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ANALYTIC DISKS WITH BOUNDARIES IN A MAXIMAL REAL SUBMANIFOLD OF $\mathbf{C}^2$

by Franc FORSTNERIC (*)

Introduction.

A real submanifold M of $\mathbf{C}^n$ of class $\mathbf{C}^1$ is called *totally real* if for all p in M the tangent space $T_p M$ contains no nontrivial complex subspace. A totally real submanifold $M \subset \mathbf{C}^n$ of real dimension n is called *maximal real.*

Let $\Delta = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$ be the open unit disk, $\overline{\Delta}$ the closed unit disk, and $b\Delta$ the unit circle $\{|\zeta| = 1\}$. A nonconstant continuous map $F : \overline{\Delta} \longrightarrow \mathbf{C}^n$ that is holomorphic on Δ and maps the circle $b\Delta$ to M is called an *analytic disk with boundary in* M. Assuming that M is sufficiently smooth and the pullback $F^{0*}TM$ of the tangent bundle of M by the restriction of F^0 to $b\Delta$ is a trivial bundle on $b\Delta$, we shall describe the behavior of analytic disks with boundary in M near an initial immersed analytic disk F^0 in terms of an integer $\mathrm{Ind}_{F^0} M$, called the *index of* M *along* F^0 (Definition 1), that can be calculated from a parametrization of M along the curve $F^0(b\Delta)$.

We recall the known results. A special case of the problem we are treating here is the classical *Riemann boundary problem* : Find a continuous function $f = u + iv : \overline{\Delta} \longrightarrow \mathbf{C}$ which is holomorphic on Δ and whose boundary values on $b\Delta$ satisfy a linear equation $a(z)u(z) - b(z)v(z) = c(z)$, where a, b and c are continuous real functions on $b\Delta$ and $a^2 + b^2 > 0$. For each fixed $z \in b\Delta$ this equation determines a real line $l(z)$ in the complex plane. If

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f is a solution of this problem, the corresponding map

$$F(z) = (z, f(z))$$

is an analytic disk in $\mathbf{C}^2$ with boundary in the totally real submanifold $M = \bigcup_{z \in b\Delta} \{z\} \times l(z)$ of $\mathbf{C}^2$. This problem was first solved by Hilbert [18] using Fredholm integral equations. It can also be solved using the Cauchy type integrals; see [23, Chapter 16]. The number of solutions depends on an integer called the *index* which measures the rotation of the line $l(z)$ in $\mathbf{C}$ as the point z traces the circle $b\Delta$ once in the positive direction. If we replace the right hand side of the equation by a more general expression $\lambda c(z, u, v)$, we obtain the nonlinear Riemann problem whose solutions for small values of the parameter $\lambda \in \mathbf{R}$ were obtained by Schauder's fixed point theorem [23, pp. 591-601].

In a different direction Bishop [7] solved a functional equation by method of convergent iterations in order to construct families of disks with boundaries in a given C-R submanifold of $\mathbf{C}^n$ of dimension bigger than n. These disks are small in the sense that they can be shrunk to a point in M. In the case when dim $M = n$ he constructed such disks in a neighborhood of a point $p \in M$ for which the tangent space $T_p M$ contains a nontrivial complex subspace. Several authors improved his results by working in different spaces of functions. Later Hill and Taiani [19] solved Bishop's equation using the implicit function theorem in Banach spaces. Very precise results in this direction were obtained by Kenig and Webster [20, 21] and Bogges and Pitts [8]. No such small disks can exist if M is a totally real submanifold of $\mathbf{C}^n$ since M is then locally polynomially convex [17, p. 301]. However, if M is a compact maximal real submanifold of $\mathbf{C}^n$, the polynomial hull $\hat{M}$ has topological dimension at least $n + 1$ [2, 9, 14], and the question arises whether the hull contains any images of analytic disks or, more generally, analytic varieties with boundaries in M. To our knowledge no counterexample is known.

Alexander [1] and Bedford [4] proved that the families of disks that form the topological boundary of the polydisk Δ^n are stable under small perturbations of the distinguished boundary $(b\Delta)^n$ by solving a generalized Bishop's equation using the implicit function theorem. A similar method was used by Bedford and Gaveau, who defined the *index* of an analytic disk [6, p. 992] and constructed a family of nearby disks in a special case in order to compute the envelope of holomorphy of certain two-spheres in $\mathbf{C}^2$. In [5, Theorem 2.1] Bedford proved a weaker version of our Theorem 1 in the case when the index of the initial disk F^0 with boundary in $M \subset \mathbf{C}^2$ is non negative, and he showed by an example that the initial disk may disappear under small perturbations of M in the case when the index is negative. Lempert also constructed a family of neighboring disks in a particular case [22].

In this paper we obtain a rather complete description of analytic disks with boundary in $M \subset \mathbf{C}^2$ near an original immersed disk F^0. Our main tool is the implicit function theorem in Banach spaces, and our method is similar to the one used in [4, 5, 6]. We first define the index of a maximal real submanifold $M \subset \mathbf{C}^n$ around any closed curve in M (Definition 1). In Section 2 we show that the index only depends on the homology class of the curve in M and it induces a homomorphism $\mathrm{H}_1 M \longrightarrow \mathbf{Z}$ when M is orientable. Our index differs by one from the one used by Bedford and Gaveau in [5, 6]. Suppose that F^0 is an immersed analytic disk with boundary in a submanifold $M \subset \mathbf{C}^2$ of the form (1.1). If the index m of M around the curve $F^0(b\Delta)$ is at least one, the analytic disks with boundary in M that are close to F^0 in some Lipschitz space form a $(2m+2)$-parameter family that is stable under small smooth perturbations of M (Theorem 1). If $m = 1$, these disks form a Levi flat hypersurface Σ with boundary in M (Theorem 3). If $m \geqslant 2$, the disks fill an open subset of $\mathbf{C}^2$. If on the other hand $m \leqslant 0$, every disk with boundary in M that is sufficiently close to F^0 in a Lipschitz space is just a reparametrization of the initial disk. Moreover, there is a one parameter family $M_s, s \geqslant 0$, of deformations of M such that no disks with boundary in M_s for $s > 0$ are close to F^0 (Theorem 2).

As an application of our method we obtain a regularity theorem for immersed Levi flat hypersurfaces Σ in $\mathbf{C}^2$ with boundary in a maximal real submanifold M in $\mathbf{C}^2$. If M is of class $\mathbf{C}^k$, $k \geqslant 4$, then Σ is of class $\left[\frac{k-1}{2}\right]$ (Corollary 5). We also prove a strong uniqueness result for an embedded Levi flat hypersurface (Theorem 6).

The paper is organized as follows. In Section 1 we state our results. In Section 2 we analyze the properties of the index homomorphism. In Section 3 we obtain a normal form of the tangent bundle of M along the boundary of an analytic disk. In Sections 4 through 7 we prove the main results. Section 8 contains five examples.

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1. Statement of the results.

Let M be a $\mathbf{C}^1$ maximal real submanifold of $\mathbf{C}^n$ and let $F : b\Delta \longrightarrow M$ be a closed path in M such that the pullback $F^* \mathrm{T}M$ of the tangent bundle of M is a trivial bundle. Thus, there exist continuous maps $X_1, X_2, \ldots, X_n : b\Delta \longrightarrow \mathbf{C}^n$ such that, for each $\zeta \in b\Delta$, the vectors $X_1(\zeta), \ldots, X_n(\zeta)$ form a basis of the tangent space $\mathrm{T}_{F(\zeta)} M$. Let $d(\zeta)$ be the determinant of the complex $n \times n$ matrix $X(\zeta) = (X_1(\zeta), \ldots, X_n(\zeta))$. Since M is totally real at the point $F(\zeta)$, the vectors $X_1(\zeta), \ldots, X_n(\zeta)$ are linearly independent over $\mathbf{C}$ whence $d(\zeta) \neq 0$. The nonvanishing function $d : b\Delta \longrightarrow \mathbf{C} \setminus \{0\}$ has a well-defined winding number $\mathrm{W}(d)$ around the origin 0. If $Y = (Y_1, \ldots, Y_n)$ is another n-tuple of maps from $b\Delta$ to $\mathbf{C}^n$ such that, for each $\zeta \in b\Delta$, the vectors $Y_1(\zeta), \ldots, Y_n(\zeta)$ form a basis of the tangent space $\mathrm{T}_{F(\zeta)} M$, there is a map $A : b\Delta \longrightarrow \mathrm{GL}(n, \mathbf{R})$ such that $Y = AX$ on $b\Delta$. Hence $\det Y = \det A \, \det X$ and

$$\mathrm{W}(\det X) = \mathrm{W}(\det A) + \mathrm{W}(\det X).$$

Since $\det A$ is real-valued and nonvanishing on $b\Delta$, its winding number equals 0 whence $\det Y$ and $\det X$ have equal winding numbers. This justifies the following definition:

DEFINITION 1. – *If M, F and d are as above, then the winding number $\mathrm{W}(d) \in \mathbf{Z}$ is called the index of M along F and is denoted by $\mathrm{Ind}_F M$.*

If the pullback $F^* \mathrm{T}M$ is not trivial, it is isomorphic to a direct sum of a trivial bundle of rank $n-1$ and a Möbius band $\mathfrak{M} \longrightarrow b\Delta$ [25, p. 134]. The second possibility occurs if and only if M is not orientable along the curve $F : b\Delta \longrightarrow M$. If M is orientable and totally real, then the index is defined for every closed curve in M. In Section 2 we shall show that the index $\mathrm{Ind}_F M$ only depends on the homology class of F in $\mathrm{H}_1 M$. Moreover, we shall extend the definition of index to maximal real immersed submanifolds $\iota : M^n \longrightarrow \mathbf{C}^n$ that are not necessarily orientable such that the index is a homomorphism $I_\iota : \mathrm{H}_1 M \longrightarrow 1/2\,\mathbf{Z}$ depending only on the regular homotopy class of the totally real immersion ι.

In the case when M^2 is a compact orientable $\mathbf{C}^1$ submanifold of $\mathbf{C}^2$ there is a close connection between the index homomorphism and the topology of M. (See Theorem 9 in Section 2.)

Assume now that M is a maximal real submanifold of $\mathbf{C}^2$ of class $\mathbf{C}^k$, $k \geqslant 2$, and let $F^0 : \bar{\Delta} \longrightarrow \mathbf{C}^2$ be an immersed analytic disk with boundary in M. (F^0 is a $\mathbf{C}^1$ immersion on $\bar{\Delta}$.) We shall assume that there is an open neighborhood U of $F^0(b\Delta)$ in M with trivial tangent bundle $\mathrm{T}U$. This implies in particular that the index $\mathrm{Ind}_{F^0} M$ is an integer. Since M is totally real, the normal bundle $\mathrm{N}U$ is isomorphis to the tangent bundle $\mathrm{T}U$ whence $\mathrm{N}U$ is also trivial. Therefore there is a small neighborhood V of $F^0(b\Delta)$ in $\mathbf{C}^2$ such that $M \cap V$ is a *transversal intersection*, i.e., there are real valued $\mathbf{C}^k$ functions r_1^0, r_2^0 on V satisfying

$$M \cap V = \{z \in V \mid r_1^0(z) = r_2^0(z) = 0\}\ ,\quad dr_1^0 \wedge dr_2^0 \neq 0. \tag{1.1}$$

Denote by $\mathbf{C}^k(V)$ the Banach space of real-valued $\mathbf{C}^k$ functions on V with the standard norm

$$\|r\| = \sum_{|\alpha| \leqslant k} \sup\{|\mathrm{D}^\alpha r(z)| : z \in V\}\ ,$$

where the differentiation is with respect to the real coordinates on $\mathbf{C}^2$. If $r = (r_1, r_2) \in \mathbf{C}^k(V)^2$ is sufficiently close to r^0 in this norm, the set

$$M_r = \{z \in V \mid r_1(z) = r_2(z) = 0\} \tag{1.2}$$

is a maximal real $\mathbf{C}^k$ submanifold of V that is close to $M_{r^0} = M$ in the $\mathbf{C}^k$ sense. Moreover, every sufficiently small $\mathbf{C}^k$ perturbation of M near $F^0(b\Delta)$ is of this form.

Let $\mathbf{C}^{1/2}(b\Delta)$ be the Banach algebra of real-valued functions on $b\Delta$ with finite Lipschitz norm of exponent $1/2$:

$$\|f\|_{1/2} = \sup_{\theta \in \mathbf{R}} |f(e^{i\theta})| + \sup_{\theta,\tau \in \mathbf{R}} \frac{|f(e^{i\theta}) - f(e^{i\tau})|}{|\theta - \tau|^{1/2}}. \tag{1.3}$$

Let $\mathbf{C}^{1/2}_{\mathbf{C}}(b\Delta)$ be the algebra of complex-valued functions with finite norm (1.3). Denote by $\mathcal{A}^{1/2}$ the subset of all functions in $\mathbf{C}^{1/2}_{\mathbf{C}}(b\Delta)$ which extend to holomorphic functions on Δ. The space $\mathcal{A}^{1/2}$ is a Banach algebra in the topology induced from $\mathbf{C}^{1/2}_{\mathbf{C}}(b\Delta)$. Each $f \in \mathcal{A}^{1/2}$ extends to a holomorphic function on Δ that satisfies the Lipschitz condition of order $1/2$ on $\bar{\Delta}$ [15, p. 74].

Our main results are the following two theorems:

THEOREM 1. – *Let $M \subset \mathbf{C}^2$ be a maximal real submanifold of $\mathbf{C}^2$ of the form (1.1) and of class $\mathbf{C}^k, k \geqslant 2$, and let $F^0 : \bar{\Delta} \longrightarrow \mathbf{C}^2$ be an immersed analytic disk with boundary in M. If the index $\mathrm{Ind}_{F^0} M = m$ is at least one, then there is an open neighborhood $B = B_1 \times B_2$ of the point $(r,t) = (r^0, 0)$ in the Banach space $\mathbf{C}^k(V)^2 \times \mathbf{R}^{2m+2}$ and a map*

$$F = (F_1, F_2) : B \longrightarrow (\mathcal{A}^{1/2})^2$$

of class $\mathbf{C}^{k-1}$ satisfying:

(a) $F(r^0, 0) = F^0$,

(b) $F(r,t)(b\Delta) \subset M_r$ for all $(r,t) \in B_1 \times B_2$,

(c) for each $r \in B_1$ and $t_1, t_2 \in B_2, t_1 \neq t_2$, the analytic disks $F(r,t_1)$ and $F(r,t_2)$ are distinct as maps from $b\Delta$ to $\mathbf{C}^2$, and

(d) if $F^* \in (\mathcal{A}^{1/2})^2$ is any analytic disk sufficiently close to F^0 in the $\mathbf{C}^{1/2}$ norm such that $F^*(b\Delta) \subset M_r$ for some $r \in B_1$, then $F^* = F(r,t)$ for some $t \in B_2$.

