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THE TRACE INEQUALITY AND EIGENVALUE ESTIMATES FOR SCHRODINGER OPERATORS

by R. KERMAN (1) and E. SAWYER (2)

1. Introduction.

This paper deals with potential operators T_Φ given at Lebesgue measurable f on $\mathbf{R}^n$ by a convolution integral

$$(T_\Phi f)(x) = \int_{\mathbf{R}^n} \Phi(x-y)f(y)\,dy,$$

provided this integral exists for almost all $x \in \mathbf{R}^n$. The kernels $\Phi(y)$ are radially decreasing (r.d.) functions; that is, they are nonnegative, locally integrable radial functions on $\mathbf{R}^n$, which are nonincreasing in $|y|$. These T_Φ include the Riesz potential operator I_α whose kernel K_α is defined directly as

$$K_\alpha(y) = |y|^{\alpha-n}, \qquad 0 < \alpha < n$$

and the Bessel potential operator J_α with kernel G_α defined in terms of its Fourier transform $\hat{G}_\alpha$ by

$$\hat{G}_\alpha(\zeta) = \int_{\mathbf{R}^n} G_\alpha(x)e^{-i\zeta.x}\,dx = (1+|\zeta|^2)^{-\frac{\alpha}{2}}, \qquad 0 < \alpha < n.$$

Given an r.d. kernel Φ and $1 < p < \infty$, we wish to characterize the (possibly singular) positive Borel measures μ on $\mathbf{R}^n$ for which there exists $C > 0$ such that

$$(1.1) \qquad \int_{\mathbf{R}^n} (T_\Phi f)(x)^p\,d\mu(x) \leqslant C \int_{\mathbf{R}^n} f(x)^p\,dx$$

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for all nonnegative measurable f. Clearly this will be true if and only if T_Φ is a bounded linear operator between the Lebesgue spaces $L^p(\mathbf{R}^n)$ and $L^p(\mathbf{R}^n,\mu)$. An important special case, with $p=2$ and $\Phi=G_1$, arises in estimating the spectrum of Schrödinger operators and will be considered in detail below. Another special case is treated in Stein [19], where it is shown that (1.1) holds for J_α when $\mu = \mu_k, \alpha > \frac{n-k}{p}$, where

$$\mu_k(E) \equiv m_k(E \cap \mathbf{R}^k),$$

m_k being k-dimensional Lebesgue measure on $\mathbf{R}^k$ considered as a subset of $\mathbf{R}^n$. The inequality of [19] can be stated in the equivalent form

$$\int_{\mathbf{R}^n} (J_\alpha f)(x_1,\ldots,x_k,0,\ldots,0)^p\, dx_1,\ldots,dx_k \leqslant C \int_{\mathbf{R}^n} f(x_1,\ldots,x_n)^p\, dx_1,\ldots,dx_n.$$

It is thus a statement about the restriction, or trace, of $J_\alpha f$. For this reason we follow other authors in referring to (1.1) as « the trace inequality ».

Generalizing results of Adams [1] and Maz'ya [14], K. Hansson in [12] has characterized the μ satisfying (1.1) in terms of capacities (see also B. Dahlberg [8]). He shows the trace inequality holds if and only if $K > 0$ exists for which

$$\mu(E) \leqslant K \operatorname{cap}(E) \tag{1.2}$$

whenever E is a compact subset of $\mathbf{R}^n$. Here cap (E) denotes the L^p capacity associated with the kernel Φ,

$$\operatorname{cap}(E) = \inf\left\{\int_{\mathbf{R}^n} f(x)^p\, dx : f \geqslant 0 \text{ and } T_\Phi f \geqslant 1 \text{ on } E\right\}.$$

A criterion such as (1.2) can be difficult to verify for all compact sets E. On the other hand if one only requires (1.2) to hold for a class of simple sets such as all cubes Q with sides parallel to the coordinate axes, the resulting condition is no longer sufficient (D. Adams [2]). For example, when $n = p = 2$, $I_{\frac{1}{2}}$ doesn't satisfy (1.1) with μ_1, yet inequality (1.2) for cubes, which amounts to $\mu_1(Q) \leqslant K|Q|^{\frac{1}{2}}$, holds. In fact, with $f(x) = x_2^{-\frac{1}{2}} |\ln x_2|^{-1} \chi_{\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]}(x_1,x_2)$, $I_{\frac{1}{2}} f$ is infinite on

$\left\{(x_1,0) : 0 \leqslant x_1 \leqslant \frac{1}{2}\right\}$ and thus the left side of (1.1) is infinite while the right side is finite. Examples of this nature were first pointed out in [2].

Theorem 2.3 below gives a necessary and sufficient condition for (1.1) that involves testing an inequality over dyadic cubes Q, namely

$$(1.3) \qquad \int_Q (\mathrm{M}_\Phi \mathrm{X}_Q \mu)(x)^{p'}\, dx \leqslant \mathrm{K} \int_Q d\mu < \infty$$

where $p' = \frac{p}{p-1}$, the constant $\mathrm{K} > 0$ is independent of Q, and

$$(\mathrm{M}_\Phi f\mu)(x) = \sup_{x \in Q} \left[\frac{1}{|Q|} \int_{|y| \leqslant |Q|^{\frac{1}{n}}} \Phi(y)\, dy\right] \int_Q f(y)\, d\mu(y).$$

Alternatively, (1.1) is equivalent to

$$(1.4) \qquad \int_{\mathbf{R}^n} (\mathrm{T}_\Phi \chi_Q \mu)(x)^{p'}\, dx \leqslant \mathrm{K} \int_Q d\mu < \infty \quad \text{for all dyadic cubes } Q.$$

To compare (1.2) and (1.4), we note that (1.2) is equivalent by an elementary argument (see Theorem 4 in [2]) to testing the inequality in (1.4) over all compact sets Q. The reduction in (1.4) to testing over dyadic cubes Q is essential in obtaining sharp estimates for the higher eigenvalues of Schrödinger operators in § 3. For a different characterization involving test functions see Stromberg and Wheeden [21].

In the special case where $\mathrm{T}_\Phi = \mathrm{I}_\alpha$, the equivalence of (1.1) and (1.3) can be established by dualizing inequality (1.1), using the « good λ inequality » of B. Muckenhoupt and R. L. Wheeden [15] in order to replace I_α by its associated maximal operator M_α, and then using the characterization of the weighted inequality for M_α in [18]. The general case of the theorem is proved along similar lines, the crucial new estimate being an extension (Theorem 2.2) of the « good λ inequality » in [15].

As an application of Theorem 2.3 we obtain a sharpened form of recent results of C. L. Fefferman and D. H. Phong on the distribution of eigenvalues of Schrödinger operators, $\mathrm{H} = -\Delta - v$, $v \geqslant 0$ ([10]; Theorem 5, 6 and 6′ in Chapter II). Roughly speaking, their results show that for many $v \geqslant 0$, the negative eigenvalues of $\mathrm{H} = -\Delta - v$ are approximately given by $-|Q|^{-\frac{2}{n}}$ as Q varies over the minimal dyadic cubes satisfying $|Q|^{\frac{2}{n}-1}\int_Q v \geqslant C$. Theorem 3.3 below shows, as suggested by condition (1.3), that this picture extends to arbitrary $v \geqslant 0$ if the fractional average, $|Q|^{\frac{2}{n}-1}\int_Q v$, is replaced by

$$\frac{1}{|Q|_v}\int [I_1(\chi_Q v)(x)]^2\,dx = \frac{1}{|Q|_v}\int_Q I_2(\chi_Q v)(x)v(x)\,dx,$$

the v-average over Q of the Newtonian potential of $\chi_Q v$. Certain of the results in [10] have been generalized by S. Y. A. Chang, J. M. Wilson and T. H. Wolff ([5]) and by S. Chanillo and R. L. Wheeden ([6]). This is discussed in more detail in § 3. Further applications of Theorem 2.3 have been announced in [13].

