ANNALES DE L'INSTITUT FOURIER

GERALD W. SCHWARZ

On classical invariant theory and binary cubics

Annales de l'institut Fourier, tome 37, nº 3 (1987), p. 191-216

<http://www.numdam.org/item?id=AIF_1987__37_3_191_0>

Ann. Inst. Fourier, Grenoble
37, 3 (1987), 191-216

ON CLASSICAL INVARIANT THEORY AND BINARY CUBICS

by
Gerald W. SCHWARZ*

0. Introduction.

(0.0) Throughout this paper, G denotes a reductive complex algebraic group and $\phi : \mathrm{G} \longrightarrow \mathrm{GL}(\mathrm{V})$ a k-dimensional representation of G. A *first main theorem* (FMT) for ϕ gives generators for the algebras $\mathbf{C}[n\mathrm{V}]^{\mathrm{G}}, n \geqslant 0$, where $n\mathrm{V}$ denotes the direct sum of n copies of V. A *second main theorem* (SMT) for ϕ is a determination of the relations of these generators. Classical invariant theory provides FMT'*s* and SMT'*s* for the standard representations of the classical groups, and in [14] we provide ones for the standard representations of G_2 and Spin_7.

(0.1) There are classical [21] and recent ([19], [29]) results on how to bound the computations involved in establishing FMT'*s* and SMT'*s*. Our work in [14] required improved bounds, and we present them in this paper. As an application, we compute the FMT and SMT for SL_2 acting on binary cubics. Perhaps these last results can be of help in the enumerative problem of twisted cubics.

(0.2) Let $m \in \mathbf{N}$. Then from generators and relations for $\mathbf{C}[m\,\mathrm{V}]^{\mathrm{G}}$, one obtains, by polarization, a partial set of generators and relations for $\mathbf{C}[n\,\mathrm{V}]^{\mathrm{G}}, n > m$. Let gen (ϕ) (resp. rel (ϕ)) denote the smallest m such that this process yields generators (resp. generators and relations) for all $n > m$. It is classical that

(*) Research partially supported by the NSF. AMS (MOS) subject classification 1980. Primary 14L30, 20G05.

Key-works: Classical invariant theory – Binary cubics – First main theorem – Second main theorem.

$\text{gen}(\phi) \leqslant k = \dim V$, and we show that $\text{rel}(\phi) \leqslant k + \text{gen}(\phi)$. Vust [19] showed that the relations of $\mathbf{C}[nV]^G$ are generated by polarizations of the relations of $\mathbf{C}[kV]^G$ and by relations of degree at most $k+1$ in the generators of $\mathbf{C}[nV]^G$. We improve upon the bound $k+1$.

(0.3) In § 1 we recall facts about integral representations of GL_n and apply them to invariant theory. We give bounds on $\text{gen}(\phi)$, mostly due to Weyl. For example, if ϕ is symplectic, then $\text{gen}(\phi) \leqslant k/2$. Something similar is true if ϕ is orthogonal.

In § 2 we establish the (new) results on SMT'*s* described in (0.2). We show how one uses them to easily recover the SMT'*s* for the classical groups. In § 3 we recall properties of the Poincaré series of $\mathbf{C}[V]^G$ (or any $\mathbf{C}[nV]^G$). If one knows a homogeneous sequence of parameters for $\mathbf{C}[V]^G$, then one easily bounds the degrees of its generators and relations. The bound on degrees of relations was essential to the work described in [13]. In § 4 we apply the techniques developed to obtain the FMT and SMT for binary cubics.

(0.4) I thank Th. Vust and C. DeConcini for their aid and encouragement, and I thank J. Dixmier, H. Kraft and P. Littelmann for helpful comments.

1. First Main Theorems.

(1.0) We first recall properties of integral representations of GL_n (i.e. those representations lying in tensor powers of the standard representation on $\mathbf{C}^n$). Our presentation is a variation of that of Vust ([19], [20]). We then recall Cauchy's formula and its applications to FMT'*s*. We end by giving results estimating $\text{gen}(\phi)$.

(1.1) Let $\psi_1(n)$ denote the standard representation of GL_n on $\mathbf{C}^n$, and let $\psi_i(n) = \Lambda^i(\psi_1(n))$, $i \geqslant 0$. Note that $\psi_i(n) = 0$ for $i > n$ and that $\psi_0(n)$ is the 1-dimensional trivial representation. Let $\mathbf{N}^\infty$ denote the sequences of natural numbers which are eventually zero. If $(a) = (a_1, a_2, \ldots) \in \mathbf{N}^\infty$, let $\psi_{(a)}(n)$ denote the highest weight (Cartan) component in $S^{a_1}(\psi_1(n)) \otimes \ldots \otimes S^{a_m}(\psi_m(n))$

where m is minimal such that $a_j = 0$ for $j > m$. If $m \leqslant n$ (hence $\psi_{(a)}(n) \neq 0$), we will also use the notation $\psi_1^{a_1} \ldots \psi_n^{a_n}(n)$ or $\psi_1^{a_1} \ldots \psi_m^{a_m}(n)$ for $\psi_{(a)}(n)$. If (a) is the zero sequence, then $\psi_{(a)}(n) = \psi_0(n)$. We will confuse the $\psi_{(a)}(n)$ with their corresponding representation spaces, and similarly for representations $\psi_{(a)}$ defined below.

(1.2) We include $\mathbf{C}^n$ in $\mathbf{C}^{n+1}$ as the subspace with last co-ordinate zero. For any $(a) \in \mathbf{N}^\infty$, this induces inclusions $\psi_{(a)}(n) \subseteq \psi_{(a)}(n+1) \subseteq \ldots$ compatible with the actions of $\mathrm{GL}_n \subseteq \mathrm{GL}_{n+1} \subseteq \ldots$. Thus $\mathrm{GL} = \varinjlim \mathrm{GL}_n$ acts linearly on $\psi_{(a)} = \varinjlim \psi_{(a)}(n)$. Let U_n denote the subgroup of GL_n consisting of upper triangular matrices with $1's$ on the diagonal, and set $\mathrm{U} = \varinjlim \mathrm{U}_n$. We identify GL_n, U_n and $\psi_{(a)}(n)$ with their images in GL, U and $\psi_{(a)}$, respectively. If $\psi_{(a)}(n) \neq 0$, then $\psi_{(a)}^{\mathrm{U}} = \psi_{(a)}(n)^{\mathrm{U}_n}$ is the space of highest weight vectors of $\psi_{(a)}(n)$.

(1.3) Let $(a) \in \mathbf{N}^\infty$. We define

$$\deg(a) = \Sigma\, ia_i\,, \ \mathrm{width}\,(a) = \Sigma\, a_i\,,$$

and $\mathrm{ht}\,(a)$ (the height of (a)) is the least $j \geqslant 0$ such that $a_i = 0$ for $i > j$. The height, degree etc. of $\psi_{(a)}$ and $\psi_{(a)}(n)$ are defined to be the height, degree, etc. of (a).

Let $(b) \in \mathbf{N}^\infty$. Then $(a) + (b)$ denotes $(a_1 + b_1, \ldots)$ and $\psi_{(a)}\psi_{(b)}$ denotes $\psi_{(a)+(b)}$. We order $\mathbf{N}^\infty$ lexicographically from the right, i.e. we write $(a) < (b)$ (and also $\psi_{(a)} < \psi_{(b)}$) if there is a $j \in \mathbf{N} - \{0\}$ such that $a_j < b_j$ and $a_i = b_i$ for $i > j$.

(1.4) We say that $\psi_{(c)}$ *occurs* in $\psi_{(a)} \otimes \psi_{(b)}$ if $\psi_{(a)} \otimes \psi_{(b)}$ contains a subspace isomorphic to $\psi_{(c)}$, and similarly for representations $0 \neq \psi_{(c)}(n)$ of GL_n. We identify isomorphic representations of GL and GL_n.

(1.5) PROPOSITION. – *Suppose that* $0 \neq \psi_{(c)}(n)$ *occurs in* $\psi_{(a)}(n) \otimes \psi_{(b)}(n)$. Then

(1) $\deg \psi_{(c)} = \deg \psi_{(a)} + \deg \psi_{(b)}$.

(2) $\mathrm{ht}\, \psi_{(a)}, \mathrm{ht}\, \psi_{(b)} \leqslant \mathrm{ht}\, \psi_{(c)} \leqslant \mathrm{ht}\, \psi_{(a)} + \mathrm{ht}\, \psi_{(b)}$.

(3) width $\psi_{(a)}$, width $\psi_{(b)} \leqslant$ width $\psi_{(c)} \leqslant$ width $\psi_{(a)}$ + width $\psi_{(b)}$.

(4) *The multiplicity of* $\psi_{(c)}(n)$ *in* $\psi_{(a)}(n) \otimes \psi_{(b)}(n)$ *is independent of* n *as long as* $\psi_{(c)}(n) \neq 0$.

Proof. – One can use the Littlewood-Richardson rule [9], or one can use the methods of Vust [20] together with standard Lie algebra results on tensor products.

□

(1.6) COROLLARY. – *Let* $(a), (b) \in \mathbf{N}^\infty$. *Then there are* $(c^1), \ldots, (c^r) \in \mathbf{N}^\infty$, *not necessarily distinct, such that*

$$\psi_{(a)} \otimes \psi_{(b)} = \bigoplus_{i=1}^{r} \psi_{(c^i)},$$

i.e.

$$\psi_{(a)}(n) \otimes \psi_{(b)}(n) = \bigoplus_{i=1}^{r} \psi_{(c^i)}(n)$$

for all n.