The map $\sigma : B \times \bar{\Delta} \longrightarrow \mathbf{C}^2$ defined by $\sigma(r,t,\zeta) = F(r,t)(\zeta)$ is of class $\mathbf{C}^l$, where l is the integer part of $\dfrac{k-1}{2}$. If $m \geqslant 2$, there is a nonempty open subset $E \subset \mathbf{C}^2$ contained in the union $\bigcup_{t \in B_2} F(r,t)(\Delta)$ for each $r \in B_1$.

THEOREM 2. – *Let* $M \subset \mathbf{C}^2$ *be a maximal real* C^4 *submanifold of* $\mathbf{C}^2$ *of the form (1.1) and let* $F^0 : \overline{\Delta} \longrightarrow \mathbf{C}^2$ *be an immersed analytic disk with boundary in* M. *If* $\operatorname{Ind}_{F^0} M \leqslant 0$, *then there is an open neighborhood* Ω *of* F^0 *in* $(\mathcal{A}^{1/2})^2$ *such that the only disks* $F \in \Omega$ *with boundary in* M *are those of the form* $F^0 \circ \phi$, *where* ϕ *in an automorphism of the unit disk. If* F^0 *is an embedding, there is a path* $r(s)$ *in* $\mathbf{C}^4(V)^2$ *starting at* $r(0) = r^0$ *such that there is no disk* $F \in \Omega$ *with boundary in* $M_{r(s)}$ *for* $s > 0$.

The smoothness hypothesis on M in Theorems 1 and 2 can be weakened somewhat. Several remarks are appropriate at this point.

Remark 1. – Our choice of the spaces $\mathbf{C}^{1/2}$ and $\mathcal{A}^{1/2}$ is not very important. If $s \leqslant k - 2$ is an integer and $0 < \epsilon < 1$, then Theorem 1 holds if we replace the space $\mathbf{C}^{1/2}(b\Delta)$ by the space $\mathbf{C}^{s,\epsilon}(b\Delta)$ of functions on $b\Delta$ whose derivatives of order s satisfy the Lipschitz condition of order ϵ. The corresponding map $F : \mathbf{C}^k(V)^2 \times \mathbf{R}^{2m+2} \longrightarrow (\mathcal{A}^{s,\epsilon})^2$ is of class $\mathbf{C}^{k-s-1}$.

Remark 2. – Čirka proved [12, p. 293] that every holomorphic map $F : \Delta \longrightarrow \mathbf{C}^n$ whose boundary values lie in a totally real $\mathbf{C}^k$ submanifold M of $\mathbf{C}^n$ $(k > 1)$ can be extended to a map of class $\mathbf{C}^{k-0} = \bigcap_{0<\epsilon<1} \mathbf{C}^{k-1,\epsilon}$ on $\overline{\Delta}$. Therefore each individual map $F(r,t) : \overline{\Delta} \longrightarrow \mathbf{C}^2$ in Theorem 1 is of class $\mathbf{C}^{k-0}$ on $\overline{\Delta}$. A similar result was proved under stronger initial assumptions of F also in [6, Theorem 4.5] and [22]. Čirka's result will be used in the proof of Theorem 1.

Remark 3. – If $F : \overline{\Delta} \longrightarrow \mathbf{C}^2$ is an analytic disk with boundary in M, so is $F \circ \phi$ for each automorphism ϕ of Δ. Since the group $\operatorname{Aut}\Delta$ has dimension 3, this suggests that we could parametrize the disks close to F^0 with boundaries in M_r by a smooth map $F' : B \subset \mathbf{C}^k(V)^2 \times \mathbf{R}^{2m-1} \longrightarrow (\mathcal{A}^{1/2})^2$, the additional three parameters coming by composing the disks $F'(r,t)$ with the automorphisms of Δ. We shall see in Section 6 that such a parametrization exists at least when $k \geqslant 4$.

Remark 4. – The assumption that M be parallelizable in an open neighborhood of the set $F^0(b\Delta)$ is not necessary for Theorem 1 to hold, although it simplies the statement of the results. It suffices to assume instead that M is orientable along the curve $F^0 : b\Delta \longrightarrow M$. (See Section 4.) If $F^0 : \overline{\Delta} \longrightarrow M$ is an embedding, the two assumptions are equivalent.

Theorem 2 gives a partial answer to a question raised by Bishop [7]: If M is a totally real submanifold of the form $\{(z, h(z)) \in \mathbf{C}^2 \mid z \in \mathbf{C}\}$, where $h : \mathbf{C} \longrightarrow \mathbf{C}$ is a function of class $\mathbf{C}^4$, then the set of all simple closed curves $\gamma : b\Delta \longrightarrow \mathbf{C}$ of class $\mathbf{C}^{1/2}$ such that the curve $\theta \longrightarrow \Gamma(\theta) = (\gamma(\theta), h \circ \gamma(\theta))$ in M bounds an analytic disk in $\mathbf{C}^2$ is a discrete subset of $\mathbf{C}_{\mathbf{C}}^{1/2}(b\Delta)$, provided that we identify two curves which differ only by an automorphism of Δ. To see this, observe that $\mathrm{Ind}_\Gamma M = 0$ since Γ is contractible in M and apply Theorem 2.

In the case when the index of M along F^0 is one, the next theorem implies that F^0 generates an immersed Levi flat hypersurface with boundary in M that is stable under small perturbations of M. A similar result was proved in [5] and [6].

THEOREM 3. – *Let $M \subset \mathbf{C}^2$ be a maximal real $\mathbf{C}^k$ submanifold of the form (1.1), $k \geqslant 3$, let $F^0 : \overline{\Delta} \longrightarrow \mathbf{C}^2$ be an immersed analytic disk with boundary in M such that $\mathrm{Ind}_{F^0} M = 1$, and let $I = (-1, 1) \subset \mathbf{R}$. There exist a neighborhood B of the point r^0 in $\mathbf{C}^k(V)^2$ and a map $\sigma : B \times I \times \overline{\Delta} \longrightarrow \mathbf{C}^2$ of class $\mathbf{C}^l, l = [(k-1)/2]$, satisfying*

(a) For each $r \in B$ the map $\sigma(r, \cdot, \cdot) : I \times \overline{\Delta} \longrightarrow \mathbf{C}^2$ is a $\mathbf{C}^l$ immersion that is of class $\mathbf{C}^{k-1}$ on $I \times \Delta$.

(b) For each $r \in B$ and $t \in I$ the map $\sigma(r, t, \cdot) : \overline{\Delta} \longrightarrow \mathbf{C}^2$ is an analytic disk with boundary in M_r.

(c) If $k \geqslant 4$, there is a neighborhood U of the disk $\sigma^0 = \sigma(r^0, 0, \cdot)$ in the space $(\mathfrak{A}^{1/2})^2$ such that every analytic disk $f \in U$ with boundary in M_r for some $r \in B$ equals $\sigma(r, t, \cdot) \circ \phi$ for a $t \in (-1, 1)$ and an automorphism ϕ of Δ.

The next lemma shows that there are no immersed families of disks with boundary in M through F^0 unless the index $\mathrm{Ind}_{F^0} M$ equals one.

LEMMA 4. – *Let M be a maximal real submanifold of $\mathbf{C}^2$ and let $I = (-1, 1) \subset \mathbf{R}$. If $\sigma : I \times \overline{\Delta} \longrightarrow \mathbf{C}^2$ is a $\mathbf{C}^1$ immersion* *such that for each* $t \in I$ *the map* $\sigma^t = \sigma(t, \cdot)$ *is an analytic disk with boundary in* M, *then* $\mathrm{Ind}_{\sigma^t} \mathrm{M} = 1$ *for every* $t \in I$.

Lemma 4 and the uniqueness result Theorem 3 (c) imply that, up to a reparametrization, every immersed family of analytic disks with boundary in M arises as in Theorem 3. As a consequence we obtain the following result on the regularity of immersed families of disks.

COROLLARY 5. – *Let* M *be a maximal real submanifold of* $\mathbf{C}^2$ *of class* $\mathbf{C}^k, k \geqslant 4$, *and let* $\sigma : I \times \overline{\Delta} \longrightarrow \mathbf{C}^2$ *be as in Lemma 4 above. Then there exists a* $\mathbf{C}^1$ *diffeomorphism* h *of* $I \times \overline{\Delta}$ *such that* $\sigma \circ h$ *has the same properties as* σ *and it is of class* $\mathbf{C}^l, l = [k-1)/2]$

In the case when M is the boundary of an embedded Levi flat hypersurface we obtain a stronger uniqueness result for analytic disks with boundary in M.

THEOREM 6. – *Let* M *be a maximal real submanifold of* $\mathbf{C}^2$ *of class* $\mathbf{C}^3$, *and let* $\sigma : I \times \overline{\Delta} \longrightarrow \mathbf{C}^2$ *be a* $\mathbf{C}^3$ *embedding such that for each* $t \in I = (-1, 1) \subset \mathbf{R}$ *the map* $\sigma^t = \sigma(t, \cdot)$ *is an analytic disk with boundary in* M. *Then there is an open neighborhood* U *of* $\sigma^0(\overline{\Delta})$ *in* $\mathbf{C}^2$ *with the following property: If* $f : \overline{\Delta} \longrightarrow U$ *is an analytic disk with boundary in* M, *then* $f(\Delta) = \sigma^t(\Delta)$ *for some* $t \in I$.

If F^0 is an analytic disk with boundary in M, then by the maximum principle $F^0(\Delta)$ is contained in the polynomially convex hull $\hat{M}$ of M. Therefore our results give some information on the hull of M. If M and F^0 satisfy the hypotheses of Theorem 1 and $\mathrm{Ind}_{F^0} M \geqslant 2$, then F^0 generates an open subset of $\hat{M}$ which is stable under small $\mathbf{C}^2$ perturbations of M. See also the examples in Section 8.

If we only assume that the initial disk F^0 with boundary in M is an immersion in a neighborhood of $b\Delta$ in Δ, a similar result holds. Let M be orientable along $F^0 : b\Delta \longrightarrow M$. Suppose that the derivatives $F_1^{0'}$ and $F_2^{0'}$ have precisely p common zeroes on Δ. If the number

$$k = \operatorname{Ind}_{F^0} M - p - 1$$

is nonnegative, then the nearby disks with boundary in M form a $(2\operatorname{Ind}_{F^0} M + 2)$-parameter family that is stable under small $\mathbf{C}^2$ perturbations of M. This generalization is especially interesting in the case when M is not orientable along the curve $F^0(b\Delta)$, since we may then replace F^0 by $G^0(z) = F^0(z^2)$. We omit the details.

One would also like to know the answer to this problem when $F^0 : b\Delta \longrightarrow \mathbf{C}^n$ is an analytic disk with boundary in a maximal real $\mathbf{C}^2$ submanifold M of $\mathbf{C}^n$ for $n > 2$. Suppose that there exist n linearly independent vector fields $X_1, \ldots, X_n : b\Delta \longrightarrow \mathbf{C}^n$ of class $\mathbf{C}^{1/2}$ such that for each $j \in \{1, \ldots, n\}$ and $\theta \in \mathbf{R}$, we have

a) $X_j(e^{i\theta})$ is tangent to M at $F^0(e^{i\theta})$,

and

b) the components of X_j extend to holomorphic functions on Δ.

A straightforward generalization of our methods gives an $(n-1)$-parameter family of analytic disks with boundaries in M whose images are pairwise distinct (although not necessarily disjoint), and the family is stable under small $\mathbf{C}^2$ perturbations of M. The precise number of parameters may be bigger than $n-1$ if $\operatorname{Ind}_{F^0} M$ is at least two. On the other hand, if the maximal number of linearly independent vector fields satisfying (a) and (b) above is less than n, the initial disk F^0 may not be stable and there may be no nearby disks. (See Example 5 in Section 8.)

In conclusion we mention some open problems related to our discussion.

Problem 1. – If M is a closed compact $\mathbf{C}^1$ maximal real submanifold of $\mathbf{C}^n$, are there any positive dimensional analytic varieties with boundaries in M? If so, do such varieties fill the polynomial hull $\hat{M}$ of M?

It is well-known that if M is not smooth, the set $\hat{M}/M$ need not contain any positive dimensional varieties [26, 30]. However, if M is a smooth manifold, it seems that no counterexamples are known. Recently Alexander and Wermer [3] obtained rather precise description of the hull in special cases. See also [32, 33].

Problem 2. – Describe the global behavior of analytic disks with index one with boundary in a compact maximal real submanifold $M \subset \mathbf{C}^2$. In the examples in Section 8 below such disks form a compact immersed Levi flat hypersurface. Is this always true ? For partial results see [32, 33].

Problem 3. – Suppose that $M \subset \mathbf{C}^n$ is a maximal real submanifold, $n \geqslant 3$, and F^0 is an immersed disk satisfying the conditions (a) and (b) above. Find all analytic disks with boundary in M near F^0. A special case is the nonlinear vector valued Riemann problem whose solution can be found in [23, p. 591-601].

2. Properties of the index.

Let $G = G(n)$ denote the Grassman manifold of oriented real n-dimensional subspaces of $\mathbf{C}^n$. If $g \in G$, we choose n vectors $X_1, \ldots, X_n$ in $\mathbf{C}^n$ that form a positively oriented real orthonormal basis of g and denote by $X = (X_1, \ldots, X_n)$ the complex $n \times n$ matrix whose columns are the vectors X_j. We define

$$\pi(g) = \det(X_1, \ldots, X_n). \tag{2.1}$$

If $Y = (Y_1, \ldots, Y_n)$ is another positively oriented orthonormal basis of g, there is a real matrix $A \in \mathrm{SL}(n, \mathbf{R})$ such that $Y = AX$ whence $\det Y = \det A \det X = \det X$. Thus the formula (2.1) gives a well-defined map $\pi : G \longrightarrow \mathbf{C}$. Its zero set consists of precisely those n-dimensional subspaces of $\mathbf{C}^n$ that contain a nontrivial complex subspace. In other words, $G_{tr} = G \backslash \pi^{-1}\{0\}$ is the set of all oriented totally real n-dimensional subspaces of $\mathbf{C}^n$.