2. The trace inequality.

We begin by deriving the basic properties of r.d. kernels Φ and Borel measures μ for which the trace inequality holds. For the sake of completeness, we consider here and in § 3 the more general trace inequality

$$(2.1)\qquad \left[\int_{\mathbf{R}^n}(T_\Phi f)(x)^q\,d\mu(x)\right]^{\frac{1}{q}} \leqslant C\left[\int_{\mathbf{R}^n} f(x)^p\,dx\right]^{\frac{1}{p}}$$

for all nonnegative measurable f, where $1 < p \leqslant q < \infty$. For $p < q$ and many r.d. kernels Φ, the trace inequality (2.1) can be characterized in terms of very simple conditions – see e.g. [12]. However, many applications, such as that in the next section, require the case $p = q$.

PROPOSITION 2.1. – *If* (2.1) *holds for a non-trivial r.d. kernel* Φ *and a non-trivial Borel measure* μ, *then* (*i*) μ *is locally finite, that is,* $\int_Q d\mu < \infty$ *for all cubes* Q, *and* (*ii*) Φ *satisfies*

$$(2.2)\qquad \int_{|y|\geqslant r}\Phi(y)^{p'}\,dy < \infty \qquad \textit{for all } r > 0.$$

Proof. – Choose $\varepsilon > 0$ so that $\Phi(2\varepsilon) > 0$. If B is any ball of radius ε, and if γ_n denotes the measure of the surface of the unit ball in

$\mathbf{R}^n$, then

$$\gamma_n \varepsilon^n \Phi(2\varepsilon)\left(\int_B d\mu\right)^{\frac{1}{q}} \leqslant \left[\int_B (T_\Phi \chi_B)^q \, d\mu\right]^{\frac{1}{q}} \leqslant [\gamma_n \varepsilon^n]^{\frac{1}{p}} \|T_\Phi\|_{0p} < \infty .$$

Hence $\int_B d\mu < \infty$ and this proves that μ is locally finite.

To obtain (2.2), fix $R > 0$ so that $\int_B d\mu > 0$ where B is the ball of radius R centred at the origin. Momentarily fix $S > 2R$ and let $f(x) = \Phi(x)^{p'-1}\chi_{\{2R \leqslant |y| \leqslant S\}}(x)$. For $|x| \leqslant R$, we have $T_\Phi f(x) = \int_{2R \leqslant |y| \leqslant S} \Phi(x-y)\Phi(y)^{p'-1} \, dy \geqslant C \int_{2R \leqslant |y| \leqslant S} \Phi(y)^{p'} \, dy$. Indeed, $\Phi(x-y) \geqslant \Phi(y)$ for all y satisfying $|x-y| \leqslant |y|$ and this in turn holds provided $|x| \leqslant R, |y| \geqslant 2R$ and the distance between $\frac{x}{|x|}$ and $\frac{y}{|y|}$ is sufficiently small. With this estimate, (2.1) yields

$$C \int_{2R \leqslant |y| \leqslant S} \Phi(y)^{p'} \, dy \left(\int_B d\mu\right)^{\frac{1}{a}} \leqslant \left[\int (T_\Phi f)^q \, d\mu\right]^{\frac{1}{q}} \leqslant C\left[\int_{2R \leqslant |y| \leqslant S} \Phi(y)^{p'} \, dy\right]^{\frac{1}{p}}.$$

Letting $S \to \infty$ yields $\int_{|y| \geqslant 2R} \Phi(y)^{p'} \, dy < \infty$ and this proves (2.2).

To obtain a criterion for (2.1) to hold, we look at the inequality dual to it. A standard argument shows this dual is, with the same $C > 0$,

$$\text{(2.3)} \qquad \left[\int_{\mathbf{R}^n} (T_\Phi f\mu)(x)^{p'} \, dx\right]^{\frac{1}{p'}} \leqslant C\left[\int_{\mathbf{R}^n} f(x)^{q'} \, d\mu(x)\right]^{\frac{1}{q'}},$$

where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$, and

$$(T_\Phi f\mu)(x) = \int_{\mathbf{R}^n} \Phi(x-y) f(y) \, d\mu(y).$$

The behaviour of T_Φ in (2.3) is determined by that of the maximal operator M_Φ given at a positive Borel measure ν by

$$(M_\Phi \nu)(x) = \sup_{x \in Q} \left[\frac{1}{|Q|} \int_{|y| \leqslant |Q|^{\frac{1}{n}}} \Phi(y)\, dy \right] \int_Q d\nu .$$

Note that the first factor on the right side is the average of Φ over the ball of radius $|Q|^{\frac{1}{n}}$ centred at the origin. In the case when Φ is the kernel K_α for the Riesz potential operator, then M_Φ is the usual fractional maximal operator M_α (see e.g. [3] or [15]).

THEOREM 2.2. — *Let* Φ *be an r.d. kernel and* ν *a positive locally finite Borel measure on* $\mathbf{R}^n$. *Then*

(a) $$(M_\Phi \nu)(x) \leqslant C_n M(T_\Phi \nu)(x), \quad x \in \mathbf{R}^n$$

where M *denotes the usual Hardy-Littlewood maximal operator and the constant* $C_n > 0$ *depends only on the dimension* n.

(b) *There exists* $\gamma > 1$ *and a positive constant* C_n *depending only on* n *so that for all* $\lambda > 0$ *and all* $\beta \in (0,1]$,

$$|\{T_\Phi \nu > \gamma\lambda \textit{ and } M_\Phi \nu \leqslant \beta\gamma\}| \leqslant C_n \frac{\beta}{\gamma} |\{M(T_\Phi \nu) > \lambda\}| .$$

Proof. — To a given cube Q in $\mathbf{R}^n$ associate the cube Q^* having the same centre as Q but edges $7\sqrt{n}$ times as long as those of Q.

To prove (a) fix $x \in \mathbf{R}^n$ and a cube Q containing x. Then

$$\begin{aligned} \int_{Q^*} (T_\Phi \nu)(y)\, dy &\geqslant \int_{Q^*} dy \int_Q \Phi(y-z)\, d\nu(z) \\ &\geqslant \int_Q d\nu(z) \int_{Q^*} \Phi(y-z)\, dy \\ &\geqslant \int_{|y| \leqslant |Q|^{\frac{1}{n}}} \Phi(y)\, dy \int_Q d\nu(y) \end{aligned}$$

since $\{y ; |y-z| \leqslant |Q|^{\frac{1}{n}}\} \subset Q^*$, whenever $z \in Q$. Hence,

$$M(T_\Phi \nu)(x) \geqslant \frac{7^{-n} n^{-\frac{n}{2}}}{|Q|} \int_{|y| \leqslant |Q|^{\frac{1}{n}}} \Phi(y)\, dy \int_Q d\nu(y)$$

and so

$$M_\Phi\nu)(x) \geqslant 7^n n^{\frac{n}{2}}\ M(T_\Phi\nu)(x),\ x \in \mathbf{R}^n.$$

