ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 36, nº 3 (1986), p. 183-205

<http://www.numdam.org/item?id=AIF_1986__36_3_183_0>

Ann. Inst. Fourier, Grenoble
36, 3 (1986), 183-205

VANISHING THEOREMS FOR COMPACT HESSIAN MANIFOLDS

by Hirohiko SHIMA

Let M be a flat affine manifold with a locally flat affine connection D. Among the Riemannian metrics on M there is an important class of Riemannian metrics which are compatible with the flat affine structure on M. A Riemannian metric g on M is said to be *Hessian* if g has an expression $g = D^2 u$ where u is a local C^∞-function. A flat affine manifold provided with a Hessian metric is called a *Hessian manifold.* A certain geometry of Hessian manifolds has been studied in Shima [10]-[14]. See also Cheng and Yau [2] and Yagi [15].

Hessian manifolds have in a certain sense some analogy with Kählerian manifolds. In this paper, being motivated by the theory of cohomology for Kählerian manifolds we study cohomology groups for Hessian manifolds.

Let F be a locally constant vector bundle over M. We denote by $\Omega^{p,q}(F)$ the space of all sections of $(\overset{p}{\wedge} T^*) \otimes (\overset{q}{\wedge} T^*) \otimes F$, where T^* is the cotangent bundle over M. Since the vector bundle $(\overset{q}{\wedge} T^*) \otimes F$ is locally constant, we can naturally define a complex

$$\dots \xrightarrow{\partial} \Omega^{p-1,q}(F) \xrightarrow{\partial} \Omega^{p,q}(F) \xrightarrow{\partial} \Omega^{p+1,q}(F) \xrightarrow{\partial} \dots .$$

We denote by $H^{p,q}(F)$ the p-th cohomology group of the complex. Then we have the following duality theorem analogous to that of Serre [9].

THEOREM. – *Let* M *be a compact oriented flat affine manifold of dimension* n. *Then we have*

Key-words : Hessian manifolds – Cohomology – Vanishing theorems.

$$H^{p,q}(F) \cong H^{n-p,n-q}((K \otimes F)^*),$$

where K *is the canonical line bundle over* M *and* $(K \otimes F)^*$ *is the dual bundle of* $K \otimes F$.

Let F be a locally constant line bundle over M. Choose an open covering $\{U_\lambda\}$ of M such that the local triviality holds on each U_λ. Denote by $\{f_{\lambda\mu}\}$ the constant transition functions with respect to $\{U_\lambda\}$. A fiber metric $a = \{a_\lambda\}$ on F is a collection of positive C^∞-functions a_λ on U_λ such that

$$a_\mu = f_{\lambda\mu}^2 a_\lambda .$$

Using this we can define a globally defined closed 1-form A and a symmetric bilinear form B by

$$A = - D \log a_\lambda ,$$

$$B = - D^2 \log a_\lambda ,$$

and we call them the *first Koszul form* and the *second Koszul form* of F with respect ot the fiber metric $a = \{a_\lambda\}$ respectively.

A locally constant line bundle F is said to be *positive* (resp. *negative*) if the second Koszul form is positive (resp. negative) definite with respect to a certain fiber metric. It should be remarked that if a compact connected flat affine manifold M admits a locally constant positive (resp. negative) line bundle, then by a theorem of Koszul [6] M is a hyperbolic affine manifold, that is, the universal covering of M is an open convex cone not containing any full straight line.

Kodaira-Nakano's vanishing theorem for compact Kählerian manifolds plays an essential role in the theory of compact Kählerian manifolds. In this paper we prove the following vanishing theorem for a compact Hessian manifold analogous to that of Kodaira-Nakano.

THEOREM. – *Let* M *be a compact connected oriented Hessian manifold. Denote by* K *the canonical line bundle over* M. *Let* F *be a locally constant line bundle over* M.

(i) *If* 2F + K *is positive, then*

$$H^{p,q}(F) = 0 \quad \textit{for} \quad p + q > n.$$

(ii) *If* $2F + K$ *is negative, then*

$$H^{p,q}(F) = 0 \quad \textit{for} \quad p + q < n.$$

As to vanishing theorem for compact hyperbolic affine manifolds we should mention the following theorem due to Koszul [7].

THEOREM. – *Let* M *be a compact oriented hyperbolic affine manifold. Then we have*

$$H^{p,q}(1) = 0 \quad \textit{for} \quad p, q > 0,$$

where 1 *is the trivial line bundle over* M.

In § 1 and § 2 a Riemannian metric g is not assumed to be Hessian. We define in § 1 fundamental operators $e(g)$, $i(g)$, Π, $*$, ∂, δ and $\Box$. In § 2 we define the Laplacian $\Box_a$ on $\Omega^{p,q}(F)$, and prove the duality theorem $H^{p,q}(F) \cong H^{n-p,n-q}((K \otimes F)^*)$ and the cohomology isomorphisms $\mathcal{H}^{p,q}(F) \cong H^{p,q}(F) \cong H^p(P^q(F))$. In § 3 we give the local expressions for geometric concepts on Hessian manifolds. In § 4 and § 5 the formulae of Weitzenböck type for $\Box$ and $\Box_a$ are obtained. In § 6 we prove a vanishing theorem analogous to that of Kodaira-Nakano. In § 7 we mention a vanishing theorem of Koszul type.

The author would like to thank Professor J.L. Koszul for his kind suggestions.

1. The Laplacian $\Box$ on $\Omega^{p,q}$.

Let M be a flat affine manifold with a locally flat affine connection D. Then there exist local coordinate systems $\{x^1, \ldots, x^n\}$ such that $Ddx^i = 0$, which will be called *affine local coordinate systems.* Throughout this paper the local expressions for geometric concepts on M will be given in terms of affine local coordinate system. From now on we assume further that M is compact, connected and oriented.