The map π induces a homomorphism of homology groups

$$\mathrm{H}_1 \pi : \mathrm{H}_1(G_{tr}) \longrightarrow \mathrm{H}_1(\mathbf{C}\backslash\{0\}) = \mathbf{Z}. \tag{2.2}$$

Let $\iota : M \longrightarrow \mathbf{C}^n$ be a $\mathbf{C}^1$ immersion of an oriented n-dimensional manifold M to $\mathbf{C}^n$. By mapping each point $p \in M$ to the oriented n-plane through 0 parallel to the tangent plane $d\iota(\mathrm{T}_p M)$ we define a mapping $\iota_* : M \longrightarrow G$. If the immersion ι is totally real, the image $\iota_*(M)$ is contained in the set G_{tr} of the totally real n-planes, and we may compose the map $\mathrm{H}_1(\iota_*)$ with the map (2.2) to obtain a homomorphism

$$I_\iota = \mathrm{H}_1(\pi) \circ \mathrm{H}_1(\iota_*) : \mathrm{H}_1(M) \longrightarrow \mathbf{Z}. \tag{2.3}$$

For each closed path $F : b\Delta \longrightarrow M$ we denote by $[F] \in H_1 M$ the cycle generated by F. If $\iota : M \hookrightarrow \mathbf{C}^n$ is the inclusion of an oriented maximal real submanifold $M \subset \mathbf{C}^n$ into $\mathbf{C}^n$, we claim that

$$I_\iota [F] = \mathrm{Ind}_F M$$

where $\mathrm{Ind}_F M$ is given by Definition 1 in Section 1. To see this, let $X_1, \ldots, X_n : b\Delta \longrightarrow \mathbf{C}^n$ be continuous maps such that, for each point $\zeta \in b\Delta$, the vectors $X_1(\zeta), \ldots, X_n(\zeta)$ form a positively oriented real orthonormal basis of the tangent space $T_{F(\zeta)} M$. These vectors then represent the n-plane $\iota_*(F(\zeta)) \subset G_{tr}$. By (2.1) we have

$$\pi \circ \iota_*(F(\zeta)) = \det(X_1(\zeta), \ldots, X_n(\zeta)). \tag{2.4}$$

By (2.3) $I_\iota [F]$ is the winding number of the map

$$\zeta \longrightarrow \pi \circ \iota_*(F(\zeta))$$

around 0. Definition 1 and (2.4) imply that this equals $\mathrm{Ind}_F M$ and the claim is proved.

Although the map $\iota_* : M \longrightarrow G$ depends on the choice of orientation on M, the induced homomorphism I_ι does not. Namely, if $\iota' : M \longrightarrow \mathbf{C}^n$ is the same immersion with the opposite choice of orientation on M, then $\pi \circ \iota'_*(p) = -\pi \circ \iota_*(p)$ for every $p \in M$. The map $z \longrightarrow -z$ on $\mathbf{C} \setminus \{0\}$ induces the identity map on $H_1(\mathbf{C} \setminus \{0\}) = \mathbf{Z}$ whence $I_\iota = I_{\iota'}$.

We shall extend the definition of index to maximal real immersed manifolds that are not necessarily orientable. Let $\widetilde{G}$ be the Grassman manifold of (nonoriented) real n-planes in $\mathbf{C}^n$ and let $\sigma : G \longrightarrow \widetilde{G}$ be the two-to-one projection that forgets the orientation. Let $\tau : \mathbf{C} \setminus \{0\} \longrightarrow \mathbf{P}^1$ be the canonical projection onto the real projective space. If $X_1, \ldots, X_n$ is an orthonormal basis of a totally real n-plane $\widetilde{g} \in \widetilde{G}$, we define

$$\widetilde{\pi}(\widetilde{g}) = \tau(\det(X_1, \ldots, X_n)).$$

This gives a well-defined map $\widetilde{\pi} : \widetilde{G}_{tr} \longrightarrow \mathbf{P}^1$, where $\widetilde{G}_{tr}$ denotes the set of totally real n-planes. Clearly we have $\tau \circ \pi = \widetilde{\pi} \circ \sigma$ on G_{tr}.

If $\iota : M \longrightarrow \mathbf{C}^n$ is a maximal real immersion, we let $\widetilde{\iota} : M \longrightarrow \widetilde{G}_{tr}$ be the induced map that sends each $p \in M$ to the tangent plane $d\iota(T_p M)$. We define a homomorphism

$$\widetilde{I}_\iota : H_1 M \longrightarrow 1/2\ H_1\ \mathbf{P}^1 = 1/2\ \mathbf{Z}$$

by

$$\widetilde{I}_\iota = 1/2\ H_1\ (\widetilde{\pi}) \circ H_1\ (\widetilde{\iota}).$$

We have to prove that $\widetilde{I}_\iota = I_\iota$ if the manifold M is orientable. In this case we also have the map $\iota_* : M \longrightarrow G_{tr}$ such that $\sigma \circ \widetilde{\iota}_* = \widetilde{\iota}$. Thus

$$\widetilde{\pi} \circ \widetilde{\iota} = \widetilde{\pi} \circ (\sigma \circ \iota_*) = (\widetilde{\pi} \circ \sigma) \circ \iota_* = (\tau \circ \pi) \circ \iota_* = \tau \circ (\pi \circ \iota_*)$$

whence

$$\widetilde{I}_\iota = \frac{1}{2} H_1\ (\widetilde{\pi} \circ \widetilde{\iota}) = \frac{1}{2} H_1\ (\tau) \circ H_1\ (\pi \circ \iota_*) = \frac{1}{2} H_1\ (\tau) \circ I_\iota\,.$$

Since $H_1\ (\tau) : H_1(\mathbf{C}\setminus\{0\}) = \mathbf{Z} \longrightarrow H_1 \mathbf{P}^1 = \mathbf{Z}$ is the multiplication by 2, $1/2\, H_1\ (\tau)$ is the identity and $\widetilde{I}_\iota = I_\iota$. Thus the definitions agree for orientable manifolds.

If the maximal real immersions $\iota_1 , \iota_2 : M \longrightarrow \mathbf{C}^n$ are regularly homotopic through maximal real immersions, then the induced maps $\iota_1^* , i_2^* : M \longrightarrow \widetilde{G}_{tr}$ are homotopic, hence $H_1\ (\iota_1) = H_1\ (\iota_2)$ and $I_{\iota_1} = I_{\iota_2}$.

The following lemma describes the behavior of index under biholomorphic change of coordinates.

LEMMA 7. – *Let $M \subset \mathbf{C}^n$ be a $\mathbf{C}^1$ maximal real submanifold and $F : b\Delta \longrightarrow M$ a closed path in M. If $\Phi : U \longrightarrow \Phi(U) \subset \mathbf{C}^n$ is a biholomorphic map on an open neighborhood U of $F(b\Delta)$ in $\mathbf{C}^n$ and if $g(z) = \det d\Phi\,(F(z))$ for all $z \in b\Delta$, then*

$$\mathrm{Ind}_{\Phi \circ F}\ \Phi(M) = \mathrm{W}(g) + \mathrm{Ind}_F\, M.$$

Proof. – Suppose first that M is orientable along F. Then

$$\mathrm{Ind}_F\, M = \mathrm{W}(\det(X_1 , \ldots , X_n))$$

for some vector fields $X_1 , \ldots , X_n : b\Delta \longrightarrow \mathbf{C}^n$ tangent to M along $F(b\Delta)$. If $A(z) = d\,\Phi\,(F(z))$, then the vector fields

$$AX_1 , \ldots , AX_n : b\Delta \longrightarrow \mathbf{C}^n$$

are tangent to $\Phi(M)$ along the curve $\Phi \circ F : b\Delta \longrightarrow \mathbf{C}^n$. Thus

$$\begin{aligned}\mathrm{Ind}_{\Phi\circ F}\,\Phi(M) &= \mathrm{W}(\det(AX)) = \mathrm{W}(\det A\cdot\det X)\\ &= \mathrm{W}(\det A) + \mathrm{W}(\det X)\\ &= \mathrm{W}(g) + \mathrm{Ind}_F M.\end{aligned}$$

If M is not orientable, we apply the same proof to the map $z \longrightarrow F(z^2)$. Lemma 7 is proved.

In the case when M is a compact orientable surface embedded in $\mathbf{C}^2$ there is a close connection between the index and the topology of M. Following Bishop [7] we call a point $p\in M$ *exceptional* if the tangent space $\mathrm{T}_p M$ is a complex linear subspace of $\mathbf{C}^2$. Since the set of complex subspaces is thin in the set of all real 2-planes in $\mathbf{C}^2$ through 0, a generic submanifold M^2 of $\mathbf{C}^2$ will only have isolated exceptional points. If $p\in M$ is such an isolated exceptional point, there is a contractible neighborhood U of p in M such that $U\setminus\{p\}$ is a totally real submanifold of $\mathbf{C}^2$. The natural orientation on the complex line $\mathrm{T}_p M$ induces in a natural way an orientation on U. Let $F: b\Delta \longrightarrow U\setminus\{p\}$ be any simple closed curve that winds around p in the positive direction with respect to the chosen orientation on U.

DEFINITION 2. – *If p is an exceptional point of M and if F is as above, we define the index of p to be the integer* $\mathrm{Ind}_p M = \mathrm{Ind}_F M$.

Since $\mathrm{Ind}_F M$ only depends on the homology class of F in $U\setminus\{p\}$, the index $\mathrm{Ind}_p M$ of p is well-defined. We may extend the definition of index to all points of M by letting $\mathrm{Ind}_p M = 0$ if M is totally real at p. The definition of index can be extended in an obvious way to immersed submanifolds of $\mathbf{C}^2$ that have only isolated exceptional points.

The following lemma shows how to compute the index of an isolated exceptional point in terms of a local parametrization of M.

LEMMA 8. – *Let U be an open set in $\mathbf{C}$ containing $\overline{\Delta}$, and let $g: U \longrightarrow \mathbf{C}$ be a $\mathbf{C}^1$ function such that $\dfrac{\partial g}{\partial \bar z}(z) \neq 0$ for every $z\neq 0$. If we let*

$$M = \{(z, g(z)) \in \mathbf{C}^2 \mid z \in U\},$$

then $\mathrm{Ind}_0 M$ *equals the winding number of the map*

$$\frac{\partial g}{\partial \bar{z}} : b\Delta \longrightarrow \mathbf{C} \setminus \{0\}$$

around 0.

Proof. – The vectors

$$X(z) = \begin{pmatrix} 1 \\ \partial g/\partial x\,(z) \end{pmatrix} \quad \text{and} \quad Y(z) = \begin{pmatrix} i \\ \partial g/\partial y\,(z) \end{pmatrix}$$

form a framing of $\mathrm{T}M$. By Definition 1 $\mathrm{Ind}_0 M$ is the winding number of the determinant

$$d(z) = \det(X(z), Y(z)) = \frac{\partial g}{\partial y}(z) - i\frac{\partial g}{\partial x}(z) = -2i\frac{\partial g}{\partial \bar{z}}(z)$$

around 0 which equals the winding number of $\partial g/\partial \bar{z}$. This concludes the proof of Lemma 8.

Example. – Let $k, l \in \mathbf{Z}_+, l \geqslant 1$. The manifold

$$M_{k,l} = \{(z, z^k \bar{z}^l \in \mathbf{C}^2 \mid z \in \mathbf{C}\}$$

is totally real except possibly at 0, and

$$\mathrm{Ind}_0 M = \mathrm{W}\left(z^k \quad \bar{z}^{l-1}\right) = k - l + 1.$$

This shows that every integer may arise as the index of an isolated exceptional point.

Bishop considered the local expansion of order 2 of a submanifold $M^2 \subset \mathbf{C}^2$ near an exceptional point $p \in M$ [7]. He showed that, in an appropriate system of local holomorphic coordinates (z, w) in $\mathbf{C}^2$ near p, we have

$$p = 0, \ \mathrm{T}_p M = \{w = 0\}, \ M = \{(z, g(z))\},$$

where

$$g(z) = \alpha z^2 + \beta \quad \bar{z}^2 + \gamma z \quad \bar{z} + o(|z|^2). \tag{2.5}$$

If $|\beta| \neq |\gamma|/2$ and $\gamma \neq 0$, there is a local change of coordinates at 0 such that in the new coordinates

$$g(z) = \beta \quad (z^2 + \bar{z}^2) + z \quad \bar{z} + o(|z|^2), \ \beta \geqslant 0,$$

with $\beta \neq 1/2$. The number β is a biholomorphic invariant of $\mathbf{M}$. The exceptional point $p = 0$ is elliptic if $\beta < 1/2$ and is hyperbolic if $\beta > 1/2$. Since

$$\frac{\partial g}{\partial \bar{z}} = 2\beta\bar{z} + z + o(|z|) = (1 + 2\beta)x + (1 - 2\beta)iy + o(|z|),$$

Lemma 8 implies that the index $\mathrm{Ind}_p M$ equals 1 if p is an elliptic point and equals -1 if p is a hyperbolic point. If the genericity assumptions in (2.5) do not hold, the example above shows that the index $\mathrm{Ind}_p M$ may be any integer. Bishop called such a point exceptionally exceptional.

Using a theorem of Chern and Spanier [11] Bishop proved [7, p. 12] that in the case when an embedded compact oriented surface $M \subset \mathbf{C}^2$ has only isolated nondegenerate exceptional points, the difference between the number of elliptic points and the number of hyperbolic points equals the Euler number $\chi(M)$ of M. He only gave the details in the case when M is a sphere, but the proof holds in general. His result implies the following theorem:

THEOREM 9. – *Let M be a closed compact oriented surface embedded differentiably in $\mathbf{C}^2$. If M has only isolated exceptional points, then*

$$\sum_{p \in M} \mathrm{Ind}_p M = \chi(M).$$

Initially Theorem 9 holds only when M is of class $\mathbf{C}^2$ and has nondegenerate exceptional points, but one can show using the properties of index that these additional assumptions are not necessary. Notice that the index of an isolated exceptional point does not depend on the orientation of M. Recently Webster proved [27, 28] the formula $\sum_{p \in M} \mathrm{Ind}_p M = \chi(M) + \chi(N)$ for every compact immersed surface in $\mathbf{C}^2$, where N is the normal bundle of the immersion and no exceptional point of M is exceptionally exceptional.

