ANNALES DE L'INSTITUT FOURIER

PAUL BAIRD

Harmonic morphisms onto Riemann surfaces and generalized analytic functions

Annales de l'institut Fourier, tome 37, nº 1 (1987), p. 135-173

<http://www.numdam.org/item?id=AIF_1987__37_1_135_0>

Ann. Inst. Fourier, Grenoble
37, 1 (1987), 135-173

HARMONIC MORPHISMS ONTO RIEMANN SURFACES AND GENERALIZED ANALYTIC FUNCTIONS

by Paul BAIRD

Introduction.

Let $P(w,z)$ be a polynomial in the complex variables w and z with constant coefficients, having degree n in w. It is well known from the theory of algebraic function that the equation

$$P(w,z) = 0 \tag{1}$$

locally determines n function elements $w_1, w_2, \ldots, w_n$ which vary analytically as functions of z. These n function elements determine an algebraic function w together with its associated Riemann surface. This is the starting point for the theory of compact Riemann surfaces [19]. A generalization of these ideas to higher dimensional domains was considered by Jacobi [18].

Let $\pi(x,y,z,\varphi)$ be an analytic function in the real variables x,y,z and the complex variable φ, such that π satisfies the equations

$$\frac{\partial^2\pi}{\partial x^2} + \frac{\partial^2\pi}{\partial y^2} + \frac{\partial^2\pi}{\partial z^2} = 0 \tag{2}$$

$$\left[\frac{\partial\pi}{\partial x}\right]^2 + \left[\frac{\partial\pi}{\partial y}\right]^2 + \left[\frac{\partial\pi}{\partial z}\right]^2 = 0\,. \tag{3}$$

Suppose that the equation

$$\pi(x,y,z,\varphi) = 0\,, \tag{4}$$

Key-words: Harmonic morphism – Horizontally conformal – Holomorphic – Harmonic – Riemann surface – Conformal vector field – Generalized analytic function – Hopf fibration.

locally determines φ as a complex valued function of x, y and z. Then an easy calculation shows that φ must also satisfy equations (2) and (3). This property was observed by Jacobi in [18].

We will regard equation (4) as the analogue of the algebraic equation (1), and the function φ as a *generalized analytic function.* In general φ must be considered as a multiple-valued function.

Independantly of these ideas, harmonic morphisms were introduced in 1965 by Constantinescu and Cornea [5] as a natural generalization of the holomorphic mappings between Riemann surfaces. These mappings were defined in terms of harmonic spaces as defined by Brelot [3].

Harmonic morphisms between Riemannian manifolds were first studied in some detail by Fuglede [14] and Ishihara [17]. They are characterized by the property that they pull back germs of harmonic functions to germs of harmonic functions. Thus if $\varphi : \mathrm{M} \longrightarrow \mathrm{N}$ is a continuous mapping between Riemannian manifolds, then φ is a *harmonic morphism* if $f \circ \varphi$ is a harmonic function on $\varphi^{-1}(\mathrm{V})$ for every function f which is harmonic on an open set $\mathrm{V} \subset \mathrm{N}$. If $\dim \mathrm{M} < \dim \mathrm{N}$ then every harmonic morphism φ is necessarily constant, and by choosing smooth local harmonic coordinates on N it follows that any harmonic morphism must in fact be smooth [14].

In [14] many of the basic properties of harmonic morphisms are established. In particular they can be characterized as mappings which are both harmonic and horizontally conformal (this result is also obtained in [17]), where we say that φ is *horizontally conformal* if for each $x \in \mathrm{M}$ where $d\varphi(x) \neq 0$, the restriction of $d\varphi(x)$ to the orthogonal complement of $\ker d\varphi(x)$ in $\mathrm{T}_x \mathrm{M}$ is conformal and surjective. In the case when φ is a complex valued function, the condition that φ be horizontally conformal is equivalent to equation (3) above. Thus in this case harmonic morphisms are precisely the generalized analytic functions considered by Jacobi in 1847.

More recently harmonic morphisms have been seen to play an important role in stochastic processes. In the paper of Bernard, Campbell and Davie [2], it is shown that a mapping between open subsets of Euclidean space is *Brownian path preserving* (see [2] for definitions) if and only if it is a harmonic morphism. They study in detail the case when M is an open subset of $\mathbf{R}^3$ and N a domain in the complex plane $\mathbf{C}$. One of the problems they pose is to classify all such harmonic morphisms. That classification at least locally is obtained here in Theorem (3.5).

In their paper, Bernard, Campbell and Davie observe that in the case when M is open in $\mathbf{R}^m$ and N is a domain in $\mathbf{C}$, then the fibres of φ are minimal submanifolds of M. The geometric properties of the fibres of a harmonic morphism have been further studied in [1]. It turns out that the case when N is a Riemann surface is special. For if $\varphi : M \longrightarrow N$ is a submersion from an oriented Riemannian manifold M onto a Riemann surface N, then

1) *If φ is horizontally conformal, then φ is harmonic and hence a harmonic morphism if and only if the fibres of φ are minimal in* M [1].

2) *The horizontal distribution, which is a distribution in the tangent bundle* TM, *is 2-dimensional, and hence has associated with it a natural almost complex structure determined by rotation through $\pi/2$ in the horizontal space. In the case when φ is a harmonic morphism this induces a natural complex structure on the space of fibres.*

In the case when M is open in $\mathbf{R}^3$ and N is a Riemann surface, then for $\varphi : M \longrightarrow N$ to be a non-constant harmonic morphism, the fibres of φ must be parts of straight lines in $\mathbf{R}^3$. Such a fibre can be expressed as the set of points $sy + c(y)$ for suitable values of $s \in \mathbf{R}$, where $y \in S^2$ determines the direction of the fibre and $c(y)$ its position in $\mathbf{R}^3$. The vector $c(y)$ can be chosen perpendicular to y in $\mathbf{R}^3$ and can thus be regarded as a vector in the tangent space $T_x S^2$. For φ to be a harmonic morphism it is necessary and sufficient that c be a *conformal vector field on* S^2 (Lemma (3.3)). This results in delightful geometric configurations for the structure of the domain M. This is described in Sections 3 and 4.

In Section 5, we return to Jacobi's original theme. We explore the idea of generalizing the concepts of analytic function theory to higher dimensional domains, showing how it is possible to regard some of the examples of Section 4 as multiple valued maps. Indeed it turns out that these maps are solutions of an algebraic equation of the form (4). In such cases we can take copies of the domain and glue them together in such a way that we obtain a harmonic morphism from a 3-dimensional C°-manifold onto the whole Riemann sphere S^2.

This is similar to the way in which the Riemann surfaces of multiple valued analytic functions are constructed.

In Section 6 we study the case when M is an open subset of S^3. The classification of such maps is obtained in Theorem (6.1) in terms of holomorphic curves in the Grassmannian of oriented 2-planes in $\mathbf{R}^4$.

The problem of finding harmonic morphisms defined globally on S^3 is considered in Section 7. By exploiting the fact that the Grassmannian of oriented 2-planes in $\mathbf{R}^4$ is biholomorphically equivalent to the product $S^2 \times S^2$, we obtain one of our main theorems:

If $\varphi : S^3 \longrightarrow N$ *is a harmonic morphism onto a Riemann surface* N, *which is also a submersion, then* φ *is the composition* $\varphi = v \circ \pi$, *where*

$\pi : S^3 \longrightarrow S^2$ *is the Hopf fibration*

and

$v : S^2 \longrightarrow N$ *is a conformal mapping.*

In particular if φ is non-constant, this shows that N must be conformally equivalent to S^2.

Our approach throughout is to reduce the classification problem to a problem in holomorphic mappings. The methods used here are inspired by the striking analogy that seems to exist between horizontally conformal submersions onto a Riemann surface and conformal immersions from a Riemann surface. Indeed much of the motivation for this article are the results of Calabi [4] and the more recent results of Din-Zakrzewski [6,7], Glaser-Stora [15] and Eells-Wood [12], where the classification of certain harmonic immersions of a Riemann surface is reduced to a problem in holomorphic mappings.

I would like to thank J. Eells and J.C. Wood, who both read through my original manuscript and suggested several improvements. I would also like to thank the referees for many helpful comments and suggestions. The historical observation that Jacobi was the first person to consider harmonic morphisms is due to B. Fuglede.

I am especially indebted to the Centre for Mathematical Analysis, Canberra, for their support during the preparation of this paper.

1. The composition properties of harmonic morphisms.

Let $\varphi : M \longrightarrow N$ be a smooth mapping between smooth Riemannian manifolds and let τ_φ denote the *tension field* of φ [11]. Thus φ is *harmonic* if and only if it has vanishing tension field. Henceforth assume all manifolds and maps are smooth unless otherwise stated.

If $x \in M$ is such that $d\varphi(x) \neq 0$, let $V_x M$ denote the subspace of $T_x M$ determined by $\ker d\varphi_x$, and let $H_x M$ denote the orthogonal complement of $V_x M$ in $T_x M$. Henceforth we will denote by HM, VM the corresponding distributions in the tangent bundle TM, which we call the *horizontal, vertical distributions* respectively.

The map φ is said to be *horizontally conformal* if, for each $x \in M$ at which $d\varphi_x \neq 0$, the restriction

$$d\varphi_x |_{HM} : H_x M \longrightarrow T_{\varphi(x)} N$$

is conformal and surjective. That is, for all $X, Y \in H_x M$, there exists a positive number $\lambda(x) \in \mathbf{R}$, such that

$$\lambda(x)^2 \, g(X, Y) = h(d\varphi(X), d\varphi(Y)),$$

where g, h denote the metrics of M, N respectively.

Let C_φ denote the critical set of the map φ. Thus C_φ consists of those points $x \in M$ where $d\varphi_x = 0$. Setting λ equal to zero on C_φ, we obtain a continuous function $\lambda : M \longrightarrow \mathbf{R}$ called the *dilation* of φ.

(1.1) THEOREM [14], [17]. – *Let $\varphi : M \longrightarrow N$ be a smooth map between Riemannian manifolds. Then φ is a harmonic morphism if and only if φ is both harmonic and horizontally conformal.*

If φ is a harmonic morphism the critical set C_φ forms a polar set in M. An important consequence of this is that C_φ cannot disconnect any open connected subset of M [14].

In general the composition of two harmonic maps need not be harmonic. However, the composition of two harmonic morphisms does yield another harmonic morphism.

(1.2) LEMMA. – *If $\varphi : M \longrightarrow N$, $\psi : N \longrightarrow P$ are both horizontally* *conformal with dilations* $\lambda : M \longrightarrow \mathbf{R}, \mu : N \longrightarrow \mathbf{R}$ *respectively, then the composition* $\psi \circ \varphi : M \longrightarrow P$ *is horizontally conformal with dilation* $\upsilon : M \longrightarrow \mathbf{R}$ *given by* $\upsilon(x) = \lambda(x)\,\mu(\varphi(x))$ *for each* $x \in M$. *If in addition* φ, ψ *are both harmonic and hence harmonic morphisms the composition* $\varphi \circ \psi$ *is also a harmonic morphism.*

Proof. – Let g, h, k denote the metrics of M, N, P respectively. For $x \in M$, Let $X, Y \in H_x M$, where $H_x M$ denotes the horizontal space with respect to $\psi \circ \varphi$, then

$$\begin{aligned}(\psi \circ \varphi)^* k(X, Y) &= \varphi^* \psi^* k(X, Y)\\ &= \mu^2(\varphi(x))\, \varphi^* h(X, Y)\\ &= \lambda^2(x)\, \mu^2(\varphi(x))\, g(X, Y),\end{aligned}$$

proving the first part of the Lemma.