Choose an arbitrary Riemannian metric g on M. Let $\Omega^{p,q}$ be the space of all sections of $(\overset{p}{\wedge} T^*) \otimes (\overset{q}{\wedge} T^*)$. We denote the local expression of $\phi \in \Omega^{p,q}$ by

$$\phi = \frac{1}{p!\,q!} \Sigma\, \phi_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} (dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \otimes (dx^{\bar{j}_1} \wedge \ldots \wedge dx^{\bar{j}_q}).$$

For simplicity let us fix some notation. We denote as follows :

$$\mathrm{I}_p = (i_1, \ldots, i_p), \quad i_1 < i_2 < \ldots < i_p, \quad \mathrm{i} \leqslant i_\sigma \leqslant n,$$

$$\mathrm{I}_{n-p} = (i_{p+1}, \ldots, i_n), \quad i_{p+1} < \ldots < i_n, \quad 1 \leqslant i_\tau \leqslant n,$$

and $(i_1, \ldots, i_p, i_{p+1}, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. Then with this notation we write

$$\phi = \sum_{\mathrm{I}_p, \bar{\mathrm{J}}_q} \phi_{\mathrm{I}_p \bar{\mathrm{J}}_q} dx^{\mathrm{I}_p} \otimes dx^{\bar{\mathrm{J}}_q},$$

where $dx^{\mathrm{I}_p} = dx^{i_1} \wedge \ldots \wedge dx^{i_p}$.

For $\phi, \psi \in \Omega^{p,q}$ we set

$$h(\phi, \psi) = \frac{1}{p!\,q!} \phi_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} \psi^{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} \,(*) \qquad (1.1)$$

$$= \phi_{\mathrm{I}_p \bar{\mathrm{J}}_q} \psi^{\mathrm{I}_p \bar{\mathrm{J}}_q}. \,(**)$$

DEFINITION 1.1. – *The inner product of* $\phi, \psi \in \Omega^{p,q}$ *is*

$$(\phi, \psi) = \int_{\mathrm{M}} h(\phi, \psi)\, v,$$

where v *is the volume element determined by* g.

DEFINITION 1.2. – *We define* $*$*-operation*

$$* : \Omega^{p,q} \longrightarrow \Omega^{n-p,n-q}$$

by $(*\phi)_{\mathrm{I}_{n-p} \bar{\mathrm{J}}_{n-q}} = (-1)^{pq} \operatorname{sgn}(\mathrm{I}_p \mathrm{I}_{n-p}) \operatorname{sgn}(\bar{\mathrm{J}}_q \bar{\mathrm{J}}_{n-q})\, \mathrm{G}\, \phi^{\mathrm{I}_p \bar{\mathrm{J}}_q}$, *where* $\operatorname{sgn}(\mathrm{I}_p \mathrm{I}_{n-p})$ *is the signature of the permutation* $(\mathrm{I}_p \mathrm{I}_{n-p})$ *of* $(1, \ldots, n)$ *and* $\mathrm{G} = \det(g_{ij})$.

(*) Throughout this paper we use Einstein's convention on indices.

(**) $\phi_{\mathrm{I}_p \bar{\mathrm{J}}_q} \psi^{\mathrm{I}_p \bar{\mathrm{J}}_q}$ means $\sum_{\mathrm{I}_p, \mathrm{J}_q} \phi_{\mathrm{I}_p \bar{\mathrm{J}}_q} \psi^{\mathrm{I}_p \bar{\mathrm{J}}_q}$.

DEFINITION 1.3. – *Let* $\phi = \Sigma\, \phi_{I_p \bar{J}_q}\, dx^{I_p} \otimes dx^{\bar{J}_q}$ *and* $\psi = \Sigma\, \psi_{K_r \bar{L}_s}\, dx^{K_r} \otimes dx^{\bar{L}_s}$.

We set $\phi \wedge \psi = \Sigma\, \phi_{I_p \bar{J}_q}\, \psi_{K_r \bar{L}_s} (dx^{I_p} \wedge dx^{K_r}) \otimes (dx^{\bar{J}_q} \wedge dx^{\bar{L}_s})$.

A straightforward calculation shows

PROPOSITION 1.1. – *Let* $\phi, \psi \in \Omega^{p,q}$. *Then*

(i) $** \phi = (-1)^{n+p+q}\, \phi$,

(ii) $\phi \wedge * \psi = (-1)^{pq}\, h(\phi, \psi)\, v \otimes v$.

DEFINITION 1.4. – *Considering the Riemannian metric* g *as an element in* $\Omega^{1,1}$ *we define*

$$e(g) : \Omega^{p,q} \longrightarrow \Omega^{p+1,q+1},$$

$$i(g) : \Omega^{p,q} \longrightarrow \Omega^{p-1,q-1},$$

by $e(g)\,\phi = g \wedge \phi$ *for* $\phi \in \Omega^{p,q}$ *and* $i(g) = (-1)^{n+p+q+1} * e(g) *$.

Then $i(g)$ is the adjoint operator of $e(g)$ with respect to the inner product given in Definition 1.1 :

$$(i(g)\,\phi, \psi) = (\phi, e(g)\,\psi) \quad \text{for} \quad \phi \in \Omega^{p,q},\ \psi \in \Omega^{p-1,q-1}.$$

DEFINITION 1.5. – *We set*

$$\Pi = \sum_{p,q} (n - p - q)\, \pi_{p,q},$$

where $\pi_{p,q}$ *is the projection from* $\sum_{r,s} \Omega^{r,s}$ *onto* $\Omega^{p,q}$.

PROPOSITION 1.2. – *We have*

$$[\Pi, e(g)] = -2e(g), \quad [\Pi, i(g)] = 2i(g), \quad [i(g), e(g)] = \Pi.$$

The proof is carried out by a direct calculation and so it is omitted.

DEFINITION 1.6. – *Define*

$$\partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}$$

by $\partial = \sum_k (e(dx^k) \otimes \mathrm{id})\, D_k$, *where* $e(dx^k)$ *is a linear map from* $\overset{p}{\wedge} T^*$ *to* $\overset{p+1}{\wedge} T^*$ *given by* $e(dx^k)\,\omega = dx^k \wedge \omega$, id *is the identity map on* $\overset{p}{\wedge} T^*$ *and* D_k *is the covariant derivation with respect to* $\partial/\partial x^k$ *for the locally flat affine connection* D.