3. Geometry near the boundary of an analytic disk.

In this section we shall find a normal form for the tangent bundle of a maximal real submanifold M in $\mathbf{C}^2$ along the boundary of an analytic disk F in terms of the index $\mathrm{Ind}_F M$. A similar result was used by Bedford and Gaveau [5, 6]. We shall identify the functions on $b\Delta$ with the 2π-periodic functions on $\mathbf{R}$ and write $f(\theta)$ instead of $f(e^{i\theta})$.

LEMMA 10. – *Let* $F : \bar{\Delta} \longrightarrow \mathbf{C}^2$ *be an immersed analytic disk with boundary in a* $\mathbf{C}^2$ *maximal real orientable submanifold* $M \subset \mathbf{C}^2$. *If* $m = \mathrm{Ind}_F M$, *there is a* 2×2 *matrix* $A(\theta)$ *with entries in* $\mathfrak{A}^{1/2}$ *and an* $\alpha \in \mathfrak{A}^{1/2}$ *such that, if we put*

$$X_0(\theta) = \begin{pmatrix} ie^{i\theta} \\ 0 \end{pmatrix} \qquad \textit{and} \qquad Y_0(\theta) = \begin{pmatrix} \alpha(\theta) \\ e^{(m-1)i\theta} \end{pmatrix},$$

the vectors $X(\theta) = A(\theta)\, X_0(\theta)$ *and* $Y(\theta) = A(\theta)\, Y_0(\theta)$ *form a real basis of the tangent space* $\mathrm{T}_{F(\theta)} M$ *for each* θ. *Moreover, the matrix* A *is invertible, and the component of* A^{-1} *are in* $\mathfrak{A}^{1/2}$.

Note. – We shall see in the course of the proof that if M is of class $\mathbf{C}^k$, $k \geqslant 2$, and $0 < \epsilon < 1$, we may choose A and

$$\alpha(\theta) = \sum_{j=0}^{\infty} \alpha_j\, e^{ij\theta}$$

to be of class $\mathbf{C}^{k-2,\epsilon}$ on $b\Delta$ and $\alpha_0 = \mathrm{Im}\,\alpha_1 = 0$.

We denote by $\tilde{f}$ the harmonic conjugate function of a function f on $b\Delta$. If $f(\theta) = \sum_{j=-\infty}^{\infty} c_j\, e^{ij\theta}$, then

$$\tilde{f}(e^{i\theta}) = i \sum_{j=-\infty}^{-1} c_j\, e^{ij\theta} - i \sum_{j=1}^{\infty} c_j\, e^{ij\theta}.$$

It is well-known that $f \longrightarrow \tilde{f}$ is a bounded linear operator of $\mathbf{C}^{1/2}(b\Delta)$ into itself [19, Proposition 3.1; 15, p. 83]. The same is true for each Lipschitz space $\mathbf{C}^{s,\epsilon}$, where $s \in \mathbf{Z}_+$ and $0 < \epsilon < 1$ [19, Proposition 3.1]. For each $f \in \mathbf{C}^{1/2}(b\Delta)$ the function $f + i\tilde{f}$ is in $\mathfrak{A}^{1/2}$.

Proof of Lemma 10. – Since $F = (F_1, F_2)$ is an immersion, the derivatives F_1' and F_2' have non common zero on $\overline{\Delta}$. By a theorem of Čirka [12, p. 293] the map F is of class $\mathbf{C}^{2-0}$ on $\overline{\Delta}$ and so the derivatives F_1', F_2' are of class $\mathbf{C}^{1-0}$. Hence there exist $\mathbf{C}^{1/2}$ functions g_1, g_2 on $\overline{\Delta}$, holomorphic on Δ, such that $F_1' g_1 + F_2' g_2 \equiv 1$ on Δ. Define

$$\widetilde{A}(z) = \begin{pmatrix} F_1'(z) & -g_2(z) \\ F_2'(z) & g_1(z) \end{pmatrix}.$$

Then $\det \widetilde{A} = F_1' g_1 + F_2' g_2 \equiv 1$, hence $\widetilde{A}$ is invertible and the components of $\widetilde{A}^{-1}$ are in $\mathcal{A}^{1/2}$.

Let $X_0(\theta) = \begin{pmatrix} ie^{i\theta} \\ 0 \end{pmatrix}$. The vector

$$X(\theta) = \widetilde{A}(\theta)\, X_0(\theta) = ie^{i\theta} \begin{pmatrix} F_1'(e^{i\theta}) \\ F_2'(e^{i\theta}) \end{pmatrix} = \frac{d}{d\theta} \begin{pmatrix} F_1(e^{i\theta}) \\ F_2(e^{i\theta}) \end{pmatrix}$$

is clearly tangent to M at the point $F(\theta)$ and is nonvanishing for every θ. Since $F^* \mathrm{T}M$ is a trivial bundle, there is another $\mathbf{C}^1$ vector field $Y(\theta)$ such that for all θ, $X(\theta)$ and $Y(\theta)$ form a basis of $\mathrm{T}_{F(\theta)} M$. Let $Y_0(\theta) = \widetilde{A}^{-1}(\theta)\, Y(\theta) = \begin{pmatrix} \alpha(\theta) \\ \beta(\theta) \end{pmatrix}$. We have

$$\begin{aligned} \det(X(\theta), Y(\theta)) &= \det \widetilde{A}(\theta) \det(X_0(\theta), Y_0(\theta)) \\ &= \det(X_0(\theta), Y_0(\theta)) \\ &= ie^{i\theta}\, \beta(\theta) \end{aligned}$$

whence $\beta(\theta) \neq 0$ for all θ. We compute the winding numbers of both sides:

$$m = \mathrm{W}(\det(X, Y)) = \mathrm{W}(ie^{i\theta}\, \beta(\theta)) = 1 + \mathrm{W}(\beta).$$

This implies that the function $\theta \longrightarrow e^{(-m+1)i\theta}\, \beta(\theta)$ has winding number 0 whence it is an exponential,

$$e^{(-m+1)i\theta}\, \beta(\theta) = e^{a(\theta) + ib(\theta)}.$$

If we set $v(\theta) = e^{-a(\theta) - \widetilde{b}(\theta)}$, then

$$
\begin{aligned}
v(\theta)\, e^{(-m+1)i\theta}\, \beta(\theta) &= e^{-\tilde{b}(\theta)+ib(\theta)} \\
&= e^{i(b(\theta)+i\tilde{b}(\theta))} \\
&= \beta^*(\theta)
\end{aligned}
$$

is an invertible element of $\mathcal{A}^{1/2}$. Since v has no zeroes on $b\Delta$, we may replace Y by vY. We extend β^* to an analytic function on Δ and define

$$
\Psi(z, w) = F(z) + w\beta^*(z) \begin{pmatrix} -g_2(z) \\ g_1(z) \end{pmatrix}, \tag{3.2}
$$

$$
A(z) = d\Psi(z, 0) = \begin{pmatrix} F_1'(z) & -g_2(z)\,\beta^*(z) \\ F_2'(z) & g_1(z)\,\beta^*(z) \end{pmatrix}. \tag{3.3}
$$

Then the vector $A^{-1}\, Y = Y_0$ equals $Y_0(\theta) = \begin{pmatrix} \alpha(\theta) \\ e^{(m-1)i\theta} \end{pmatrix}$. Since $\det A = \beta^*$ is invertible in $\mathcal{A}^{1/2}$, the components of A^{-1} are in $\mathcal{A}^{1/2}$. Notice that when $m = 1$ we may take $\beta = \beta^*$ and $v = 1$.

The first component α of Y_0 may not be in $\mathcal{A}^{1/2}$. If we let $\delta(\theta) = ie^{-i\theta}\,\alpha(\theta)$ and set $u(\theta) = \operatorname{Re}\delta(\theta) + \widetilde{(\operatorname{Im}\delta)}(\theta)$, the first component of $uX_0 + Y_0$ equals

$$
\begin{aligned}
u(\theta)\, ie^{i\theta} + \alpha(\theta) &= ie^{i\theta}\,(u(\theta) - ie^{-i\theta}\,\alpha(\theta)) \\
&= e^{i\theta}\,((\operatorname{Im}\delta)(\theta) + i\widetilde{(\operatorname{Im}\delta)}(\theta))
\end{aligned}
$$

which is an element of $\mathcal{A}^{1/2}$. If we replace Y_0 by $uX_0 + Y_0$, Lemma 10 is proved.

The proof shows that if M is of class $\mathbf{C}^k$ and $0 < \epsilon < 1$, we may choose X, Y and A as in Lemma 10 such that their components are in the algebra $\mathcal{A}^{k-2,\epsilon}(b\Delta)$.

4. Parametrization of the disks.

In this section we will find a convenient parametrization of all maps $F \in \mathbf{C}^{1/2}(b\Delta)^2$ that map $b\Delta$ to M_r for some r close to r^0. In Section 5 and 6 we will examine the conditions under which a given map in our parametrized family is the boundary value of an analytic disk.

Let $X, Y : b\Delta \longrightarrow \mathbf{C}^2$ be chosen as in Lemma 10. Denote by $\widetilde{f}$ the harmonic conjugate of f. Consider the map

$$G : \mathbf{C}^{1/2}(b\Delta)^2 \times \mathbf{C}^{1/2}(b\Delta)^2 \longrightarrow \mathbf{C}^{1/2}(b\Delta)^2,$$

given for each $u = (u_1, u_2)$ and $f = (f_1, f_2)$ in $\mathbf{C}^{1/2}(b\Delta)^2$ by

$$G(u, f) = F^0 + u_1 X + u_2 Y + i(f_1 + i\widetilde{f}_1)X + i(f_2 + i\widetilde{f}_2)Y. \tag{4.1}$$

Since the vectors $X(\theta)$, $iX(\theta)$, $Y(\theta)$ and $iY(\theta)$ form a real basis of $\mathbf{C}^2$ for each θ, every map $F : b\Delta \longrightarrow \mathbf{C}^2$ that is of class $\mathbf{C}^{1/2}$ equals $G(u, f)$ for some uniquely determined real-valued functions $u_1, u_2, f_1, f_2 \in \mathbf{C}^{1/2}(b\Delta)$.

Let V be the open subset of $\mathbf{C}^2$ for which (1.1) holds. Choose u and f sufficiently small in the $\mathbf{C}^{1/2}(b\Delta)$ norm such that $G(u, f)(b\Delta) \subset V$. The condition that the image $G(u, f)(b\Delta)$ lies in the manifold M_r is equivalent to

$$\Phi(r, u, f) = r(G(u, f)) = 0. \tag{4.2}$$

The map Φ is defined on an open neighborhood $B_1 \times B_2 \times B_3$ of the point $(r, u, f) = (r^0, 0, 0)$ in the Banach space

$$\mathbf{C}^k(V)^2 \times \mathbf{C}^{1/2}(b\Delta)^2 \times \mathbf{C}^{1/2}(b\Delta)^2$$

and maps it into $\mathbf{C}^{1/2}(b\Delta)^2$.

LEMMA 11. – *The equation (4.2) has a unique solution*

$$f = f(r, u), \quad f(r^0, 0) = 0 \tag{4.3}$$

in a neighborhood of the initial point $(r^0, 0)$ satisfying

$$\Phi(r, u, f(r, u)) \equiv 0.$$

Proof. – By the implicit function theorem in Banach spaces [10, p. 61] it suffices to prove that

(i) the map Φ is of class $\mathbf{C}^1$, and

(ii) the partial derivative

$$D_f \Phi(r^0, 0, 0) : \mathbf{C}^{1/2}(b\Delta)^2 \longrightarrow \mathbf{C}^{1/2}(b\Delta)^2$$

is a linear isomorphism.

Observe that the map G defined by (4.1) is affine in both variables whence it is smooth in these variables. The map Φ is clearly linear in r. Lemma 5.1 in [19, p. 340] implies that for each fixed $r \in \mathbf{C}^k(V)^2$ the map

$$(u, f) \longrightarrow r(G(u, f))$$

is of class $\mathbf{C}^{k-1}$. More generally, if $r \in \mathbf{C}^k(V)^2$ and if we replace $\mathbf{C}^{1/2}(b\Delta)$ by the space $\mathbf{C}^{s,e}(b\Delta)$, Φ is of class $\mathbf{C}^{k-s-1}$.

To compute $\mathrm{D}_f\Phi$ we identify $\mathbf{C}^2$ with $\mathbf{R}^4$ in the standard way. Let X^* and Y^* be the vectors that correspond to iX and iY under this identification, and denote by $\nabla r = \begin{pmatrix} \nabla r_1 \\ \nabla r_2 \end{pmatrix}$ the real gradient of r represented by the 2×4 matrix of partial derivatives. By the chain rule we have for each $h = (h_1, h_2) \in \mathbf{C}^{1/2}(b\Delta)^2$

$$\mathrm{D}_f\Phi(r, u, f)h = \nabla r(G(u, f)) \cdot (h_1 X^* + h_2 Y^* - \widetilde{h}_1 X - \widetilde{h}_2 Y) .$$

Since the vector fields X and Y are tangent to $M_{r^0} = M$ along $F^0(b\Delta)$, we have

$$\nabla r^0(F^0) \cdot X = 0 \qquad \text{and} \qquad \nabla r^0(F^0) \cdot Y = 0. \tag{4.4}$$

Therefore

$$\mathrm{D}_f\Phi(r^0, 0, 0)h = \nabla r^0(F^0) \cdot (h_1 X^* + h_2 Y^*) = C \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{4.5}$$

where C is the real 2×2 matrix function $\nabla r^0(F^0) \cdot (X^*, Y^*)$ on $b\Delta$ with entries in $\mathbf{C}^{1/2}(b\Delta)$.