We now show (b). Given $\lambda > 0$, let

$$\Omega_\lambda = \{M(T_\Phi\nu) > \lambda\}.$$

Decompose Ω_λ into disjoint Whitney cubes Q with $Q^* \cap \Phi_\lambda^c \neq \varnothing$. See De Guzman [11]. Let $\{Q_k\}$ be those Whitney cubes for which there is an $x_k \in Q_k$ satisfying $(M_\Phi\nu)(x_k) \leqslant \beta\lambda$. Fixing attention on such a Q_k, which we'll denote simply by Q, we define ν_1 and ν_2 to be restrictions of the measure ν; the first to Q^*, the second to $\mathbf{R}^n - Q^*$. We claim it is enough to obtain a dimensional constant $C_n > 0$ such that

$$T_\Phi\nu_2 \leqslant C_n\lambda \tag{2.4}$$

on Q. Suppose for the moment that (2.4) has been proved and take $\gamma > 2C_n$. Then

$$\{x \in Q; (T_\Phi\nu)(x) > \gamma\lambda\} \subset \left\{x \in Q; (T_\Phi\nu_1)(x) > \frac{\gamma\lambda}{2}\right\}.$$

Now,

$$\int_Q \Phi(x-z)\,dx \leqslant \int_{|y| \leqslant |\frac{\sqrt{n}}{2}|Q|^{\frac{1}{n}}} \Phi(y)\,dy. \tag{2.5}$$

This means

$$\begin{aligned}\int_Q (T_\Phi\nu_1)(x)\,dx &= \int_Q dx \int_{Q^*} \Phi(x-y)\,d\nu(y)\\ &= \int_{Q^*} d\nu(y) \int_Q \Phi(x-y)\,dx \leqslant \int_{|y| \leqslant |\frac{\sqrt{n}}{2}|Q|^{\frac{1}{n}}} \Phi(y)\,dy \int_{Q^*} d\nu(y)\\ &\leqslant (7\sqrt{n})^n |Q| (M_\Phi\nu)(x_k) \leqslant (7\sqrt{n})^n \beta\lambda |Q|.\end{aligned}$$

Thus with $C = 2(7\sqrt{n})^n$,

$$\left|\left\{x \in Q; (T_\Phi\nu_1)(x) > \frac{\gamma\lambda}{2}\right\}\right| \leqslant \frac{2}{\gamma\lambda} \int_Q (T_\Phi\nu_1)(x)\,dx > C\frac{\beta}{\gamma}|Q|.$$

Therefore,

$$\begin{aligned}|\{T_\Phi\nu > \gamma\lambda \text{ and } M_\Phi\nu \leqslant \beta\lambda\}| &= \sum_k |\{x \in Q_k; (T_\Phi\nu)(x) > \gamma\lambda\}|\\ &\leqslant \frac{C\beta}{\gamma} \sum_k |Q_k| \leqslant C\frac{\beta}{\gamma}\, |\{M(T_\Phi\nu) > \lambda\}|.\end{aligned}$$

To prove (2.4) we'll require the fact that $C'_n > 0$ exists with

$$\Phi(y) \leqslant \frac{C'_n}{r^n} \int_{|y-z| \leqslant r} \Phi(z)\, dz\,, \quad 0 < r \leqslant |y|\,. \tag{2.6}$$

As Φ is nonincreasing, this would be true if it were known to hold whenever Φ is the characteristic function of a ball centred at the origin. For this it suffices to know that the set of z in the ball $|y-z| \leqslant r$ satisfying $|z| \leqslant |y|$ occupies at least a fixed fraction of the ball. The change of variable $z = |y|v$, followed by the rotation that sends $\frac{y}{|y|}$ to $e_1 = (1,0,\dots 0)$, reduces the problem to the relative size of the intersection of the balls $|v| \leqslant 1$ and $|v-e_1| \leqslant s$, $0 < s < 1$, to the size of the ball $|v-e_1| \leqslant s$ itself. But for these sets the result in clear.

If $x \in Q$ (where Q denotes some fixed Q_k) and $y \in \mathbf{R}^n - Q^*$, then $|x-y| \geqslant |Q|^{\frac{1}{n}}$. Thus taking $r = |Q|^{\frac{1}{n}}$ in (2.6), we get

$$\begin{aligned}(T\nu_2)(x) &= \int_{\mathbf{R}^n - Q^*} \Phi(x-y)\, d\nu(y) \\ &\leqslant \frac{C'_n}{r^n} \int_{\mathbf{R}^n - Q^*} d\nu(y) \int_{|z| \leqslant r} \Phi(x-y-z)\, dz\,.\end{aligned}$$

Making the substitution $v = x - z$, the last expression becomes

$$\frac{C'_n}{r^n} \int_{|x-v| \leqslant r} (T_\Phi \nu_2)(v)\, dv \leqslant \frac{C'_n}{r^n} \int_{Q^*} (T_\Phi \nu)(x)\, dx \leqslant \frac{C'_n}{r^n} \lambda |Q^*| = C_n \lambda$$

with $C_n = (7\sqrt{n})^n C'_n$, since Q^* intersects $\mathbf{R}^n - \Omega_\lambda = \{M(T_\Phi \nu) \leqslant \lambda\}$ by the Whitney condition. This completes the proof.

THEOREM 2.3. — *Suppose* Φ *is a nonnegative, locally integrable radially decreasing function satisfying* (2.2). *Then for* $1 < p \leqslant q < \infty$ *and* μ *a positive locally finite Borel measure on* $\mathbf{R}^n$, *the following statements are equivalent :*

1. *There exists* $C > 0$ *so that whenever* f *is a nonnegative measurable function on* $\mathbf{R}^n$

$$\left[\int_{\mathbf{R}^n} (T_\Phi f)(x)^q\, d\mu(x) \right]^{\frac{1}{q}} \leqslant C \left[\int_{\mathbf{R}^n} f(x)^p\, dx \right]^{\frac{1}{p}}.$$

2. *There exists* $C' > 0$ *so that for all dyadic cubes* Q

$$\left[\int_{\mathbf{R}^n} T_\Phi(\chi_Q\mu)(x)^{p'}\,dx\right]^{\frac{1}{p'}} \leqslant C'[\mu(Q)]^{\frac{1}{q'}} < \infty$$

where $p' = \dfrac{p}{p-1}$, $q' = \dfrac{q}{q-1}$.

3. *There exists* $K > 0$ *so that for all dyadic cubes* Q

$$\left[\int_Q (M_\Phi\chi_Q\mu)(x)^{p'}\,dx\right]^{\frac{1}{p'}} \leqslant K[\mu(Q)]^{\frac{1}{q'}} < \infty .$$

Moreover, the least possible C, C' *and* K *in the above are all within constant multiples of one another, the constants being independent of* Φ *and* μ.