□

We give examples of tensor product decompositions which play a role in classical invariant theory (see § 2). They are actually disguised versions of the Clebsch-Gordan formula.

(1.7) LEMMA. – *Let* $p, q \in \mathbf{N}$ *with* $p \leqslant q$. *Then*

(1) $\psi_p \otimes \psi_q = \psi_p \psi_q + \psi_{p-1} \psi_{q+1} + \ldots + \psi_{p+q}$.

(2) $S^2 \psi_p = \psi_p^2 + \psi_{p-2} \psi_{p+2} + \ldots$.

(3) $\Lambda^2 \psi_p = \psi_{p-1} \psi_{p+1} + \psi_{p-3} \psi_{p+3} + \ldots$.

Proof. – Let $n = p + q$. As representations of $\mathrm{SL}_n \subseteq \mathrm{GL}_n$, $\psi_p(n)$ and $\psi_q(n)$ are dual and irreducible, hence the trivial SL_n-representation $\psi_n(n)$ occurs once in $\psi_p(n) \otimes \psi_q(n)$. Thus $\psi_p \otimes \psi_q$ equals ψ_n together with representations of height $\leqslant n-1$. Relative to the action of SL_{n-1}, $\psi_p(n-1) \otimes \psi_q(n-1)$ is dual to $\psi_r(n-1) \otimes \psi_s(n-1)$ where

$$r = n - 1 - q, s = n - 1 - p \quad \text{and} \quad r + s = n - 2.$$

By induction, $\psi_r \otimes \psi_s$ has a decomposition as in (1), hence so does $\psi_p(n-1) \otimes \psi_q(n-1)$ by duality, and (1) follows. The proofs of (2) and (3) are similar.

□

(1.8) We recall Cauchy's theorem on the decomposition of the symmetric algebra of a tensor product: We consider groups of the form $\mathrm{GL}_n \times \mathrm{GL}_m$ (or $\mathrm{GL} \times \mathrm{GL}$) and irreducible representations $\psi_{(a)}(n) \otimes \psi'_{(b)}(m)$ (or $\psi_{(a)} \otimes \psi'_{(b)}$) where we use the prime to distinguish between representations of the first and second copies of the general linear group.

(1.9) THEOREM ([9], [10]).

(1) $$\mathrm{S}^d(\psi_1 \otimes \psi'_1) = \bigoplus_{\deg(a)=d} \psi_{(a)} \otimes \psi'_{(a)} .$$

(2) $$\mathrm{S}^d(\psi_1(n) \otimes \psi'_1(m)) = \bigoplus_{\substack{\deg(a)=d \\ \mathrm{ht}(a)\leqslant m,n}} \psi_{(a)}(n) \otimes \psi'_{(a)}(m) .$$

(1.10) *Remark.* – Most proofs of (2) are combinatorial in nature. However, as in [18], one can use Frobenius reciprocity to show that $\mathbf{C}[\psi_1(n) \otimes \psi'_1(m)]$ contains $\psi'_{(a)}(m)^*$ with multiplicity $\dim \psi_{(a)}(n)$ when $n < m$. (GL_m then has an orbit in $\psi_1(n) \otimes \psi'_1(m)$ whose complement has codimension $\geqslant 2$.) One easily shows that $\mathrm{S}^d(\psi_1(n) \otimes \psi'_1(m))$ contains every

$$\psi_{(a)}(n) \otimes \psi'_{(a)}(m) \quad \text{with} \quad \deg(a) = d ,$$

hence (2) is true when $n < m$. The case $n = m$ follows by taking fixed points of a copy of GL_{m-n}, and (2) implies (1).

(1.11) COROLLARY. – *Let* $(a), (b), (c) \in \mathbf{N}^\infty$ *and suppose that* $\psi_{(c)}$ *occurs in* $\psi_{(a)} \otimes \psi_{(b)}$. *Then* $\psi_{(c)} \otimes \psi'_{(c)}$ *is contained in the product of* $\psi_{(a)} \otimes \psi'_{(a)}$ *and* $\psi_{(b)} \otimes \psi'_{(b)}$ in $\mathrm{S}^\cdot(\psi_1 \otimes \psi'_1)$.

Proof. – Let $\ell = \mathrm{ht}(a), m = \mathrm{ht}(b), n = \ell + m$. Then there is a copy of

$$\psi_{(c)}(n) \subseteq \psi_{(a)}(n) \otimes \psi_{(b)}(n) \subseteq \mathrm{S}^\cdot(\ell\psi_1(n)) \otimes \mathrm{S}^\cdot(m\psi_1(n)) \subseteq \mathrm{S}^\cdot(n\psi_1(n)) .$$

Now use (1.9).

□

(1.12) We apply the results above to invariant theory: Let $\phi : G \longrightarrow GL(V)$ be our k-dimensional representation of the reductive group G. We will also denote ϕ by (V, G) and we will sometimes confuse ϕ with V, so, for example, $\mathbf{C}[\phi]^G = \mathbf{C}[V]^G$. If $(a) \in \mathbf{N}^\infty$, we let $\psi_{(a)}(V)$ denote the representation (or representation space) of $GL(V)$ as defined in (1.1), e.g. $\psi_2(V) = \Lambda^2 V$. Via $\phi : G \longrightarrow GL(V)$, we obtain a representation $\phi_{(a)}$ of G on $\psi_{(a)}(V)$.

Let $P = S^\bullet(\psi_1 \otimes V^*)$ and $P(n) = S^\bullet(\psi_1(n) \otimes V^*) \subseteq P$. Then P (resp. $P(n)$) is a graded direct sum of $GL \times G$ (resp. $GL_n \times G$) representations. Let $R = P^G$ and $R(n) = P(n)^G$. Note that $P(n) \simeq \mathbf{C}[nV]$, $R(n) \simeq \mathbf{C}[nV]^G$ and that $P = \varinjlim P(n)$, $R = \varinjlim R(n)$. By (1.9) we have

$$(1.13)\quad P = \bigoplus_{\mathrm{ht}(a) \leqslant k} \psi_{(a)} \otimes \psi_{(a)}(V^*),$$

$$(1.14)\quad R = \bigoplus_{\mathrm{ht}(a) \leqslant k} \psi_{(a)} \otimes \psi_{(a)}(V^*)^G,$$

and similarly for $R(n)$ and $P(n)$.

Let $R(n)^+$ (resp. R^+) denote the elements of $R(n)$ (resp. R) with zero constant term. Since $R(n)$ is finitely generated, $R(n)^+/(R(n)^+)^2$ is a finite-dimensional GL_n-representation. We can thus find elements $0 \neq f_i \in \psi_{(a^i)}(V^*)^G$, $i = 1, \ldots, p$, such that the representation spaces $\psi_{(a^i)}(n) \otimes f_i \subseteq R(n)$ minimally generate $R(n)$, i.e. bases of these subspaces are a minimal set of generators of $R(n)$ and map onto a basis of $R(n)^+/(R(n)^+)^2$. From (1.14) we see that $\mathrm{ht}(a^i) \leqslant k$ for all i, hence:

(1.15) THEOREM. – *Let $f_i \in \psi_{(a^i)}(V^*)^G$, and suppose that the subspaces $\psi_{(a^i)}(k) \otimes f_i$ minimally generate $R(k)$, $i = 1, \ldots, p$. Then the subspaces $\psi_{(a^i)}(n) \otimes f_i$ minimally generate $R(n)$ for any n.*

□

(1.16) Let $\psi_{(a^i)}(n) \otimes f_i$, $i = 1, \ldots, p$, minimally generate $R(n)$. We say that the generators lying in $\psi_{(a^i)}(n) \otimes f_i$ *transform by* $\psi_{(a^i)}(n)$, and their height, degree, etc. are defined to be that of (a^i). We say that the *minimal generators of* $R(n)$ *transform by* $\psi_{(a^1)}(n), \ldots, \psi_{(a^p)}(n)$.

Suppose that $n \geqslant k$. Then R is generated by the $\psi_{(a^i)} \otimes f_i$, and we say that the *minimal generators of* R *transform by* $\psi_{(a^1)}, \ldots, \psi_{(a^p)}$. Let λ_i be a highest weight vector of $\psi_{(a^i)}$. We call $h_i = \lambda_i \otimes f_i$ a *(minimal) highest weight generator* of R (and of $\mathrm{R}(m), m \geqslant \mathrm{ht}(a^i)$). All elements of $\psi_{(a^i)} \otimes f_i$ can be obtained from h_i via the action of the Lie algebra of strictly lower triangular matrices (acting as polarization operators, in Weyl's language [21]).

(1.17) Let $h = \lambda \otimes f \in \psi_{(a)}(n)^{\mathrm{U}_n} \otimes \psi_{(a)}(\mathrm{V}^*)^{\mathrm{G}} \subseteq \mathrm{R}(n)$. Identifying $\mathrm{R}(n)$ with $\mathbf{C}[n\mathrm{V}]^{\mathrm{G}}$ in the standard way, one sees that h corresponds to an invariant homogeneous of degree $a_i + a_{i+1} + \ldots$ in the ith copy of V.

(1.18) *Remark.* – Let $\tau : \mathrm{G} \longrightarrow \mathrm{GL}(\mathrm{W})$ be an irreducible representation, and let $\mathrm{P}(n)_\tau$ (resp. P_τ) denote the sum of the G-irreducible subspaces of $\mathrm{P}(n)$ (resp. P) isomorphic to τ. Then $\mathrm{P}(n)_\tau$ is isomorphic to the invariants of $\mathrm{S}^\bullet(\psi_1(n) \otimes \mathrm{V}^* \oplus \mathrm{W}^*)$ which are homogeneous of degree 1 in W^*. We can find finitely many subspaces $\psi_{(c^j)}(n) \otimes g_j$, where $g_j \in (\psi_{(c^j)}(\mathrm{V}^*) \otimes \mathrm{W}^*)^{\mathrm{G}}$, which minimally generate $\mathrm{P}(n)_\tau$ as an $\mathrm{R}(n)$-module. Moreover, $\mathrm{ht}(c^j) \leqslant k$ for all j. Analogous results hold for P_τ.