We claim that $\det C(\theta) \neq 0$ for every θ. If not, there are real numbers $a, b \neq 0$ such that, for some θ_0,

$$\begin{aligned} 0 = C(\theta_0) \cdot \begin{pmatrix} a \\ b \end{pmatrix} &= \nabla r^0(F^0(\theta_0)) \cdot (X^*(\theta_0), Y^*(\theta_0)) \cdot \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \nabla r^0(F^0(\theta_0)) \cdot (aX^*(\theta_0) + bY^*(\theta_0)). \end{aligned}$$

This says that the vector $aX^*(\theta_0) + bY^*(\theta_0) = i(aX(\theta_0) + bY(\theta_0))$ lies in $\mathrm{T}_{F^0(\theta_0)}M$ which is a contradiction since M is totally real. Therefore $\det C$ is a nonvanishing function in $\mathbf{C}^{1/2}(b\Delta)$ whence C^{-1} exists and has components in $\mathbf{C}^{1/2}(b\Delta)$. This shows that the partial derivative (4.5) is a linear isomorphism with the inverse $h \longrightarrow C^{-1} \cdot h$ and the implicit function theorem applies. The solution $f(r, u)$ is of class $\mathbf{C}^{k-1}$. This proves Lemma 11.

If f is the solution (4.3) of (4.2), let $\widetilde{F}$ be the composition map

$$\widetilde{F}(r, u) = G(u, f(r, u)) \tag{4.6}$$

defined on an open neighborhood $B_1 \times B_2$ of $(r^0, 0)$ in

$$\mathbf{C}^2(V)^2 \times \mathbf{C}^{1/2}(b\Delta)^2 .$$

It follows that, for each $r \in B_1$ and $u \in B_2$, $\widetilde{F}(r, u)(b\Delta) \subset M_r$.

We shall compute the partial derivative $\mathrm{D}_u f(r, u)$ of the map (4.3) at the point $(r^0, 0)$. Applying the chain rule to the map Φ defined by (4.2) we get

$$0 = \mathrm{D}_u \Phi(r^0, 0, 0) = \mathrm{D}_f \Phi(r^0, 0, 0) \circ \mathrm{D}_u f(r^0, 0) + \mathrm{D}_u \Phi(r^0, 0, 0). \tag{4.7}$$

The last term in (4.7), evaluated on $v = (v_1, v_2)$, is

$$\begin{aligned}\mathrm{D}_u \Phi(r^0, 0, 0) v &= \nabla r^0(F^0) \cdot \mathrm{D}_u G(0, 0) v \\ &= \nabla r^0(F^0) \cdot (v_1 X + v_2 Y)\end{aligned}$$

which is 0 in view of (4.4). On the other hand, we have seen above that $\mathrm{D}_f \Phi(r^0, 0, 0)$ is a linear isomorphism. Thus (4.7) implies

$$\mathrm{D}_u f(r^0, 0) = 0. \tag{4.8}$$

We can use (4.8) to find the partial derivative $\mathrm{D}_u \widetilde{F}$ of the map $\widetilde{F}$ defined by (4.6) at the point $r = r^0$, $u = 0$:

$$\mathrm{D}_u \widetilde{F}(r^0, 0) v = \mathrm{D}_u G(0, 0) v = v_1 X + v_2 Y = (X, Y) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Since the matrix $(X(\theta), Y(\theta))$ is nondegenerate for every θ, it follows that the derivative $\mathrm{D}_u \widetilde{F}(r^0, 0)$ is a linear isomorphism of $\mathbf{C}^{1/2}_{\mathbf{C}}(b\Delta)^2$ onto itself.

If we write a map $F^* \in \mathbf{C}^{1/2}_{\mathbf{C}}(b\Delta)^2$ that is close to F^0 in the form $F^* = G(u, f)$, then u and f are small in the $\mathbf{C}^{1/2}(b\Delta)$-norm. Hence, if $F^*(b\Delta) \subset M_r$ for some $r \in B_1$, the map $F^* = G(u, f)$ is a solution of (4.2) that is close to the initial solution $F^0 = G(0, 0)$.

The uniqueness part of the implicit function theorem implies that $f = f(r, u)$ whence $F^* = \widetilde{F}(r, u)$. This shows that the range of the $\mathbf{C}^1$ map $\widetilde{F}$ defined by (4.6) contains all elements of $\mathbf{C}^{1/2}_{\mathbf{C}}(b\Delta)^2$ that are close to F^0 and map $b\Delta$ to a small perturbation M_r of M.

If we replace $\mathbf{C}^{1/2}(b\Delta)$ in the analysis above by $\mathbf{C}^{s,\epsilon}(b\Delta)$ for some integer $s \leqslant k-2$ and $0 < \epsilon < 1$, we obtain a map

$$\widetilde{F} : \mathbf{C}^k(V)^2 \times \mathbf{C}^{s,\epsilon}(b\Delta)^2 \longrightarrow \mathbf{C}^{s,\epsilon}(b\Delta)^2$$

of class $\mathbf{C}^{k-s-1}$ that parametrizes all maps of class $\mathbf{C}^{s,\epsilon}$ from $b\Delta$ to small perturbations M_r of M.

It remains to show how one can remove the hypothesis that M be parallelizable in an open neighborhood of the set $F^0(b\Delta)$. We shall assume instead that M is orientable along the closed curve F^0. We extend F^0 to a $\mathbf{C}^1$ immersion $\Psi : \widetilde{V} \longrightarrow V$ from an open tube $\widetilde{V}$ in $\mathbf{C}^2$ containing the circle $b\Delta \times \{0\}$ onto an open neighborhood V of $F^0(b\Delta)$ in $\mathbf{C}^2$. There is a $\mathbf{C}^1$ closed submanifold $\widetilde{M}$ in $\widetilde{V}$ that is mapped by Ψ onto an open neighborhood of $F^0(b\Delta)$ in M. Since M is orientable along F^0, $\widetilde{M}$ is a transversal intersection in the tube $\widetilde{V}$ provided that $\widetilde{V}$ is sufficiently small. Moreover, every map $F : b\Delta \longrightarrow V$ that is sufficiently close to F^0 uniformly on $b\Delta$ has a unique lift $\widetilde{F}$ to $\widetilde{V}$ such that $\Psi \circ \widetilde{F} = F$ and $\widetilde{F}(1)$ is close to $(1, 0) \in \widetilde{V}$. If we approximate Ψ sufficiently close by a $\mathbf{C}^k$ map, we may assume that $\widetilde{M}$ is of class $\mathbf{C}^k$ and the lifting property still holds. Since we have not used the analyticity of vector fields X and Y so far, we can work with $\widetilde{M}$ instead of M, thus obtaining a parametrization (4.6) of all $\mathbf{C}^{1/2}$ maps $F : b\Delta \longrightarrow M \cap V$ near F^0 in terms of parameters $u \in \mathbf{C}^{1/2}(b\Delta)^2$ and $r \in \mathbf{C}^k(\widetilde{V})^2$.

5. Proof of Theorem 1.

Assuming that $m = \mathrm{Ind}_{F^0} M \geqslant 1$ we shall now analyze the conditions on u under which the map $\widetilde{F}(r, u)$ define by (4.6) is an element of $(\mathcal{A}^{1/2})^2$, i.e., it is the boundary value of an analytic disk. Let $f = f(r, u) = (f_1(r, u), f_2(r, u))$ and $\widetilde{f}_j = \widetilde{f}_j(r, u)$ for $j = 1, 2$. Denote the k-th Fourier coefficient of a function f by $\hat{f}(k)$. Recall that $\hat{\widetilde{f}}(0) = 0$.

Since $F^0 \in (\mathcal{A}^{1/2})^2$, we must have

$$u_1 X + u_2 Y + i(f_1 + i\widetilde{f_1})X + i(f_2 + i\widetilde{f_2})Y \in (\mathcal{A}^{1/2})^2. \quad (5.1)$$

Let A, X_0 and Y_0 be as in Lemma 10. Recall that the inverse matrix A^{-1} has components in $\mathcal{A}^{1/2}$. Multiplying (5.1) by A^{-1} we conclude that the expression

$$u_1 X_0 + u_2 Y_0 + i(f_1 + \widetilde{f_1})X_0 + i(f_2 + i\widetilde{f_2})Y_0 \quad (5.2)$$

must be in $(\mathcal{A}^{1/2})^2$.

The second component of (5.2) equals

$$(u_2(\theta) + i(f_2(\theta) + i\widetilde{f_2}(\theta)))\, e^{i(m-1)\theta}. \quad (5.3)$$

The function (5.3) is in $\mathcal{A}^{1/2}$ if and only if the Fourier coefficients of order less than $-m+1$ of the function

$$u_2 + i(f_2 + i\widetilde{f_2}) = (u_2 - \widetilde{f_2}) + if_2 \quad (5.4)$$

all vanish. Since $m \geqslant 1$ by assumption and $f_2 + i\widetilde{f_2}$ is holomorphic, this condition is equivalent to $\hat{u}_2(l) = 0$ for $l < -m+1$. The most general form of the real valued function u_2 satisfying this condition is a trigonometric polynomial

$$u_2 = P(a, b) = a_0 + 2\,\mathrm{Re} \sum_{j=1}^{m-1} (a_j - ib_j)\, e^{ij\theta} \quad (5.5)$$

depending on $2m-1$ real parameters

$$a = (a_0, a_1, \ldots, a_{m-1}),\ b = (b_1, \ldots, b_{m-1}).$$

Let $\alpha = \sum_{j=0}^{\infty} \alpha_j e^{ij\theta} \in \mathcal{A}^{1/2}$ be the first component of Y_0. The Fourier expansion of the product $P(a, b)\alpha$ is of the form

$$P(a, b)\alpha = \sum_{j=1}^{m-1} \delta_j(a, b)\, e^{-ij\theta} + \sum_{j=0}^{\infty} \epsilon_j(a, b)\, e^{ij\theta},$$

where each coefficient $\delta_j(a, b)$ is a linear combination of the parameters a, b. If we define

$$\gamma_j(a, b) = -i\,\overline{\delta_{j-1}(a, b)},\ 2 \leqslant j \leqslant m$$

and set

$$\gamma(a,b)(\theta) = \sum_{j=2}^{m} \gamma_j(a,b)e^{ij\theta} + \overline{\gamma_j(a,b)}\, e^{-ij\theta}, \qquad (5.6)$$

then the function

$$\theta \longrightarrow ie^{i\theta}\gamma(a,b)(\theta) + P(a,b)(\theta)\alpha(\theta) = \widetilde{P}(a,b)(\theta)$$

is an element of $\mathfrak{A}^{1/2}$ for each $(a,b) \in \mathbf{R}^{2m-1}$.

The first component of the map (5.2) may now be written in the form

$$(u_1 - \gamma(a,b))ie^{i\theta} + \widetilde{P}(a,b) + i(f_1 + i\widetilde{f_1})ie^{i\theta} + i(f_2 + i\widetilde{f_2})\alpha.$$

Each term after the first one is in $\mathfrak{A}^{1/2}$ whence so is

$$(u_1 - \gamma(a,b))e^{i\theta}.$$

Since the function $u_1 - \gamma(a,b)$ is a real-valued, its most general form is

$$(u_1 - \gamma(a,b))(\theta) = Q(c,d)(\theta) = c_0 + 2\,\mathrm{Re}(c_1 - id_1)e^{i\theta}, \quad (5.7)$$

where $c = (c_0, c_1) \in \mathbf{R}^2$ and $d = d_1 \in \mathbf{R}$. Therefore the most general form of the function u_1 is $u_1 = \gamma(a,b) + Q(c,d)$.

The map $u = (u_1, u_2) : \mathbf{R}^{2m+2} \longrightarrow \mathbf{C}^{1/2}(b\Delta)^2$ determined by (5.5) and (5.7) is a linear one-one function of the $2m+2$ real parameters $t = (a,b,c,d) \in \mathbf{R}^{2m+2}$. Our construction implies that the range of the $\mathbf{C}^{k-1}$ map

$$(r,t) \longrightarrow F(r,t) = \widetilde{F}(r, u(t)), \qquad (5.8)$$

which is defined on a neighborhood of $(r,t) = (r^0, 0)$ in the Banach space $\mathbf{C}^2(V)^2 \times \mathbf{R}^s$, contains all analytic disks that have boundary in M_r and are close to F^0 in the space $(\mathfrak{A}^{1/2})^2$. This proves (a), (b) and (d) of Theorem 1. To prove (c) it suffices to show that the partial derivative $\mathrm{D}_t F(r^0, 0)$ is nonsingular. By the chain rule this follows immediately from the fact that $\mathrm{D}_u \widetilde{F}$ is nonsingular at the point $r = r^0$, $u = 0$.

If we replace $\mathfrak{A}^{1/2}$ by the algebra $\mathfrak{A}^{s,\epsilon}(b\Delta)$ for some integer $s \leqslant k-2$ and $0 < \epsilon < 1$, the map $\widetilde{F}(r,u)$ is of class $\mathbf{C}^{k-s-1}$ in (r,u) and consequently $F(r,t)$ is of class $\mathbf{C}^{k-s-1}$.

In order to prove the claim about smoothness of the map $\sigma : B \times \overline{\Delta} \longrightarrow \mathbf{C}^2$, $\sigma(r, t, \zeta) = F(r, t)(\zeta)$, observe that σ is the composition of the C^{k-s-1} map

$$B \times \overline{\Delta} \longrightarrow \mathcal{A}^{s,\epsilon} \times \overline{\Delta}, \qquad ((r, t), \zeta) \longrightarrow (F(r, t), \zeta)$$

and the evaluation map

$$E : \mathcal{A}^{s,\epsilon} \times \overline{\Delta} \longrightarrow \mathbf{C}^2 \qquad , E(F, \zeta) = F(\zeta).$$

The map E is clearly linear in F, and for each fixed F it is of class $\mathbf{C}^s$ in ζ. Therefore E is of class $\mathbf{C}^s$ whence σ is of class

$$l = \min \{s, k - s - 1\}.$$

This value is maximal when $s = l = \left[\dfrac{k-1}{2}\right]$.

It remains to show that in the case $m \geqslant 2$ there is a nonempty open subset $E \subset \mathbf{C}^2$ contained in the union $\bigcup_{t \in B_2} F(r, t)(\Delta)$ for all r close to r^0. Choose a point $\zeta_0 \in \Delta$ such that the linear map $\mathbf{R}^{2m-1} \longmapsto \mathbf{C}$, $(a, b) \longmapsto P(a, b)(\zeta_0)\zeta_0^{m-1}$ has real rank two. This is possible if $m \geqslant 2$; almost any choice of ζ_0 works. The construction of $F(r, t)$ implies that the derivative of the map $\mathbf{R}^{2m-1} \times \Delta \longmapsto \mathbf{C}^2$,

$$((a, b), \zeta) \longmapsto F(r, (a, b, 0, 0))(\zeta) \tag{5.9}$$

has real rank four at the point $a = 0$, $b = 0$, $\zeta = \zeta_0$, $r = r^0$. The calculation is similar to (6.5) and (6.6) in the Proof of Theorem 3 below, and we omit the details. The implicit function theorem implies that the map (5.9) is a submersion at $(a, b) = (0, 0)$, $\zeta = \zeta_0$, $r = r^0$, and Theorem 1 is proved.