The tension field $\tau_{\psi \circ \varphi}$ satisfies [11]

$$\begin{aligned}\tau_{\psi \circ \varphi} &= d\psi(\tau_\varphi) + \operatorname{trace} \nabla\, d\psi(d\varphi, d\varphi)\\ &= d\psi(\tau_\varphi) + \lambda^2\, \tau_\psi\,.\end{aligned}$$

Thus $\psi \circ \varphi$ is harmonic if both φ and ψ are, and by the first part of the Lemma the composition is a harmonic morphism.

□

In particular, if $\varphi : M \longrightarrow N$ is a harmonic morphism onto a surface N, and $\psi : N \longrightarrow P$ is a weakly conformal map between surfaces, then ψ is harmonic [9] and the composition $\psi \circ \varphi$ is a harmonic morphism. Thus the notion of a harmonic morphism onto an oriented surface does not depend on the conformal structure of that surface and it makes sense to talk about a harmonic morphism onto a Riemann surface N.

(1.3) LEMMA. – *Let* $\varphi : M \longrightarrow N$ *be a horizontally conformal map with dilation* $\lambda : M \longrightarrow \mathbf{R}$, *such that* $\varphi = \zeta \circ \pi$ *where* $\pi : M \longrightarrow P$ *is a map onto a Riemann surface* P, *and* $\zeta : P \longrightarrow N$ *is a map between Riemann surfaces. Let* k, h *denote the metrics of* P, N *respectively and let* υ_1, υ_2 *denote the (non zero) eigenvalues of* $\pi^* k$ *with respect to* g *and* μ_1, μ_2 *those of* $\zeta^* h$ *with respect to* k. *Then, either*

(i) $v_1 \mu_1 = v_2 \mu_2$, *or*

(ii) $v_1 \mu_2 = v_2 \mu_1$.

Consequently if ζ *is weakly conformal, then* π *is horizontally conformal.*

Proof. – Let U, V be open sets on M, P respectively, such that $\pi(U) \subset V$ and such that v_i is non-zero on U and μ_i is non-zero on V, for $i = 1, 2$.

By simultaneous diagonalization of the symmetric 2-tensors $\pi^* k$ and g, we can choose a horizontal orthonormal basis (e_1, e_2) at each point $x \in U$, such that $d\pi(e_1), d\pi(e_2)$ are orthogonal and of lengths $v_1^{1/2}, v_2^{1/2}$ respectively. Then $d\zeta(d\pi(e_1)), d\zeta(d\pi(e_2))$ are orthogonal and of equal lengths $(v_1 \mu_1)^{1/2}, (v_2 \mu_2)^{1/2}$ or $(v_1 \mu_2)^{1/2}, (v_2 \mu_1)^{1/2}$ respectively, and hence either equation (i) or (ii) holds at points of U.

By continuity one of equations (i) or (ii) holds at points x where $d\varphi(x) = 0$. At such points either $v_1 = v_2 = 0$ or $\mu_1 = \mu_2 = 0$.

□

(1.4) *Example.* – Let $\pi : \mathbf{R}^3 \setminus \{z - axis\} \longrightarrow S^1 \times \mathbf{R}$ be the map $\pi(x, y, z) = (x/\sqrt{(x^2 + y^2)}, y/\sqrt{(x^2 + y^2)}, z)$ onto the cylinder. Then no map $\zeta : S^1 \times \mathbf{R} \longrightarrow N$ between Riemann surfaces can produce a horizontally conformal map

$$\varphi = \zeta \circ \pi : \mathbf{R}^3 \setminus \{z - axis\} \longrightarrow N .$$

For the eigenvalues v_1, v_2 of $\pi^* k$, where k denotes the metric of $S^1 \times \mathbf{R}$, are given by $v_1 = 1/(x^2 + y^2), v_2 = 1$. Thus clearly v_2/v_1 is non-constant along the fibres of π.

Let $\varphi : M \longrightarrow N$ be a harmonic morphism from an open subset M of $\mathbf{R}^m$ onto a Riemann surface N. In order to develop our study further, we will put a number of restrictions on φ.

1) Suppose that $d\varphi_x \neq 0$ for each $x \in M$, and that

2) The fibres of φ are totally geodesic. Thus each connected component of a fibre of φ is part of an $(m - 2)$-plane in $\mathbf{R}^m$.

We note that we can associate an orientation to each connected component of a fibre of φ as follows. For each $x \in \mathrm{M}$ give $\mathrm{H}_x \mathrm{M}$ a natural orientation by requiring that $d\varphi_x|_{\mathrm{HM}} : \mathrm{H}_x \mathrm{M} \longrightarrow \mathrm{T}_{\varphi(x)} \mathrm{N}$ be orientation preserving. Then we require the orientation on $\mathrm{V}_x \mathrm{M}$ together with that on $\mathrm{H}_x \mathrm{M}$ gives the standard orientation on $\mathbf{R}^m$.

3) Let F be a component of a fibre of φ together with its induced orientation. We require that no other oriented fibre component is part of the oriented plane determined by F. (Thus if G is a component of a fibre of φ distinct from F contained in the plane determined by F, the orientation of G must differ from the one induced by F).

We will clarify these conditions with a number of remarks.

a) In general the Gauss map γ defined in Section 2 will not extend over the critical set of a harmonic morphism (c.f. Remark (2.1)).

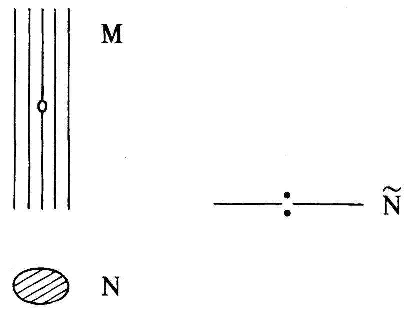
b) The fibres of a harmonic morphism onto a Riemann surface are always minimal [1]. Thus in the case when $m = 3$, the components of the fibres are parts of straight lines in $\mathbf{R}^3$, and are therefore totally geodesic. Thus 2) is always satisfied in this case.

c) The restriction of a harmonic morphism to any open subset is also a harmonic morphism. This allows arbitrary closed sets to be removed from the domain M. We wish to consider, in some sense, the maximal domain associated to a given harmonic morphism. Furthermore, as we will show, Lemma (1.5) which follows will not hold unless condition 3) is satisfied.

Let $\widetilde{\mathrm{N}}$ denote the space of connected components of the fibres of φ. Thus $\widetilde{\mathrm{N}}$ is the quotient space obtained from M under the equivalence relation "$\sim$", where $x \sim y$ if and only if x and y lie in the same connected component of a fibre of φ. Give $\widetilde{\mathrm{N}}$ the quotient topology induced from M. Then φ factors, $\varphi = \zeta \circ \widetilde{\varphi}$, where $\widetilde{\varphi}$ is the natural projection and $\zeta(e) = \varphi(x)$, for each $e \in \widetilde{\mathrm{N}}$, where x is a representative of e.

(1.5) Lemma. – *The topological space* $\widetilde{\mathrm{N}}$ *is Hausdorff.*

Remark. – In general $\widetilde{\mathrm{N}}$ will not be Hausdorff if condition 3) is not satisfied. For example, let $\varphi : \mathbf{R}^3 \setminus \{0\} \longrightarrow \mathbf{R}^2$ be the projection mapping $\varphi(x, y, z) = (x, y)$.



Then the fibre over the origin in $\mathbf{R}^2$ consists of two disconnected components which correspond to two distinct points of $\widetilde{\mathrm{N}}$. No two neighbourhoods of these points in $\widetilde{\mathrm{N}}$ can be disjoint.

Proof (of Lemma). – Let $y_1, y_2 \in \widetilde{\mathrm{N}}$ be distinct points. We will show that y_1, y_2 have disjoint neighbourhoods.

Let $\mathrm{G}(m-2, \mathbf{R}^m)$ denote the Grassmannian of oriented $(m-2)$-planes in $\mathbf{R}^m$, and define $\gamma : \mathrm{M} \longrightarrow \mathrm{G}(m-2, \mathbf{R}^m)$ by assigning to each $x \in \mathrm{M}$, the oriented $(m-2)$-plane $\gamma(x)$ determined by the connected component of the fibre of φ which passes through x.

Define $\Gamma : \mathrm{M} \longrightarrow \mathrm{G}(m-2, \mathbf{R}^m) \times \mathbf{R}^m$ by

$$\Gamma(x) = (\gamma(x), c(x)),$$

where $c(x)$ is the orthogonal projection of the origin in $\mathbf{R}^m$ onto the plane determined by the fibre through x.

Then Γ is smooth and is constant along each fibre component. Let $\mathrm{F}_1, \mathrm{F}_2$ denote the fibre components corresponding to y_1, y_2 respectively. Then by condition 3), $\Gamma(\mathrm{F}_1), \Gamma(\mathrm{F}_2)$ are distinct points in $\mathrm{G}(m-2, \mathbf{R}^m) \times \mathbf{R}^m$. Let $\mathrm{V}_1, \mathrm{V}_2$ be disjoint neighbourhoods of $\Gamma(\mathrm{F}_1), \Gamma(\mathrm{F}_2)$ in $\mathrm{G}(m-2, \mathbf{R}^m) \times \mathbf{R}^m$. Then $\Gamma^{-1}(\mathrm{V}_1), \Gamma^{-1}(\mathrm{V}_2)$ are disjoint neighbourhoods of $\mathrm{F}_1, \mathrm{F}_2$ in M which project to disjoint neighbourhoods of y_1, y_2 in $\widetilde{\mathrm{N}}$. □

(1.6) LEMMA. – *The space $\widetilde{\mathrm{N}}$ can be given the structure of a Riemann surface with respect to which $\zeta : \widetilde{\mathrm{N}} \longrightarrow \mathrm{N}$ is conformal. Hence, with respect to this structure, $\widetilde{\varphi} : \mathrm{M} \longrightarrow \widetilde{\mathrm{N}}$ is a harmonic morphism with connected fibres.*

Proof. – If x is a point of M, let $W \subset M$ be a slice about x. That is W is part of a 2-dimensional plane in M containing x, which is everywhere bounded away from the vertical.

Consider the restriction $\psi = \varphi|_W : W \longrightarrow N$. Then

$$d\psi_x : T_x W \longrightarrow T_{\psi(x)} N$$

is an isomorphism, and by the inverse function theorem, ψ is a local diffeomorphism at x. Let $U \subset W$ be a neighbourhood of x in W over which $\psi : U \longrightarrow \psi(U)$ is a diffeomorphism. Then U is homeomorphic to a neighbourhood $\widetilde{U}$ in $\widetilde{N}$ about the equivalence class e determined by x, and $\widetilde{U}$ inherits its differentiable structure from U. In fact identifying U with an open subset of $\mathbf{R}^2$ gives a local chart about e in $\widetilde{N}$. In this way $\widetilde{N}$ can be given a differentiable structure with respect to which both $\widetilde{\varphi}$ and ζ are smooth maps.

The conformal structure on $\widetilde{N}$ is induced by pulling back the conformal structure on N under ζ. Thus if J is the complex structure on N, we define a complex structure $\widetilde{J}$ on $\widetilde{N}$ by

$$\widetilde{J}v = (d\zeta)^{-1} J(d\zeta(v))$$

at each point $e \in \widetilde{N}$ and for each $v \in T_e\widetilde{N}$. Clearly ζ is holomorphic with respect to the complex structure $\widetilde{J}$.