Then we have

$$\partial\partial = 0. \tag{1.2}$$

DEFINITION 1.7. – *Define*

$$\delta : \Omega^{p,q} \longrightarrow \Omega^{p-1,q}$$

by $\delta = (-1)^{n+1} \sqrt{G} * \partial \left(\frac{1}{\sqrt{G}} *\right)$.

PROPOSITION 1.3. – δ *is the adjoint operator of* ∂ *with respect to the inner product given in Definition 1.1;*

$$(\partial\phi, \psi) = (\phi, \delta\psi) \quad \text{for} \quad \phi \in \Omega^{p,q}, \ \psi \in \Omega^{p+1,q}.$$

In Proposition 2.1 we prove the above fact in more general situation and so we omit the proof.

DEFINITION 1.8. – *We define*

$$\square : \Omega^{p,q} \longrightarrow \Omega^{p,q}$$

by $\square = \partial\delta + \delta\partial$, *and call it the Laplacian.* $\phi \in \Omega^{p,q}$ *is said to be* $\square$-*harmonic if* $\square\,\phi = 0$.

2. The Laplacian $\square_a$ on $\Omega^{p,q}(F)$.

Let F be a locally constant vector bundle over M. Choose an open covering $\{U_\lambda\}$ of M such that the local triviality holds on each U_λ. Let $\{\xi_\lambda^1, \ldots, \xi_\lambda^m\}$ be fiber coordinate systems such that the transition functions $\{f_{\lambda\mu}\}$ defined by

$$\xi_\lambda^i = \sum_j f_{\lambda\mu}{}^i{}_j \, \xi_\mu^j$$

are constants. A fiber metric $a = \{a_\lambda\}$ on F is a collection of $m \times m$ positive definite symmetric matrices $a = (a_{\lambda ij})$ such that each $a_{\lambda ij}$ is a C^∞-function on U_λ and

$$a_\lambda = {}^t f_{\mu\lambda} \, a_\mu \, f_{\mu\lambda}$$

holds.

Let $\Omega^{p,q}(F)$ denote the space of all sections of $(\overset{p}{\wedge} T^*) \otimes (\overset{q}{\wedge} T^*) \otimes F$.

Using fiber coordinate systems $\{\xi_\lambda^i\}$ we express an element $\phi \in \Omega^{p,q}(F)$ as $\phi = \{\phi_\lambda^i\}$.

DEFINITION 2.1. – *Define*

$$\partial : \Omega^{p,q}(F) \longrightarrow \Omega^{p+1,q}(F)$$

by $\partial \{\phi^i\} = \{\partial \phi^i\}$. (*)

We have then

$$\partial\partial = 0. \tag{2.1}$$

DEFINITION 2.2. – *The inner product of* $\phi, \psi \in \Omega^{p,q}(F)$ *is*

$$(\phi, \psi) = \int_M \Sigma \, a_{ij} h(\phi^i, \psi^j) \, v.$$

DEFINITION 2.3 – *Define*

$$\delta_a : \Omega^{p,q}(F) \longrightarrow \Omega^{p-1,q}(F)$$

by $\delta_a \{\phi^i\} = \left\{ (-1)^{n+1} \sum_{j,k} \sqrt{G}\, a^{ij} * \partial \left(\frac{a_{jk}}{\sqrt{G}} * \phi^k \right) \right\}$, *where* a^{ij} *is the* (i,j)*-component of* $(a_{ij})^{-1}$.

(*) For brevity the subscripts $\lambda, \mu, \ldots$ are droped where no confusion will arise.

PROPOSITION 2.1. – δ_a *is the adjoint operator of* ∂ *with respect to the inner product given in Definition 2.2;*

$$(\partial\phi, \psi) = (\phi, \delta_a \psi) \quad \textit{for} \quad \phi \in \Omega^{p-1,q}(\mathrm{F}), \quad \psi \in \Omega^{p,q}(\mathrm{F}).$$

Proof. – Since $\sum_{i,j} a_{ij}\, \phi^i \wedge * \psi^j$ is globally defined on M, there exists $(n-1)$-form ω on M such that $\omega \otimes v = \Sigma\, a_{ij} \phi^i \wedge * \psi^j$. Then

$$\partial(\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,$$

where $\alpha = d \log \sqrt{\mathrm{G}}$, and

$$\partial(\Sigma\, a_{ij}\, \phi^i \wedge * \psi^j)$$
$$= (-1)^{pq}\, \Sigma\, a_{ij}\, h(\partial\phi^i, \psi^j)\, v \otimes v + (-1)^{n-q}\, \Sigma\, \phi^i \wedge ** \partial(a_{ij} * \psi^j).$$

Since

$$\delta_a \psi^i = -(-1)^{n+1} * (\alpha \wedge * \psi^i) + (-1)^{n+1}\, \Sigma\, a^{ij} * \partial(a_{jk} * \psi^k),$$

we have

$$(\alpha \wedge \omega + d\omega) \otimes v$$
$$= (-1)^{pq}\, \Sigma\, a_{ij}\, h(\partial\phi^i, \psi^j)\, v \otimes v + (-1)^{n-q}\, \Sigma\, a_{ij}\phi^i \wedge ** (\alpha \wedge * \psi^j)$$
$$+ (-1)^{q+1}\, \Sigma\, a_{ij} \phi^i \wedge * \delta_a \psi^j$$
$$= (-1)^{pq}\, \Sigma\, a_{ij} h(\partial\phi^i, \psi^j)\, v \otimes v + (\alpha \wedge \omega) \otimes v$$
$$+ (-1)^{pq-1}\, \Sigma\, a_{ij} h(\phi^i, \delta_a \psi^j)\, v \otimes v,$$

and so

$$d\omega = (-1)^{pq}(\Sigma\, a_{ij}\, h(\partial\phi^i, \psi^j) - \Sigma\, a_{ij}\, h(\phi^i, \delta_a \psi^j))\, v.$$

Therefore

$$0 = \int_{\mathrm{M}} d\omega = (-1)^{pq}((\partial\phi, \psi) - (\phi, \delta_a \psi)).$$

Q.E.D.