Proof. – Let M_Φ^{dy} denote the dyadic analogue of M_Φ given by

$$M_\Phi^{dy}\nu(x) = \sup_{x\in Q \text{ dyadic}} \left[\frac{1}{|Q|}\int_{|y|\leqslant |Q|^{\frac{1}{n}}} \Phi(y)\,dy\right]\int_Q d\nu$$

for $x \in \mathbf{R}^n$ and ν a locally finite positive measure. We claim that for all such ν,

$$\int_{\mathbf{R}^n} |M_\Phi^{dy}\nu|^{p'} \leqslant \int_{\mathbf{R}^n} |M_\Phi\nu|^{p'} \leqslant C_1\int_{\mathbf{R}^n} |T_\Phi\nu|^{p'}, \tag{2.7}$$

$$\int_{\mathbf{R}^n} |T_\Phi\nu|^{p'} \leqslant C_2\int_{\mathbf{R}^n} |M_\Phi\nu|^{p'} \leqslant C_3\int_{\mathbf{R}^n} |M_\Phi^{dy}\nu|^{p'}, \tag{2.8}$$

where the constants C_1, C_2, C_3 depend only on n and $p (1<p<\infty)$. The first inequality in (2.7) is trivial and the second inequality follows from part (*a*) of Theorem 2.2 and the classical $L^{p'}$ inequality for M ([18]). The first inequality in (2.8) follows from part (*b*) of Theorem 2.2 as in [6]. Finally, to prove the second inequality in (2.8), we apply a standard covering argument to $\{M_\Phi\nu>\lambda\}$ (where $\lambda>0$) to obtain the existence of cubes $(Q_k)_k$ with disjoint triples satisfying

(i) $$\left[\frac{1}{|Q_k|}\int_{|y|\leqslant |Q_k|^{\frac{1}{n}}} \Phi(y)\,dy\right]\int_{Q_k} d\nu > \lambda \quad \text{for all } k$$

(ii) $$|\{M_\Phi\nu>\lambda\}| \leqslant C\sum_k |Q_k|.$$

Now each Q_k is covered by at most 2^n dyadic cubes $(I_k^j)_{1\leqslant j\leqslant 2^n}$ with

$2^{-n}|Q_k| \leqslant |I_k^j| \leqslant |Q_k|$. There is at least one of these dyadic cubes, say $I_k = I_k^j$, with $\int_{I_k} d\nu \geqslant 2^{-n} \int_{Q_k} d\nu$. Then, since Φ is r.d. and $|I_k| \leqslant |Q_k|$,

$$\left[\frac{1}{|I_k|}\int_{|y|\leqslant |I_k|^{\frac{1}{n}}} \Phi(y)\, dy\right]\int_{I_k} d\nu > 2^{-n}\lambda \quad \text{for all } k$$

and so $\bigcup_k I_k \subset \{M_\Phi^{dy}\nu > 2^{-n}\lambda\}$. Since the I_k 's are pairwise disjoint, we have

$$\begin{aligned}|\{M_\Phi \nu > \lambda\}| &\leqslant C\sum_k |Q_k| \leqslant C\sum_k |I_k| \\ &\leqslant C|\{M_\Phi^{dy}\nu > 2^{-n}\lambda\}|\end{aligned}$$

and (2.8) follows upon multiplying this inequality by $\lambda^{p'-1}$ and then integrating over $(0,\infty)$.

From (2.3), (2.7) and (2.8) we obtain that the trace inequality in 1. holds if and only if there is $C > 0$, comparable to the one in (2.1), for which

$$\text{(2.9)} \qquad \left[\int_{\mathbf{R}^n} (M_\Phi^{dy} f\mu)(x)^{p'}\, dx\right]^{\frac{1}{p'}} \leqslant C\left[\int_{\mathbf{R}^n} f(x)^{q'}\, d\mu(x)\right]^{\frac{1}{q'}}, \qquad \text{for all } f.$$

Theorem A of [16] (with M_Φ^{dy} in place of $M_{\mu,\alpha}$, the proof is unchanged) shows that (2.9) holds if and only if there is $C > 0$, comparable to that in (2.9), for which

$$\text{(2.10)} \qquad \left[\int_{\mathbf{R}^n} [M_\Phi^{dy}(\chi_Q\, d\mu)]^{p'}\right]^{\frac{1}{p'}} \leqslant C\mu(Q)^{\frac{1}{q'}} < \infty$$

for all dyadic cubes Q. Theorem 2.3 now follows easily. The trace inequality 1. implies its dual (2.3) which in turn implies 2. upon taking $f = \chi_Q$. Inequality 2. implies 3. by (2.7) and finally, 3. $\Rightarrow$ (2.10) $\Rightarrow$ (2.9) $\Rightarrow$ 1.

3. Schrödinger operators.

In this section, Theorem 2.3 is used to refine the estimates for eigenvalues of a Schrödinger operator $H = -\Delta - v$ given in Theorem 5, Chapter II, of [10]. By eigenvalues, we mean the numbers

$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N \ldots$ where λ_N is the maximum over all $N-1$ tuples $\Phi_1, \ldots \Phi_{N-1}$ of the quantity $\inf \dfrac{\langle Hu,u\rangle}{\langle u,u\rangle}$, the infimum being over all $u \in Q(H)$, $\langle u,\Phi_j\rangle = 0$, $j = 1, \ldots N-1$. Here $Q(H)$ denotes the form domain of H (see [16]) and $\langle Hu,u\rangle = \int_{\mathbf{R}^n} (|\nabla u|^2 - v|u|^2)$ for $u \in Q(H)$. Recall that $I_2 f(x) = \int_{\mathbf{R}^n} |x-y|^{2-n} f(y)\, dy$ denotes the Newtonian potential of f.

THEOREM 3.1. — *Let* $H = -\Delta - v$, *where* $v(x) \geqslant 0$ *is locally integrable on* $\mathbf{R}^n$ *and* $n \geqslant 3$. *Denote the* v *measure of* Q, $\int_Q v(x)\, dx$, *by* $|Q|_v$. *There are positive constants* C, c *depending only on the dimension* n *such that the least eigenvalue* λ_1 *of* H *satisfies* $E_{sm} \leqslant -\lambda_1 \leqslant E_{big}$ *where*

$$E_{sm} = \sup\left\{|Q|^{-2/n};\, |Q|_v^{-1} \int_Q I_2(\chi_Q v)v \geqslant C\right\}$$

$$E_{big} = \sup\left\{|Q|^{-2/n};\, |Q|_v^{-1} \int_Q I_2(\chi_Q v)v \geqslant c\right\}.$$

Example 3.2. — Consider Example V in [10]: a particle in a rectangular box $B = B_1 \times B_2 \times \cdots B_n$ with side lengths $\delta_1 \leqslant \delta_2 \leqslant \cdots \delta_n$. Let $v = \chi_B$ and let x_B denote the centre of B. Since

$$\sup_Q |Q_v|^{-1} \int_Q I_2(\chi_Q v)v \approx I_2 v(x_B) \approx \delta_1^2 + \delta_1\,\delta_2 + \delta_1\,\delta_2 \log(\delta_3/\delta_2)$$
$$\approx \delta_1\,\delta_2 \log(1 + \delta_3/\delta_2),$$

Theorem 3.1 yields the correct order of magnitude for the energy, $E_{critical}$, needed to trap a particle in B, namely

$$E_{critical} = \sup\{E > 0;\, -\Delta - Ev \geqslant 0\} = 1/\delta_1\,\delta_2 \log(1 + \delta_3/\delta_2).$$

A refinement of Theorems 6 and 6′ in Chapter II of [10], similar to the one above, is given in

THEOREM 3.3. — *Let* $H = -\Delta - v$ *where* $v(x) \geqslant 0$ *is locally integrable on* $\mathbf{R}^n$ *and* $n \geqslant 3$. *There are positive constants* C, c *depending only on the dimension* n *such that :*

(A) *Suppose* $\lambda \geqslant 0$ *and let* $Q_1, \ldots, Q_N$ *be a collection of cubes of side length at most* $\lambda^{-\frac{1}{2}}$ *whose doubles are pairwise disjoint. Suppose further that*

$|Q_j|_v^{-1}\int_{Q_j} I_2(\chi_{Q_j}v)v \geqslant C, \quad 1 \leqslant j \leqslant N$. *Then* H *has at least* N *eigenvalues* $\leqslant -\lambda$.