6. Proof of Theorems 2 and 3.

Proof of Theorem 2. – Recall that $\widetilde{F}(r, u)$ is analytic if and only if the Fourier coefficients of the function (5.4) of order less than $-m + 1$ vanish. Here $f = f(r, u)$ is given by Lemma 11. We now have $m \leqslant 0$ and hence $-m + 1 > 0$. Since $\widetilde{f}_2(0) = 0$ and both $\hat{u}_2$ and $\hat{f}_2$ are real valued, it follows that $\hat{u}_2(0) = 0 = \hat{f}_2(0)$. Further, since $f_2 + i\widetilde{f}_2$ is analytic, its negative Fourier coefficients vanish whence $\hat{u}_2(k) = 0$ for $k < 0$. Since u_2 is real valued, $\hat{u}_2(k) = \overline{\hat{u}_2(-k)} = 0$ if $k > 0$. This shows that $u_2 = 0$. As in Section 5 we conclude that u_1 is of the form (5.7) with $\gamma(a, b) = 0$. Thus (5.8) defines a $\mathbf{C}^3$ map

$$F: C^4(V)^2 \times \mathbf{R}^3 \longrightarrow \mathbf{C}_{\mathbf{C}}^{1/2}(b\Delta)^2$$

such that every analytic disk with boundary in M_r which is close to F^0 is of the form $F(r, t)$ for some $t = (c_0, c_1, d) \in \mathbf{R}^3$.

Since we must also have $\widehat{f_2(r, t)}(0) = 0$, it is no longer true that each $F(r, t)$ is analytic. Denote by Σ the set of pairs $(r, t) \in \mathbf{C}^4(V)^2 \times \mathbf{R}^3$ in the domain of F for which $F(r, t)$ is an element of $(\mathfrak{A}^{1/2})^2$.

LEMMA 12. – *There is a neighborhood $U = U_1 \times U_2$ of $(r^0, 0)$ in $\mathbf{C}^4(V)^2 \times \mathbf{R}^3$ with the property: If $(r, t) \in \Sigma \cap U$, then $(r, t') \in \Sigma$ for all $t' \in U_2$.*

Proof. – We have seen in Section 5 that $D_t F(r^0, 0)$ is nondegenerate. Choose an affine map $S: C_{\mathbf{C}}^{1/2}(b\Delta)^2 \longrightarrow \mathbf{R}^3$ which is one-one on the image $D_t F(r^0, 0)(\mathbf{R}^3)$ and $S(F^0) = 0$. Such a map exists by the Hahn Banach extension theorem.

Denote by $\mathrm{Aut}\,\Delta$ the group of automorphisms of Δ. Let $C: \mathbf{C}_{\mathbf{C}}^2(b\Delta)^2 \times \mathrm{Aut}\,\Delta \longrightarrow \mathbf{C}_{\mathbf{C}}^{1/2}(b\Delta)^2$ be the composition map $C(F, \phi) = F \circ \phi$. Clearly C is linear in F; for each fixed F the map $\phi \longrightarrow C(F, \phi)$ is of class $\mathbf{C}^1$ according to Lemma 1 in [19]. Thus C is of class $\mathbf{C}^1$.

Denote by $T: \mathbf{C}^4(V)^2 \times \mathbf{R}^3 \times \mathrm{Aut}\Delta \longrightarrow \mathbf{C}_{\mathbf{C}}^{1/2}(b\Delta)^2$ the $\mathbf{C}^1$ map

$$T(r, t, \phi) = C(F(r, t), \phi) = F(r, t) \circ \phi,$$

where F is considered as a $\mathbf{C}^1$ map into $\mathbf{C}_{\mathbf{C}}^2(b\Delta)^2$. A straightforward calculation shows that the derivative $D_\phi T(r^0, 0, id)$ is nondegenerate and its range equals the range of $D_t F(r^0, 0)$.

It follows that $S \circ F: \mathbf{C}^4(V)^2 \times \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ is a submersion with respect to $t \in \mathbf{R}^3$ at the point $(r^0, 0)$, and

$$S \circ T: C^4(V)^2 \times \mathbf{R}^3 \times \mathrm{Aut}\Delta \longrightarrow \mathbf{R}^3$$

is a submersion with respect to $\phi \in \mathrm{Aut}\Delta$ at the point $(r^0, 0, id)$. The implicit function theorem implies that for each

$$(r, t, \lambda) \in \mathbf{C}^4(V)^2 \times \mathbf{R}^3 \times \mathbf{R}^3$$

close to $(r^0, 0, 0)$ there exist unique $t' \in \mathbf{R}^3$ close to 0 and $\phi \in \mathrm{Aut}\Delta$ close to the identity such that

$$S(F(r,t')) = \lambda = S(F(r,t)\circ\phi). \tag{6.1}$$

As λ runs over a neighborhood of 0 in $\mathbf{R}^3$, so does t'. Thus (6.1) determines a correspondence $\phi = \phi(r,t,t')$ satisfying

$$S(F(r,t')) = S(F(r,t)\circ\phi(r,t,t')). \tag{6.2}$$

Suppose now that $(r,t)\in\Sigma$. This means that $F(r,t)$ is an analytic disk with boundary in M_r and hence so is

$$F(r,t)\circ\phi(r,t,t').$$

Since $F(r,\cdot)$ parametrizes all analytic disks with boundary in M_r, it follows that $F(r,t)\circ\phi(r,t,t') = F(r,t_1)$ for some $t_1\in\mathbf{R}^3$ close to 0. The relation (6.2) implies $S(F(r,t')) = S(F(r,t_1))$ and hence $t' = t_1$ by the choice of S. This means that (r,t') is an element of Σ and Lemma 12 is proved.

In fact we proved more: if $(r,t)\in\Sigma$, then there is an automorphism $\phi(t)$ of the disk such that $F(r,t) = F(r,0)\circ\phi(t)$. Thus, modulo the automorphisms of Δ, there is at most one analytic disk with boundary in M_r close to F^0.

Denote by $c_0(r)$ the 0-th Fourier coefficient of the function $f_2(r,0)$. To prove Theorem 2 we have to find a path $r = r(s)$ in the parameter space $\mathbf{C}^4(V)^2$ starting at $r(0) = r^0$ such that $c_0(r(s)) \neq 0$ for $s > 0$ since then the corresponding map $F(r,0)$ is not analytic. It suffices to find a vector $r'\in\mathbf{C}^4(V)^2$ such that the derivative $\mathrm{D}_r c_0(r^0)r'$ is nonzero.

If we denote by $h(r') = (h_1(r'), h_2(r'))$ the derivative $\mathrm{D}_r f(r^0,0)r'$, then $\mathrm{D}_r c_0(r^0)r' = \widehat{h_2(r')}(0)$. From (4.2) and (4.5) we compute using the chain rule

$$\begin{aligned} 0 = \mathrm{D}_r\Phi(r^0,0,0)r' &= r'(F^0) \\ &+ \mathrm{D}_f\Phi(r^0,0,0)\cdot \mathrm{D}_r f(r^0,0)r' = r'(F^0) + C\cdot h(r'), \end{aligned} \tag{6.3}$$

where C is an invertible matrix function on $b\Delta$, If F^0 is an embedding, one can see that the set of functions $r'(F^0)$, $r'\in\mathbf{C}^4(V)^2$, is dense in the space of continuous functions on $b\Delta$. Hence (6.3) implies that we can find an r' such that the function $h_2(r')$ has positive values on $b\Delta$ and therefore $\widehat{h_2(r')}(0) > 0$. This concludes the proof of Theorem 2.

Proof of Theorem 3. – Let $m = \operatorname{Ind}_{F^0} M$. Recall that the parameter space $t = (t', t'')$ consists of $2m - 1$ parameters $t' = (a, b)$ and three parameters $t'' = (c, d)$ (Section 5). Let $\sigma(r, t', \zeta) = F(r, (t', 0))(\zeta)$. Clearly σ satisfies Theorem 3 (*b*). To prove (*a*) assume that $m = 1$ so that $t' = a \in \mathbf{R}$ and let $A(\zeta)$ be the matrix function (3.3). We have

$$D_\zeta \, \sigma(r^0, 0, \zeta) = D_\zeta \, F(r^0, 0)(\zeta) = D_\zeta \, F^0(\zeta) = A(\zeta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6.4)$$

and

$$D_a \, \sigma(r^0, 0, \zeta) = D_a \, F(r^0, 0)(\zeta) = A(\zeta) \begin{pmatrix} \alpha(\zeta) \\ 1 \end{pmatrix}. \quad (6.5$$

This shows that the map $(t, \zeta) \longrightarrow \sigma(r^0, t, \zeta)$ is a $\mathbf{C}^l$ immersion $(l = [(k - 1/2)])$ of $(-\epsilon, \epsilon) \times \overline{\Delta}$ into $\mathbf{C}^2$ for a small $\epsilon > 0$. The same is then true for each r in a neighborhood B of r^0. After a change in the t variable we may assume that $\epsilon = 1$.

For a fixed $r \in \mathrm{B}$ the map

$$I = (-1, 1) \longrightarrow \mathcal{A}^{1/2}(\Delta)^2, \ t \longrightarrow \sigma(r, t, \cdot)$$

is of class $\mathbf{C}^{k-1}$. (See the Remark 1 following Theorem 1.) For each $0 < \delta < 1$ the restriction map $\mathcal{A}^{1/2}(\Delta) \longrightarrow \mathbf{C}^k(\overline{\Delta}(0, \delta))$ is smooth. Finally, the evaluation $\mathbf{C}^k(\overline{\Delta}(0, \delta)) \times \overline{\Delta}(0, \delta) \longrightarrow \mathbf{C}$ given by $(f, \zeta) \longrightarrow f(\zeta)$ is of class $\mathbf{C}^k$. Hence $(t, \zeta) \longrightarrow \sigma(r, t, \zeta)$ is of class $\mathbf{C}^{k-1}$ on $I \times \overline{\Delta}(0, \delta)$. Since this holds for each $\delta < 1$, the map $\sigma(r, ., .)$ is $\mathbf{C}^{k-1}$ on $I \times \Delta$. This proves (*a*).

To prove (*c*) notice that when $m = 1$ and $k \geqslant 4$ the proof of Lemma 12 shows that every disk $F(r, t)$ is of the form

$$F(r, (t', 0)) \circ \varphi$$

where φ is an automorphism of Δ. Theorem 3 is proved.

7. Proof of Lemma 4 and Theorem 6.

Proof of Lemma 4. – Let $\sigma : I \times \overline{\Delta} \longrightarrow \mathbf{C}^2$ be a $\mathbf{C}^1$ immersion and $\sigma^t = \sigma(t, \cdot)$. The vectors

$$X_1(t,z) = iz \frac{\partial}{\partial z} \sigma(t,z),$$

$$X_2(t,z) = \frac{\partial}{\partial t} \sigma(t,z)$$

are tangent to M at the point $\sigma(t,z)$ for all $z \in b\Delta$ and $t \in I$, and for each fixed $t \in I$ we have

$$\det(X_1(t,z), X_2(t,z)) = iz \det\left(\frac{\partial}{\partial z} \sigma(t,z), \frac{\partial}{\partial t} \sigma(t,z)\right).$$

Since σ is an immersion, the determinant on the right hand side is nonvanishing for all $z \in \overline{\Delta}$ whence its winding number is zero. Therefore by Definition 1

$$\mathrm{Ind}_{\sigma t} M = \mathrm{W}(\det(X_1, X_2)) = \mathrm{W}(iz) = 1.$$

Lemma 4 is proved.

Proof of Theorem 6. – Choose a relatively compact subinterval $\mathrm{J} \subset (-1,1)$. The set $\Sigma = \sigma(\mathrm{J} \times \overline{\Delta})$ is an embedded hypersurface of class $\mathbf{C}^3$ in $\mathbf{C}^2$ with boundary $b\Sigma = \sigma(\mathrm{J} \times b\Delta)$. Choose a neighborhood V of Σ in $\mathbf{C}^2$ such that Σ is a closed subset of V. If V is sufficiently small, there are real $\mathbf{C}^3$ functions F and k on V such that $\Sigma_0 = \{F = 0\}$ is a closed $\mathbf{C}^3$ hypersurface in V containing Σ, $\{k = 0\}$ is a closed $\mathbf{C}^3$ hypersurface intersecting Σ_0 transversely along $b\Sigma = \sigma(\mathrm{J} \times b\Delta)$, and $k \leqslant 0$ on Σ. We may assume that $dF \neq 0$ on V.

For $c \in \mathrm{R}$ we define

$$F_c(z) = \begin{cases} F(z) + c k^3(z) & \text{if} \quad k(z) > 0, \\ F(z) \quad \text{if} \quad k(z) \leqslant 0, \end{cases}$$

and $\Sigma_c = \{z \in V : F_c(z) = 0\}$. There is an open neighborhood U of Σ contained in V such that $U \cap \Sigma_c$ is a real $\mathbf{C}^2$ hypersurface containing Σ.

LEMMA 13. – *There is a $c > 0$ and a neighborhood U_c of Σ such that $\Sigma_c \cap \mathrm{U}_c$ is pseudoconvex from the side $\{F_c < 0\}$ and $\Sigma_{-c} \cap \mathrm{U}_c$ is pseudoconvex from the side $\{F_{-c} > 0\}$.*

Proof. – We shall prove the pseudoconvexity of Σ_c $(c > 0)$ since the proof for Σ_{-c} is analoguous. Choose a neighborhood $U_c \subset U$ of Σ such that $ck \leqslant 1$ on U_c. It suffices to consider the set $W_c = U_c \cap \{k > 0\}$ since Σ_c coincides with Σ where $k \leqslant 0$. On W_c we have

$$|dF_c - dF| \leqslant |3\, ck^2\, dk| \leqslant 3\, |ck|\, |k|\, |dk| = O(k),$$

where the bound in $O(k)$ is independent of c on the set W_c. Similarly we show that $|\partial F_c - \partial F| = O(k)$ on W_c. Let $\partial F_c = a_c\, dz + b_c\, dw$ and define a $(1, 0)$ vector field X_c on U_c by $X_c = -b_c\, \partial/\partial z + a_c\, \partial/\partial w$. Then $\langle \partial F_c, X_c \rangle = 0$ which means that X_c is complex tangent to the hypersurface Σ_c.