DEFINITION 2.4. – *We define*

$$\square_a : \Omega^{p,q}(\mathrm{F}) \longrightarrow \Omega^{p,q}(\mathrm{F})$$

by $\Box_a = \partial\delta_a + \delta_a\partial$, *and call it the Laplacian.* $\phi \in \Omega^{p,q}(F)$ *is said to be* $\Box_a$*-harmonic if* $\Box_a \phi = 0$.

DEFINITION 2.5. – *We set*

$$\mathcal{H}^{p,q}(F) = \{\phi \in \Omega^{p,q}(F) \mid \Box_a \phi = 0\}.$$

THEOREM 2.2. – *We have the following duality:*

$$\mathcal{H}^{p,q}(F) \cong \mathcal{H}^{n-p,n-q}((K \otimes F)^*),$$

where K *is the canonical line bundle over* M *and* $(K \otimes F)^*$ *is the dual bundle of* $K \otimes F$.

Proof. – For $\psi = \{\psi^j\} \in \Omega^{p,q}(F)$ we set

$$\psi_i^* = \sum_j \frac{a_{ij}}{\sqrt{G}} * \psi^j. \tag{2.2}$$

Then we have $\psi^* = \{\psi_i^*\} \in \Omega^{n-p,n-q}((K \otimes F)^*)$. It follows from Proposition 1.1 (i)

$$\psi^j = (-1)^{n+p+q} \sum_i \sqrt{G}\, a^{ji} * \psi_i^*. \tag{2.3}$$

Thus the map $\psi \longrightarrow \psi^*$ is a linear isomorphism from $\Omega^{p,q}(F)$ onto $\Omega^{n-p,n-q}((K \otimes F)^*)$.

Let $\phi \in \Omega^{p,q}(F)$ and $\psi^* \in \Omega^{n-p,n-q}((K \otimes F)^*)$. Then $\sum_i \sqrt{G}\,\phi^i \wedge \psi_i^*$ is globally defined on M. Hence there exists a C^∞-function $k(\phi, \psi^*)$ on M such that

$$\sum_i \sqrt{G}\,\phi^i \wedge \psi_i^* = k(\phi, \psi^*)\, v \otimes v.$$

We set

$$\langle \phi, \psi^* \rangle = (-1)^{pq} \int_M k(\phi, \psi^*)\, v.$$

Since

$$k(\phi, \psi^*)\, v \otimes v = \sum_{i,j} a_{ij}\, \phi^i \wedge * \psi^j = (-1)^{pq} \sum_{i,j} a_{ij}\, h(\phi^i, \psi^j)\, v \otimes v,$$

we have

$$\langle \phi, \psi^* \rangle = (\phi, \psi) \quad \text{for} \quad \phi, \psi \in \Omega^{p,q}(F).$$

Define the inner product of $\psi^*, \phi^* \in \Omega^{n-p,n-q}((K \otimes F)^*)$ by

$$(\psi^*, \phi^*) = \int_M \Sigma\, G a^{ij} h(\psi_i^*, \phi_j^*)\, v.$$

Since

$$\sum_{i,j} G a^{ij} h(\psi_i^*, \phi_j^*)\, v \otimes v = \sum_{i,j} a_{ij} h(*\psi^i, *\phi^j)\, v \otimes v$$

$$= (-1)^{pq} \sum_{i,j} a_{ij} \phi^j \wedge *\psi^i = \sum_{i,j} a_{ij} h(\phi^j, \psi^i)\, v \otimes v,$$

we obtain

$$(\psi^*, \phi^*) = (\phi, \psi) \quad \text{for} \quad \phi, \psi \in \Omega^{p,q}(F).$$

Let $\phi \in \Omega^{p-1,q}(F)$ and $\psi^* \in \Omega^{n-p,n-q}((K \otimes F)^*)$. Then $\sum_i \sqrt{G}\, \phi^i \wedge \psi_i^*$ is globally defined on M and hence there exists $(n-1)$-form ω on M such that

$$\sum_i \sqrt{G} \phi^i \wedge \psi_i^* = \omega \otimes v.$$

Since

$$\partial\Big(\sum_i \sqrt{G}\, \phi^i \wedge \psi_i^*\Big)$$

$$= \sum_i \{\alpha \wedge \sqrt{G}\, \phi^i \wedge \psi_i^* + \sqrt{G}\, \partial\phi^i \wedge \psi_i^* + (-1)^{p-1} \sqrt{G}\, \phi^i \wedge \partial\phi_i^*\}$$

$$= (\alpha \wedge \omega) \otimes v + \sum_i \{k(\partial\phi^i, \psi_i^*) + (-1)^{p-1} k(\phi^i, \partial\psi_i^*)\}\, v \otimes v,$$

and

$$\partial(\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,$$

we obtain

$$d\omega = \sum_i \{k(\partial\phi^i, \psi_i^*) + (-1)^{p-1} k(\phi^i, \partial\psi_i^*)\}\, v.$$

Therefore

$$0 = \int_M d\omega$$

$$= (-1)^{pq} \langle \partial\phi, \psi^* \rangle + (-1)^{p-1+(p-1)q} \langle \phi, \partial\psi^* \rangle.$$

This implies

$$\langle \partial\phi , \psi^* \rangle = (-1)^{p+q} \langle \phi , \partial \psi^* \rangle .$$

Using these facts we obtain

$$(\phi^* , \partial \psi^*) = \langle \phi , \partial \psi^* \rangle = (-1)^{p+q} \langle \partial\phi , \psi^* \rangle = (-1)^{p+q} (\partial\phi , \psi)$$
$$= (-1)^{p+q} (\phi , \delta_a \psi) = (-1)^{p+q} (\phi^* , (\delta_a \psi)^*) ,$$

hence

$$\partial \psi^* = (-1)^{p+q} (\delta_a \psi)^* \quad \text{for} \quad \psi \in \Omega^{p,q}(\mathrm{F}) . \tag{2.4}$$

By the same way we have

$$(\psi^* , \delta_a \phi^*) = (\partial \psi^* , \phi^*) = \langle \phi , \partial \psi^* \rangle = (-1)^{p+q} \langle \partial\phi , \psi^* \rangle$$
$$= (-1)^{p+q} (\partial\phi , \psi) = (-1)^{p+q} ((\partial\phi)^* , \psi^*) ,$$

hence

$$\delta_a \phi^* = (-1)^{p+q} (\partial\phi)^* .$$

Thus

$$\delta_a \psi^* = (-1)^{p+q+1} (\partial \psi)^* \quad \text{for} \quad \psi \in \Omega^{p,q}(\mathrm{F}) . \tag{2.5}$$

(2.4) and (2.5) imply that ψ^* is harmonic if and only if ψ is harmonic.