(B) *Conversely, suppose* $\lambda \geqslant 0$ *and that* H *has at least* CN *eigenvalues* $\leqslant -\lambda$. *Then there is a collection of pairwise disjoint (dyadic) cubes* $Q_1, \ldots, Q_N$ *of side lengths at most* $\lambda^{-\frac{1}{2}}$ *that satisfy* $|Q_j|_v^{-1}\int_{Q_j} I_2(\chi_{Q_j}v)v \geqslant c, \quad 1 \leqslant j \leqslant N$.

Roughly speaking, Theorem 3.3 says that the negative eigenvalues of H are approximately given by $-|Q|^{-2/n}$ as Q ranges over the minimal dyadic cubes satisfying $|Q|_v^{-1}\int_Q I_2(\chi_Q v)v \geqslant C$.

In [10], results corresponding to Theorems 3.1 and 3.3 were obtained with the quantity $|Q|_v^{-1}\int_Q I_2(\chi_Q v)v$ replaced by the simpler average $C|Q|^{\frac{2}{n}-1}\int_Q v$ in part (A) of Theorem 3.3 and by $C_p|Q|^{\frac{2}{n}-\frac{1}{p}}\left(\int_Q v^p\right)^{\frac{1}{p}}$ in part (B). A comparison of these quantities is made in Remark 3.5 at the end of this section. Chang, Wilson, and Wolff [5] show part (B) of Theorem 3.3 holds for v if $\sup_Q |Q|^{\frac{2}{n}-1}\int_Q v(x)\Phi(|Q|^{\frac{2}{n}}v(x))\,dx < \infty$, where $\Phi\colon\ [0,\infty] \to [1,\infty]$ is increasing and $\int_1^\infty \frac{dx}{x\Phi(x)} < \infty$. See also Chanillo and Wheeden [6].

Proof of Theorem 3.1. — The Schwartz class S is dense in Q(H) and thus we have

$$-\lambda_1 = -\inf_{u\in Q(H)} \frac{\langle Hu,u\rangle}{\langle u,u\rangle} = \sup_{u\in S} \frac{\int |u|^2 v - \int |\nabla u|^2}{\int |u|^2}$$

$$= \inf\{\alpha > 0; \int |u|^2 v \leqslant \int |\nabla u|^2 + \alpha |u|^2$$

$$= \int (|\xi|^2 + \alpha)|\hat{u}(\xi)|^2\,d\xi,\ u \in S\}$$

$$= \inf\{\alpha>0; \int (I_1^\alpha f)^2 v \leqslant \int f^2, f \geqslant 0\}$$

where I_1^α is the operator with r.d. kernel K_1^α defined by $(K_1^\alpha)^\wedge(\xi) = (|\xi|^2+\alpha)^{-\frac{1}{2}}$. Thus $K_1^1(x) = G_1(x)$ and

$$K_1^\alpha(x) = \alpha^{\frac{n-1}{2}} G_1(\alpha^{\frac{1}{2}}x).$$

If we let C_α denote the least constant such that

$$\int (I_1^\alpha f)^2 v \leqslant C_\alpha \int f^2 \qquad \text{for all } f \geqslant 0,$$

then $-\lambda_1 = \inf\{\alpha; C_\alpha \leqslant 1\}$. By Theorem 2.3,

$$C_\alpha \approx \sup_Q \frac{1}{|Q|_v} \int [I_1^\alpha(\chi_Q v)]^2 \tag{3.1}$$

in the sense that the ratio of the left and right sides is bounded between two constants independent of α and v. We now show that, in fact, the supremum in (3.1) need only be taken over those cubes Q with $|Q|^{\frac{1}{n}} \leqslant \alpha^{-\frac{1}{2}}$. To this end, set $M = \sup_{\substack{Q \\ |Q|^{1/n} \leqslant \alpha^{-1/2}}} \frac{1}{|Q|_v} \int [I_1^\alpha(\chi_Q v)]^2$ and suppose Q is a cube with $|Q|^{\frac{1}{n}} > \alpha^{-\frac{1}{2}}$. Express Q as a union of congruent cubes, Q_j, having pairwise disjoint interiors and common sidelengths, $|Q_j|^{\frac{1}{n}}$, satisfying $\frac{1}{2}\alpha^{-\frac{1}{2}} \leqslant |Q_j|^{\frac{1}{n}} \leqslant \alpha^{-\frac{1}{2}}$. Then, we claim

$$\begin{aligned} \int [I_1^\alpha(\chi_Q v)]^2 &= \sum_{i,j} \int I_1^\alpha(\chi_{Q_i} v) I_1^\alpha(\chi_{Q_j} v) \\ &\leqslant C \sum_i \int [I_1^\alpha(\chi_{Q_i} v)]^2 \\ &\leqslant CM \sum_i |Q_i|_v = CM|Q|_v. \end{aligned} \tag{3.2}$$

The second inequality holds by definition of M and since $|Q_i|^{\frac{1}{n}} \leqslant \alpha^{-\frac{1}{2}}$. To prove the first inequality, we consider two cases. First, when Q_i and Q_j are adjacent, we simply use

$$\int I_1^\alpha(\chi_{Q_i} v) I_1^\alpha(\chi_{Q_j} v) \leqslant \frac{1}{2} \int [I_1^\alpha(\chi_{Q_i} v)]^2 + \frac{1}{2} \int [I_1^\alpha(\chi_{Q_j} v)]^2.$$

To treat the case when Q_i and Q_j have a distance of roughly k

sidelengths between them, $k \geqslant 1$, we require the facts that $K_2^\alpha(x) \approx |x|^{2-n}$ if $|x| \leqslant \alpha^{-\frac{1}{2}}$ and $K_2^\alpha(x) \leqslant C\alpha^{\frac{n-2}{2}} e^{-\sqrt{\alpha}|x|}$ if $|x| > \alpha^{-\frac{1}{2}}$, for which see [4]. We then have

$$\int I_1^\alpha(\chi_{Q_i}v) I_1^\alpha(\chi_{Q_j}v) = \int_{Q_i} I_2^\alpha(\chi_{Q_j}v)(x)v(x)\,dx \leqslant C\alpha^{\frac{n-2}{2}} e^{-k}|Q_i|_v|Q_j|_v .$$

However, $I_1^\alpha(\chi_{Q_i})(x) \geqslant C\alpha^{-\frac{1}{2}}$ for $x \in Q_i$ and so

$$|Q_i|_v \leqslant \frac{\alpha^{\frac{1}{2}}}{C}\int_{Q_i} I_1^\alpha(\chi_{Q_i})v = \frac{\alpha^{\frac{1}{2}}}{C}\int_{Q_i} I_1^\alpha(\chi_{Q_i}v)(x)\,dx .$$

Thus

$$\begin{aligned} 2|Q_i|_v|Q_j|_v &\leqslant |Q_i|_v^2 + |Q_j|_v^2 \\ &\leqslant C\alpha\left(\left[\int_{Q_i} I_1^\alpha(\chi_{Q_i}v)\right]^2 + \left[\int_{Q_j} I_1^\alpha(\chi_{Q_j}v)\right]^2\right) \\ &\leqslant C\alpha^{1-\frac{n}{2}}\left(\int_{Q_i}[I_1^\alpha(\chi_{Q_i}v)]^2 + \int_{Q_j}[I_1^\alpha(\chi_{Q_j}v)]^2\right). \end{aligned}$$

Now, for a fixed cube Q_i, there are at most Ck^{n-1} cubes Q_j at a distance of roughly k sidelengths from Q_i. Combining all of the above, we obtain

$$\sum_{\substack{i,j \\ i \neq j}} \int I_1^\alpha(\chi_{Q_i}v) I_1^\alpha(\chi_{Q_j}v) \leqslant C\left[1 + \sum_{k=1}^{\infty} k^{n-1}e^{-k}\right]\sum_i \int [I_1^\alpha(\chi_{Q_i}v)]^2$$

which yields the first inequality in (3.2). From (3.1) and (3.2), we have $C_\alpha \approx M$ and since $\int [I_1^\alpha(\chi_Q v)]^2 = \int I_2^\alpha(\chi_Q v)v \approx \int I_2(\chi_Q v)v$ when $|Q|^{\frac{1}{n}} \leqslant \alpha^{-\frac{1}{2}}$, we finally have

$$C_\alpha \approx \sup_{\substack{Q \\ |Q|^{1/n} \leqslant \alpha^{-1/2}}} \frac{1}{|Q|_v}\int_Q I_2(\chi_Q v)v$$

and Theorem 3.1 follows readily.