The last part of the Lemma follows immediately from Lemma (1.3). □

2. Harmonic morphisms and associated holomorphic immersions into Grassmannians.

Let $\varphi : M \longrightarrow N$ be a harmonic morphism from an open subset M of $\mathbf{R}^m$ onto a Riemann surface N, satisfying conditions 1), 2) and 3) of Section 1.

To the map φ we associate the Gauss map

$$\gamma : M \longrightarrow G(m-2, \mathbf{R}^m),$$

into the Grassmannian of oriented $(m-2)$-planes in $\mathbf{R}^m$, where for each $x \in M$, the point $\gamma(x) \in G(m-2, \mathbf{R}^m)$ is determined by the oriented $(m-2)$-plane $V_x M$ translated to the origin in $\mathbf{R}^m$.

(2.1) *Remark.* – In general the Gauss map will not extend over the critical set of a harmonic morphism. For example,

$$\varphi : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

given by $\varphi(z, w) = zw$ is a harmonic morphism with critical set the origin $z = w = 0$. The Gauss map does not extend over C_φ.

However, Bernard, Campbell and Davie have shown that in the case when $m = 3, \gamma$ extends continuously over C_φ [2].

For $y \in \widetilde{N}$, let $F_y = \widetilde{\varphi}^{-1}(y)$ denote the fibre of $\widetilde{\varphi}$ over y. Since F_y is planar, γ is locally constant along F_y. Since F_y is connected, it is contained in a fibre of γ and hence γ factors, $\gamma = \psi \circ \widetilde{\varphi}$,

$$\begin{tikzcd} M \arrow[d, "\widetilde{\varphi}"'] \arrow[dr, "\gamma"] & \\ \widetilde{N} \arrow[r, "\psi"] & G(m-2, \mathbf{R}^m) \end{tikzcd}$$

for some map $\psi : \widetilde{N} \longrightarrow G(m-2, \mathbf{R}^m)$. In fact, for each $y \in \widetilde{N}$, $\psi(y)$ determines the fibre over y as follows.

Locally about y we can lift ψ to a map

$$\widetilde{\psi} : \widetilde{N} \longrightarrow S(m-2, \mathbf{R}^m)$$

into the Stiefel manifold of orthonormal $(m-2)$-planes in $\mathbf{R}^m$. Thus if $\pi : S(m-2, \mathbf{R}^m) \longrightarrow G(m-2, \mathbf{R}^m)$ is the canonical projection given by $\pi(e_3, \ldots, e_m) =$ (the $(m-2)$-plane spanned by $e_3, \ldots, e_m$), then $\psi = \pi \circ \widetilde{\psi}$. Writing

$$\widetilde{\psi}(y) = (e_3(y), \ldots, e_m(y)),$$

the fibre over $y \in \widetilde{N}$ has an expression of the form

$$(s_3, \ldots, s_m) \longrightarrow \sum_{r=3}^{m} s_r e_r(y) + c(y),$$

where $c(y) \in \mathbf{R}^m$ satisfies $\langle e_r(y), c(y) \rangle = 0$ for each $r = 3, \ldots, m$, where $\langle\ ,\ \rangle$ denotes the Euclidean inner product and $s_3, \ldots, s_m$ range over suitable values.

The Grassmannian $G(m-2, \mathbf{R}^m)$ has a natural almost complex structure J defined as follows. For each $p \in G(m-2, \mathbf{R}^m)$, the tangent space $T_p G(m-2, \mathbf{R}^m)$ can be identified with $V_p^* \otimes H_p$, where V_p is the subspace of $\mathbf{R}^m$ determined by p, and H_p is the orthogonal complement of V_p in $\mathbf{R}^m$. Since H_p is 2-dimensional, it has a natural almost complex structure J^H associated with it, given by rotation through $\pi/2$ in H_p (the direction of rotation is well defined in terms of the orientation induced on H_p). If

$$a \in T_p G(m-2, \mathbf{R}^m),$$

we define Ja by $(Ja)(v) = J^H(av)$ for each $v \in V_p$.

The complex structure J is the standard one induced from the canonical identification of $G(m-2, \mathbf{R}^m)$ with the complex quadric hypersurface Q_{m-2} of complex projective space $\mathbf{CP}^{m-1}$ [8]. With respect to this structure $G(m-2, \mathbf{R}^m)$ is a Kähler manifold.

Let $\mathcal{H}$, $\mathcal{V}$ denote the projections onto HM, VM respectively. For simplicity assume that φ has connected fibres and replace $\widetilde{N}$ by N. We may do this on account of Lemma (1.6).

(2.2) LEMMA. – *Let $\varphi : M \longrightarrow N$ be a harmonic morphism with connected fibres satisfying conditions 1), 2) and 3) and let $(e_3, \ldots, e_m)$ denote an orthonormal frame field spanning* VM *about a point $x \in M$, and (Y_1, Y_2) an orthonormal frame field about $\varphi(x)$ on* N. *Then*

i) $\langle d\psi(Y_1)e_r, d\psi(Y_2)e_t\rangle + \langle d\psi(Y_2)e_r, d\psi(Y_1)e_t\rangle = 0$

ii) $\langle d\psi(Y_1)e_r, dc(Y_2)\rangle + \langle d\psi(Y_2)e_r, dc(Y_1)\rangle = 0$

iii) $\langle \mathcal{H}dc(Y_1), \mathcal{H}dc(Y_2)\rangle = 0$,

for all $r, t = 3, \ldots, m$.

Proof. – Taking a representation

$$(s_3, \ldots, s_m) \longrightarrow \sum_{r=3}^{m} s_r e_r + c$$

of the fibres as before. The vector field Y_i $(i = 1, 2)$ lifts to

$$X_i = \sum_{r=3}^{m} s_r de_r(Y_i) + dc(Y_i).$$

Since $\varphi(\Sigma s_r e_r(y) + c(y)) = y$, we see that the horizontal projection $\mathcal{H} X_i$ is the horizontal lift of Y_i. The condition that φ be horizontally conformal is therefore equivalent to

$$\langle \mathcal{H} X_1 , \mathcal{H} X_2 \rangle = 0 \tag{2.3}$$

for all choices of orthonormal frame fields (Y_1 , Y_2). Now

$$\mathcal{H} X_i = \Sigma s_r \ \mathcal{H} de_r (Y_i) + \mathcal{H} dc (Y_i), \quad (i = 1 , 2)$$

thus (2.3) becomes

$$\begin{aligned} 1/2 \sum_{r,t} \ & s_r s_t [\langle \mathcal{H} de_r (Y_1) , \mathcal{H} de_t (Y_2)\rangle \\ & + \langle \mathcal{H} de_r (Y_2) , \mathcal{H} de_t (Y_1) \rangle] \\ & + \sum_r s_r [\langle \mathcal{H} de_r (Y_1) , \mathcal{H} dc(Y_2)\rangle \\ & + \langle \mathcal{H} de_r (Y_2) , \mathcal{H} dc (Y_1)\rangle] \\ & + \langle \mathcal{H} dc (Y_1) , \mathcal{H} dc(Y_2)\rangle = 0 . \end{aligned} \tag{2.4}$$

Equation (2.4) is a polynomial in $s_3 , \ldots , s_m$ whose coefficients are independant of $s_3 , \ldots , s_m$ and valid for all $(s_3 , \ldots , s_m)$ in a certain neighbourhood. Thus each coefficient must vanish.

From the identification of $TG(m-2, \mathbf{R}^m)$ with $V^* \otimes H$, we see that $d\psi(Y_i) : VM \longrightarrow HM$ is given by

$$d\psi(Y_i)\, e_r = \mathcal{H} \ de_r (Y_i) ,$$

for each $i = 1 , 2$; $r = 3 , \ldots , m$. Hence we obtain equations (i), (ii) and (iii).

□

Let J^N denote the complex structure of N. Then ψ *is* $\pm$ *holomorphic* if and only if $J d\psi(Y) = \pm d\psi(J^N Y)$ for each vector field Y over N.

(2.5) PROPOSITION. – *If* $\varphi : M \longrightarrow N$ *is a harmonic morphism with connected fibres satisfying conditions 1), 2) and 3), then the induced mapping* $\psi : N \longrightarrow G(m-2, \mathbf{R}^m)$ *is* $\pm$ *holomorphic.*

Proof. – Consider equation (i) of Lemma (2.2). If we write $Y_1 = Y$ and $Y_2 = J^N Y$ we obtain

(i) $\langle d\psi(Y)\, e_r, d\psi(J^N Y)\, e_t \rangle + \langle d\psi(Y)\, e_t, d\psi(J^N Y)\, e_r \rangle = 0$,

for all vector fields Y, and for all choices $(e_3, \ldots, e_m)$. Hence, for each $r = 3, \ldots, m$

$$\langle d\psi(Y)\, e_r, d\psi(J^N Y)\, e_r \rangle = 0\,, \tag{2.6}$$

that is $d\psi(J^N Y)\, e_r$ is proportional to $J^H d\psi(Y)\, e_r$. If we now let $Y \longrightarrow (Y + J^N Y)/\sqrt{2}$, $J^N Y \longrightarrow (-Y + J^N Y)/\sqrt{2}$, then (2.6) becomes

$$|d\psi(Y)\, e_r|^2 = |d\psi(J^N Y)\, e_r|^2\,. \tag{2.7}$$

Hence

$$d\psi(J^N Y)\, e_r = \pm J^H d\psi(Y)\, e_r\,,$$

and ψ is $\pm$ holomorphic.

□

Henceforth we will assume that ψ is holomorphic. Similar considerations apply to the case when ψ is -holomorphic.

(2.8) Corollary. – *The Gauss map* $\gamma : M \longrightarrow G(m-2, \mathbf{R}^m)$ *is harmonic.*

Proof. – This follows since ψ is $\pm$ holomorphic and $(G(m-2, \mathbf{R}^m), J)$ is Kähler, hence ψ is harmonic [9]. Furthermore φ is a harmonic morphism, so the composition $\gamma = \psi \circ \varphi$ is harmonic.

□

Conversely, suppose we are given a map

$$\psi : N \longrightarrow G(m-2, \mathbf{R}^m)$$

and a map $c : N \longrightarrow \mathbf{R}^m$, where N is a Riemann surface. Then for each $y \in N$, the plane

$$(s_3, \ldots, s_m) \longrightarrow \sum_r s_r\, e_r(y) + c(y) \tag{2.9}$$

is determined, where $(e_3(y), \ldots, e_m(y))$ is a local lift of ψ. We may attempt to form the union of these planes, by removing points where fibres intersect. More precisely, suppose

$$\varphi : \mathrm{M} \longrightarrow \mathrm{N}, \mathrm{M} \subset \mathbf{R}^m,$$

is a smooth mapping such that the fibre of φ over y is given by (2.9), where $s_3, \ldots, s_m$ range over suitable values.