(1.19) Let ϕ, τ and the (a^i) and (c^j) be as above. Then (see (0.2)) $\mathrm{gen}(\phi) = \max\limits_i \mathrm{ht}(a^i)$, and we set $\mathrm{gen}(\phi, \tau) = \max\limits_j \mathrm{ht}(c^j)$. We find situations where the estimates $\mathrm{gen}(\phi), \mathrm{gen}(\phi, \tau) \leqslant k$ can be improved.

We say that a representation $\psi_{(a)}(n)$ is *irrelevant* (for ϕ) if $\psi_{(a)}(n) = 0$ or $\psi_{(a)}(n)$ does not occur as a subrepresentation of $\mathrm{P}(n)/\mathrm{R}(n)^+\mathrm{P}(n)^+$. One similarly defines when $\psi_{(a)}$ is irrelevant, and if $\mathrm{ht}(a) \leqslant n$, then $\psi_{(a)}$ is irrelevant if and only if $\psi_{(a)}(n)$ is. By definition, no minimal generators of $\mathrm{R}(n)$ or any P_τ transform by an irrelevant representation.

From corollary (1.11) we obtain :

(1.20) PROPOSITION. – (1) *If* $\psi_{(a)}$ *is irrelevant and* $(b) \in \mathbf{N}^\infty$, *then any irreducible representation occurring in* $\psi_{(a)} \otimes \psi_{(b)}$ *is irrelevant. In particular,* $\psi_{(a)+(b)}$ *is irrelevant.*

(2) *If* ψ_m *is irrelevant, then* ψ_n *is irrelevant for* $n > m$, *and* $\mathrm{gen}(\phi)$, $\mathrm{gen}(\phi, \tau) < m$.

(3) *If* $\psi_k(V^*)^G \neq 0$ (i.e. $G \subseteq SL(V)$), *then any representation of height* k, *except perhaps for* ψ_k, *is irrelevant.*

□

(1.21) PROPOSITION. – *The representation* ψ_m *is irrelevant if and only if*

$$\Lambda^m V^* = \sum_{1 \leqslant i < m} (\Lambda^i V^*)^G \wedge \Lambda^{m-i} V^* .$$

In particular, ψ_k *is irrelevant if and only if* $\Lambda^i (V^*)^G \neq 0$ *for some* i *with* $1 \leqslant i < k$.

Proof. – One sees directly that the product of $\psi_i(m) \otimes \Lambda^i(V^*)^G$ and $\psi_{m-i}(m) \otimes \Lambda^{m-i}(V^*)$ in $S^m(\psi_1(m) \otimes V^*)$ projects to $\psi_m(m) \otimes \Lambda^i(V^*)^G \wedge \Lambda^{m-i}(V^*) \subseteq \psi_m(m) \otimes \Lambda^m(V^*)$.

□

(1.22) THEOREM ([21]). – (1) *Suppose that* $k = 2m \geqslant 4$ *and that* V *admits a non-degenerate skew form* $\omega \in (\Lambda^2 V^*)^G$ *(i.e.* ϕ *is symplectic). Then* ψ_{m+1} *is irrelevant.*

(2) *Suppose that* $k \geqslant 2$ *and that* V *admits a non-degenerate symmetric* G*-invariant bilinear form (i.e.* ϕ *is orthogonal). Then* $\psi_p \psi_q$ *is irrelevant whenever* $p + q > k$.

Proof (See ([21] p. 154) for (2)). – Part (1) follows from (1.21) and the well-known fact that $\omega \wedge \Lambda^{m-1}(V^*) = \Lambda^{m+1}(V^*)$.

□

(1.23) *Remarks.* – (1) Let $V = V_1 \oplus \ldots \oplus V_r$ where the V_i are irreducible representations of G. Then the homogeneous invariants of $n_1 V_1 \oplus \ldots \oplus n_r V_r$ transform by sums of representations $\psi_{(a^1)}(n_1) \otimes \ldots \otimes \psi_{(a^r)}(n_r)$ of

$GL_{n_1} \times \ldots \times GL_{n_r}$, and $\psi_{(a^1)}(n_1) \otimes \ldots \otimes \psi_{(a^r)}(n_r)$

is irrelevant (obvious definition) if any $\psi_{(a^j)}(n_j)$ is irrelevant for (V_j, G). In particular, the representation is irrelevant if $\mathrm{ht}(a^j) > \dim V_j$ for some j.

(2) Let $V = W \oplus W^*$ where W is an m-dimensional representation of G. Then (V, G) has a symplectic structure, and

$$\Lambda^{m+1}(V) = \bigoplus_{i=0}^{m+1} \Lambda^i W \otimes \Lambda^{m+1-i} W^* .$$

Thus a representation $\psi_{(a)}(n_1) \otimes \psi_{(b)}(n_2)$ is irrelevant if $\mathrm{ht}(a) + \mathrm{ht}(b) > m$. In other words, modulo polarization, generators of $\mathbf{C}[n_1 W \oplus n_2 W^*]^G$ occur in subspaces $\mathbf{C}[rW \oplus sW^*]^G$ where $r \leqslant n_1$, $s \leqslant n_2$ and $r + s \leqslant m$.

(3) Let $\phi = (V, G) = (\mathbf{C}^m \oplus (\mathbf{C}^m)^*, SL_m)$. Then one cannot improve upon the bound $\mathrm{gen}(\phi) \leqslant m$ since there are generators (determinant invariants) of height m.

(4) Let (V, G) be orthogonal, and let

$$h = \lambda \otimes f \in \psi_{(a)}(k) \otimes \psi_{(a)}(V^*)^G$$

be a highest weight generator. Write $\psi_{(a)} = \psi_{(b)} \psi_\ell$ where $\ell = \mathrm{ht}(a) \geqslant m = \mathrm{ht}(b)$. Then $\ell + m \leqslant k$ by (1.22). As an element of $\mathbf{C}[\ell V]^G$, h is linear and skew symmetric in the last $\ell - m$ copies of V (see (1.17)). Thus h maps non-trivially to

$$(M = \bigoplus_j P(m)_{\tau_j})/R^+ M^+ , \quad \text{where} \quad \Lambda^{\ell - m} V = \oplus \tau_j .$$

In other words, we can obtain the minimal highest weight generators of R from minimal generators of $R(m)$-modules $P(m)_\tau$, where $m \leqslant k/2$ and τ is a subrepresentation of some $\Lambda^r V$ with $2m + r \leqslant k$.

(1.24) For later reference and as examples we now state the FMT's for the orthogonal and symplectic groups (see (2.22) for SL_k). Given our results so far, one can establish these FMT's using the Luna-Richardson theorem [8], the methods of [11], or the standard approach [21]. (Using (1.22) one can even improve upon the standard approach in the symplectic case.)

(1.25) *Example.* – Let $G = Sp_{2k}$ act standardly on $V = \mathbf{C}^{2k}$, $k \geqslant 2$. Let $\omega \in (\Lambda^2 V^*)^G$ be the usual G-invariant. Then $S^2(\psi_1(n) \otimes V^*)^G \simeq \Lambda^2 \psi_1(n) \otimes (\Lambda^2 V^*)^G = \psi_2(n) \otimes \omega \simeq \psi_2(n)$ generates $R(n)$. The generator ω_{ij} of $\mathbf{C}[nV]^G$ corresponding to the usual basis element $e_i \wedge e_j$ of $\psi_2(n)$ has value $\omega(v_i, v_j)$ on $(v_1, \ldots, v_n) \in nV$. A highest weight generator is ω_{12}.

(1.26) *Example.* – Let $G = O_k$ act standardly on $V = \mathbf{C}^k$, and let $\eta \in (S^2 V^*)^G$ be the usual G-invariant. Then $S^2 (\psi_1 (n) \otimes V^*)^G \simeq S^2 \psi_1 (n) \otimes (S^2 V^*)^G = \psi_1^2 (n) \otimes \eta \simeq \psi_1^2 (n)$ generates $R(n)$. In other words, $\mathbf{C}[nV]^G$ has generators $\eta_{ij}, 1 \leqslant i \leqslant j \leqslant n$, where $\eta_{ij}(v_1, \ldots, v_n) = \eta(v_i, v_j)$, and η_{11} is a highest weight generator.

2. Second Main Theorems.

(2.0) Let $\phi = (V, G)$ and $k = \dim V$ as in § 1, and let $R = S^{\bullet} (\psi_1 \otimes V^*)^G$ be minimally generated by subspaces

$$\psi_{(a^1)} \otimes f_1, \ldots, \psi_{(a^p)} \otimes f_p .$$

Let $T = S^{\bullet} (\oplus\, \psi_{(a^i)})$, and let $\pi : T \longrightarrow R$ be the canonical GL-equivariant surjection (canonical given our choice of the f_i). Define $T(n) = S^{\bullet} (\oplus\, \psi_{(a^i)} (n)) \subseteq T$. Then π induces

$$\pi (n) : T(n) \longrightarrow R(n), \quad \text{and} \quad I(n) = \operatorname{Ker} \pi (n)$$

lies in $I = \operatorname{Ker} \pi$. We give elements of $\psi_{(a^i)} \supseteq \psi_{(a^i)} (n)$ their natural degree ($= \deg (a^i)$), in which case π and $\pi (n)$ are degree preserving homomorphisms of graded algebras.