Proof of Theorem 3.3, *part* (A). — As in [10], it suffices by elementary functional analysis to construct an N-dimensional subspace $\Omega \subset Q(H)$ so that $\langle Hu,u\rangle \leqslant -\lambda\int|u|^2$ for u in Ω. Our hypothesis implies

$$\frac{1}{|Q_j|_v}\int_{Q_j} I_2^\lambda(\chi_{Q_j}v)v \geqslant C \qquad \text{for } j = 1, \ldots N.$$

Since $\int_Q I_2^\lambda(\chi_Q v)v \leqslant \left(\int_Q [I_2^\lambda(\chi_Q v)]^2 v\right)^{\frac{1}{2}}|Q|_v^{\frac{1}{2}}$ by Holder's inequality, we actually have

$$\int_{Q_j}[I_2^\lambda(\chi_{Q_j}v)]^2 v \geqslant C\int_{Q_j} I_2^\lambda(\chi_{Q_j}v)v, \quad 1 \leqslant j \leqslant N.$$

This suggests we let Ω be the linear span of $\{f_j\}_{j=1}^N$ where $f_j = \Phi_j I_2^\lambda(\chi_{Q_j}v)$ and $\Phi_j = 1$ on $\frac{3}{2}Q_j$ with supp Φ_j contained in $2Q_j$. Here the Φ_j are dilates and translates of a fixed $\Phi \in C_c^\infty(\mathbf{R}^n)$. We have immediately that

$$(3.3) \qquad \int f_j^2 v \geqslant C\int_{Q_j} I_2^\lambda(\chi_{Q_j}v)v \qquad \text{for} \quad 1 \leqslant j \leqslant N.$$

By hypothesis, the supports of the f_j are pairwise disjoint and so we need only establish

$$(3.4) \qquad \langle(-\Delta+\lambda)f_j,f_j\rangle \leqslant \int (f_j)^2 v \qquad \text{for } 1 \leqslant j \leqslant N$$

in order to conclude $\langle Hu,u\rangle \leqslant -\lambda\int|u|^2$ for u in Ω, as required. To prove (3.4), we let $G_j = 2Q_j - \frac{3}{2}Q_j$ and compute that

$$\begin{aligned}(-\Delta+\lambda)f_j &= (-\Delta+\lambda)[\Phi_j I_2^\lambda(\chi_{Q_j}v)]\\ &= \chi_{Q_j}v + \chi_{G_j}(-\Delta+\lambda)[\Phi_j I_2^\lambda(\chi_{Q_j}v)]\\ &= A_j + B_j\end{aligned}$$

since $I_2^\lambda = (-\Delta+\lambda)^{-1}$. Now

$$\langle A_j,f_j\rangle = \int_{Q_j} I_2^\lambda(\chi_{Q_j}v)v \leqslant \frac{1}{C}\int f_j^2 v \text{ (by 4.3))} \leqslant \frac{1}{2}\int f_j^2 v$$

provided C is chosen $\geqslant 2$. It remains to verify $\langle B_j,f_j\rangle \leqslant C'\int_{Q_j} I_2^\lambda(\chi_{Q_j}v)v$ for all j since then (3.4) will follow from (3.3) and the previous estimate provided $C \geqslant 2C'$. Now

$$(3.5)\quad |B_j| \leqslant \chi_{G_j}[\Phi_j|\Delta I_2^\lambda(\chi_{Q_j}v)| + 2|\nabla\Phi_j||\nabla I_2^\lambda(\chi_{Q_j}v)| + (\lambda + |\Delta\Phi_j|)[I_2^\lambda(\chi_{Q_j}v)] = D_j + E_j + F_j .$$

Using the estimates $|D^s K_2^\lambda(x)| \leqslant C|x|^{2-n-s}$, for $s \geqslant 0$ and $|x| \leqslant C\lambda^{-\frac{1}{2}}$ (see [4]) we obtain that on G_j,

$$I_2^\lambda(\chi_{Q_j}v)(x) \leqslant C|Q_j|^{\frac{2}{n}-1}\int_{Q_j} v$$

$$|\nabla I_2^\lambda(\chi_{Q_j}v)(x)| \leqslant C|Q_j|^{\frac{1}{n}-1}\int_{Q_j} v$$

$$|\Delta I_2^\lambda(\chi_{Q_j}v)(x)| \leqslant C|Q_j|^{-1}\int_{Q_j} v .$$

These inequalities, together with $|\Phi_j| \leqslant 1$, $|\nabla\Phi_j| \leqslant C|Q_j|^{-\frac{1}{n}}$, $|\Delta\Phi_j| \leqslant C|Q_j|^{-\frac{2}{n}}$ and the hypothesis $\lambda \leqslant |Q_j|^{-\frac{2}{n}}$, yields

$$(3.6)\qquad D_j, E_j, F_j \leqslant C|Q_j|^{-1}|Q_j|_v .$$

Since $f_j(x) \leqslant C|Q_j|^{\frac{2}{n}-1}\int_{Q_j} v$ on G_j, (3.5) and (3.6) imply

$$(3.7)\qquad \langle B_j, f_j\rangle \leqslant C|Q_j|^{\frac{2}{n}-1}|Q_j|_v^2 .$$

Finally,

$$|Q_j|^{\frac{2}{n}-1}\left(\int_{Q_j} v\right)^2 \leqslant C(\min_{x\in Q_j} I_2^\lambda(\chi_{Q_j}v))\left(\int_{Q_j} v\right) \leqslant C\int_{Q_j} I_2^\lambda(\chi_{Q_j}v)v$$

and this, combined with (3.7), shows that $\langle B_j, f_j\rangle \leqslant C'\int_{Q_j} I_2^\lambda(\chi_{Q_j}v)v$ and completes the proof of part (A) of Theorem 3.3.

