k+j} &&\text{if } i\in\mathbf{Z}, && j = 1,\ldots,l. \end{aligned}$$

The $\mathbf{Z}_2$-gradation on Ψ induces a $\mathbf{Z}_2$-gradation on $\mathbf{C}^{k+l}$, viz, $\deg\varphi_j=\bar{0}$ (resp. $\deg\varphi_j=\bar{1}$) if $j\leqslant k$ (resp. $j>k$). The identification (9.1) gives us an imbedding of associative algebras $T : \mathrm{Mat}_{k+l}(\mathscr{L}) \to \bar{a}_{\infty|\infty}$, where the matrix algebra $\mathrm{Mat}_{k+l}(\mathscr{L})$ acts in the usual way on $\mathscr{L}^{k+l}$. Moreover, T also gives rise to the embedding of the corresponding Lie superalgebras $T : gl_{k|l}(\mathscr{L}) \to \bar{a}_{\infty|\infty}$. Explicitly

(9.2) $T(t^m e_{ij})$

$$\begin{aligned}
&= \sum_{n\in\mathbf{Z}} E_{k(n-m)+i,\,kn+j} &&\text{if}\quad i,j\in 1,\ldots,k,\\
&= \sum_{n\in\mathbf{Z}} E_{l(n-m)-k+i-\frac{1}{2},\,kn+j} &&\text{if}\quad i\in k+1,\ldots,k+l \text{ and } j\in 1,\ldots,k,\\
&= \sum_{n\in\mathbf{Z}} E_{l(n-m)-k+i-\frac{1}{2},\,ln-k+j-\frac{1}{2}} &&\text{if}\quad i,j\in k+1,\ldots,k+l,\\
&= \sum_{n\in\mathbf{Z}} E_{k(n-m)+i,\,ln-k+j-\frac{1}{2}} &&\text{if}\quad i\in 1,\ldots,k \text{ and } j\in k+1,\ldots,k+l,
\end{aligned}$$

where e_{ij} is the $(k+l) \times (k+l)$-matrix with an (i,j)-entry 1 and zeros elsewhere.

The cocycle C on $\bar{a}_{\infty|\infty}$. defined by (2.4), induces a cocycle on $gl_{k|l}(\mathscr{L})$, which we also denote by C and which can be easily calculated using (9.2) :

$$\mathrm{C}(\mathrm{A}(t), \mathrm{B}(t)) = \mathrm{Res}_{t=0} \mathrm{Str}\left(\frac{d\mathrm{A}(t)}{dt} \mathrm{B}(t)\right).$$

Equivalently, this defines a central extension

$$\tilde{g}l_{k|l} = gl_{k|l}(\mathscr{L}) \oplus \mathbf{C}c,$$

where the bracket is defined by $(\mathrm{A,B} \in gl_{k|l}(\mathbf{C}))$:

$$[t^m\mathrm{A}, t^n\mathrm{B}] = t^{m+n}[\mathrm{A,B}] + m\delta_{m,-n} \mathrm{Str}(\mathrm{AB})c.$$

The superalgebra $\tilde{g}l_{k|l}$ is called the *affine superalgebra* associated to the finite-dimensional Lie superalgebra $gl_{k|l}(\mathbf{C})$.

Since the principal subalgebra $\tilde{\mathscr{g}} \subset a_{\infty|\infty}$ is contained in $\mathrm{T}(\tilde{g}l_{k|k})$, we get the following simple consequence:

PROPOSITION 9.1. — *V_m is an irreducible $\tilde{g}l_{k|k}$-module.* □

Note that $|m\rangle$ is the highest weight vector of the $\tilde{g}l_{k|k}$. Now the Cartan subalgebra $\hat{h}$ of $\tilde{g}l_{k|k}$ has as basis c, $e_{11}, \ldots, e$ the elements $2k, 2k$.

Using the relations in the Clifford superalgebra, and its action on V_m, we calculate the highest weight of the $\tilde{g}l_{k|k}$-module V_m :

(9.3*a*) $\hat{\pi}(c)\,|m\rangle = |m\rangle$,

(9.3*b*) $\hat{\pi}(e_{ii})\,|m\rangle = 0$ for $i \neq k+1, 2k$ and $m \in \mathbf{Z}$,

(9.3*c*) $\hat{\pi}(e_{k+1,k+1})\,|m\rangle = m\,|m\rangle$ if $m \geqslant 0$ and $= 0$ if $m < 0$,

(9.3*d*) $\hat{\pi}(e_{2k,2k})\,|m\rangle = 0$ if $m \geqslant 0$ and $= (m+1)\,|m\rangle$ if $m < 0$.

Notice that since V_m is not unitary as an $a_{\infty|\infty}$-module, the same is true for V_m as a $\tilde{g}l_{k|k}$-module. Unitarity and integrability, which play an important role in the representation theory of affine Lie algebras [10], seem to play a minor role in the representation theory of affine Lie superalgebras. Some support for this statement is given by the following. In [14], the contragredient Lie superalgebras (introduced in [6]) of finite growth with symmetrizable Cartan matrix were classified. They all turned out to be non-twisted and twisted affine superalgebras. Only for a few of those superalgebras, viz., $B^{(1)}(0,n)$, $A^{(2)}(0,2n-1)$, $C^{(2)}(n+1)$ and $A^{(n)}(0,2n)$, there exist non-trivial integrable highest weight representations (see [7]). In all other cases (resp. in all cases) one can prove, using the methods of [8], that there are no integrable (resp. unitary) highest weight representations except the 1-dimensional. We hope that our construction of the $\tilde{g}l_{k|k}$-modules V_m will give an indication which « new » conditions will take over the role of unitary and integrability in the representation theory of affine superalgebras.

The principal gradation of $a_{\infty|\infty}$ (see (2.5)) induces a gradation on $\tilde{g}l_{k|k}$:

$$\begin{aligned}
\deg(t^m e_{ij}) &= 2(km+j-i) \quad \text{if} \quad i,j \in \ldots, k \quad \text{or} \quad i,j \in k+1, \ldots, 2k \\
&= 2\left(k(m+1)+j-i+\frac{1}{2}\right) \\
&\qquad\qquad \text{if} \quad i \in k+1, \ldots, 2k \quad \text{and} \quad j \in 1, \ldots, k \\
&= 2\left(k(m-1)+j-i-\frac{1}{2}\right) \\
&\qquad\qquad \text{if} \quad i \in 1, \ldots, k \quad \text{and} \quad j \in k+1, \ldots, 2k.
\end{aligned}$$

This gives a principal gradation on the $\tilde{g}l_{k|k}$-module V_m which is exactly the same as its principal gradation as $a_{\infty|\infty}$-module. Hence the q-dimension of the $\tilde{g}l_{k|k}$-module V_m is given by formula (6.2).

Added in proof. We have constructed recently a super boson-fermion correspondence for the Lie superalgebra $b_{\infty|\infty}$. In this case the Clifford superalgebra is generated by neutral free superfermions φ_i, $i \in 1/2\mathbf{Z}$, satisfying relations

$$\varphi_i\varphi_j + (-1)^{4ij}\varphi_j\varphi_i = (-1)^{[i]}\delta_{i,-j}.$$

The corresponding SBKP operator is

$$S = 2 \sum_{j \in 1/2\mathbf{Z}} (-1)^{[j]}\varphi_{-j} \otimes \varphi_j.$$

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