(2.1) To solve the SMT for ϕ is, of course, to find generators of I. We show that one knows generators of I, up to polarization, if one knows $I(k + \operatorname{gen}(\phi))$. Vust showed that I is generated by elements of T of degree at most $k + 1$ in the $\psi_{(a^i)}$, along with polarizations of elements of $I(k)$. We refine his result, and we use it to easily rederive the SMT'*s* for the classical groups.

(2.2) It will be convenient for us to use the term relation not only for element of I, but also for irreducible subspaces of I: A *relation* (of $\pi : T \longrightarrow R$) is an equivariant injection $\nu : \psi_{(b)} \longrightarrow I$ for some (b). Note that $\nu : \psi_{(b)} \longrightarrow T$ has image in I if and only if $\nu (h) \in I$ where h is a highest weight vector of $\psi_{(b)}$ (we call $\nu (h)$ a *highest weight relation*). We also refer to equivariant injections $\sigma : \psi_{(c)} (n) \longrightarrow I(n)$ as relations (of $\pi (n) : T(n) \longrightarrow R(n)$). Clearly a relation $\nu : \psi_{(b)} \longrightarrow I$ induces relations

$$\nu(n) : \psi_{(b)}(n) \longrightarrow \mathrm{I}(n)$$

by restriction, and if $\sigma : \psi_{(c)}(n) \longrightarrow \mathrm{I}(n)$ is a relation with $\psi_{(c)}(n) \neq 0$, then there is a unique relation $\nu : \psi_{(c)} \longrightarrow \mathrm{I}$ with $\nu(n) = \sigma$. We use the notation $(\psi_{(b)}, \nu)$ to denote relations $\nu : \psi_{(b)} \longrightarrow \mathrm{I}$, and similarly for relations in $\mathrm{I}(n)$.

(2.3) Let $\nu : \psi_{(b)} \longrightarrow \mathrm{T}$ be an equivariant inclusion. If $\mathrm{ht}(b) > k$, then $\mathrm{Im}\,\nu \subseteq \mathrm{I}$ by (1.14), and we call $(\psi_{(b)}, \nu)$ a *general relation.* We call a relation *special* if it is not general. Roughly, the special relations are the ones one already sees in $\mathrm{I}(k)$, and the general relations are those which occur for dimensional reasons.

(2.4) Let $(\psi_{(b^j)}, \nu_j), j = 1, 2, \ldots$ be a minimal set of generators for I. For any j, $\nu_j(\psi_{(b^j)})$ lies in the image in T of $\sum_i \psi_{(a^i)} \otimes \mathrm{T}_{d_i}$, where $d_i = \deg(b^j) - \deg(a^i)$ and T_{d_i} denotes the elements of T of degree d_i. Any subrepresentation of T_{d_i} of height $> k$ is in I, hence by minimality, $\psi_{(b^j)}$ injects into a sum $\Sigma\, \psi_{(a^i)} \otimes \psi_{(c^\ell)}$ where $\mathrm{ht}(c^\ell) \leqslant k$ for all ℓ. One then easily obtains:

(2.5) Theorem. – *Let* $\mathrm{T} = \mathrm{S}^{\bullet}(\psi_{(a^1)} \oplus \ldots \oplus \psi_{(a^p)})$, *etc. be as above.*

(1) I *is minimally generated by relations*

$$(\psi_{(b^1)}, \nu_1), \ldots, (\psi_{(b^q)}, \nu_q)$$

where $\mathrm{rel}(\phi) := \max_j \mathrm{ht}(b^j) \leqslant k + \mathrm{gen}(\phi)$.

(2) *If* $\{(\psi_{(c^\ell)}, \eta_\ell)\}$ *are relations such that the* $(\psi_{(c^\ell)}(n), \eta_\ell(n))$ *generate* $\mathrm{I}(n)$ *for some* $n \geqslant \mathrm{rel}(\phi)$ *(e.g. for* $n = 2k$*), then the* $(\psi_{(c^\ell)}, \eta_\ell)$ *generate* I.

□

(2.6) *Example.* – To solve the SMT for $\mathrm{V}, \mathrm{G}) = (\mathbf{C}^k, \mathrm{O}_k)$ it suffices to find generators of $\mathrm{I}(k+1)$.

(2.7) Let J_r (or $\mathrm{J}_r(\bigoplus_{i=1}^{p} \psi_{(a^i)})$) denote the direct sum of the irreducible subspaces of T transforming by representations of height

$\geqslant r$. By (1.5), J_r is an ideal of T, and $I = \mathrm{Spc} + J_{k+1}$, where Spc is the subideal of I generated by the special relations. We bound the degrees of minimal generators of the ideals J_r.

(2.8) THEOREM. –*Let* $T = S^{\bullet}(\psi_{(a^1)} \oplus \ldots \oplus \psi_{(a^p)})$. *Assume that* $\mathrm{ht}(a^i) \leqslant r$ *for* $i \leqslant s$ *and* $\mathrm{ht}(a^i) > r$ *for* $s < i \leqslant p$. *Then* J_r *is generated by the* $\psi_{(a^i)}$ *with* $s < i \leqslant p$ *and by the subspaces* $J_r \cap (S^{d_1} \psi_{(a^1)} \otimes \ldots \otimes S^{d_s} \psi_{(a^s)})$ *where* $d_1 + \ldots + d_s \leqslant r - m + 1$ *and* $m = \max\{\mathrm{ht}(a^i) : d_i \neq 0\}$.

(2.9) COROLLARY (Vust [19]). – J_r *is generated by the subspaces* $J_r \cap (S^{d_1} \psi_{(a^1)} \otimes \ldots \otimes S^{d_p} \psi_{(a^p)})$ *with* $d_1 + \ldots + d_p \leqslant r$.

(2.10) *Example.* The ideal $J_6(\psi_1^2 \oplus \psi_2 \oplus \psi_3 \psi_4)$ is generated by subspaces $J_6 \cap (S^a \psi_1^2 \otimes S^b \psi_2 \otimes S^c \psi_3 \psi_4)$ with $a \leqslant 6$, $a + b \leqslant 5$ if $b \neq 0$ and $a + b + c \leqslant 3$ if $c \neq 0$.

(2.11) *Remarks.* – (1) One can usually improve our estimates in specific cases. For example, (2.8) says that $J_r(\psi_2)$ is generated by elements of degree $\leqslant r - 1$ in ψ_2. But

$$S^2 \psi_2 = \psi_2^2 + \psi_4 \,,\; S^3 \psi_2 = \psi_2^3 + \psi_2 \psi_4 + \psi_6 \,,$$

etc. (see (2.20) below), hence $J_r(\psi_2)$ is generated by elements of degree $\leqslant (r+1)/2$ in ψ_2. In example (2.10) we may add the condition $a + 2b + c \leqslant 6$.

(2) In general, one cannot improve upon (2.8) even when there are several representations of large height: Let $\psi = \psi_2 \oplus \psi_2 \oplus \psi_2$ and consider $J_4(\psi)$. There is a copy of $\psi_1^2 \psi_4$ in $\Lambda^3 \psi_2 \subseteq S^3(\psi)$. Now $J_4 \cap S^2(\psi)$ consists of copies of ψ_4, and $\psi_1^2 \psi_4 \not\subseteq \psi_2 \otimes \psi_4$. Hence $J_4 \cap S^2(\psi)$ does not generate J_4, and the estimate of (2.8) is sharp.

(2.12) We consider a multilinear version of (2.8). Let $j_r(\overset{m}{\underset{i=1}{\otimes}} \psi_{(c^i)})$ (or just j_r) denote the subspace of $\otimes \psi_{(c^i)}$ spanned by subrepresentations of height $\geqslant r$. If $A \subseteq \{1, \ldots, m\}$, let $|A|$ denote the cardinality of A and A^c its complement. Let

$j_{r,\mathrm{A}}(\otimes\, \psi_{(c^i)})$ (or just $j_{r,\mathrm{A}}$) denote $j_r(\bigotimes_{i\in \mathrm{A}} \psi_{(c^i)}) \otimes (\bigotimes_{j\in \mathrm{A}^c} \psi_{(c^j)})$, considered as a subspace of $j_r(\otimes\, \psi_{(c^i)})$ via the canonical isomorphism of $(\bigotimes_{i\in \mathrm{A}} \psi_{(c^i)}) \otimes (\bigotimes_{j\in \mathrm{A}^c} \psi_{(c^j)})$ with $\otimes\, \psi_{(c^i)}$.

(2.13) THEOREM. – *Let* $j_r = j_r(\bigotimes_{i=1}^{m} \psi_{(c^i)})$ *be as above. Suppose that* $r \geqslant \ell = \mathrm{ht}(c^1)$ *and that* $\ell \geqslant \mathrm{ht}(c^i), i = 2, \ldots, m$. *Then* j_r *is the sum of* $\{j_{r,\mathrm{A}} : 1 \in \mathrm{A}$ *and* $|\mathrm{A}| \leqslant r - \ell + 1\}$.

One easily deduces theorem (2.8) from theorem (2.13). We deduce theorem (2.13) from

(2.14) PROPOSITION. – *Let* $r, \ell, d \in \mathbf{N}$ *with* $\ell \leqslant r \leqslant \ell + d$. *Then* $j_r(\psi_\ell \otimes (\otimes^d \psi_1))$ *is generated by the subspaces* $j_{r,\mathrm{A}}$ *with* $1 \in \mathrm{A}$ *and* $|\mathrm{A}| = r - \ell + 1$.