Proof of Theorem 3.3, *part* (B). — We follow closely the argument of C. L. Fefferman and D. H. Phong in ([10]; proof of Theorem 6 in Chapter II), but with certain modifications designed to avoid the use of a square function. As in [10], it suffices to suppose v bounded and to show that if $Q_1, \ldots, Q_N$ are the minimal dyadic cubes satisfying

$\frac{1}{|Q_j|_v}\int_{Q_j} I_2(\chi_{Q_j}v)v \geqslant c$ and $|Q_j|^{\frac{1}{n}} \leqslant \lambda^{-\frac{1}{2}}$, then $H = -\Delta - v$ has at most CN eigenvalues $\leqslant -\lambda$ (where the constant C is of course independent of the bound on v). As usual, this will be accomplished by exhibiting a subspace $\Omega \subset L^2$ of codimension $\leqslant$ CN such that

$$(3.8) \qquad \langle Hu,u\rangle \geqslant -\lambda\int |u|^2 \qquad \text{for all } u \text{ in } \Omega.$$

We consider only the case $\lambda = 0$, the case $\lambda > 0$ requiring easy modifications. We begin by defining additional cubes $Q_{N+1}, \ldots, Q_M$ as in [10]; i.e. let B be the collection of all dyadic cubes Q with $\frac{1}{|Q|_v}\int_Q I_2(\chi_Q v)v \geqslant c$ and define the additional cubes $Q_{N+1}, \ldots, Q_M$ to consist of (i) the maximal cubes in B, (ii) the branching cubes in B and (iii) the descendents of branching cubes in B. The descendents of a cube Q in B are those $Q' \in B$ which are maximal with respect to the property of being properly contained in Q. A cube in B « branches » if it has at least two descendents. As shown in [10], $M \leqslant CN$. Still following [10] we define $E_0 = \mathbf{R}^n - \bigcup_{j=1}^{M} Q_j$ and $E_j = Q_j$ minus its descendents for $j \geqslant 1$. In analogy with estimates (i) and (ii) of [10], we shall prove that the weights $v_j = \chi_{E_j}v$ satisfy

$$(3.9) \qquad \frac{1}{|Q|_{v_j}}\int_Q I_2(\chi_Q v_j)v_j \leqslant Cc \qquad \text{for all } 0 \leqslant j \leqslant M, Q \text{ dyadic cube.}$$

In order to make use of (3.9) and the trace inequalities it implies we shall have to define the subspace Ω so that

$$(3.10) \quad |u(x)| \leqslant CI_1(\chi_{E_j}|\nabla u|)(x) \qquad \text{for } x \in E_j, 0 \leqslant j \leqslant M, u \in \Omega.$$

Indeed, if both (3.9) and (3.10) hold, then for $u \in \Omega$,

$$\begin{aligned}\int |u|^2 v &= \sum_{j=0}^{M}\int_{E_j}|u|^2 v_j \\ &\leqslant C\sum_{j=0}^{M}\int_{E_j}[I_1(\chi_{E_j}|\nabla u|]^2 v_j \qquad \text{by (3.10)} \\ &\leqslant Cc\sum_{j=0}^{M}\int_{E_j}|\nabla u|^2 \qquad \text{by (3.9) and Theorem 2.3} \\ &\leqslant \int |\nabla u|^2 \qquad \text{if } c \text{ small enough,}\end{aligned}$$

and this is (3.8) for $\lambda = 0$. Thus it remains to construct Ω of codimension $\leqslant CN$ such that (3.10) holds. In the case $1 \leqslant j \leqslant N$, E_j is a cube and (3.10) holds whenever $\int_{E_j} u = 0$ by the following inequality of E. Fabes, C. Kenig and R. Serapioni ([9]; Lemma 1.4)

$$(3.11) \qquad \left| u(x) - \frac{1}{|Q|} \int_Q u \right| \leqslant CI_1(\chi_Q |\nabla u|)(x) \qquad \text{for } x \in Q, Q \text{ a cube.}$$

For the case when E_j is not a cube we will need the following lemma.

LEMMA 3.4. – *Suppose $Q_1, \ldots, Q_k$ are pairwise disjoint dyadic subcubes of a dyadic cube Q in $\mathbf{R}^n$. Then there are (not necessarily dyadic or disjoint) cubes $I_1, \ldots, I_m$ such that $Q - \bigcup_{j=1}^{k} Q_j = \bigcup_{i=1}^{m} I_i$ and $m \leqslant Ck$ where C is a constant depending only on the dimension n. The above holds also for $Q = \mathbf{R}^n$ if we allow the cubes I_i to be infinite, i.e. of the form $J_1 \times J_2 \times \cdots J_n$ where each J_i is a semi-infinite interval.*

This lemma has been obtained independently by S. Chanillo and R. L. Wheeden [6], with a proof much simpler than that appearing in a previous version of this paper. As a result, we refer the reader to [6] for a proof of the lemma.

We can now define the subspace Ω. For each j with $j = 0$ or $N + 1 \leqslant j \leqslant M$, apply Lemma 3.4 with $Q = Q_j$ and $Q_1, \ldots, Q_k$ the descendents of Q_j (for $j=0$, take $Q=\mathbf{R}^n$ and $Q_1, \ldots, Q_k$ to be the maximal cubes in B), to obtain cubes $I_1^{(j)}, \ldots, I_{m_j}^{(j)}$ with $E_j = \bigcup_{i=1}^{m_j} I_i^{(j)}$ and $m_j \leqslant C$ (# of descendents of Q_j). Note that $E_j = Q_j$ for $1 \leqslant j \leqslant N$. Now define

$$\Omega = \{u; \int_{Q_j} u=0 \text{ for } 1 \leqslant j \leqslant N \text{ and } \int_{I_i^{(j)}} u=0 \text{ for } N+1 \leqslant j \leqslant M,\ j=0 \text{ and } 1 \leqslant i \leqslant m_j\}.$$

If $x \in E_j$, $N + 1 \leqslant j \leqslant M$ or $j = 0$, then $x \in$ some $I_i^{(j)}$ and thus for $u \in \Omega$, $|u(x)| \leqslant CI_1(\chi_{I_i^{(j)}}|\nabla u|)(x) \leqslant CI_1(\chi_{E_j}|\nabla u|)(x)$ by (3.11). Thus (3.10) holds. Finally, the codimension of Ω is at most

$$N + \sum_{\substack{j=0 \\ N+1 \leqslant j \leqslant M}} m_j \leqslant N + C \sum_{\substack{j=0 \\ N+1 \leqslant j \leqslant M}} (\#\text{ of descendents of } Q_j)$$

$$\leqslant N + C(M+1) \leqslant CM.$$

It remains now to establish (3.9). We begin with the case $j \neq 0$ of (3.9), and follow the corresponding argument in [10]. Since $\operatorname{supp} v_j \subset Q_j$, we need only check (3.9) for dyadic cubes $Q \in B$ with $Q \subset Q_j$ and in fact, only for proper dyadic subcubes of Q_j (since if $Q = \bigcup_{i=1}^{2^n} Q_i$, then

$$\begin{aligned}
\int_Q I_2(\chi_Q v) &= \int [I_1(\chi_Q v)]^2 \\
&= \sum_{i,j} \int I_1(\chi_{Q_i} v) I_1(\chi_{Q_j} v) \leqslant \frac{1}{2} \sum_{i,j} \int [I_1(\chi_{Q_j} v)]^2 \\
&\leqslant C_n \sum_{i=1}^{2^n} \int [I_1(\chi_{Q_i} v)]^2 \\
&= C_n \sum_{i=1}^{2^n} \int_{Q_i} I_2(\chi_{Q_i} v) v).
\end{aligned}$$

As in [10], the only « non-trivial » case occurs when $Q_j \in B$ is neither minimal nor branching and Q contains $Q^{\#_j}$, the unique maximal Q_i, $1 \leqslant i \leqslant M$, that is properly contained in Q_j (see the argument on p. 157-158 of [10]). To obtain (3.9) in this case we use a Whitney decomposition in place of the Calderon-Zygmund decomposition used in [10]. There is a dimensional constant C so large that we can choose pairwise disjoint dyadic subcubes $\hat{Q}_\alpha$ of $Q - Q^{\#} (= E_j \cap Q)$ such that each $\hat{Q}_\alpha$ satisfies

$$\text{(3.12)} \quad \textit{either } |\hat{Q}_\alpha| = |Q_j^{\#}| \text{ and } \operatorname{dist}(\hat{Q}_\alpha, Q_j^{\#}) \leqslant C \quad \textit{or } 2 \leqslant \frac{\operatorname{dist}(\hat{Q}_\alpha, Q_j^{\#})}{\operatorname{diam} \hat{Q}_\alpha} \leqslant 2C.$$