(2.10) PROPOSITION. – *If* $\psi : \mathrm{N} \longrightarrow \mathrm{G}(m-2, \mathbf{R}^m)$ *is non-constant holomorphic and* $c : \mathrm{N} \longrightarrow \mathbf{R}^m$ *satisfies* $\langle c(y), e_r(y)\rangle = 0$ *for each* $r = 3, \ldots, m$ *as well as equation (iii), for all choices of frame fields* $(e_3, \ldots, e_m)$ *representing* ψ. *Then* φ *is horizontally conformal, and hence a harmonic morphism.*

Proof. – If ψ is holomorphic, then for each $r = 3, \ldots, m$ and for each frame field $(\mathrm{Y}_1, \mathrm{Y}_2)$ where $\mathrm{Y}_2 = \mathrm{J}^{\mathrm{N}} \mathrm{Y}_1$,

$$\begin{aligned} \langle d\psi(\mathrm{Y}_1) e_r, d\psi(\mathrm{Y}_2) e_r\rangle &= \langle d\psi(\mathrm{Y}_1) e_r, d\psi(\mathrm{J}^{\mathrm{N}} \mathrm{Y}_1) e_r\rangle \\ &= \langle d\psi(\mathrm{Y}_1) e_r, \mathrm{J}^{\mathrm{H}} d\psi(\mathrm{Y}_1) e_r\rangle \\ &= 0. \end{aligned}$$

Now letting $e_r \longrightarrow (e_r + e_t)/\sqrt{2}$ we obtain

$$\langle d\psi(\mathrm{Y}_1) e_r, d\psi(\mathrm{Y}_2) e_t\rangle + \langle d\psi(\mathrm{Y}_1) e_t, d\psi(\mathrm{Y}_2) e_r\rangle = 0,$$

that is equation (i) is satisfied.

Assume r is chosen such that $d\psi(\mathrm{Y}_1) e_r \neq 0$. Choose a horizontal frame (e_1, e_2) such that e_1 is proportional to $d\psi(\mathrm{Y}_1) e_r$ and e_2 is proportional to $d\psi(\mathrm{Y}_2) e_r$. Then

$$\mathcal{H}\, dc(\mathrm{Y}_i) = \langle e_1, dc(\mathrm{Y}_i)\rangle e_1 + \langle e_2, dc(\mathrm{Y}_i)\rangle e_2.$$

From equation (iii)

$$\langle e_1, dc(\mathrm{Y}_1)\rangle \langle e_1, dc(\mathrm{Y}_2)\rangle + \langle e_2, dc(\mathrm{Y}_1)\rangle \langle e_2, dc(\mathrm{Y}_2)\rangle = 0.$$

Also

$$\begin{aligned} \langle e_1, dc(\mathrm{Y}_1)\rangle &= \langle \mathrm{J}^{\mathrm{H}} e_1, \mathrm{J}^{\mathrm{H}} \mathcal{H}\, dc(\mathrm{Y}_1)\rangle \\ &= \langle e_2, dc(\mathrm{J}^{\mathrm{H}} \mathrm{Y}_1)\rangle \text{ (again using (iii))} \\ &= \langle e_2, dc(\mathrm{Y}_2)\rangle, \end{aligned}$$

and (ii) is also satisfied. Thus φ is horizontally conformal and since the fibres are minimal, φ is a harmonic morphism.

□

3. The classification of harmonic morphisms from an open subset of $\mathbf{R}^3$ onto a Riemann surface.

Let $\varphi : M \longrightarrow N$ be a harmonic morphism from a connected open subset of $\mathbf{R}^3$ onto a Riemann surface N, satisfying conditions 1), 2) and 3) of Section 1.

From Lemma (1.6) we have the commutative diagram

$$\begin{tikzcd} M \arrow[d, "\widetilde{\varphi}"'] \arrow[dr, "\gamma"] & \\ \widetilde{N} \arrow[r, "\psi"'] & G(1, \mathbf{R}^3) \approx S^2 , \end{tikzcd} \tag{3.1}$$

where $\widetilde{N}$ is the space of connected components of the fibres of φ.

Suppose first that ψ is constant, then the fibres of φ consist of parallel straight lines. Assume these lines are extended maximally throughout $\mathbf{R}^3$, and let E denote a plane orthogonal to these lines. Then the fibres of φ intersect E in an open set U, and φ factors, $\varphi = \xi \circ \pi$, where $\pi : M \longrightarrow U$ is the projection map, and $\xi : U \longrightarrow N$ is conformal. Otherwise said, φ is simply the composition of the projection mapping onto the plane, followed by a conformal mapping. Henceforth we will assume that ψ is non-constant.

By Proposition (2.5) ψ is holomorphic with respect to the natural complex structure on S^2. Since ψ is non-constant, it has rank 2, except possibly at isolated points. The composition $\gamma = \psi \circ \widetilde{\varphi}$ is horizontally conformal and harmonic and hence a harmonic morphism onto its image $P \subset S^2$. If $y \in P$, then $\psi^{-1}(y)$ consists of discrete points $x_1, x_2, \ldots,$ in $\widetilde{N}$, and the fibre of γ over y is given by the components

$$s_i \longrightarrow s_i y + c(x_i),$$

for $i = 1, 2, \ldots,$ where $\langle c(x_i), y \rangle = 0$ and s_i range over suitable values. Thus, for each i, the vector $c(x_i)$ can be considered as lying in $T_y S^2$. Otherwise said c is a section of the bundle $\psi^{-1} TS^2$.

We wish to reduce the problem of classifying φ to studying the harmonic morphism γ. To this end we regard c as a multiple valued vector field over P, where, for each $y \in P$, $c(y)$ can take on any one of the values $c(x_i)$, for $x_i \in \psi^{-1}(y)$.

Suppose for the moment that on a neighbourhood $W \subset P$, we have made a smooth choice of one of the possible values of c.

We now have the situation where $\tilde{\varphi}$ is replaced by γ in (3.1), and the problem is to determine $c : P \longrightarrow \mathbf{R}^3$ satisfying equations (ii) and (iii) of Lemma (2.2). In fact, since $y = e_3$, equation ii) takes the form

ii) $\langle Y_1, dc(Y_2)\rangle + \langle Y_2, dc(Y_1)\rangle = 0$.

Letting $Y_1 \longrightarrow (Y_1 + Y_2)/\sqrt{2}$, $Y_2 \longrightarrow (Y_1 - Y_2)/\sqrt{2}$, this becomes

ii)′ $\langle Y_1, dc(Y_1)\rangle = \langle Y_2, dc(Y_2)\rangle$.

Now $\mathcal{H}\, dc(Y_i) = \langle Y_1, dc(Y_i)\rangle\, Y_1 + \langle Y_2, dc(Y_i)\rangle\, Y_2$, for $i = 1,2$. Thus equation (iii) can be written as

$$\langle Y_1, dc(Y_1)\rangle \langle Y_1, dc(Y_2)\rangle + \langle Y_2, dc(Y_1)\rangle \langle Y_2, dc(Y_2)\rangle = 0.$$

But this is implied by (ii) and (ii)′. We therefore wish to solve (ii) and (ii)′ for c.

We recall that a vector field X over a Riemannian manifold M with metric g, is *conformal* if and only if

$$1/2\,[g(\nabla_V X, W) + g(\nabla_W X, V)] = ag(V, W), \tag{3.2}$$

for all vector fields V, W over M, and for some function

$$a : M \longrightarrow \mathbf{R}.$$

(3.3) LEMMA. – *A vector field* c *over* $P \subset S^2$ *satisfies (ii) and (ii)′ if and only if* c *is a conformal vector field on* P.

Proof. – Let g denote the metric of S^2, and let (Y_1, Y_2) be an orthonormal frame field defined on some neighbourhood contained in P.

From equation (3.2), c is conformal if and only if

$$1/2\,[g(dc(Y_1), Y_2) + g(dc(Y_2), Y_2)] = 0$$
$$g(dc(Y_1), Y_1) = a$$
$$g(dc(Y_2), Y_2) = a,$$

for some function $a : \mathrm{P} \longrightarrow \mathbf{R}$. But these equations are equivalent to (ii) and (ii)$'$. □

If c is a conformal vector field over a region $\mathrm{P} \subset \mathrm{S}^2$, then under stereographic projection from $\mathrm{S}^2 \setminus \{\text{North pole}\} \longrightarrow \mathbf{C}$, c is identified with a conformal vector field s on a region Q of $\mathbf{C}$. At each point $z \in \mathrm{Q}$, $s(z) \in \mathrm{T}_z\,\mathbf{C} \approx \mathbf{C}$. Thus s can equivalently be regarded as a function $s : \mathrm{Q} \longrightarrow \mathbf{C}$. It is easy to see that s is a conformal vector field if and only if the corresponding function is analytic or conjugate analytic. Indeed multiple valued analytic functions can be considered as multiple valued conformal vector fields. We therefore have a correspondence between conformal vector fields c on regions of S^2 and analytic or conjugate analytic functions s on regions of $\mathbf{C}$. Provided s has suitable behaviour at infinity, the corresponding conformal vector field on S^2 will extend over the North pole.

We propose to simplify our discussion by modifying condition 3) of Section 1 to the following condition :

3$'$) *that no two components of fibres are parallel in an oriented sense.*

An immediate consequence of this condition is that ψ is injective and c must be single valued over $\mathrm{P} \subset \mathrm{S}^2$.

Remark. – This will not be true in general if condition 3$'$) is not satisfied. In Example (4.5) we will consider the 2-valued analytic function $s(z) = \sqrt{z}$. On a small disc about the origin in $\mathbf{C}$, s corresponds to a 2-valued conformal vector field c in a neighbourhood U of the South pole on S^2. This yields a harmonic morphism

$$\varphi : \mathrm{M} \longrightarrow \mathrm{U}$$

with corresponding fibre map $\psi : \mathrm{U} \longrightarrow \mathrm{S}^2$ having the form

$$\psi(z) = z^2$$

with respect to a suitable local chart.

(3.4) Lemma. – *Under the assumption 3$'$), the map ψ is a biholomorphic equivalence between $\widetilde{\mathrm{N}}$ and* P.

Proof. – Since ψ is injective, its inverse $\psi^{-1} : \mathrm{P} \longrightarrow \widetilde{\mathrm{N}}$ is defined and holomorphic.

Relaxing the condition that the fibres of φ be connected, we can summarize the above results in the following:

(3.5) THEOREM. – *If* $\varphi : \mathrm{M} \longrightarrow \mathrm{N}$ *is a harmonic morphism from an open subset* M *of* $\mathbf{R}^3$ *onto a Riemann surface* N, *which is a submersion everywhere and satisfies condition 3′), then* φ *is the composition* $\varphi = \zeta \circ \gamma$, *where* $\gamma : \mathrm{M} \longrightarrow \mathrm{P}$ *is a harmonic morphism onto* $\mathrm{P} \subset \mathrm{S}^2$ *and* $\zeta : \mathrm{P} \longrightarrow \mathrm{N}$ *is a conformal map between Riemann surfaces. Furthermore the fibres of* γ *have the form*

$$s \longrightarrow sy + c(y)$$

for each $y \in \mathrm{P}$, *where* s *ranges over suitable values and* c *is a conformal vector field over* P.

Conversely, every conformal vector field c *over an open subset* P *of* S^2 *yields a (not necessarily unique) harmonic morphism* γ *as above.*

4. Examples of harmonic morphisms from domains in $\mathbf{R}^3$.

In order to determine the structure of the sets $\mathrm{M} \subset \mathbf{R}^3$ and $\mathrm{P} \subset \mathrm{S}^2$, we must consider points of $\mathbf{R}^3$ where various fibres intersect. The boundary of such regions is the envelope of a family of lines in $\mathbf{R}^3$. For the moment we consider the more general situation of $(m\text{-}2)$-planes in $\mathbf{R}^m$.