Proof of (2.13). – Let $d_i = \deg(c^i), i = 1, \ldots, m$, and let Q_i be a GL-equivariant projection from $\otimes^{d_i} \psi_1$ onto $\psi_{(c^i)}, i = 2, \ldots, m$. Let Q_1 be an equivariant projection from $\psi_\ell \otimes (\otimes^{d_1 - \ell} \psi_1)$ onto $\psi_{(c^1)}$, and let

$$\mathrm{Q} = \mathrm{Q}_1 \otimes \ldots \otimes \mathrm{Q}_m : \psi_\ell \otimes (\otimes^d \psi_1) \longrightarrow \psi_{(c^1)} \otimes \ldots \otimes \psi_{(c^m)}$$

where $d = -\ell + \Sigma\, d_i$. Then

$$\mathrm{Q}(j_r(\psi_\ell \otimes (\otimes^d \psi_1))) = j_r(\psi_{(c^1)} \otimes \ldots \otimes \psi_{(c^m)}).$$

By (2.14), $j_r(\psi_\ell \otimes (\otimes^d \psi_1))$ is generated by subspaces $j_{r,\mathrm{A}}$ where $\mathrm{A} = \{1 < i_1 < \ldots < i_{r-\ell}\}$, and clearly the images $\mathrm{Q}(j_{r,\mathrm{A}})$ are contained in subspaces $j_{r,\mathrm{B}}(\psi_{(c^1)} \otimes \ldots \otimes \psi_{(c^m)})$ where $1 \in \mathrm{B}$ and $|\mathrm{B}| \leqslant |\mathrm{A}| = r - \ell + 1$.

□

(2.15) The proof of (2.14) requires some results about the symmetric group S_n: Let r, ℓ and d be as in (2.14) and set $n = \ell + d$. Let E denote the group algebra $\mathbf{C}[\mathrm{S}_n]$. If $\mathrm{A} \subseteq \{1, \ldots, n\}$, then $\mathrm{S}(\mathrm{A})$ denotes the subgroup of S_n fixing A^c and we set $p_\mathrm{A} = 1/|\mathrm{A}|! \sum_{\sigma\in \mathrm{S}(\mathrm{A})} (\mathrm{sign}\ \sigma)\, \sigma$. If $\mathrm{A} = \{1, \ldots, s\}$, we also write p_s for p_A.

Let $W = \otimes^n \psi_1$. Then W is a left E-module where

$$\sigma(x_1 \otimes \ldots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(n)}, \; \sigma \in S_n .$$

The actions of GL and E on W commute, and $p_\ell W$ is the subspace $\psi_\ell \otimes (\otimes^d \psi_1)$.

(2.16) LEMMA. – $j_r(W) = \sum_{|A|=r} p_A W$.

Proof. – There is a canonical embedding

$$W \hookrightarrow S^n(n\,\psi_1) \simeq S^n(\psi_1 \otimes \psi_1'(n)),$$

where the elements of W are of degree 1 in each copy of ψ_1. Versions of (1.9), (1.11) and (1.5) show that $J_k(\psi_1 \otimes \psi_1'(n))$ is generated by $\psi_k \otimes \psi_k'(n) \subseteq S^k(\psi_1 \otimes \psi_1'(n))$. Intersecting W and $J_k(\psi_1 \otimes \psi_1'(n))$ in $S^n(\psi_1 \otimes \psi_1'(n))$ shows that $j_r(W)$ is generated as claimed.

□

Proof of (2.14). – Note that

$$j_r(\psi_\ell \otimes (\otimes^d \psi_1)) = j_r(p_\ell(W)) = p_\ell(j_r W),$$

and by (2.16) it suffices to proves the following : Let $A \subseteq \{1, \ldots, n\}$ with $|A| = r$. Then $p_\ell p_A$ is in the right ideal of E generated by elements p_B with $\{1, \ldots, \ell\} \subseteq B$ and $|B| = r$.

Let $p'_{\ell-1}$ denote p_C where $C = \{2, \ldots, \ell\}$. Then $p_\ell p'_{\ell-1} = p_\ell$, and by induction on ℓ (the case $\ell = 0$ being trivial) we may assume that $p'_{\ell-1} p_A$ is in the right ideal generated by elements $p_{A'}$ where $|A'| = r$ and $A' \supseteq \{2, \ldots, \ell\}$. Thus it suffices to consider the case $A = \{2, \ldots, r+1\}$.

Now

$$(r+1)\, p_{r+1} = (1 - \sigma_{1,2} - \ldots - \sigma_{1,r+1})\, p_A ,$$

$$\ell\, p_\ell\, p_A = (1 - \sigma_{1,2} - \ldots - \sigma_{1,\ell})\, p_A ,$$

where $\sigma_{i,j}$ is the transposition of i and j. Hence

$$\ell p_\ell p_A = (r+1)\, p_{r+1} + \sum_{j=\ell+1}^{r+1} \sigma_{1,j}\, p_A .$$

Now $\sigma_{1,j} p_A = p_B\, \sigma_{1,j}$ where

$$B = \{1, \ldots, r+1\} - \{j\}, \; j = \ell+1, \ldots, r+1,$$

and $p_{r+1} \in p_r \mathrm{E}$. Hence $p_{\mathfrak{Q}} p_{\mathrm{A}}$ is in the desired right ideal.

□

(2.17) We now easily recapture the SMT'*s* for the classical groups. The following proposition will come in handy.

(2.18) PROPOSITION (see [5] pp. 100-101). – *Let* $\rho : \mathrm{H} \longrightarrow \mathrm{GL}(\mathrm{W})$ *be a representation of the complex algebraic group* H, *where* H^0 *is semisimple. Then*

$$\dim \mathbf{C}[\mathrm{W}]^{\mathrm{H}} = \dim \mathrm{W} - \max_{w \in \mathrm{W}} \dim \mathrm{H}w .$$

(2.19) *Example.* – Let $(\mathrm{V}, \mathrm{G}) = (\mathrm{Sp}_{2k}, \mathbf{C}^{2k})$ as in (1.25). Then $\mathrm{R} \cong \mathrm{T}/\mathrm{I}$ where $\mathrm{T} = \mathrm{S}^{\bullet}(\psi_2)$. The generic isotropy group of G acting on $(2k)\,\mathrm{V}$ is trivial (this is already true for $\mathrm{SL}(\mathrm{V})$), and then (2.18) shows that $\mathrm{R}(2k)$ and $\mathrm{R}(2k+1)$ are regular (i.e. polynomial) algebras. It follows that $\mathrm{I}(2k+2)$ is generated by $\psi_{2k+2}(2k+2) \subseteq \mathrm{S}^{k+1}(\psi_2(2k+2))$, and by theorem (2.5), we see that I is minimally generated by $\psi_{2k+2} \subseteq \mathrm{S}^{k+1}(\psi_2)$.

Let $\sigma = \sum_{1 \leqslant i < j \leqslant 2k+2} \omega_{ij}\, e_i \wedge e_j$ where the ω_{ij}, etc. are as in (1.25). Then the coefficient of $e_1 \wedge \ldots \wedge e_{2k+2}$ in the $(k+1)$ *st* exterior power of σ is a highest weight vector of $\psi_{2k+2} \subseteq \mathrm{I}$, which, up to a scalar, is the Pfaffian of the ω_{ij} see [21]).

(2.20) *Remark.* – Our arguments above show that $\mathrm{S}^{\bullet}(\psi_2)$ contains no elements of odd height and that $\mathrm{J}_{2m}(\psi_2)$ is generated by $\psi_{2m} \subseteq \mathrm{S}^m(\psi_2)$ for any m. By an easy induction we get

$$\mathrm{S}^d(\psi_2) = \oplus \{\psi_{(a)} : \deg(a) = 2d \quad \text{and} \quad a_i = 0 \quad \text{for} \quad i \text{ odd}\}.$$

(2.21) *Example.* – Let $(\mathrm{V}, \mathrm{G}) = (\mathbf{C}^k, \mathrm{O}_k)$ as in (1.26). Then $\mathrm{R} \cong \mathrm{T}/\mathrm{I}$ where $\mathrm{T} = \mathrm{S}^{\bullet}(\psi_1^2)$. Using (2.18) one sees that $\mathrm{R}(k)$ is regular and that $\mathrm{I}(k+1)$ is generated by a single element. By (1.5) and (2.9) this element lies in $\mathrm{S}^{k+1}(\psi_1^2(k+1))$, hence $\mathrm{I}(k+1)$ is generated by $\psi_{k+1}^2(k+1) \subseteq \mathrm{S}^{k+1}(\psi_1^2(k+1))$, and I is generated by $\psi_{k+1}^2 \subseteq \mathrm{S}^{k+1}(\psi_1^2)$. A corresponding highest weight relation is $\det(\eta_{ij})$ $i, j = 1, \ldots, k+1$ (see (1.26)). As in (2.20) one can prove

$$\mathrm{S}^d\,\psi_1^2 = \oplus\,\{\psi_{(a)} : \deg(a) = 2d \quad \text{and all} \quad a_i \quad \text{are even}\}.$$

(2.22) *Example.* – Let $V = \mathbf{C}^k$ and let $G = SL_k$ act standardly on V and V^*. As in (1.23) it is convenient to use two copies of GL to describe invariants of several copies of V and V^*, so we set $R = S^\cdot(\psi_1 \otimes V^* + \psi_1' \otimes V)^G$. Then $R \simeq T/I$ where $T = S^\cdot(\psi_k + \psi_k' + \psi_1 \otimes \psi_1')$ and the representations ψ_k, ψ_k' and $\psi_1 \otimes \psi_1'$ correspond to determinants of k copies of V, determinants of k copies of V^* and contractions of copies of V and V^*, respectively. Irreducible subspaces of T and I transform by representations $\psi_{(a)} \otimes \psi'_{(b)}$, and it is appropriate here to call a relation $\nu : \psi_{(a)} \otimes \psi'_{(b)} \longrightarrow I$ special (resp. general) if $\mathrm{ht}(a)$, $\mathrm{ht}(b) \leqslant k$ (resp. $\mathrm{ht}(a) > k$ or $\mathrm{ht}(b) > k$).