Q.E.D.

DEFINITION 2.6. — *We set*

$$\mathrm{H}^{p,q}(\mathrm{F}) = \{\phi \in \Omega^{p,q}(\mathrm{F}) \mid \partial\phi = 0\} / \{\partial \psi \mid \psi \in \Omega^{p-1,q}(\mathrm{F})\} .$$

A q-form ω on M is said to be *D-parallel* if $\mathrm{D}\omega = 0$. Let us denote by $\mathrm{P}^q(\mathrm{F})$ the sheaf over M of germs of F-valued D-parallel q-forms.

DEFINITION 2.7. — *We denote by* $\mathrm{H}^p(\mathrm{P}^q(\mathrm{F}))$ *the* p*-th cohomology group of* M *with coefficients on* $\mathrm{P}^q(\mathrm{F})$.

THEOREM 2.3. — *We have the following isomorphisms:*

$$\mathcal{H}^{p,q}(\mathrm{F}) \cong \mathrm{H}^{p,q}(\mathrm{F}) \cong \mathrm{H}^p(\mathrm{P}^q(\mathrm{F})) .$$

Proof. – By the theory of harmonic integral we have

$$\mathcal{H}^{p,q}(\mathrm{F}) \cong \mathrm{H}^{p,q}(\mathrm{F}).$$

Let $\mathrm{A}^{p,q}(\mathrm{F})$ denote the sheaf over M of germs of sections of $(\overset{p}{\wedge}\mathrm{T}^*) \otimes (\overset{q}{\wedge}\mathrm{T}^*) \otimes \mathrm{F}$. Then

$$0 \longrightarrow \mathrm{P}^q(\mathrm{F}) \longrightarrow \mathrm{A}^{0,q}(\mathrm{F}) \xrightarrow{\partial} \mathrm{A}^{1,q}(\mathrm{F}) \xrightarrow{\partial} \mathrm{A}^{2,q}(\mathrm{F}) \xrightarrow{\partial} \cdots$$

is a fine resolution of $\mathrm{P}^q(\mathrm{F})$. Thus we have $\mathrm{H}^{p,q}(\mathrm{F}) \cong \mathrm{H}^p(\mathrm{P}^q(\mathrm{F}))$.

Q.E.D.

3. Hessian metrics on affine local coordinate systems.

Let M be a Hessian manifold with a locally flat affine connection D and a Hessian metric g. We denote by ∇ the Riemannian connection for g. In this section we shall express various geometric concepts on the Hessian manifold M in terms of affine local coordinate systems. Let us denote by D_k and ∇_k the covariant derivations with respect to $\partial/\partial x^k$ for D and ∇ respectively. Since the Christoffel symbol Γ^i_{jk} for g is the difference between the components of affine connections ∇ and D, we may consider that Γ^i_{jk} is a tensor field. We have then

$$\Gamma^i_{jk} = \frac{1}{2} g^{is} \mathrm{D}_k g_{sj}, \tag{3.1}$$

$$\mathrm{D}_k g_{ij} = 2\Gamma_{ijk}, \qquad \mathrm{D}_k g^{ij} = -2\Gamma^{ij}_{\ \ k},$$

$$\Gamma_{ijk} = \Gamma_{jik} = \Gamma_{ikj}.$$

DEFINITION 3.1. – *We define a* 1*-form* α *and a symmetric bilinear form* β *by*

$$\alpha = \mathrm{D} \log \sqrt{\mathrm{G}},$$

$$\beta = \mathrm{D}^2 \log \sqrt{\mathrm{G}},$$

where $\mathrm{G} = \det(g_{ij})$, *and call them the first Koszul form and the second Koszul form of* M *respectively.*

Then we have

$$\alpha_i = \Gamma^r{}_{ir}, \tag{3.2}$$

$$\beta_{i\bar{j}} = D_{\bar{j}}\Gamma^r{}_{ir}.$$

DEFINITION 3.2. – *Let* γ_k *be the derivation of the algebra of tensor fields defined by*

$$\gamma_k = \nabla_k - D_k.$$

Let T^p_q be the space of tensor fields of type (p,q) defined on M.

DEFINITION 3.3. – *We define certain covariant derivations* ∇'_k, $\bar{\nabla}'_{\bar{k}}$ *on* $T^p_q \otimes T^r_s$ *by*

$$\nabla'_k = (2\gamma_k) \otimes \mathrm{id} + D_k,$$

$$\bar{\nabla}'_{\bar{k}} = \mathrm{id} \otimes (2\gamma_{\bar{k}}) + D_{\bar{k}},$$

where id *are the identity transformations.*

Notice that

$$\nabla_k = \frac{1}{2}(\nabla'_k + \bar{\nabla}'_{\bar{k}}), \quad \text{where} \quad k = \bar{k}.$$

LEMMA 3.1. – *For the Hessian metric* g *we have*

$$\nabla'_k g_{i\bar{j}} = 0, \qquad \bar{\nabla}'_{\bar{k}} g_{i\bar{j}} = 0,$$

$$\nabla'_k g^{i\bar{j}} = 0, \qquad \bar{\nabla}'_{\bar{k}} g^{i\bar{j}} = 0.$$

Proof. – By (3.1) we obtain

$$\nabla'_k g_{i\bar{j}} = D_k g_{i\bar{j}} - 2\Gamma^m{}_{ki} g_{m\bar{j}} = 2\Gamma_{i\bar{j}k} - 2\Gamma_{\bar{j}ki} = 0.$$

Similarly we can prove the other equalities.

Q.E.D.

DEFINITION 3.4. – *Considering* γ_i *as tensor fields of type* (1.1) *we define tensor fields* γ *and* S *by*

$$\gamma = \sum_i \gamma_i \otimes dx^i,$$

$$S = D\gamma.$$

The component of S is given by

$$S^i_{jkl} = D_k \Gamma^i_{jl} .$$

LEMMA 3.2. – $S_{ijkl} = S_{kjil} = S_{klij} = S_{ilkj}$.