Let $\varphi : \mathrm{M} \longrightarrow \mathrm{N}$ be a harmonic morphism from an open subset of $\mathbf{R}^m$ onto a Riemann surface N, satisfying conditions 1), 2) and 3′). Let $\gamma : \mathrm{M} \longrightarrow \mathrm{G}(m-2, \mathbf{R}^m)$ denote the Gauss map and write $\mathrm{P} = \gamma(\mathrm{M}) \subset \mathrm{G}(m-2, \mathbf{R}^m)$. Since φ is a harmonic morphism and ψ is holomorphic, the composition $\gamma = \psi \circ \varphi$ is horizontally conformal onto its image P, and we can associate a dilation

$$\mu : \mathrm{M} \longrightarrow \mathbf{R} \quad \text{with} \quad \gamma.$$

(4.1) LEMMA. – *If we extend the fibres of* γ *throughout* $\mathbf{R}^m$, *then points which lie on the envelope of fibres of* γ *are points where the dilation* μ *takes on the value infinity.*

Proof. – Let $u \longrightarrow y(u)$ be a curve in P such that the tangent vector $Y = y'(u)$ has unit length. In a suitably small neighbourhood express points y of P in the form $e_3(y) \wedge \ldots \wedge e_m(y)$. Then the fibre of γ over y is given by

$$(s_3, \ldots, s_m) \longrightarrow \sum_r s_r e_r(y) + c(y),$$

and the vector Y lifts to

$$X = \sum_r s_r de_r(Y) + dc(Y).$$

The horizontal projection $\mathcal{H}X$ is given by

$$\mathcal{H}X = \sum_r s_r de_r(Y) + dc(Y) - \sum_{r,a} s_r \langle de_r(Y), e_a \rangle e_a - \sum_a \langle dc(Y), e_a \rangle e_a,$$

and the dilation $\mu : M \longrightarrow \mathbf{R}$ of γ is determined by

$$\mu^2 = 1/|\mathcal{H}X|^2.$$

We claim that $\mathcal{H}X = 0$ along the envelope of the planes

$$(s_3, \ldots, s_m) \longrightarrow \sum_r s_r e_r(y(u)) + c(y(u)),$$

for a curve $y(u)$ in P.

The envelope is given as the curve determined by the points of intersection of infinitesimally nearby planes.

Consider a nearby point u_1 to u, and write

$$y = y(u), y_1 = y(u_1).$$

The two corresponding fibres intersect when

$$\sum_r s_r e_r(y) + c(y) = \sum_r t_r e_r(y_1) + c(y_1).$$

Then

$$\frac{\sum s_r e_r(y) - \sum s_r e_r(y_1) + \sum s_r e_r(y_1) - \sum t_r e_r(y_1)}{u - u_1} + \frac{c(y) - c(y_1)}{u - u_1} = 0.$$

Now

$$t_r = \langle \sum t_r e_r(y_1) + c(y_1), e_r(y_1)\rangle$$
$$= \langle \sum s_r e_r(y) + c(y), e_r(y_1)\rangle$$

and

$$s_r = \langle \sum s_r e_r(y) + c(y), e_r(y)\rangle,$$

hence

$$\frac{\sum_r s_r (e_r(y) - e_r(y_1)) + \sum_r \left\langle \sum_a s_a e_a(y) + c(y), e_r(y_1) - e_r(y)\right\rangle e_r(y_1)}{u - u_1} + \frac{c(y) - c(y_1)}{u - u_1} = 0.$$

Now let $u \longrightarrow u_1$, and we see that

$$\sum s_r de_r(\mathrm{Y}) + \sum \langle \sum s_a e_a(y) + c(y), de_r(\mathrm{Y})\rangle e_r(y) + dc(\mathrm{Y}) = 0$$

along the envelope.

Recall that
$\langle e_r(y), c(y)\rangle = 0$, so that $\langle de_r(\mathrm{Y}), c(y)\rangle + \langle e_r(y), dc(\mathrm{Y})\rangle = 0$.
Also $\langle e_r(y), e_a(y)\rangle = \delta_{ra}$ implies that

$$\langle de_r(\mathrm{Y}), e_a(y)\rangle + \langle e_r(y), de_a(\mathrm{Y})\rangle = 0.$$

Hence

$$\sum_r s_r de_r(\mathrm{Y}) + dc(\mathrm{Y}) - \sum_r \left\langle \sum_a s_a de_a(\mathrm{Y}), e_r(y) \right\rangle e_r(y)$$

$$- \sum_r \langle e_r(y), dc(\mathrm{Y}) \rangle e_r(y) = 0$$

along the envelope. But this is precisely the horizontal projection $\mathcal{H}\mathrm{X}$.

□

The Lie algebra of conformal vector fields over S^2 forms a vector space $\mathcal{L}$ over $\mathbf{C}$ with $\dim_{\mathbf{C}} \mathcal{L} = 3$. They arise from conformal vector fields on the complex plane $\mathbf{C}$ under the inverse of stereographic projection. A basis for $\mathcal{L}$ in terms of vector fields on $\mathbf{C}$ is given by

$$\mathrm{V}_1(z) = z$$

$$\mathrm{V}_2(z) = 1$$

$$\mathrm{V}_3(z) = z^2,$$

for each $z \in \mathbf{C}$.

Express each point $y \in \mathrm{S}^2$ in the form $y = (\cos t, \sin t\, e^{i\theta})$ and consider the orthonormal frame field $(\mathrm{Y}_1, \mathrm{Y}_2)$, where $\mathrm{Y}_1 = (0, ie^{i\theta})$ and $\mathrm{Y}_2 = (-\sin t, \cos t\, e^{i\theta})$, defined at all points of S^2 where $t \neq 0, \pi$. Regard $\mathcal{L}$ as a vector space over $\mathbf{R}$ with $\dim_{\mathbf{R}} \mathcal{L} = 6$. Then a basis for $\mathcal{L}$ is determined from $\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3$ and is given explicitly by the vector fields

$$\mathrm{U}_1 = \sin t\, \mathrm{Y}_1$$

$$\mathrm{U}_2 = \sin t\, \mathrm{Y}_2$$

$$\mathrm{U}_3 = 1/2\,(\cos t - 1)(\sin\theta\, \mathrm{Y}_1 + \cos\theta\, \mathrm{Y}_2)$$

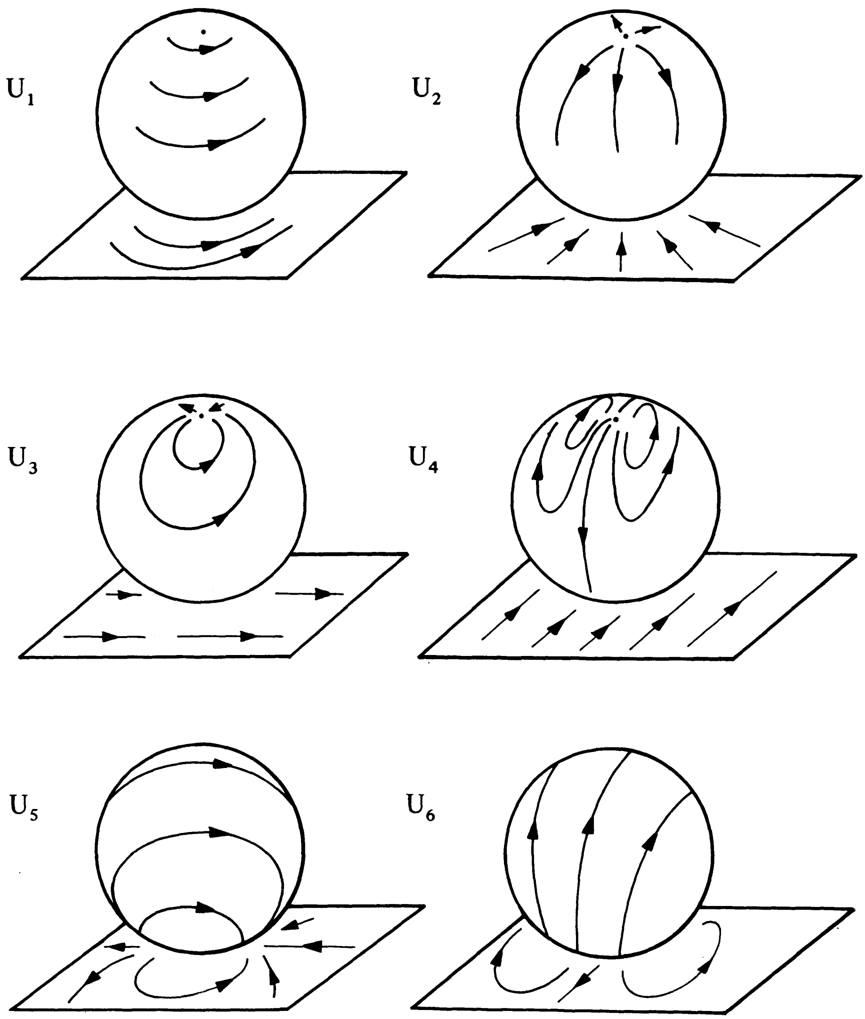
$$\mathrm{U}_4 = 1/2\,(\cos t - 1)(-\cos\theta\, \mathrm{Y}_1 + \sin\theta\, \mathrm{Y}_2)$$

$$\mathrm{U}_5 = 1/2\,(1 + \cos t)(-\sin\theta\, \mathrm{Y}_1 + \cos\theta\, \mathrm{Y}_2)$$

$$\mathrm{U}_6 = 1/2\,(1 + \cos t)(\cos\theta\, \mathrm{Y}_1 + \sin\theta\, \mathrm{Y}_2).$$

Three of these vector fields are of course obtained from the other three by rotation through $\pi/2$ in the tangent space. Since $\mathrm{Y}_1, \mathrm{Y}_2$

are not defined at $t = 0, \pi$, we define U_3, U_4 at $t = 0$ to be given by $U_3 = (0, 1, 0), U_4 = (0, 0, 1)$ and U_5, U_6 at $t = \pi$ to be given by $U_5 = (0, 1, 0), U_6 = (0, 0, 1)$.



There are several cases to consider in determining the structure of M and P. We will consider each case $c = U_r, r = 1, \ldots, 6$ separately. We will use Lemma (4.1) to determine the boundary of regions where fibres intersect.

(4.2) *Example* $c = U_1$. Express each $y \in S^2$ in the form

$$y = (\cos t\,, \sin t\, e^{i\theta}).$$

Then the fibre over y is given by

$$s \longrightarrow s\,(\cos t\,, \sin t\, e^{i\theta}) + \sin t\,(0\,, ie^{i\theta}).$$

The vector $Y_2 = (-\sin t\,, \cos t\, e^{i\theta})$ lifts to

$$X_2 = sY_2 + \cos t\,(0\,, ie^{i\theta}),$$

hence $\mathcal{H}X_2 = X_2$ and

$$|\mathcal{H}X_2|^2 = s^2 + \cos^2 t.$$

Thus the dilation $\lambda : M \longrightarrow \mathbf{R}$ is given by

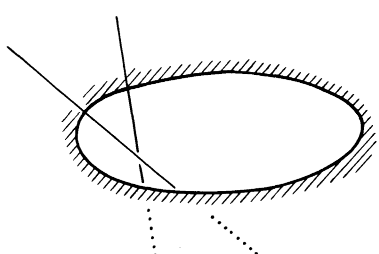
$$\lambda^2 = 1/(s^2 + \cos^2 t).$$

Now $s^2 + \cos^2 t$ is zero when $s = 0$ and $\cos t = 0$, that is when $t = \pi/2$. These are the points $(0, -\sin\theta\,, \cos\theta)$, $\theta \in [0\,, 2\,\pi)$, that is the unit circle in the plane $x = 0$.