To prove that Σ_c is pseudoconvex from the side $\{F_c < 0\}$ it suffices to show that the Levi form $L_{F_c}(X_c)$ of F_c applied to the vector X_c is nonnegative on Σ_c. From the estimate

$$|\partial F_c - \partial F| = O(k)$$

it follows that $|X_c - X_0| = O(k)$ on W_c. Computation gives

$$\begin{aligned} L_{F_c}(X_c) &= L_{F_c}(X_0) + O(k) \\ &= L_F(X_0) + 3\, ck^2 \quad L_k(X_0) + 6\, ck \quad |\langle \partial k, X_0 \rangle|^2 + O(k). \end{aligned}$$

Since Σ is Levi flat, $L_F(X_0) = 0$ on Σ and therefore

$$L_F(X_0) = O(k) \quad \text{on} \quad W_c \cap \Sigma_c.$$

The second term on the right is also $O(k)$. Since the hypersurface $\{k = 0\}$ intersects each leaf of Σ transversely and X_0 is tangent to these leaves, the quantity $|\langle \partial k, X_0 \rangle|^2$ is bounded below by a constant $K > 0$. Thus

$$L_{F_c}(X_c) \geqslant 6\, cK\; k + O(k) \geqslant (6\, cK - c_1)\, k$$

on $W_c \cap \Sigma_c$, where c_1 is a constant independent of c. If we choose $c > c_1/6K$, then the Levi form of Σ_c is strictly positive on $\Sigma_c \cap W_c$. Therefore Σ_c is strictly pseudoconvex outside Σ and Lemma 13 is proved.

Fix a $c > 0$ for which the conclusions of Lemma 13 hold. The domain $A^+ = \{z \in U_c : F_c(z) < 0\}$ is pseudoconvex along the hypersurface $\Sigma^+ = \Sigma_c \cap U_c$, the domain

$$A^- = \{z \in U_c : F_{-c}(z) > 0\}$$

is pseudoconvex along $\Sigma^- = \Sigma_c \cap U_c$, and $\bar{A}^+ \cap \bar{A}^- = \Sigma$. Since $\sigma : I \times \bar{\Delta} \longrightarrow \mathbf{C}^2$ is an embedding that is holomorphic in $\zeta \in \Delta$, the vector field $V(t, \zeta) = i \dfrac{d}{dt} \sigma(t, \zeta)$ is also holomorphic in ζ and is transversal to Σ. Let $Y(\zeta)$ be a holomorphic vector field on a neighborhood of $\sigma^0(\bar{\Delta})$ in $\mathbf{C}^2$ that approximates the holomorphic field $V(0, \zeta)$ uniformly on the disk $\sigma^0(\bar{\Delta})$. If the approximation is close enough, then Y is transversal to Σ near $\sigma^0(\bar{\Delta})$; suppose that it is pointing into the domain A^+.

Let $f : \bar{\Delta} \longrightarrow \mathbf{C}^2$ be an analytic disk with boundary in $b\Sigma$. If $f(\bar{\Delta})$ is sufficiently close to $\sigma^0(\bar{\Delta})$ but is not contained in $\bar{A}^+$, then for a suitable $\epsilon > 0$ the analytic disk

$$\zeta \longrightarrow f(\zeta) + \epsilon \cdot Y(f(\zeta))$$

has boundary in A^+ and it touches the pseudoconvex hypersurface Σ^+ in an interior point. This is a contradiction and therefore $f(\bar{\Delta})$ is contained in $\bar{A}^+$. In a similar way we prove that $f(\bar{\Delta})$ is contained in $\bar{A}^-$ whence $f(\bar{\Delta}) \subset \Sigma$. It follows that $f(\bar{\Delta})$ is a leaf of Σ. This concludes the proof of Theorem 6.

8. Examples.

Example 1. – Let $\mathbf{T}^2 = \{(e^{i\theta}, e^{i\tau}) \mid \theta, \tau \in \mathbf{R}\}$ be the standard totally real torus in $\mathbf{C}^2$. The vector fields $X = (ie^{i\theta}, 0)$ and $Y = (0, ie^{i\tau})$ parallelize the tangent bundle of $\mathbf{T}^2$. If

$$F = (F_1, F_2) : \bar{\Delta} \longrightarrow \mathbf{C}^2$$

is an analytic disk with boundary in $\mathbf{T}^2$, the index $\mathrm{Ind}_F \mathbf{T}^2$ equals the winding number of the determinant

$$\det\begin{pmatrix} iF_1(e^{i\theta}) & 0 \\ 0 & iF_2(e^{i\theta}) \end{pmatrix} = -F_1(e^{i\theta})F_2(e^{i\theta})$$

which is $\mathrm{W}(F_1) + \mathrm{W}(F_2)$. By the argument principle $\mathrm{W}(F_1)$ equals the number of zeroes of F_1 in Δ and similarly for F_2. For example, if $F(z) = (z^p, z^q)$, we have $\mathrm{Ind}_F \mathbf{T}^2 = p + q$. In particular, the index of every nonconstant analytic disk with boundary in $\mathbf{T}^2$ is positive.

By Theorem 1 there is an $s = 2(\mathrm{W}(F_1) + \mathrm{W}(F_2)) + 2$ parameter family of analytic disks with boundary in $\mathbf{T}^2$ near F. In our case we can see this directly. Since F_j is a proper holomorphic map of Δ onto itself, F_j is a finite Blaschke product

$$F_j(z) = e^{ic_j} \prod_{k=1}^{s_j} \frac{z - \alpha^j_k}{1 - \overline{\alpha}^j_k z}, \quad j = 1, 2$$

depending on $s_j = 2\mathrm{W}(F_j) + 1$ real parameters c_j, $\mathrm{Re}\alpha^j_k$, $\mathrm{Im}\alpha^j_k$. All disks with boundary in $\mathbf{T}^2$ are stable under small $\mathbf{C}^2$ perturbations of $\mathbf{T}^2$.

The disks with index one are either of the form

$z \longrightarrow (\phi(z), c)$ or $z \longrightarrow (c, \phi(z))$, where ϕ is an automorphism of Δ and $|c| = 1$. The images of these disks form the topological boundary of the bidisk

$$\Delta^2 = \{(z, w) \in \mathbf{C}^2 : |z| < 1, |w| < 1\}.$$

By Theorem 3 the two families of disks are stable under small $\mathbf{C}^3$ perturbations of $\mathbf{T}^2$, a result that has been proved already by Alexander [1] and Bedford [4]. Lemma 4 implies that there are no other immersed families of disks with boundary in $\mathbf{T}^2$.

Example 2. – Although it is known that the torus is the only compact orientable two-manifold that can be embedded totally real to $\mathbf{C}^2$, Rudin showed [24] that the (nonorientable) Klein bottle also embeds totally real in $\mathbf{C}^2$. Here is a slight modification of his example. Choose constants $a > b > 0$ and let $\Phi : \mathbf{R}^2 \longrightarrow \mathbf{C}^2$ be defined by

$$\begin{aligned} \Phi_1(\phi, \theta) &= (a + b\cos\phi)\, e^{2i\theta}, \\ \Phi_2(\phi, \theta) &= (\sin\phi + i\sin 2\phi)\, e^{i\theta}. \end{aligned} \tag{8.1}$$

One can check easily that Φ is one-to-one on the region

$$-\pi \leqslant \phi < \pi, \ 0 \leqslant \theta < \pi$$

and that it satisfies the conditions

$$\Phi(\phi + 2\pi, \theta) = \Phi(\phi, \theta) = \Phi(-\phi, \theta + \pi).$$

The complex Jacobian of Φ equals

$$\det d\Phi(\phi, \theta) = ie^{3i\theta}(b\cos^2\phi + 2a\cos\phi + b \qquad (8.2)$$
$$+ i(6b\cos^3\phi + 8a\cos^2\phi - 2b\cos\phi - 4a)).$$

One can check that the function in parenthesis has no zeroes on $\mathbf{R}^2$ and therefore $\Phi(\mathbf{R}^2) = K$ is a totally real embedded Klein bottle in $\mathbf{C}^2$.

For each $\phi \in \mathbf{R}$ we define a map $F_\phi : \mathbf{C} \longrightarrow \mathbf{C}^2$ by

$$F_\phi(z) = ((a + b\cos\phi)z^2, (\sin\phi + i\sin 2\phi)z).$$

Clearly F_ϕ is an analytic disk with boundary in K. From (8.2) we see that the Jacobian determinant of Φ along the curve $\theta \longrightarrow F_\phi(e^{i\theta})$ is a constant multiple of $e^{3i\theta}$ and therefore $\operatorname{Ind}_{F_\phi} K = 3$ for each $\phi \in \mathbf{R}$. Thus there is an 8-parameter family of disks with boundary in K near F_ϕ, and each F_ϕ generates an open subset contained in the hull of K according to Theorem 1. One parameter is ϕ, and three parameters come from compositions of F_ϕ with automorphisms of Δ. This leaves four other parameters. It seems far from obvious how to find these remaining disks without using Theorem 1.

For $\phi = 0$ we have $F_0(z) = ((a + b)z^2, 0)$. The disk $f(z) = ((a + b)z, 0)$ also has boundary in K. Since the cycle $[F_0] \in \mathrm{H}_1 K$ determined by F_0 equals $2[f]$ and index is a homomorphism on $\mathrm{H}_1 K$, we have $\operatorname{Ind}_f K = \frac{1}{2}\operatorname{Ind}_{F0} K = \frac{3}{2}$. Recall that the fractional indices can occur only in the case of nonorientable submanifolds.

For each integer k we define a map

$$\Psi_k : \mathbf{C}^* \times \mathbf{C} \longrightarrow \mathbf{C}^* \times \mathbf{C}$$

by

$$\Psi_k(z, w) = (z, z^k w).$$

Clearly Ψ_k is a biholomorphic with the inverse $(z, w) \longrightarrow (z, z^{-k} w)$. If K is the Klein bottle considered above, then $K_k = \Psi_k(K)$ is also a totally real Klein bottle. The Jacobian of Ψ_k equals

$$d(z) \det \Psi_k(f(z)) = (a+b)^k z^k .$$

Lemma 7 implies

$$\operatorname{Ind}_f K_k = \operatorname{Ind}_f K + \mathrm{W}(d) = \frac{3}{2} + k.$$

This shows that every number in $\dfrac{1}{2}\mathbf{Z}\backslash\mathbf{Z}$ can arise as the index of an analytic disk with boundary in a totally real Klein bottle in $\mathbf{C}^2$.

Similarly, the map $\widetilde{\Phi} : \mathbf{R}^2 \longrightarrow \mathbf{C}^2$,

$$\widetilde{\Phi}(\phi, \theta) = (\cos\phi\, e^{2i\theta}, \sin\phi\, e^{i\theta})$$

induces a totally real *immersion* of the Klein bottle into the boundary of the unit ball in $\mathbf{C}^2$, and $F_\phi(z) = (\cos\phi \cdot z^2, \sin\phi \cdot z)$ is an analytic disk with boundary in $\widetilde{\Phi}(\mathbf{R}^2)$ and with index 3.

Example 3. – In this example we show that the results of Theorem 1 hold only for sufficiently small perturbations of a maximal real manifold $M \subset \mathbf{C}^2$. We shall construct a regular homotopy of totally real embedded tori T_ϵ, $\epsilon \geqslant 0$, starting at the torus

$$T_0 = \{|z| = 1, |w| = 1\}$$

in $\mathbf{C}^2$, such that at $\epsilon = 1$ the structure of analytic disks with boundary in T_ϵ and the polynomially convex hull $\hat{T}_\epsilon$ change catastrophically. This example is of interest in connection with the classical nonlinear Riemann problem [23, p. 591; 31].

For $\epsilon \geqslant 0$ and $k \in \mathbf{Z}$ we define a map $\Phi_{\epsilon,k} : (\mathbf{R}^2) \longrightarrow \mathbf{C}^2$ by

$$\Phi_{\epsilon,k}(\theta, \tau) = (e^{i\theta}, e^{ik\theta}(\epsilon + e^{i\tau})). \tag{8.3}$$

We have $\Phi_{\epsilon,k}(\theta, \tau) = \Phi_{\epsilon,k}(\theta', \tau')$ if and only if $\theta \equiv \theta' \pmod{2\pi}$ and $\tau \equiv \tau' \pmod{2\pi}$. The Jacobian of $\Phi_{\epsilon,k}$ equals

$$\det d\Phi_{\epsilon,k}(\theta, \tau) = -e^{i(k+1)\theta} e^{i\tau} \tag{8.4}$$

which has no zeroes on $\mathbf{R}^2$. Hence $\Phi_{\epsilon,k}(\mathbf{R}^2) = T_{\epsilon,k}$ is a totally real torus in $\mathbf{C}^2$ for each ϵ and k. The map $\Phi_{\epsilon,k}$ is smooth in ϵ, and $T_{0,k}$ is the standard torus $b\Delta \times b\Delta$.

To find the index of $T_{\epsilon,k}$ along a closed curve

$$F = (F_1, F_2) : b\Delta \longrightarrow T_{\epsilon,k}$$

we evaluate the Jacobian of $\Phi_{\epsilon,k}$ at the points $F(z), z \in b\Delta$. If $\Phi_{\epsilon,k}(\theta, \tau) = (z, w)$, it follows from (8.3) and (8.4) that

$$\det d\Phi_{\epsilon,k}(\theta, \tau) = -zw + \epsilon z^{k+1}.$$

Therefore $\mathrm{Ind}_F T_{\epsilon,k}$ equals the winding number of the function

$$d(z) = -F_1(z) F_2(z) + \epsilon F_1(z)^{k+1}, \quad z \in b\Delta.$$

In particular, if $\epsilon = 1$ and $F^0(z) = (z, 0)$, then

$$\mathrm{Ind}_{F^0} T_{1,k} = \mathrm{W}(z^{k+1}) = k+1.$$

This shows that there exist totally real tori in $\mathbf{C}^2$ with disks of arbitrary integral indices, both positive and negative.