Then

$$\begin{aligned}
\int_Q I_2(\chi_Q v_j) v_j &= \sum_{\alpha,\beta} \int_{\hat{Q}_\alpha} I_2(\chi_{\hat{Q}_\beta} v) v \\
&\leqslant C \sum_{\{\alpha,\beta;\, \hat{Q}_\alpha \text{ touches } \hat{Q}_\beta\}} \int I_1(\chi_{\hat{Q}_\alpha} v) I_1(\chi_{\hat{Q}_\beta} v) \\
&\quad + C \sum_{\substack{\{\alpha,\beta;\, |\hat{Q}_\beta| \leqslant |\hat{Q}_\alpha| \\ \text{and } \hat{Q}_\alpha, \hat{Q}_\beta \text{ do not touch}\}}} \int_{\hat{Q}_\alpha} I_2(\chi_{\hat{Q}_\beta} v) v = D + E.
\end{aligned}$$

Now (3.12) shows that the number of $\hat{Q}_\beta$ touching a given $\hat{Q}_\alpha$ doesn't exceed a dimensional constant and so

$$D \leqslant C \sum_{\alpha} \int [I_1(\chi_{\hat{Q}_\alpha} v)]^2 = C \sum_{\alpha} \int_{\hat{Q}_\alpha} I_2(\chi_{\hat{Q}_\alpha} v) v \leqslant Cc \sum_{\alpha} \int_{\hat{Q}_\alpha} v_j = Cc \int_Q v_j$$

since the $\hat{Q}_\alpha$ are not in B. Condition (3.12) also shows that if $|\hat{Q}_\beta| \leqslant |\hat{Q}_\alpha|$ and $\hat{Q}_\beta$, $\hat{Q}_\alpha$ do not touch, then dist $(\hat{Q}_\beta, \hat{Q}_\alpha) \geqslant c|\hat{Q}_\alpha|^{\frac{1}{n}}$. Thus

$$E \leqslant C \sum_{\alpha} \left(\int_{\hat{Q}_\alpha} v \right) |\hat{Q}_\alpha|^{\frac{2}{n}-1} \sum_{\beta; |\hat{Q}_\beta| \leqslant |\hat{Q}_\alpha|} \left[\int_{\hat{Q}_\beta} v \right].$$

But $|\hat{Q}_\beta|^{\frac{2}{n}-1} \int_{\hat{Q}_\beta} v \leqslant \frac{1}{|\hat{Q}_\beta|_v} \int_{\hat{Q}_\beta} I_2(\chi_{\hat{Q}_\beta} v) v \leqslant c$ since $\hat{Q}_\beta \notin B$ and, by (3.12), the number of $\hat{Q}_\beta$ of a given size does not exceed a dimensional constant. Thus

$$\begin{aligned} E &\leqslant Cc \sum_{\alpha} \left(\int_{\hat{Q}} v \right) |\hat{Q}_\alpha|^{\frac{2}{n}-1} \sum_{\{k; 2^{kn} \leqslant |\hat{Q}_\alpha|\}} \left[\sum_{|\hat{Q}_\beta| = 2^{kn}} |\hat{Q}_\beta|^{1-\frac{2}{n}} \right] \\ &\leqslant Cc \sum_{\alpha} \int_{\hat{Q}_\alpha} v = Cc \int_Q v_j \quad (\text{since } n \geqslant 3) \end{aligned}$$

and this completes the verification of (3.9) for $j \neq 0$. For $j = 0$, we again suppose Q dyadic in B. If $Q \subset$ some $Q_1, \dots Q_M$, then $\operatorname{supp} v_0 \cap Q = \varnothing$ and (3.9) holds trivially. Otherwise, Q contains a unique maximal $Q_i (1 \leqslant i \leqslant M)$, say $Q^\#$, and we may argue as above to obtain (3.9). This completes the proof of Theorem 3.3.

Remark 3.5. – In [10] it is shown that $\sup_Q |Q|^{\frac{2}{n}-1} \int_Q v \leqslant C$ is necessary and $\sup_Q |Q|^{\frac{2}{n}-\frac{1}{p}} \left(\int_Q v^p \right)^{1/p} \leqslant C_p, p > 1$, sufficient for the L^2 trace inequality (1.1) with $T_\Phi = I_1$. We give here a direct proof that

$$\begin{aligned} (3.20) \quad \sup_Q |Q|^{\frac{2}{n}-1} \int_Q v &\leqslant C \sup_Q |Q|_v^{-1} \int_Q I_2(\chi_Q v) v \\ &\leqslant C_p \sup_Q |Q|^{\frac{2}{n}-\frac{1}{p}} \left(\int_Q v^p \right)^{1/p}, \qquad p > 1. \end{aligned}$$

The first inequality in (3.20) follows from the observation that $I_2(\chi_Q v)(x) \geqslant C|Q|^{\frac{2}{n}-1} \int_Q v$ for x in a cube Q.

Let $B_p = \sup_Q |Q|^{\frac{2}{n}-\frac{1}{p}}\left(\int_Q v^p\right)^{1/p}$. Suppose first that v satisfies the A_∞ condition of B. Muckenhoupt. Choose p so close to 1 that the reverse Hölder condition $\left(|Q|^{-1}\int_Q v^p\right)^{1/p} \leqslant C_p|Q|^{-1}\int_Q v$ holds for all cubes Q. Let $M_\alpha f(x) = \sup_{x\in Q} |Q|^{\frac{\alpha}{n}-1}\int_Q |f|$. Since $M_2(\chi_Q v) \leqslant B_p$ on Q,

$$(3.21)\quad \int_Q I_2(\chi_Q v)v \leqslant \left(\int_Q I_2(\chi_Q v)^{p'}\right)^{\frac{1}{p'}}\left(\int_Q v^p\right)^{1/p}$$

$$\leqslant C_p\left(\int_Q M_2(\chi_Q v)^{p'}\right)^{1/p'}\left(\int_Q v^p\right)^{1/p} \quad \text{(see [15])}$$

$$\leqslant C_pB_p|Q|^{1/p'}\left(\int_Q v^p\right)^{1/p} \leqslant C_pB_p\int_Q v\,.$$

For the general case, we use the observations in [10] that $v^+(x) = \sup_{x\in Q}\left(|Q|^{-1}\int_Q v^p\right)^{1/p}$ satisfies the A_∞ condition and $M_2 v^+ \leqslant C_pB_p$ ([10]; p. 153). The above argument then yields (3.21) with v^+ in place v. Since $v \leqslant v^+$, (3.20) follows. This is of course obvious from Theorem 2.3, but can also be proved directly. Finally, we point out that the condition $M_{2p}(v^p) \leqslant C_p$ is equivalent to the boundedness of M_p from L^2 to $L^2(v^p)$ ([17]). Together with the inequality $|I_1 f(x)| \leqslant C_pM_p|f|(x)^{1/p}Mf(x)^{1/p'}$ of D. R. Adams, this yields another proof that $M_{2p}(v^p) \leqslant C_p$ is sufficient for the L^2 trace inequality (1.1) with $T_\Phi = I_1$. J. M. Wilson has recently communicated to us yet another proof.

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