The domain M is given by $M = \mathbf{R}^3 \backslash K$, where

$$K = \{(x\,, y\,, z) \in \mathbf{R}^3 \mid x = 0,\, y^2 + z^2 \geqslant 1\}.$$

The image P of $\gamma : M \longrightarrow S^2$ is the upper hemisphere (or lower one) $x^2 + y^2 + z^2 = 1$, $x > 0$. This is to avoid multiple valuedness of γ, which we will discuss again in Section 5. The fibres of γ are lines which twist through the disc $x = 0$, $y^2 + z^2 < 1$ and fill out M.



The harmonic morphism γ has a functional expression as follows.

Consider the family of ellipsoids given by the equations

$$\frac{x^2}{s^2} + \frac{y^2 + z^2}{s^2 + 1} = 1,$$

where $s > 0$ denotes a variable parameter. Note that in the limit as s tends to 0, the ellipsoid tends to the disc of radius 1 in the plane $x = 0$. Let $\mathcal{R}$ be the upper half space of $\mathbf{R}^3$ determined by $x > 0$. Then for each point $(x, y, z) \in \mathcal{R}$, there is a unique $s > 0$ such that (x, y, z) lies on the ellipsoid. Define $\pi : \mathcal{R} \longrightarrow \mathrm{P}$ by

$$\pi(x, y, z) = \left(\frac{x}{s}, \frac{sy + z}{s^2 + 1}, \frac{-y + sz}{s^2 + 1}\right).$$

Then π is a harmonic morphism with dilation μ given by

$$\mu^2(x, y, z) = s^2/(s^4 + x^2),$$

and π extends to the harmonic morphism γ determined above.

(4.3) *Example* $c = \mathrm{U}_2$.

The fibre over $y \in \mathrm{S}^2$ has the form

$$s \longrightarrow s(\cos t, \sin t\, e^{i\theta}) + \sin t(-\sin t, \cos t\, e^{i\theta}),$$

and the vector Y_2 lifts to

$$\mathrm{X}_2 = s\mathrm{Y}_2 + \cos t\, \mathrm{Y}_2 - \sin t\, y.$$

Thus

$$\mathcal{H}\mathrm{X}_2 = s\,\mathrm{Y}_2 + \cos t\, \mathrm{Y}_2,$$

and

$$\lambda^2 = 1/(s + \cos t)^2.$$

This becomes infinite when $s = -\cos t$, that is the point $(-1, 0, 0)$. The set M is given by $\mathrm{M} = \mathbf{R}^3\backslash\{(-1, 0, 0)\}$, and $\mathrm{P} = \mathrm{S}^2$. The fibres are half-line segments to the point $(-1, 0, 0)$. In fact γ has a functional expression

$$\gamma(x, y, z) = (x + 1, y, z)/[(x + 1)^2 + y^2 + z^2]^{1/2}.$$

(4.4) *Example* $c = \mathrm{U}_3$.

The fibre over $y \in \mathrm{S}^2$ has the form

$$s \longrightarrow s(\cos t, \sin t\, e^{i\theta}) + 1/2(\cos t - 1)(\sin\theta\, \mathrm{Y}_1 + \cos\theta\, \mathrm{Y}_2).$$

As before we compute the dilation λ, and

$$1/\lambda^2 = (s - (1/2)\sin t \cos\theta)^2 + (1/4)\sin^2 t\, \sin^2\theta.$$

This is zero when $\sin t = 0$ and $s = 0$, giving the point $(0,0,0)$ and $(0,-1,0)$. It is also zero when $\sin\theta = 0$ and $s = (1/2)\sin t \cos\theta$.

If $\theta = 0$, then $s = (1/2)\sin t$, giving the curve

$$1/2\,(\sin t\,,\,1-\cos t\,,\,0)\,.$$

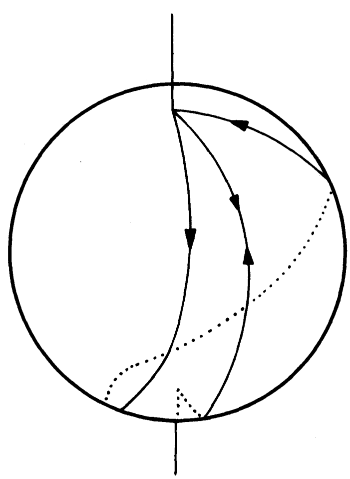
This is the circle of radius $1/2$ in the (x,y)-plane, centre $(0,1/2,0)$. If $\theta = \pi$, then $s = -(1/2)\sin t$, and we obtain the same curve traversed in the opposite direction. The set K to be removed is the exterior to this circle, $K = \{(x,y,z) \in \mathbf{R}^3 \mid z = 0,\ x^2 + (y-1/2)^2 \geqslant 1/4\}$. The image P of γ is a hemisphere on one side of the great circle $\theta = 0, \pi$. The fibres twist through the interior of the circle.

The cases $c = U_4, U_5$ and U_6 are obtained from $c = U_3$ by performing an isometry of S^2. The associated harmonic morphisms are identical up to an isometry of $\mathbf{R}^3$.

(4.5) *Example.* – Let $s : \mathbf{C} \longrightarrow \mathbf{C}$ be the 2-valued analytic function given by $s(z) = \sqrt{z}$. Under the inverse of stereographic projection this corresponds to the 2-valued conformal vector field on S^2 given by

$$c = -2\,r^{1/2}\sin^2(t/2)\,[\sin(\theta/2)\,Y_1 + \cos(\theta/2)\,Y_2],$$

where $r = \cot(t/2)$, $\theta \in [0, 4\pi)$.



The vector field Y_2 has horizontal lift

$$\mathcal{H}X_2 = sY_2 - \left[-\frac{1}{r^{1/2}} + 2\,r^{1/2}\sin\frac{t}{2}\cos\frac{t}{2}\right]\left[\sin\frac{\theta}{2}\,Y_1 + \cos\frac{\theta}{2}\,Y_2\right].$$

The coefficient of Y_1 is zero when $\sin(\theta/2) = 0$, i.e. $\theta = 0, 2\pi$, or when $1 - r\sin t = 0$. That is $t = \pi/2$. If $\theta = 0, 2\pi$ the coefficient of Y_2 is

$$s \mp \left[-\frac{1}{r^{1/2}} + 2\, r^{1/2} \sin\frac{t}{2} \cos\frac{t}{2} \right],$$

which must be zero along envelope points. When $\theta = 0$ this gives the curve in $\mathbf{R}^3$

$$(1/r^{1/2}, 0, 0), \qquad t \in [0, \pi].$$

This is the positive x-axis. When $\theta = 2\pi$, we obtain the negative x-axis

$$(-1/r^{1/2}, 0, 0), \qquad t \in [0, \pi].$$

If $t = \pi/2$, then the coefficient of Y_2 is zero provided $s = 0$. This corresponds to the closed curve in $\mathbf{R}^3$ given by

$$(\cos(\theta/2), \sin(\theta/2)\sin\theta, -\sin(\theta/2)\cos\theta), \ \ \theta \in [0, 4\pi),$$

which can equally be described as one of the components of the intersection of the unit sphere

$$x^2 + y^2 + z^2 = 1,$$

and the surface

$$(1 - x^2)(2x^2 - 1)^2 = z^2.$$

It is a question of some interest as to what a maximal domain $M \subset \mathbf{R}^3$ is for this harmonic morphism. The image P will be one of the two connected regions of S^2 separated by the equator $t = \pi/2$ with the half-circle $\theta = 0$ removed.

We can compute the fibre map ψ described in Section 2. So

$$\begin{tikzcd} M \arrow[d, "\gamma"'] & \\ P \arrow[r, "\psi"] & S^2 \end{tikzcd}$$

With respect to coordinate charts given by stereographic projection, locally about the South pole ψ is the mapping $\psi : \mathbf{C} \longrightarrow \mathbf{C}$ given by $\psi(z) = z^2$. Thus, referring back to Lemma (3.4) we see that in general we cannot expect ψ to be injective.

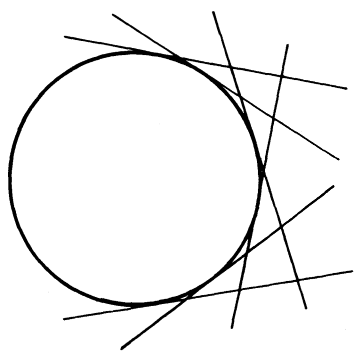
5. Generalized analytic functions.

In this section we show how it is possible to regard some of the examples of the last section as multiple valued maps. In such a case we can take copies of the domain and glue them together in such a way that we obtain a harmonic morphism from a 3-dimensional manifold (possibly with singularities) onto the whole Riemann sphere S^2. This is similar to the way in which the Riemann surfaces of multiple valued analytic functions are constructed [19].

Consider Example (4.2) of the last section. Here the fibre over $y \in S^2$ has an expression

$$s \longrightarrow sy + c(y),$$

where $c = U_1$. The envelope of intersecting fibres is the circle $x = 0$, $y^2 + z^2 = 1$.

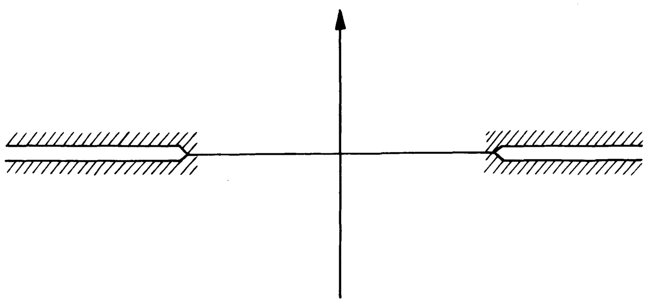


The set K to be removed is given by

$$K = \{(x, y, z) \in \mathbf{R}^3 \mid x = 0,\ y^2 + z^2 \geqslant 1\}.$$

Then $M = \mathbf{R}^3 \backslash K$, and M is filled out by the fibres over the upper hemisphere. In fact each point of M lies on two fibres, one being the fibre over a point of the upper hemisphere and the other being the fibre over a point of the lower hemisphere. In this sense we can regard $\gamma : M \longrightarrow S^2$ as a multiple valued function.

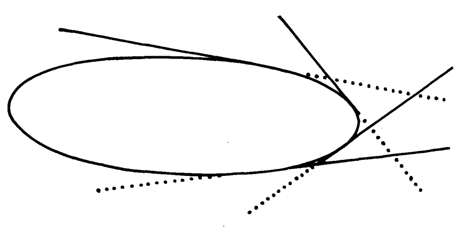
Suppose we now take $\mathbf{R}^3$, and cut it along the set K. The diagram below is a cross section in the (x, y)-plane representing this procedure.



The points $(0, y, z), y^2 + z^2 > 1$ can now be regarded as lying on one of two sheets.