Using (2.18) one can see that the special relations are generated by a copy of $\psi_k \otimes \psi_k'$: the copies of $\psi_k \otimes \psi_k'$ in $S^2(\psi_k \oplus \psi_k') \subseteq T$ and in $S^k(\psi_1 \otimes \psi_1') \subseteq T$ have the same image in R. Applying (2.8) and (1.7) one immediately sees that the general relations are generated by

(2.22.1) $$\psi_{k-2}\,\psi_{k+2} + \psi_{k-4}\,\psi_{k+4} + \ldots \subseteq S^2(\psi_k).$$

(2.22.2) $$\psi'_{k-2}\,\psi'_{k+2} + \psi'_{k-4}\,\psi'_{k+4} + \ldots \subseteq S^2(\psi'_k).$$

(2.22.3) $$\psi_{k+1} \otimes \psi_1' \subseteq \psi_k \otimes (\psi_1 \otimes \psi_1').$$

(2.22.4) $$\psi_1 \otimes \psi'_{k+1} \subseteq \psi'_k \otimes (\psi_1 \otimes \psi_1').$$

(2.22.5) $$\psi_{k+1} \otimes \psi'_{k+1} \subseteq S^{k+1}(\psi_1 \otimes \psi_1').$$

A minimal set of relations does not include (2.22.5) since it results from (2.22.3) or (2.22.4) and the special relation.

3. Bounds using Poincaré Series.

(3.0) We briefly recall some of the main properties of the Poincaré series of an algebra of invariants. In case one knows the degrees of a homogeneous sequence of parameters, then one can estimate the degrees of minimal generating sets and their relations. We have applied such estimates in [13] and [14].

(3.1) Let $\tau : H \longrightarrow GL(W)$ be a representation of the reductive complex algebraic group H. Let $A = \mathbf{C}[W]^H$ and $d = \dim A$. By Noether normalization there are always homogeneous sequences of parameters (HSOP's) for A, i.e. sequences $f_1, \ldots, f_d$ of non-constant homogeneous elements of A such that A is a finite $\mathbf{C}[f_1, \ldots, f_d]$-module. Using results of Hochster and Roberts [3] (or Boutot [1]) and the Nullstellensatz we have

(3.2) PROPOSITION. – *Let* $f_1, \ldots, f_d$ *be non-constant homogeneous element of* A. *The following are equivalent:*

(1) *The* f_i *are an* HSOP *for* A.

(2) A *is a graded finite free* $\mathbf{C}[f_1, \ldots, f_d]$*-module.*

(3) $\{w \in W : f_i(w) = 0\, i = 1, \ldots, d\} = \{w \in W : f(w) = f(0)$ *for every* $f \in A\}$.

(3.3) Recall that the Poincaré series $P_t(A)$ of a finitely generated graded $\mathbf{C}$-algebra $A = \bigoplus_{n \geq 0} A_n$ is $\sum_{n \geq 0} (\dim_{\mathbf{C}} A_n)\, t^n$. If $A = \mathbf{C}[W]^H$ and $f_1, \ldots, f_d$ are an HSOP for A, then it follows from (3.2) that $A \cong \mathbf{C}[f_1, \ldots, f_d] \otimes_{\mathbf{C}} A^0$ as graded $\mathbf{C}[f_1, \ldots, f_d]$-module, where $A^0 = A/(f_1 A + \ldots + f_d A)$. Thus

$$P_t(A) = \prod_{i=1}^{d} (1 - t^{e_i})^{-1}\, P_t(A^0) \tag{3.4}$$

where $e_i = \deg f_i$, $i = 1, \ldots, d$. Since A^0 is a finite dimensional algebra,

$$P_t(A^0) = \sum_{i=0}^{\ell} a_i t^i, \tag{3.5}$$

for some a_i and ℓ, where we assume that $a_\ell \neq 0$.

Construct a surjection $\rho : F \longrightarrow A$ of graded algebras, where $F = \mathbf{C}[X_1, \ldots, X_p]$ for some p and where the $\rho(X_i)$ minimally generate A. Let $r \in \mathbf{N}$ be minimal such that $J = \operatorname{Ker} \rho$ is generated by elements of degree $\leq r$, and set $m = \max_j \deg X_j$.

(3.6) THEOREM. – *Let* A, m, ℓ, *etc. be as above. Then*

(1) $m \leq \max\{\ell, e_1, \ldots, e_d\}$

(2) $r \leq m + \ell$.

Proof. – Part (1) is obvious from (3.4) and (3.5). Let $a_1, \ldots, a_s$ be homogeneous elements of A mapping onto a basis of A^0. Choose homogeneous preimages $a'_1, \ldots, a'_s, f'_1, \ldots, f'_d$ of $a_1, \ldots, a_s, f_1, \ldots, f_d$ in F. We will use symbols b_t and b_{ijt} to denote elements of $\mathbf{C}[f_1, \ldots, f_d]$, and b'_t and b'_{ijt} will denote the unique elements of $\mathbf{C}[f'_1, \ldots, f'_d]$ such that $\rho(b'_t) = b_t, \rho(b'_{ijt}) = b_{ijt}$. (Note that the f_i are algebraically independent, hence so are the f'_i.) Now $\rho(X_i) a_j$ can be uniquely written as a sum $\sum_t b_{ijt} a_t$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant s$. Thus J contains elements

$$h_{ij} = X_i a'_j - \sum_t b'_{ijt} a'_t \, , 1 \leqslant i \leqslant p \, , 1 \leqslant j \leqslant s \tag{3.6.3}$$

of degree $\leqslant m + \ell$.

We may assume that $a_1 = a'_1 = 1$. Let $M = X_1^{n_1} \ldots X_p^{n_p}$ be a monomial in F. By induction on Σn_i one can show that there is an expression $E = \Sigma\, b'_t a'_t$ such that $M - E$ lies in the ideal of the h_{ij}. (One begins the induction with the cases $M = X_i = X_i a'_1$.) There is a canonical linear section σ for ρ, where σ sends $\Sigma\, b_t a_t$ to $\Sigma\, b'_t a'_t$. Our argument above shows that $\mathrm{Im}\, \sigma$ and the ideal of the h_{ij} span F. Hence J is generated by the h_{ij}.

□

(3.7) THEOREM. – *Assume that* H *is connected and semisimple. Then*

(1) A *and* A^0 *are Gorenstein*: $\dim (A^0)_\ell = 1$, *and the bilinear map* $(A^0)_i \times (A^0)_{\ell - i} \longrightarrow (A^0)_\ell \cong \mathbf{C}$ *is a non-degenerate pairing,* $0 \leqslant i \leqslant \ell$. *In particular,* $a_i = a_{\ell - i}, 0 \leqslant i \leqslant \ell$.

(2) $\dim A \leqslant -\ell + \Sigma e_i \leqslant \dim W$.

(3) $\ell = -\dim W + \Sigma e_i$ *if* $\operatorname{codim}_W (W - W') \geqslant 2$,

where W' *is the union of the orbits in* W *with finite isotropy.*

Proof. – Part (1) is due to Murthy; see [15]. Parts (2) and (3) are recent work of Knop [4] (c.f. [16]).

□

We note here that the representation of SL_2 on one or more copies of the space of binary cubics satisfies the hypothesis of (3.7.3.). In § 4 we apply the results above to this situation.

(3.8) *Example.* – Let $(W, H) = (k\mathbf{C}^k, SO_k), k \geqslant 2$. Then the η_{ij} of (1.26), $1 \leqslant i \leqslant j \leqslant k$, are an HSOP, and one can check that the hypothesis of (3.7.3) is satisfied. Theorem (3.6) then gives estimates of degree k for generators and degree $2k$ for relations, both of which are sharp. (The determinant det and the η_{ij} generate A, and det satisfies a quadratic relation over the η_{ij}.)

(3.9) *Example.* – Let $(W, H) = ((2k+2)\mathbf{C}^{2k}, Sp_{2k})$. Again, (3.7.3) applies, and $\ell = 2k$. Since $m = 2$, theorem (3.6) gives an estimate of degree $2k+2$ for the relations, which is sharp. The estimate $m \leqslant 2k$ is not sharp unless $k = 1$.

4. Binary Cubics.

(4.0) We use the results of §§ 1-3 to find the FMT and SMT for the representation (V, G) of SL_2 on binary cubics. The generators were known classically, but not the relations (c.f. [2] pp. 323-326, [17]). We quickly rederive the generators, and we indicate the form and degree of the relations.

(4.1) Let $R(n) = S^{\cdot}(\psi_1(n) \otimes V^*)^G$, etc. be as usual. We begin by calculating $R(1)$ and $R(2)$.