We shall analyse more carefully the analytic disks of the form

$$G(z) = (z, g(z)), \quad z \in \overline{\Delta} \tag{8.5}$$

with boundaries in $T_\epsilon = T_{\epsilon,-1}$ and show that their nature changes discontinuously at the point $\epsilon = 1$. The condition $G(b\Delta) \subset T_\epsilon$ is equivalent to

$$|zg(z) - \epsilon| = 1, \quad z \in b\Delta.$$

Therefore there is a finite Blaschke product $B(z)$ such that

$$zg(z) = \epsilon + B(z), \quad z \in \Delta. \tag{8.6}$$

Evaluating the equation (8.6) at $z = 0$ gives $B(0) = -\epsilon$. Since $|B| \leqslant 1$ on $\overline{\Delta}$, we must have $\epsilon \leqslant 1$ or else there are no disks of the form (8.5) with boundaries in T_ϵ. If $\epsilon = 1$, then $B(0) = -1$ whence $B \equiv -1$ on Δ. Therefore the only disk of the form (8.5) with boundary in T_1 is $F^0(z) = (z, 0)$.

If $0 \leqslant \epsilon < 1$, the equation (8.6) has several solutions. We shall only consider those for which B is a single Blaschke factor,

$$B(z) = e^{ic} \frac{z-a}{1-\bar{a}z}, \quad a \in \Delta, c \in \mathbf{R}. \tag{8.7}$$

The conditions $B(0) = -\epsilon$ implies $\epsilon = ae^{ic}$. For each $\epsilon \in [0, 1)$ we have a family of solutions of (8.6)

$$g(c,\epsilon)(z) = \frac{(1-\epsilon^2)\, e^{ic}}{1-\epsilon\, e^{ic} z}\,, \tag{8.8}$$

depending on one real parameter c, such that the corresponding analytic disks

$$G(c,\epsilon)(z) = (z, g(c,\epsilon)(z)) \tag{8.9}$$

have boundaries in T_ϵ. A simple calculation shows that the index of T_ϵ along $G(c,\epsilon)$ equals one.

Similarly there exist disks of the form (8.5) with index bigger than one provided that $0 \leqslant \epsilon < 1$. They correspond to the solutions of (8.6) where the Blaschke product B consists of more than one factor.

A catastrophic change occurs at the point $\epsilon = 1$. The disks with positive indices that exist for $\epsilon < 1$ disappear and all that is left is the "singular" disk F^0 of index 0 with boundary in T_1. For $\epsilon > 1$ there are no disks of the form (8.5) with boundary in T_ϵ.

There is a family of analytic disks

$$F_c(z) = (e^{ic}, z + \epsilon\, e^{ic}), \quad z \in \bar{\Delta}, c \in \mathbf{R}$$

with boundary in T_ϵ for each ϵ. A simple calculation shows that $\mathrm{Ind}_{F_c} T_\epsilon = 1$ and by Theorem 1 this family is stable under small $\mathbf{C}^2$ perturbations of T_ϵ.

We shall show that for each $\epsilon \geqslant 0$ the polynomial hull of T_ϵ is the union of images of all analytic disks with boundaries in it and therefore the hull also changes catastrophically at $\epsilon = 1$. Denote by M_ϵ the hypersurface

$$M_\epsilon = \{(e^{i\theta}, w) \mid \theta \in \mathbf{R}, \mid w - \epsilon\, e^{-i\theta} \mid \leqslant 1\}$$

with boundary T_ϵ. If $\epsilon < 1$ and $G(c,\epsilon)$ is given by (8.9), we denote by N_ϵ the hypersurface

$$N_\epsilon = \{G(c,\epsilon)(z) \mid z \in \bar{\Delta}, c \in \mathbf{R}\}$$

with boundary in T_ϵ. The union $M_\epsilon \cup N_\epsilon$ bounds a pseudoconvex domain Ω_ϵ when $\epsilon < 1$. Since M_ϵ and N_ϵ are contained in the polynomial hull $\hat{T}_\epsilon$, the same is true of $\bar{\Omega}_\epsilon$. One can see that Ω_ϵ is the union images of analytic disks of the form (8.5).

PROPOSITION 14. – *The polynomially convex hull of* T_ϵ *equals*

(a) *the closure of the domain* Ω_ϵ *if* $0 \leqslant \epsilon < 1$,

(b) *the union of* M_ϵ *and the disk* $\Delta \times \{0\}$ *if* $\epsilon = 1$, *and*

(c) *the hypersurface* M_ϵ *if* $\epsilon > 1$.

Proof of (a). – Fix an $\epsilon \in [0, 1)$ and let $\Omega = \Omega_\epsilon$. Observe that for each $z \in \Delta$ the fiber

$$\Omega_z = \{w \in \mathbf{C} \mid (z, w) \in \Omega\}$$

is a region in $\mathbf{C}$ bounded by the closed curve

$$\gamma : \mathbf{R} \longrightarrow \mathbf{C}, \gamma(c) = g(c, \epsilon)(z),$$

where

$$g(c, \epsilon)(z) = \frac{(1 - \epsilon^2)\, e^{ic}}{1 - \epsilon\, e^{ic} \cdot z} = \frac{(1 - \epsilon^2)}{e^{-ic} - \epsilon \cdot z}$$

is given by (8.8). The above formula for g shows that the image of γ equals the image of the circle $|w| = 1$ under the conformal map $w \longrightarrow (1 - \epsilon^2)/(w - \epsilon z)$. Hence the boundary of Ω_z is a circle containing 0 in the interior.

Let $A_t : \mathbf{C}^2 \longrightarrow \mathbf{C}^2$ be multiplication by $t > 0$ in the second coordinate, i.e., $A_t(z, w) = (z, tw)$. The family of pseudoconvex domains $\Omega^t = A_t(\Omega)$ in $\mathbf{C}^2$ has the following properties:

(i) $\Omega^t \subset \Omega^s$ if $t < s$,

(ii) $\bigcup_{t<s} \Omega^t = \Omega^s$ and $\bigcup_{t>0} \Omega^t = \Delta \times \mathbf{C}$,

(iii) $\bigcap_{t>s} = \overline{\Omega}^s$, where the closure is in $\Delta \times \mathbf{C}$, and

(iv) $\mathrm{Int}\,(\bigcap_{t>s} \Omega^t) = \Omega^t$.

A theorem of Docquier and Grauert [13] implies that $\overline{\Omega}$ is holomorphically convex in the Stein manifold $\Delta \times \mathbf{C}$ and therefore it is polynomially convex. This proves Part (a).

Proof of (b). – As ϵ tends to 1 from below, the domains Ω_ϵ converge to $M_1 \cup (\Delta \times \{0\})$. By part (a) it suffices to show that no point $(z, w) \in \Delta \times \mathbf{C}^*$ lies in the hull of T_1. From (8.8) we see immediately that

$$\lim_{\substack{\epsilon \to 1 \\ \epsilon < 1}} g(c, \epsilon)(z) = 0$$

uniformly in $c \in \mathbf{R}$ and uniformly on the compact subsets of Δ. Fix a $w \in \mathbf{C}^*$ and choose an $\epsilon < 1$ such that $2\,|g(c, \epsilon)(z)| < |w|$ for all c. Then the point (z, w) does not lie in $\overline{\Omega}_\epsilon$. Since

$$T_1 \subset A_2(M_\epsilon) \subset A_2(\overline{\Omega}_\epsilon)$$

for each $0 < \epsilon < 1$, part (a) implies $\hat{T}_1 \subset \overline{\Omega}_\epsilon$. This shows that (z, w) does not lie in $\hat{T}_1$ and part (b) is proved.

Proof of (c). – Observe that for $\epsilon > 1$ the set T_ϵ is contained in $A_t(M_1)$ if $t > 0$ is sufficiently large. For example, $t = 2\epsilon$ would do. Therefore by part (b) we have

$$\hat{T}_\epsilon \subset A_t(\hat{M}_1) = A_t(M_1 \cup \Delta \times \mathbf{C}) .$$

Let $M = M_\epsilon$ and denote, for each $z \in b\Delta$, by M_z the fiber of M over $z : M_z = \{w \in \mathbf{C} \mid (z, w) \in M\}$. Since each fiber M_z is a convex disk in $\{z\} \times \mathbf{C}$, it follows that

$$\hat{T}_\epsilon \subset M \cup (\Delta \times \{0\}) .$$

It remains to prove that no point $(z, 0)$, $z \in \overline{\Delta}$, is in the hull of T_ϵ.

By Runge's theorem we can find a holomorphic polynomial $P(w)$ arbitrarily close to the function $-\dfrac{1}{w}$ on the fiber $M_1 \subset \mathbf{C}$. The polynomial $P(zw)$ is then close to $-\dfrac{z}{w}$ on the fiber M_z for each $z \in b\Delta$ and so the polynomial

$$h(z, w) = z + wP(zw)$$

is small on M. Since $h(z, 0) = z$, this implies that no point $(z, 0)$ for $z \in \mathbf{C}^*$ is in the hull of T_ϵ. To show the same for $(0, 0)$ we choose an automorphism B of Δ such that $B(0) \neq 0$ and approximate the function

$$B(z) + wP(B(z)w)$$

by a holomorphic polynomial on a neighborhood of $\overline{\Delta} \times \mathbf{C}$. This proves (c) and concludes the proof of Proposition 14.

Example 4. – We shall use the previous example to construct totally real three dimensional tori in $\mathbf{C}^3$ with rather bizzare polynomial hulls.

Choose a real $\mathbf{C}^\infty$ function $\epsilon \geqslant 0$ on $b\Delta$ and define the following subsets of $\mathbf{C}^3$:

$$T_\epsilon = \{(e^{i\theta}, e^{-i\theta}(2 + e^{i\phi}), e^{-i(\theta+\phi)}(1 + \epsilon(e^{i\theta}) + e^{i\tau})) \mid \theta, \phi, \tau \in \mathbf{R}\},$$

$$M_\epsilon = \{(e^{i\theta}, e^{-i\theta}(2 + e^{i\phi}), e^{-i(\theta+\phi)}(1 + \epsilon(e^{i\theta}) + re^{i\tau})) \mid \theta, \phi, \tau \in \mathbf{R}, 0 \leqslant r \leqslant 1\}.$$

The set T_ϵ is a smooth totally real three-torus in $\mathbf{C}^3$, and M_ϵ is a smooth four-dimensional submanifold with boundary T_ϵ. We fix a function ϵ and drop ϵ in our notation.

Denote, for each $z \in b\Delta$, by T_z the fiber of T over z, and similarly for M_z. Considering the projection onto $\mathbf{C}^2 \times \{0\}$ and applying Proposition 14 (c) we conclude that the hull $\hat{T}$ is obtained by taking the polynomially convex hull of each fiber $T_z, z \in b\Delta$. Parts (b) and (c) of Proposition 14 imply that $\hat{T}_z = M_z$ if $\epsilon(z) > 0$ and $\hat{T}_z = M_z \cup \Delta_z$ if $\epsilon(z) = 0$, where

$$\Delta_z = \{(\zeta, 0) \in \mathbf{C}^2 : |\zeta - 2\bar{z}| < 1\}.$$

Choose a closed subset $K \subset b\Delta$ and let ϵ be a smooth nonnegative function on $b\Delta$ that vanishes precisely on K. The above implies

$$\hat{T}_\epsilon = M_\epsilon \cup \left(\bigcup_{z \in K} \{z\} \times \Delta_z \right).$$

For example, if we choose K to be a Cantor set, then the hull of the torus T_ϵ is the smooth manifold with boundary M_ϵ together with a Cantor set of disjoint disks.

Example 5. – In this example we show that our results in dimension $n = 2$ do not have straightforward generalizations for $n \geqslant 3$.

For each $k \in \mathbf{R}$ let $\Phi_k : \mathbf{R}^3 \longrightarrow \mathbf{C}^3$ be the smooth map

$$\Phi_k(\theta, t, s) = (e^{i\theta}, te^{ik\theta}, (s + i\chi(t^2))\, e^{-i\theta}),$$

where $\chi(t)$ is a smooth function which equals 0 for $t \leqslant 0$ and is positive for $t > 0$. One can check easily that Φ_k is a totally real embedding of $\mathbf{R}^3$ into $\mathbf{C}^3$. The set $M_k = \Phi_k(\mathbf{R}^3)$ contains the boundary of the analytic disk $F^0(z) = (z, 0, 0)$ (set $t = s = 0$). The vectors

$$X_1(\theta) = \frac{\partial}{\partial \theta} \Phi_k(\theta, 0, 0) = \begin{pmatrix} ie^{i\theta} \\ 0 \\ 0 \end{pmatrix},$$

$$X_2(\theta) = \frac{\partial}{\partial t} \Phi_k(\theta, 0, 0) = \begin{pmatrix} 0 \\ e^{ik\theta} \\ 0 \end{pmatrix},$$

$$X_3(\theta) = \frac{\partial}{\partial s} \Phi_k(\theta, 0, 0) = \begin{pmatrix} 0 \\ 0 \\ e^{-i\theta} \end{pmatrix}$$

form a real basis of $\mathrm{T}_{F^0(\theta)} M_k$ for each θ. Notice that $\mathrm{Ind}_{F^0} M_k = k$. The components of the first two vectors extend holomorphically to Δ if $k \geqslant 0$, but we cannot find three independent vector fields tangent to M_k along $F^0(b\Delta)$ with this property.

The analytic disks close to F^0 can be written in the form

$$F(z) = (z, a(z), b(z)). \tag{8.10}$$

The third component is of the form

$$b(e^{i\theta}) = (s(e^{i\theta}) + i\chi \circ t^2(e^{i\theta}))\, e^{-i\theta}. \tag{8.11}$$

If b is the boundary value of an analytic function on Δ, then the 0-th Fourier coefficient of s and $\chi(t^2)$ vanishes. Since $t^2 \geqslant 0$, the definition of χ implies that t is identically zero, and (8.11) implies that $s = 0$ as well.

It follows that F^0 is the only analytic disk of the form (8.10) with boundary in M_k. Notice however that along the circle $F^0\,(b\Delta)$ the manifold M_k has infinite order of contact with the manifold

$$M'_k \doteq \{(e^{i\theta}, te^{ik\theta}, se^{-i\theta}) \mid t, s \in \mathbf{R}\}$$

which bounds a $2k+1$ parameter family of disks of the form (8.10). This shows the subtlety of our problem in dimensions bigger than two.

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