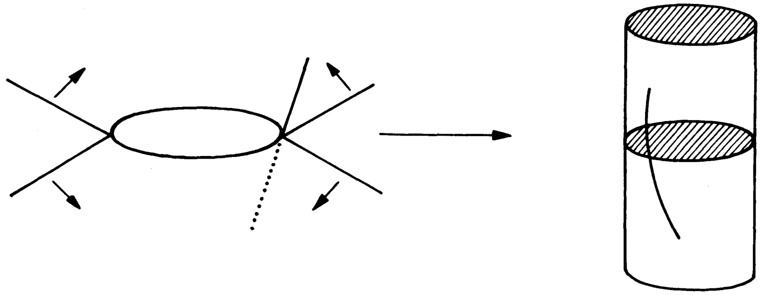
Consider the fibres which lie over points of the equatorial circle $C = \{(x, y, z) \in S^2 \mid x = 0\} \subset S^2$. In the original construction these fibres intersect in points exterior to the envelope and are all tangent to the envelope, the point of tangency being given by $s = 0$.

Suppose now, in our cut manifold, we say the fibres lie on the lower sheet for $s < 0$, and on the upper sheet for $s > 0$. Thus the fibres pass from one sheet to the other. Notice now that distinct fibres never intersect.

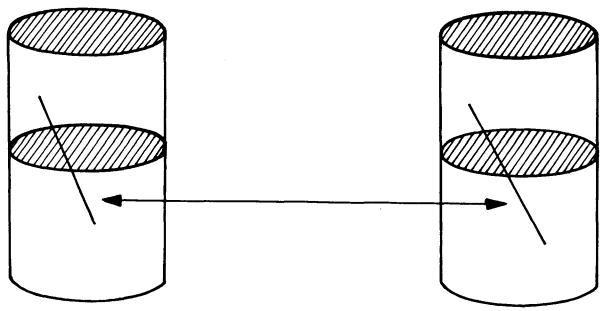


Now take two copies of the cut manifold and glue these together along the edges created by the cuts. We do this in such a way as to identify fibres over points of the circle C.

This is best pictured by opening the cut manifold out into an infinite solid cylinder.



Take two copies and identify



It is clear that after identification we obtain a manifold $\widetilde{M}$ which is homeomorphic to $S^2 \times \mathbf{R}$. The map $\gamma : M \longrightarrow P$ can now be extended in a continuous way to a map $\widetilde{\gamma} : \widetilde{M} \longrightarrow S^2$. This is defined by mapping points x of the boundary of each solid cylinder (both boundaries are of course identified) onto the appropriate point of the equatorial circle C in S^2 determined by the fibre x lies on. Points in the interior of one cylinder are then mapped onto the upper hemisphere, while points in the interior of the other cylinder are mapped onto the lower hemisphere.

The space $\widetilde{M}$ is a C^0-manifold, and both $\widetilde{M}$ and $\widetilde{\gamma}$ are smooth apart from points corresponding to the circle $x = 0 , y^2 + z^2 = 1$ in $\mathbf{R}^3$.

Returning to Jacobi's original theme, consider the functional expression for the harmonic morphism of Example (4.2); we see that the harmonic morphism π is related to the coordinates (x , y , z) by an "algebraic equation" as follows.

Compose π with stereographic projection from $S^2 \backslash \{(1 , 0 , 0)\}$ onto the complex plane. We obtain the map σ, given by

$$\sigma(x, y, z) = \frac{sy + z}{(s^2 + 1)(1 - x/s)} + i\,\frac{-y + sz}{(s^2 + 1)(1 - x/s)},$$

where we recall x, y, z and s are related by the equation

$$\frac{x^2}{s^2} + \frac{y^2 + z^2}{s^2 + 1} = 1.$$

If we now perform the following change of coordinates:

$$\tilde{x} = x/s, \qquad w = (y + iz)/s + i),$$

we see that σ, $\tilde{x}$ and w are related by the equation

$$\sigma^2 \overline{w} - 2\sigma + w = 0.$$

The fact that there is no dependance on $\tilde{x}$ reflects the fact that $\tilde{x}$ is constant along the fibres of σ.

This equation is a polynomial in σ of degree 2, and hence its solution $\sigma = \sigma(w)$ is to be considered as a two valued mapping.

6. Harmonic morphisms from an open subset of S^3 onto a Riemann surface.

Let $\upsilon : Q \longrightarrow N$ be a harmonic morphism from an open subset Q of S^3 onto a Riemann surface N. Then υ determines a harmonic morphism $\varphi : M \longrightarrow N$ with totally geodesic fibres, where M is open in $\mathbf{R}^4$, as follows.

Regarding Q as a subset of $\mathbf{R}^4$ lying in the unit 3-sphere, let M be the set $\mathbf{R}^+ Q = \{\lambda x \in \mathbf{R}^4 \mid x \in Q, \lambda \in \mathbf{R}, \lambda > 0\}$, and let $\pi : M \longrightarrow Q$ be defined by $\pi(x) = x/|x|$ for all $x \in M$. Define $\varphi : M \longrightarrow N$ to be the composition $\varphi = \upsilon \circ \pi$.

Since the fibres of υ are minimal, they form parts of geodesics in S^3. Thus the fibres of φ form parts of 2-planes in $\mathbf{R}^4$ which all extend through the origin in $\mathbf{R}^4$. From Lemma (1.1) the composition $\varphi = \upsilon \circ \pi$ is horizontally conformal. Hence φ is a harmonic morphism with totally geodesic fibres. In the notation of Section 2, the vector field c which determines the positions of the fibres is identically zero.

Conversely, suppose we are given a harmonic morphism $\varphi : M \longrightarrow N$, where M is open in $\mathbf{R}^4$, and φ has totally geodesic fibres such that the associated vector field c is identically zero. Then define v to be the restriction $\varphi|_{S^3} : Q \longrightarrow N$, where $Q = M \cap S^3$ is open in S^3.

Since the fibres of φ are parts of 2-planes passing through the origin in $\mathbf{R}^4$, the fibres of v are parts of geodesics in S^3 and hence are minimal. Furthermore, for each $x \in Q$, the horizontal space $H_x M$ of φ satisfies $H_x M \subset T_x S^3$, so that v is horizontally conformal and hence a harmonic morphism.

We therefore have a correspondence between harmonic morphisms $v : Q \longrightarrow N, Q$ open in S^3, and harmonic morphisms $\varphi : M \longrightarrow N$ where M is open in $\mathbf{R}^4$ and φ has fibres which are parts of 2-planes passing through the origin in $\mathbf{R}^4$. Assuming that φ (and hence v) satisfies conditions 1), 2) and 3) of Section 1, we have a commutative diagram.

$$\begin{tikzcd}
M \arrow[d, "\tilde{\varphi}"'] \arrow[dr, "\gamma"] \arrow[dd, bend right=60, "\varphi"'] & \\
\tilde{N} \arrow[r, "\psi"] \arrow[d, "\zeta"'] & P \subset G(2, \mathbf{R}^4) \\
N &
\end{tikzcd}$$

Since $c \equiv 0$, condition 3′) of Section 3 is also satisfied and ψ is injective. Otherwise said, ψ is a holomorphic curve in the Grassmannian $G(2, \mathbf{R}^4)$.

Conversely, given a holomorphic curve $\psi : N \longrightarrow G(2, \mathbf{R}^4)$, we can construct a (not necessarily unique) open set $M \subset \mathbf{R}^4$ and a harmonic morphism $\varphi : M \longrightarrow N$ as in Section 2. Thus M is the union of the planes $\psi(x), x \in N$, with intersection points removed. We summarize this in the following:

(6.1) THEOREM. – *If* $v : Q \longrightarrow N$ *is a harmonic morphism from an open subset* Q *of* S^3 *onto a Riemann surface* N, *with empty critical set and subject to condition 3), then* v *is the composition,* $v = \eta \circ \rho$, *where* $\rho : Q \longrightarrow P$ *is a harmonic morphism onto a holomorphic curve* $P \subset G(2, \mathbf{R}^4)$ *and* $\eta : P \longrightarrow N$ *is a weakly conformal map between Riemann surfaces. Furthermore, a holomorphic curve* P *in the Grassmannian* $G(2, \mathbf{R}^4)$ *determines a harmonic morphism* $\rho : Q \longrightarrow P$ *for some (not necessarily unique) open subset* Q *of* S^3.

(6.2) *Remark.* – It has been pointed out to me by J. Jost that there are holomorphic curves of arbitrary high genus in the Grassmannian $G(2, \mathbf{R}^4)$. This follows from [16, Chapter 5, Corollary 2.18 and Chapter 2, Theorem 8.18]. For the Grassmannian $G(2, \mathbf{R}^4)$ can be regarded as a ruled complex surface in $\mathbf{CP}^3$.

The Grassmannian $G(2, \mathbf{R}^4)$ of oriented 2-planes in $\mathbf{R}^4$ can be described as the set $\{\alpha \in \Lambda^2 \mathbf{R}^4 \mid |\alpha|^2 = 1 \text{ and } \alpha \wedge \alpha = 0\}$. The de Rham-Hodge star operator acts on $\Lambda^2 \mathbf{R}^4$ as an involution with eigenspaces $\Lambda_\pm$ corresponding to the eigenvalues ± 1. Thus $\Lambda^2 \mathbf{R}^4$ decomposes, $\Lambda^2 \mathbf{R}^4 = \Lambda_+ \oplus \Lambda_-$ as the direct sum of 3-dimensional Euclidean subspaces. Each $\alpha \in G(2, \mathbf{R}^4)$ therefore has a decomposition $\alpha = \alpha_+ + \alpha_-$, where $\alpha_\pm \in S_\pm$, the spheres of radius $1/\sqrt{2}$ in $\Lambda_\pm$. We therefore have a diffeomorphism $h : G(2, \mathbf{R}^4) \longrightarrow S_+ \times S_-$ given by $h(\alpha) = (\alpha_+, \alpha_-)$. In fact h is an isometric biholomorphic equivalence with respect to the natural Kähler structure on each space (see, for example [10]).

If $(e_i)_{i=1,\dots,4}$ is an orthonormal basis for $\mathbf{R}^4$ and we set $e_{ij} = e_i \wedge e_j$, then $(e_{12} \pm e_{34})/\sqrt{2}, (e_{13} \mp e_{24})/\sqrt{2}, (e_{14} \pm e_{23})/\sqrt{2}$ is an orthonormal basis for $\Lambda_\pm$.

We consider some examples illustrating the correspondence between the holomorphic immersion ψ and the harmonic morphism v.

(6.3) *Example.* – Let $\psi : S^2 \longrightarrow S^2 \times S^2$ be given by $\psi(x) = (x, (1,0,0))$, for $x \in S^2$. Then writing

$$x = (x_1, x_2, x_3) \in \mathbf{R}^3, \psi(x)$$

$\psi(x)$ decomposes as

$$\psi(x) = x_1(e_{12} + e_{34})/\sqrt{2} + x_2(e_{13} - e_{24})/\sqrt{2} + x_3(e_{14} + e_{23})/\sqrt{2} + (e_{12} - e_{34})/\sqrt{2}.$$

Express points of S^2 in the form $x = (\cos 2t, \sin 2t\, e^{i\theta})$, where $t \in [0, \pi/2], \theta \in [0, 2\pi)$. Then ψ has an expression

$$\psi(x) = f_3 \wedge f_4/\sqrt{2},$$

where $f_3 = (\cos t\, e^{i\xi}, \sin t\, e^{i\eta}), f_4 = (\cos t\, ie^{i\xi}, -\sin t\, ie^{i\eta})$ and ξ, η satisfy $\xi + \eta = \theta + \pi/2$.