Proof. – Let $g_{ij} = D_i D_j u$. By (3.1) we have

$$\begin{aligned} S_{ijkl} &= g_{ip} D_k \Gamma^p_{jl} = g_{ip} D_k (g^{pq} \Gamma_{qjl}) = g_{ip} (D_k g^{pq}) \Gamma_{qjl} + g_{ip} g^{pq} D_k \Gamma_{qjl} \\ &= - 2\Gamma_i{}^q{}_k \Gamma_{qjl} + D_k \Gamma_{ijl} = - 2g^{qr} \Gamma_{irk} \Gamma_{qjl} + D_k \Gamma_{ijl} \\ &= \frac{1}{2} D_i D_j D_k D_l u - \frac{1}{2} g^{qr} (D_r D_i D_k u)(D_q D_j D_l u) . \end{aligned}$$

This proves the Lemma.

Q.E.D.

LEMMA 3.3. – $\beta_{ij} = S^r{}_{rij} = S_{ij}{}^r{}_r$.

Proof. – $\beta_{ij} = D_j \alpha_i = D_i \alpha_j = D_i \Gamma^r{}_{rj} = S^r{}_{rij}$. By Lemma 3.2 we have $S^r{}_{rij} = g^{rp} S_{prij} = g^{rp} S_{ijpr} = S_{ij}{}^r{}_r$.

Q.E.D.

4. The local expression for □.

From now on we always assume that M is a compact connected oriented Hessian manifold.

PROPOSITION 4.1. – *Let* $\phi \in \Omega^{p,q}$. *Then we have*

$$(\partial\phi)_{i_1 \dots i_{p+1} \bar{j}_q} = \sum_\sigma (-1)^{\sigma-1} \nabla'_{i_\sigma} \phi_{i_1 \dots \hat{i}_\sigma \dots i_{p+1} \bar{j}_q} ,$$

where $\hat{i}_\sigma$ *means "omit* i_σ*".*

Proof. – By Definition 1.6 we have

$$(\partial\phi)_{I_{p+1} \bar{J}_q} = \sum_{\sigma=1}^{p+1} (-1)^{\sigma-1} D_{i_\sigma} \phi_{i_1 \dots \hat{i}_\sigma \dots i_{p+1} \bar{j}_q} . \tag{4.1}$$

Using this and (3.1) we obtain the proposition.

Q.E.D.

PROPOSITION 4.2. – *Let* $\phi \in \Omega^{p,q}$. *Then we have*

$$(\delta\phi)_{I_{p-1}\bar{J}_q} = -g^{s\bar{r}}\,\bar{\nabla}'_{\bar{r}}\phi_{sI_{p-1}\bar{J}_q} + \alpha^s\,\phi_{sI_{p-1}\bar{J}_q}.$$

Proof. – Let $\psi \in \Omega^{p-1,q}$. By (4.1) and Green's theorem we have

$$(\phi, \partial\psi) = -\int_M D_r(\phi^{rI_{p-1}\bar{J}_q}\sqrt{G})\frac{1}{\sqrt{G}}\,\psi_{I_{p-1}\bar{J}_q}\,v.$$

Thus we obtain

$$\begin{aligned}(\delta\phi)^{I_{p-1}\bar{J}_q} &= -D_r\,\phi^{rI_{p-1}\bar{J}_q} - \alpha_r\,\phi^{rI_{p-1}\bar{J}_q}\\ &= -\nabla_r\,\phi^{rI_{p-1}\bar{J}_q} + \alpha_r\,\phi^{rI_{p-1}\bar{J}_q}.\end{aligned}$$

This completes the proof.

Q.E.D.

THEOREM 4.1. – *Let* $\phi \in \Omega^{p,q}$. *Then we have*

$$(\Box\phi)_{I_p\bar{J}_q} = -g^{s\bar{r}}\,\bar{\nabla}'_{\bar{r}}\nabla'_s\phi_{I_p\bar{J}_q} + \alpha^s\nabla'_s\phi_{I_p\bar{J}_q} - \sum_{\sigma}\beta^s{}_{i_\sigma}\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q}$$
$$+ 2\sum_{\sigma,\tau} S^{\bar{t}s}{}_{i_\sigma\bar{j}_\tau}\,\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{j}_1\ldots(\bar{t})_\tau\ldots\bar{j}_q},$$

where $(s)_\sigma$ *means "substitute* s *for* σ*-th place".*

Proof. – Using Proposition 4.1, Proposition 4.2 and $\nabla'_i\alpha^j = \beta^j{}_i$ we obtain

$$(\partial\delta\phi)_{I_p\bar{J}_q} = -g^{s\bar{r}}\sum_{\sigma}\nabla'_{i_\sigma}\bar{\nabla}'_{\bar{r}}\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q} + \sum_{\sigma}\beta^s{}_{i_\sigma}\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q}$$
$$+ \sum_{\sigma}\alpha^s\nabla'_{i_\sigma}\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q};$$

$$(\delta\partial\phi)_{I_p\bar{J}_q} = -g^{s\bar{r}}\bar{\nabla}'_{\bar{r}}(\nabla'_s\phi_{I_p\bar{J}_q} - \sum_{\sigma}\nabla'_{i_\sigma}\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q})$$
$$+ \alpha^s(\nabla'_s\phi_{I_p\bar{J}_q} - \sum_{\sigma}\nabla'_{i_\sigma}\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q}),$$

and so

$$(\Box\phi)_{I_p\bar{J}_q} = - g^{s\bar{r}}\,\overline{\nabla}'_{\bar{r}}\nabla'_s\,\phi_{I_p\bar{J}_q} + \alpha^s\,\nabla'_s\,\phi_{I_p\bar{J}_q}$$

$$- g^{s\bar{r}}\sum_{\sigma}\,[\nabla'_{i_\sigma},\overline{\nabla}'_{\bar{r}}]\,\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q}$$

$$+ \sum_{\sigma}\,\beta^s_{i_\sigma}\,\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q}\,.$$