Let $\{e_1, e_2\}$ be the standard basis of $\mathbf{C}^2$. Then $W_m = S^m\mathbf{C}^2$ has basis $\left\{\binom{m}{i} e_1^i e_2^{m-i}, i = 0, \ldots, m\right\}, m \geqslant 0$. The W_m are (all the) irreducible representations of SL_2, and by counting weights one obtains:

(4.1.1) $$S^2W_3 = W_6 + W_2.$$

(4.1.2) $$S^3W_3 = W_9 + W_5 + W_3.$$

(4.1.3) $$S^4W_3 = W_{12} + W_8 + W_6 + W_4 + W_0.$$

We think of V as W_3^*, so a typical element $f \in V$ can be written

(4.2) $$f = ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

where $\{x, y\}$ is the dual basis to $\{e_1, e_2\}$. We may factor f as a product of 3 linear forms, $f = \ell_1\ell_2\ell_3$. Since SL_2 acts transitively on triples of points on the projective line, a non-zero f has one of three normal forms:

(4.2.1) $f = 3b\,(x^2y + xy^2)\,, b \neq 0\,.$

(4.2.2) $f = 3x^2y\,.$

(4.2.3) $f = x^3\,.$

The isotropy group of the form in (4.2.1) is isomorphic to $\mathbf{Z}/3\mathbf{Z}$, hence $\dim \mathbf{C}\,[\mathrm{V}]^{\mathrm{G}} = 1$ and $\mathbf{C}\,[\mathrm{V}]^{\mathrm{G}} \cong \mathrm{R}\,(1)$ is generated by a non-zero invariant D of minimal degree, namely 4 (see (4.1) and [5] p. 103). We choose

(4.3) $\mathrm{D}\,(f) = a^2d^2 + 4ac^3 - 6abcd + 4b^3d - 3b^2c^2\,,$

where f is as in (4.2). Then D is a multiple of the discriminant of f (see [7].

(4.4) We now consider R(2): By (2.18), $\dim \mathrm{R}\,(2) = 5$. Let $f\,,h \in \mathrm{V}\,, t \in \mathbf{C}$. Then $\mathrm{D}\,(f + th) = \sum_{i+j=4} \alpha_{ij}\,(f,h)\,t^{4-i}$ where $\alpha_{ij} \in \mathbf{C}\,[2\mathrm{V}]^{\mathrm{G}}$ and $\alpha_{40}\,(f\,,h) = \mathrm{D}\,(f)$. The α_{ij} are a basis of the copy of $\psi_1^4\,(2)$ in R (2) with highest weight vector D (where α_{ij} corresponds to $\binom{4}{i} e_1^i e_2^j \in \mathrm{S}^4\,\psi_1\,(2))$. As Hilbert already knew we have:

(4.5) LEMMA. – *The* α_{ij} *are an* HSOP *for* R(2).

Proof. – Let $(f\,,h) \in 2\mathrm{V}$ and suppose that $\alpha_{ij}\,(f\,,h) = 0$, $i + j = 4$. By (3.2) it suffices to show that the orbit S of $(f\,,h)$ has the origin in its closure. We may assume that f has the form (4.2.2) or (4.2.3), and let

(1) $h = a'x^3 + 3b'x^2y + 3c'xy^2 + d'y^3\,.$

Then $\mathrm{D}\,(f + th) = 0$ for all t forces $c' = d' = 0$, and clearly $0 \in \overline{\mathrm{S}}$.

□

Let $f\,,h$ be as in (4.2) and (4.5.1), respectively. Set

(4.6) $\beta\,(f\,,h) = ad' - 3bc' + 3cb' - da'\,.$

Then $\beta \in \psi_2\,(2) \otimes (\Lambda^2\,\mathrm{V}^*)^{\mathrm{G}} \subseteq \mathrm{R}\,(2)$, and β is a non-degenerate skew form on V. Thus (V, G) is symplectic.

(4.7) Since R(2) is finite over $\mathbf{C}[\alpha_{ij}]$, the Noether normalization lemma shows that it is also finite over $\mathbf{C}[\beta, \alpha'_1, \ldots, \alpha'_4]$ where the α'_i are linear combinations of the α_{ij}. Thus

$$P_t(R(2)) = (1-t^2)^{-1}(1-t^4)^{-4} P_t(R(2)^0),$$

where, by (3.7.3),

$$P_t(R(2)^0) = 1 + a_1 t + \ldots + a_9 t^9 + t^{10}$$

for some $a_1, \ldots, a_9 \in \mathbf{N}$.

No odd tensor power of V contains the trivial representation, hence all a_i with i odd are zero. Clearly $a_2 = 0$. Using (4.1.1), etc. one easily sees that $\dim R(2)_4 = 6$, which forces $a_4 = 1$. Applying (3.7.1) we obtain

(4.8) $P_t(R(2)^0) = 1 + t^4 + t^6 + t^{10}$.

From (4.8) we see that there is an element $\gamma \in R(2)$ of degree 6 whose image $\overline{\gamma} \in R(2)^0$ is non-zero. Let α be some α_{ij} not in the span of $\alpha'_1, \ldots, \alpha'_4$. Then α has non-zero image $\overline{\alpha} \in R(2)^0$. Clearly $\overline{\alpha}^2 = \overline{\gamma}^2 = 0$, while $\overline{\alpha}\overline{\gamma} \neq 0$ by (3.7.1). Hence

(4.9) PROPOSITION. – (1) R(2) *has generators* α_{ij}, β *and* γ.

(2) *The relations are generated by one in degree* 8 *and one in degree* 12.

□

We make the relations explicit below.

(4.10) We normalize γ as follows: Let $f, h \in V$. Then their resultant $\mathrm{Res}(f,h)$ (see [7]) is an invariant transforming by $\psi_2^3(2)$. Degree arguments (or computations as below) show that Res is not a multiple of β^3, hence we may set $\gamma = \mathrm{Res}$.

From (1.22) and (4.9) we obtain

(4.11) THEOREM. – R *has minimal generators transforming by representations* ψ_1^4, ψ_2 *and* ψ_2^3 *with corresponding highest weight generators* α_{11}, β *and* γ, *respectively.*

□

(4.12) The rest of this section is devoted to describing generators of I, where $R \simeq T/I$ and $T = S^{\bullet}(\psi_1^4 + \psi_2 + \psi_2^3)$. Let

$J_m = J_m(\psi_1^4 + \psi_2 + \psi_2^3)$, let K_m denote the subideal of I generated by subrepresentations of height $< m$, and let

$$I_m = (I + J_{m+1})/(K_m + J_{m+1}).$$

To find generators for I is equivalent to finding subrepresentations which project to generators of the $T/(K_m + J_{m+1})$ — ideals I_m for $m \leqslant 6$.

We use the notation $\psi_{(a)}(\alpha^k \beta^\ell \gamma^m)$ to denote a copy of $\psi_{(a)}$ lying in $S^k \psi_1^4 \otimes S^\ell \psi_2 \otimes S^m \psi_2^3 \subseteq T$ (in all cases considered the multiplicity will be one), and $\lambda(\psi_{(a)}(\alpha^k \beta^\ell \gamma^m))$ denotes a corresponding highest weight vector.

We need to use the following tensor product decompositions. They follow from the Littlewood-Richardson rule and the techniques in [6].

(4.12.1) $$S^2 \psi_1^4 = \psi_1^8 + \psi_1^4 \psi_2^2 + \psi_2^4 .$$

$$S^3 \psi_1^4 = \psi_1^{12} + \psi_1^8 \psi_2^2 + \psi_1^6 \psi_2^3 + \psi_1^4 \psi_2^2 + \psi_2^6$$

(4.12.2) $$+ \psi_1^6 \psi_3^2 + \psi_1^2 \psi_2^2 \psi_3^2 + \psi_1^3 \psi_2^3 \psi_3 + \psi_3^4 .$$

(4.12.3) $$S^2 \psi_2^3 = \psi_2^6 + \psi_1^2 \psi_2^2 \psi_3^2 + \psi_2^4 \psi_4 + \psi_2^2 \psi_4^2 + \psi_1^2 \psi_3^2 \psi_4 + \psi_4^3 .$$

(4.12.4) $$\psi_1^4 \otimes \psi_2^3 = \psi_1^4 \psi_2^3 + \psi_1^3 \psi_2^2 \psi_3 + \psi_1^2 \psi_2 \psi_3^2 + \psi_1 \psi_3^3 .$$

(4.12.5) $$S^2 \psi_1^4 \otimes \psi_2 \supseteq \psi_1^4 \psi_2^2 \otimes \psi_2 \supseteq \psi_1^4 \psi_2 \psi_4 .$$

(4.12.6) $$\psi_2^3 \otimes \psi_2 \supseteq \psi_2^2 \psi_4 .$$

(4.13) *Generator of* I *of height* 2: From (4.9) we see that I_2 is generated by relations of degrees 8 and 12 which must transform by ψ_2^4 and ψ_2^6, respectively. Using (2.20) and (4.12.1), etc. one easily determines that the copies of ψ_2^4 and ψ_2^6 in T are

(4.13.1) $$\psi_2^4(\alpha^2), \psi_2^4(\beta^4), \psi_2^4(\beta\gamma),$$

(4.13.2) $$\psi_2^6(\alpha^3), \psi_2^6(\beta^6), \psi_2^6(\beta^3\gamma), \psi_2^6(\gamma^2),$$

where we set

(4.13.3) $$\lambda(\psi_2^4(\alpha^2)) = \alpha_{22}^2 - 3\alpha_{31}\alpha_{13} + 12\alpha_{40}\alpha_{04} ,$$

$$\lambda(\psi_2^6(\alpha^3)) = 2\alpha_{22}^3 - 9\alpha_{31}\alpha_{22}\alpha_{13} + 27\alpha_{40}\alpha_{13}^2$$

(4.13.4) $$- 72\alpha_{40}\alpha_{22}\alpha_{04} + 27\alpha_{31}^2\alpha_{04} ,$$

and $\lambda(\psi_2^4(\beta^4)) = \beta^4$, $\lambda(\psi_2^4(\beta\gamma)) = \beta\gamma$, etc.