The Hopf map $v: S^3 \longrightarrow S^2$ is given by

$$v(\cos t\, e^{i\xi}, \sin t\, e^{i\eta}) = (\cos 2t, \sin 2t\, e^{i(\xi+\eta)}).$$

The fibres are given by $\xi + \eta =$ constant, and the vector field tangent to these fibres is given by $(\cos t\, ie^{i\xi}, -\sin t\, ie^{i\eta})$. Thus the 2-plane in $\mathbf{R}^4$ determined by the fibres of the induced map $\varphi: \mathbf{R}^4 \setminus \{0\} \longrightarrow S^2$ is none other than $f_3 \wedge f_4$. Hence, up to an isometry of S^3, the immersion ψ corresponds to the Hopf map $v: S^3 \longrightarrow S^2$.

(6.4) *Example.* – Let $\psi: S^2 \longrightarrow S^2 \times S^2$ be given by

$$\psi(x) = (x, x),$$

for $x \in S^2$. Then, writing $x = (x_1, x_2, x_3)$

$$\begin{aligned}\sqrt{2}\,\psi(x) &= x_1(e_{12} + e_{34}) + x_2(e_{13} - e_{24}) + x_3(e_{14} + e_{23}) \\ &\quad + x_1(e_{12} - e_{34}) + x_2(e_{13} + e_{24}) + x_3(e_{14} + e_{23}) \\ &= 2e_1 \wedge (x_1 e_2 + x_2 e_3 + x_3 e_4).\end{aligned}$$

These planes all intersect along the e_1 – axis. The corresponding harmonic morphism $v: S^3 \setminus \{(\pm 1, 0, 0, 0)\} \longrightarrow S^2$ is given by

$$v(\cos t, \sin t\, x) = x,$$

where $x \in S^2$ and $t \in (0, \pi)$.

7. Harmonic morphisms defined globally on S^3.

Let $v: S^3 \longrightarrow N$ be a harmonic morphism onto a Riemann surface N which is a submersion everywhere. As before we let $\varphi: M \longrightarrow N$ denote the corresponding harmonic morphism from $M = \mathbf{R}^4 \setminus \{0\}$. Then φ will also be a submersion at each point of M. We first of all show that condition 3) of Section 1 is satisfied.

(7.1) LEMMA. – *Let* F *be a component of a fibre of* υ *together with its induced orientation, and let* Π *denote the corresponding oriented great circle in* S^3 *determined by* F. *Then* $\Pi = F$ *as oriented submanifolds.*

Proof. – Each fibre is closed in S^3, hence each connected component F is closed in S^3 and hence F is closed in the great circle Π containing it. Since υ is a submersion, F is a manifold and must also be open in Π and so must equal Π.

□

We can therefore form the Riemann surface $\widetilde{N}$ of the connected components of the fibres of φ, and we have the diagram

$$\begin{tikzcd}
M \arrow[d, "\widetilde{\varphi}"'] \arrow[dd, bend right=60, "\varphi"'] & \\
\widetilde{N} \arrow[r, "\psi"] \arrow[d, "\zeta"'] & G(2, \mathbf{R}^4) \\
N &
\end{tikzcd}$$

where ψ is an injective mapping. Note that $\widetilde{N}$ is compact by the compactness of S^3.

Conversely, given a holomorphic curve $\psi : \widetilde{N} \longrightarrow G(2, \mathbf{R}^4)$, we have associated a harmonic morphism $\widetilde{\varphi} : M \longrightarrow N$ as described in Section 2. Let $\widetilde{\upsilon} : Q \longrightarrow \widetilde{N}$ denote the restriction of $\widetilde{\varphi}$ to S^3. The fibre of $\widetilde{\upsilon}$ over each $y \in \widetilde{N}$, is given as the intersection of the plane $\psi(y)$ with S^3, with intersection points (with other fibres) removed. If the domain Q is to be the whole of S^3, then a necessary condition is that distinct fibres do not intersect in S^3. Thus if $x, y \in \widetilde{N}, x \neq y$, then the corresponding planes $\psi(x), \psi(y)$ must intersect only in the origin of $\mathbf{R}^4$. This is satisfied if and only if $\psi(x) \wedge \psi(y) \neq 0$.

Identifying $G(2, \mathbf{R}^4)$ with $S^2 \times S^2$, let $\pi_i : S^2 \times S^2 \longrightarrow S^2$

denote projection onto the i'th factor $(i = 1, 2)$, and write $\psi_i = \pi_i \circ \psi$. Thus $\psi = (\psi_1, \psi_2)$. Define the function

$$\beta : \widetilde{\mathrm{N}} \times \widetilde{\mathrm{N}} \longrightarrow \mathbf{R}$$

by

$$\beta(x, y) = \langle \psi_1(x), \psi_1(y) \rangle - \langle \psi_2(x), \psi_2(y) \rangle,$$

for each $x, y \in \widetilde{\mathrm{N}}$.

(7.2) LEMMA. – *For each* $x, y \in \widetilde{\mathrm{N}}$, *the exterior product* $\psi(x) \wedge \psi(y)$ *is equal to* $\beta(x, y)$.

Proof. – Since $\psi_1(x), \psi_1(y)$ are both self-dual 2-forms and $\psi_2(x), \psi_2(y)$ are both anti self-dual 2-forms, the result follows.

□

(7.3) COROLLARY. – *For each pair of distinct points* $x, y \in \widetilde{\mathrm{N}}$, *suppose that* $\beta(x, y)$ *is non-zero. Then it is either always negative or always positive.*

Proof. – This follows from the connectedness of $\widetilde{\mathrm{N}}$ and the continuity of ψ_1 and ψ_2.

□

Henceforth we will assume that $\beta(x, y) < 0$ for each pair of distinct points $x, y \in \widetilde{\mathrm{N}}$.

(7.4) LEMMA. – *The maps* $\psi_1, \psi_2 : \widetilde{\mathrm{N}} \longrightarrow \mathrm{S}^2$ *are both holomorphic.*

Proof. – By Proposition (2.5), ψ_1, ψ_2 are the composition of holomorphic mappings and hence holomorphic. □

(7.5) LEMMA. – *The map* ψ_2 *is constant, whereas* ψ_1 *is a conformal diffeomorphism. In particular* $\widetilde{\mathrm{N}}$ *is conformally equivalent to* S^2.

Proof. – First note that ψ_1 is non-constant. For otherwise

$$\beta(x, y) = 1 - \langle \psi_2(x), \psi_2(y) \rangle \geqslant 0.$$

Suppose ψ_2 is also non-constant, then by the compactness of $\widetilde{N}$ and since ψ_2 is holomorphic, ψ_2 covers S^2. Thus there exists $x, y \in \widetilde{N}, x \neq y$, with $\psi_2(x) = (1, 0, 0)$ and $\psi_2(y) = (-1, 0, 0)$ and then

$$\beta(x, y) = \langle \psi_1(x), \psi_2(y) \rangle + 1 \geqslant 0,$$

a contradiction. Thus ψ_2 is constant.

Suppose ψ_1 has a branch point at $x_0 \in \widetilde{N}$. Then locally about x_0, there exist $x, y, x \neq y$, such that $\psi_1(x) = \psi_1(y)$, and

$$\beta(x, y) = 0,$$

again a contradiction. Thus ψ_1 has no branch points and is a covering map. But then $\widetilde{N}$ is conformally equivalent to S^2 and ψ_1 is a conformal diffeomorphism. □

(7.6) THEOREM. – *If $\upsilon : S^3 \longrightarrow N$ is a submersive harmonic morphism from the Euclidean 3-sphere onto a Riemann surface N, then υ is the composition $\upsilon = \eta \circ \rho$ where $\rho : S^3 \longrightarrow S^2$ is the Hopf fibration, and $\eta : S^2 \longrightarrow N$ is a conformal mapping.*

Proof. – We have seen from Example (6.3) that the Hopf fibration arises from the holomorphic curve $\widetilde{\psi} : S^2 \longrightarrow S^2 \times S^2$ given by $\widetilde{\psi}(x) = (x, (1, 0, 0))$ for each $x \in S^2$. It therefore suffices to show that if $\upsilon : S^3 \longrightarrow S^2$ is a harmonic morphism with connected fibres, then the corresponding holomorphic curve

$$\psi : S^2 \longrightarrow S^2 \times S^2$$

is given by $\widetilde{\psi}$.

By choosing suitable coordinates on $\widetilde{N} = S^2$, we may assume from Lemma (7.5) that ψ_1 is the identity map $\psi_1(x) = x$. Also by an appropriate isometry of S^3 we may assume from the same Lemma that $\psi_2(x) = (1, 0, 0)$, hence proving the theorem. □

BIBLIOGRAPHY

[1] P. Baird & J. Eells, A conservation law for harmonic maps, Geometry Symp. Utrecht 1980, *Springer Notes,* 894 (1981), 1-25.

[2] A. Bernard, E.A. Campbell & A.M. Davie, Brownian motion and generalized analytic and inner functions, *Ann. Inst. Fourier,* 29-1 (1979), 207-228.

[3] M. Brelot, Lectures on potential theory, *Tata Institute of Fundamental Research, Bombay* (1960).

[4] E. Calabi, Minimal immersions of surfaces in Euclidan spheres, *J. Diff. Geom.,* 1 (1967), 111-125.

[5] C. Constantinescu & A. Cornea, Compactifications of harmonic spaces, *Nagoya Math. J.,* 25 (1965), 1-57.

[6] A.M. Din & W.J. Zakrzewki, General classical solutions in the CP^{n-1} model, *Nucl. Phys. B.,* 174 (1980), 397-406.

[7] A.M. Din & W.J. Zakrzewski, Properties of the general classical CP^{n-1} model, *Phys. Lett.,* 95B (1980), 419-422.

[8] J. Eells, Gauss maps of surfaces, *Perspectives in Mathematics, Birkhäuser Verlag, Basel* (1984), 111-129.

[9] J. Eells & L. Lemaire, A report on harmonic maps, *Bull. London Math. Soc.,* 10 (1978), 1-68.

[10] J. Eells & L. Lemaire, On the construction of harmonic and holomorphic maps between surfaces, *Math. Ann.,* 252 (1980), 27-52.

[11] J. Eells & J.H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.,* 86 (1964), 109-160.

[12] J. Eells & J.C. Wood, Harmonic maps from surfaces to complex projective spaces, *Advances in Math.,* 49 (1983), 217-263.

[13] O. Forster, Lectures on Riemann Surfaces, *Springer* (1981).

[14] B. Fuglede, Harmonic morphisms between Riemannian manifolds, *Ann. Inst. Fourier,* 28-2 (1978), 107-144.

[15] V. Glaser & R. Stora, Regular solutions of the CP^n models and further generalizations, *CERN* (1980).

[16] R. HARTSHORNE, Algebraic Geometry, *Springer* (1980).

[17] T. ISHIHARA, A mapping of Riemannian manifolds which preserves harmonic functions, *J. Math. Kyoto Univ.*, 19 (1979), 215-229.

[18] C.G.J. JACOBI, Uber Eine Particuläre Lösung der Partiellen Differential Gleichung $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$, *Crelle Journal für die reine und angewandte Mathematik*, 36 (1847), 113-134.

[19] C.L. SIEGEL, Topics in complex function theory I, *Wiley* (1969).

[20] J.C. WOOD, Harmonic morphisms, foliations and Gauss maps, *Contemporary Mathematics*, Vol. 49 (1986), 145-183.

Manuscrit reçu le 27 décembre 1984
révisé le 4 septembre 1986.

Paul BAIRD,
University of Melbourne
Department of Mathematics
Parkville, Victoria 3052 (Australia).