Let us calculate the third term on the right-hand of the above formula. Since $[\nabla'_i, \overline{\nabla}'_{\bar{j}}]$ is a derivation of the algebra of tensor fields which maps every function to 0 and since

$$[\nabla'_i, \overline{\nabla}'_{\bar{j}}]\,\xi_k = 2S^p{}_{i\bar{j}k}\,\xi_p\,,$$

$$[\nabla'_i, \overline{\nabla}'_{\bar{j}}]\,\xi_{\bar{k}} = -\,2S^{\bar{p}}{}_{\bar{j}i\bar{k}}\,\xi_{\bar{p}}\,,$$

we have

$$[\nabla'_{i_\sigma}, \overline{\nabla}'_{\bar{r}}]\,\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q} = \sum_{\tau} 2S^m{}_{i_\sigma\bar{r}i_\tau}\,\phi_{i_1\ldots(s)_\sigma\ldots(m)_\tau\ldots i_p\bar{J}_q}$$

$$+ 2S^m{}_{i_\sigma\bar{r}s}\;\;\phi_{i_1\ldots(m)_\sigma\ldots i_p\bar{J}_q}$$

$$- \sum_{\tau} 2S^{\bar{m}}{}_{\bar{r}i_\sigma\bar{j}_\tau}\,\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{j}_1\ldots(\bar{m})_\tau\ldots\bar{j}_q}\,.$$

Thus, by Lemma 3.2 and 3.3 we obtain

$$g^{s\bar{r}}\sum_{\sigma}\,[\nabla'_{i_\sigma}, \overline{\nabla}'_{\bar{r}}]\,\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{J}_q} = 2\sum_{\sigma}\,\beta^m{}_{i_\sigma}\phi_{i_1\ldots(m)_\sigma\ldots i_p\bar{J}_q}$$

$$-\,2\sum_{\sigma,\tau} S^{\bar{m}s}{}_{i_\sigma\bar{j}_\tau}\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{j}_1\ldots(\bar{m})_\tau\ldots\bar{j}_q}\,.$$

This completes the proof.

Q.E.D.

Example. – For the Hessian metric g we have

$$(\Box g)_{i\bar{j}} = -\,\beta_{i\bar{j}}\,.$$

Thus the Hessian metric g is $\square$-harmonic if and only if the second Koszul form $\beta = 0$. Therefore, by [12] the following conditions are equivalent:

(i) g is $\square$-harmonic.

(ii) The first Koszul form $\alpha = 0$.

(iii) The second Koszul form $\beta = 0$.

(iv) g is locally flat.

5. The local expression for $\square_a$.

Let F be a locally constant line bundle over a compact connected oriented Hessian manifold M, and let a be a fiber metric on F.

PROPOSITION 5.1. – *We have*

$$\delta_a = \delta + i(\mathrm{A}),$$

where $\mathrm{A} = -\mathrm{D}\log a$ *and* $(i(\mathrm{A})\,\phi)_{\mathrm{I}_{p-1}\bar{\mathrm{J}}_q} = \mathrm{A}^r\,\phi_{r\mathrm{I}_{p-1}\bar{\mathrm{J}}_q}$ *for* $\phi \in \Omega^{p,q}(\mathrm{F})$.

Proof. – By Definition 1.2, 1.7 and 2.3 we have

$$\begin{aligned}\delta_a &= (-1)^{n+1}\frac{\sqrt{\mathrm{G}}}{a} * \partial\left(\frac{a}{\sqrt{\mathrm{G}}}*\right)\\ &= (-1)^n * e(\mathrm{A}) * + (-1)^{n+1}\sqrt{\mathrm{G}} * \partial\left(\frac{1}{\sqrt{\mathrm{G}}}*\right)\\ &= i(\mathrm{A}) + \delta,\end{aligned}$$

where

$$(e(\mathrm{A})\,\phi)_{i_1\ldots i_{p+1}\bar{\mathrm{J}}_q} = \sum_\sigma (-1)^{\sigma-1}\mathrm{A}_{i_\sigma}\,\phi_{i_1\ldots\hat{i}_\sigma\ldots i_{p+1}\bar{\mathrm{J}}_q}$$

for $\phi \in \Omega^{p,q}(\mathrm{F})$.

Q.E.D.

DEFINITION 5.1. – *For* $\phi \in \Omega^{p,q}(\mathrm{F})$ *we set*

$$\overline{\nabla}^{\prime(a)}_{\bar r}\,\phi = \frac{1}{a}\,\overline{\nabla}^{\prime}_{\bar r}(a\phi)\,.$$

THEOREM 5.1. – *Let* $\phi \in \Omega^{p,q}(\mathrm{F})$. *Then we have*

$$(\underset{a}{\Box}\phi)_{\mathrm{I}_p\bar{\mathrm{J}}_q} = -\,g^{s\bar r}\,\overline{\nabla}^{\prime(a)}_{\bar r}\,\nabla'_s\,\phi_{\mathrm{I}_p\bar{\mathrm{J}}_q} + \alpha^s\,\nabla'_s\,\phi_{\mathrm{I}_p\bar{\mathrm{J}}_q}$$

$$+\sum_{\sigma}\,(-\,\beta^s_{i_\sigma} + \mathrm{B}^s_{i_\sigma}\;\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar{\mathrm{J}}_q}$$

$$+\,2\sum_{\sigma,\tau}\,\mathrm{S}^{\bar r s}_{i_\sigma\bar j_\tau}\,\phi_{i_1\ldots(s)_\sigma\ldots i_p\bar j_1\ldots(\bar r)_\tau\ldots\bar j_q}\,.$$

Proof. – By Proposition 5.1 we have

$$\underset{a}{\Box} = \Box + i(\mathrm{A})\,\partial + \partial\, i(\mathrm{A})\,.$$

A straightforward calculation shows

$$(i(\mathrm{A})\,\partial\phi)_{\mathrm{I}_p\bar{\mathrm{J}}_q} + (\partial\, i(\mathrm{A})\,\phi)_{\mathrm{I}_p\bar{\mathrm{J}}_q}$$

$$= g^{s\bar r}\,\mathrm{A}_{\bar r}\,\nabla'_s\,\phi_{\mathrm{I}_p\bar{\mathrm{J}}_q} + \sum_{\sigma=1}^{p}\,\mathrm{B}^r_{i_\sigma}\,\phi_{i_1\ldots(r)_\sigma\ldots i_p\bar{\mathrm{J}}_q}\,.$$

Thus our assertion follows from the above facts and Theorem 4.1.