Evaluating the λ's in case $f = ax^3 + 3bx^2y$ and $h = 3cxy^2 + dy^3$ one sees that I_2 is generated by relations with highest weight vectors

$$9\lambda(\psi_2^4(\alpha^2)) - \lambda(\psi_2^4(\beta^4)) - 8\lambda(\psi_2^4(\beta\gamma)). \tag{4.13.5}$$

$$27\lambda(\psi_2^6(\alpha^3)) + 2\lambda(\psi_2^6(\beta^6)) - 40\lambda(\psi_2^6(\beta^3\gamma)) - 16\lambda(\psi_2^6(\gamma^2)). \tag{4.13.6}$$

(4.14) *Generators of* I *of height* 3 : We will not be so specific as to the relations, but rather just indicate their form and degree. Our computations are aided by the following general fact:

(4.15) THEOREM ([12] Table 3). – *Let* $H = Sp_m$ *act standardly on* $W = (m+1)\mathbf{C}^{2m}$. *Then* $\mathbf{C}[W]$ *is a free graded* $\mathbf{C}[W]^H$*-module.*

□

Returning to binary cubics, we see that $\mathbf{C}[3V]$ is a free $\mathbf{C}[\beta_{12}, \beta_{13}, \beta_{23}]$-module, where the β_{ij} are a basis of the copy of $\psi_2(3) \subseteq R(3)$ ($\beta = \beta_{12}$ is a highest weight vector.) Projecting to G-invariants, we see that $R(3)$ is free over $S^\bullet\psi_2(3)$.

By theorem (1.22), any representation in T of height $\geqslant 3$ is, modulo I, in the ideal of ψ_2. Since $R(3)$ is free over $S^\bullet\psi_2(3)$, we have a recipe for finding generators of I_3 : Compute generators of $J_3(\psi_1^4(\alpha) + \psi_2^3(\gamma))$ and express the ones of height 3 as elements of the ideal of $\psi_2(\beta)$. For example, using (4.12.4) with $\psi_2^3 = \psi_2^3(\gamma)$ and $\psi_2^3(\beta^3)$, one finds representations $\psi_1\psi_3^3(\alpha\gamma)$ and $\psi_1\psi_3^3(\alpha\beta^3)$ in T. In fact, T contains $\psi_1\psi_3^3$ with multiplicity two, hence I contains a relation showing that $\psi_1\psi_3^3(\alpha\gamma)$ and $\psi_1\psi_3^3(\alpha\beta^3)$ have the same image in R.

Using theorem (2.8), one can see that $J_3(\psi_1^4(\alpha) + \psi_2^3(\gamma))$ is generated by the representations of height $\geqslant 3$ in (4.12.2) through (4.12.4), of which 8 are of height 3. Thus the corresponding 8 elements of I generate I_3, but not minimally : One can show (by computing highest weight vectors) that the image

$$\psi_2^4 \otimes \psi_1^4 \subseteq S^2\psi_1^4 \otimes \psi_1^4 \longrightarrow S^3\psi_1^4$$

contains $\psi_1^2\psi_2^2\psi_3^2 + \psi_1^3\psi_2^3\psi_3 + \psi_3^4$. Using the relation with highest weight (4.13.5) we see that the elements of I corresponding to $\psi_1^2\psi_2^2\psi_3^2(\alpha^3)$, etc. are not needed to generate I_3. Thus I_3 is generated by relations corresponding to $\psi_1^6\psi_3^2(\alpha^3)$, $\psi_1^2\psi_2^2\psi_3^2(\gamma^2)$,

$\psi_1^3\psi_2^2\psi_3(\alpha\gamma)$, $\psi_1^2\psi_2\psi_3^2(\alpha\gamma)$ and $\psi_1\psi_3^3(\alpha\gamma)$.

(4.16) *Generator of* I *of height* 4 : Modulo I, J_4 is generated by $\psi_4(\beta^2)$, and R (4) is a free **C** [det]-module, where det is the image in R (4) of a highest weight vector of $\psi_4(\beta^2)$. Thus, as (4.14), we obtain generators of I_4 by expressing the height 4 generators of $J_4(\psi_1^4 + \psi_2 + \psi_2^3)$ as elements of the ideal of $\psi_4(\beta^2)$. We claim that the 6 height 4 representations $\psi_2^4\psi_4(\gamma^2), \dots, \psi_2^2\psi_4(\beta\gamma)$ in (4.12.3) through (4.12.6) suffice :

By theorem (2.8), J_4 has generators in $S^k\psi_1^4 \otimes S^\ell\psi_2 \otimes S^m\psi_2^3$ with $k = 4$ and $\ell = m = 0$, or $k + \ell + m \leqslant 3$. Using our six height 4 relations and those of height $\leqslant 3$ one eliminates the following cases completely, or in favor of cases with a larger value of ℓ : $k = 4, \ell = m = 0 ; m \geqslant 2 ; k \geqslant m \geqslant 1$. It follows that our list is complete.

(4.17) *Generators of* I *of height* > 4 : Modulo the generators of I described so far, elements of J_5 lie in the ideal of $\psi_4(\beta^2)$. Hence the remaining generators of I required are among

(4.17.1) $$\psi_1^3\psi_5(\alpha\beta^2) \subseteq \psi_1^4(\alpha) \otimes S^2\psi_2(\beta).$$

(4.17.2) $$\psi_1\psi_2^2\psi_5(\beta^2\gamma) + \psi_2^2\psi_6(\beta^2\gamma) \subseteq \psi_2^3(\gamma) \otimes S^2\psi_2(\beta).$$

(4.17.3) $$\psi_6(\beta) \subseteq S^3\psi_2(\beta).$$

We only add (4.17.1) and (4.17.3) to our list, since (4.17.2) is a consequence of the height 4 relation transforming by $\psi_2^2\psi_4$.

(4.18) THEOREM. – I *is minimally generated by special relations transforming by*

ψ_2^4, ψ_2^6, $\psi_1^6\psi_3^2$, $\psi_1^2\psi_2^2\psi_3^2$, $\psi_1^3\psi_2^2\psi_3$, $\psi_1^2\psi_2\psi_3^2$, $\psi_1\psi_3^3$,

$\psi_2^4\psi_4$, $\psi_2^2\psi_4^2$, $\psi_1^2\psi_3^2\psi_4$, ψ_4^3, $\psi_1^4\psi_2\psi_4$

and $\psi_2^2\psi_4$, *and by general relations transforming by* $\psi_1^3\psi_5$ *and* ψ_6.

Proof. – We know there are generators of I as described, and degree and height considerations easily establish minimality.

□

BIBLIOGRAPHY

[1] J.-F. BOUTOT, Singularités rationnelles et quotients par les groupes réductifs, *Inv. Math.*, 88 (1987), 65-68.

[2] J. GRACE and A. YOUNG, *The Algebra of Invariants*, Cambridge University Press, Cambridge, 1903.

[3] M. HOCHSTER and J. ROBERTS, Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay, *Adv. in Math.*, 13 (1974), 115-175.

[4] F. KNOP, Über die Glattheit von Quotientenabbildungen, *Manuscripta Math.*, 56 (1986), 419-427.

[5] H. KRAFT, *Geometrische Methoden in der Invariantentheorie*, Viehweg, Braunschweig, 1984.

[6] M. KRÄMER, Eine Klassifikation bestimmter Untergruppen kompakter zusammenhängender Liegruppen, *Comm. in Alg.*, 3 (1975), 691-737.

[7] S. LANG, *Algebra*, Addison-Wesley, Reading, 1965.

[8] D. LUNA and R. RICHARDSON, A generalization of the Chevalley restriction theorem, *Duke Math. J.*, 46 (1979), 487-496.

[9] I.G. MAC DONALD, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.

[10] C. PROCESI, *A Primer of Invariant Theory*, Brandeis Lecture Notes 1, Department of Mathematics, Brandeis University, 1982.

[11] G. SCHWARZ, Representations of simple Lie groups with regular rings of invariants, *Inv. Math.*, 49 (1978), 167-191.

[12] G. SCHWARZ, Representations of simple Lie groups with a free module of covariants, *Inv. Math.*, 50 (1978), 1-12.

[13] G. SCHWARZ, Invariant theory of G_2, *Bull. Amer. Math. Soc.*, 9 (1983), 335-338.

[14] G. SCHWARZ, Invariant theory of G_2 and Spin_7, to appear.

[15] R.P. STANLEY, Invariants of finite groups and their applications to combinatorics, *Bull. Amer. Math. Soc.*, 1 (1979), 475-511.

[16] R.P. STANLEY, Combinatorics and invariant theory, *Proc. Symposia Pure Math.*, Vol. 34, Amer. Math. Soc., Providence, R.I., 1979, 345-355.

[17] F. VON GALL, Das vollständige Formensystem dreier cubischen binären Formen, *Math. Ann.*, 45 (1894), 207-234.

[18] Th. VUST, Sur la théorie des invariants des groupes classiques, *Ann. Inst. Fourier*, 26-1 (1976), 1-31.

[19] Th. VUST, Sur la théorie classique des invariants, *Comm. Math. Helv.*, 52 (1977), 259-295.

[20] Th. VUST, Foncteurs polynomiaux et théorie des invariants, in Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin, *Springer Lecture Notes*, No. 725, Springer Verlag, New York, 1980, pp. 330-340.

[21] H. WEYL, *The Classical Groups*, 2nd edn., Princeton Univ. Press, Princeton, N.J., 1946.

Manuscrit reçu le 25 novembre 1986.

Gerald W. SCHWARZ,
Brandeis University
Dept. of Mathematics
Waltham, Mass. 02254-9110 (USA).