Q.E.D.

6. A vanishing theorem of Kodaira-Nakano type.

Let θ be a symmetric covariant tensor field of degree 2. Considering θ as an element in $\Omega^{1,1}$ we define

$$e(\theta) : \Omega^{p,q} \longrightarrow \Omega^{p+1,q+1}\,,$$

$$i(\theta) : \Omega^{p,q} \longrightarrow \Omega^{p-1,q-1}\,,$$

by $e(\theta)\,\phi = \theta \wedge \phi$ for $\phi \in \Omega^{p,q}$ and $i(\theta) = (-1)^{n+p+q+1} * e(\theta) *$.

Then $i(\theta)$ is the adjoint operator of $e(\theta)$ with respect to the inner product in Definition 1.1 and 2.2.

In this section we always assume that F is a locally constant line bundle over M.

PROPOSITION 6.1. – *We have*

(i) $[\Box_a, e(g)] = e(B + \beta)$,

(ii) $[\Box_a, i(g)] = -i(B + \beta)$.

The proof follows from a straightforward calculation and so it is omitted.

PROPOSITION 6.2. – *Suppose* $\Box_a \phi = 0$. *Then we have*

(i) $(e(B + \beta)\, i(g)\, \phi, \phi) \leqslant 0$.

(ii) $(i(g)\, e(B + \beta)\, \phi, \phi) \geqslant 0$.

(iii) $([i(g), e(B + \beta)]\, \phi, \phi) \geqslant 0$.

Proof. – By Proposition 6.1 (i) we have $\Box_a e(g)\, \phi = e(B + \beta)\, \phi$. Thus we have

$$0 \leqslant (\Box_a e(g)\, \phi, e(g)\, \phi) = (e(B + \beta)\, \phi, e(g)\, \phi) = (i(g)\, e(B + \beta)\, \phi, \phi),$$

which implies (ii). By the same way, since $\Box_a i(g)\, \phi = -i(B + \beta)\, \phi$ we obtain

$$0 \leqslant (\Box_a i(g)\, \phi, i(g)\, \phi) = (-i(B + \beta)\, \phi, i(g)\, \phi)$$
$$= (\phi, -e(B + \beta)\, i(g)\, \phi),$$

which shows (i). (iii) follows from (i) and (ii).

Q.E.D.

THEOREM 6.1. – *Let* M *be a compact connected oriented Hessian manifold. Denote by* K *the canonical line bundle over* M. *Let* F *be a locally constant line bundle over* M.

(i) *If* 2F + K *is positive, then*

$$H^{p,q}(F) = 0 \quad \textit{for} \quad p + q > n.$$

(ii) *If* 2F + K *is negative, then*

$$H^{p,q}(F) = 0 \quad \textit{for} \quad p + q < n.$$

Proof. – Suppose $2F + K$ is negative. Then $B + \beta$ is negative definite. Therefore $g' = -(B + \beta)$ gives a Hessian metric on M. If we denote by β' the Koszul form on M with respect to g', then there exists a positive C^∞-function f on M such that

$$\beta' \doteq \beta + D^2 \log f.$$

If B is a Koszul form of F with respect to a fiber metric $a = \{a_\lambda\}$, then the Koszul form B′ of F with respect to the fiber metric $a' = \{fa_\lambda\}$ satisfies

$$B' + \beta' = B + \beta = -g'.$$

Therefore if we use $-(B + \beta)$ as a Hessian metric, the formula in Proposition 6.2 (iii) is reduced to

$$([i(g), -e(g)]\phi, \phi) \geqslant 0 \quad \text{for} \quad \phi \in \mathcal{H}^{p,q}(F).$$

Thus by Proposition 1.2 we have

$$(n - p - q)(\phi, \phi) \leqslant 0 \quad \text{for} \quad \phi \in \mathcal{H}^{p,q}(F).$$

Therefore, if $n - p - q > 0$ then $\phi = 0$. Hence (ii) is proved. (i) follows from (ii) and Theorem 2.2

Q.E.D.

7. A vanishing theorem of Koszul type.

In this section we mention a vanishing theorem of Koszul type. Let M be a compact oriented hyperbolic affine manifold. Then there exists a canonical Hessian metric g and a unique Killing vector field H on M such that

$$D_X H = X, \tag{7.1}$$

for all vector field X on M [7]. The following theorem is essentially due to Koszul.

THEOREM 7.1. – *Let* F *be a locally constant vector bundle over a compact hyperbolic affine manifold. If there exist a fiber metric* $a = \{(a_{ij})\}$ *and a constant* $c\ (\neq -2q)$ *such that*

$$Ha_{ij} = ca_{ij},$$

then we have

$$H^{p,q}(F) = 0, \quad \textit{for} \quad p > 0 \quad \textit{and} \quad q \geqslant 0.$$

The proof of this theorem is nearly the same as Koszul [7], and so we omit the proof.

COROLLARY 7.1. – *Let* M *be a compact oriented hyperbolic affine manifold. Then we have*

$$H^{p,q}(1) = 0, \quad \textit{for} \quad p, q > 0,$$

where 1 *is the trivial vector bundle over* M.

The tensor bundle $\overset{r}{\otimes} T \overset{s}{\otimes} T^*$ satisfies the condition of Theorem 7.1 if $q - r + s \neq 0$.

We give another example of locally constant vector bundle over M which satisfies the conditions of Theorem 7.1. Let Ω be an open convex cone in $\mathbf{R}^n$ with vertex 0 not containing any full straight line. Suppose that a discrete subgroup Γ of $GL(n, \mathbf{R})$ acts properly discontinuously and freely on Ω such that $M = \Gamma \backslash \Omega$ is compact. Assume further that there exist a linear mapping from Ω to the space of all $m \times m$ positive definite real symmetric matrices and a homomorphism from Γ to $GL(m, \mathbf{R})$, which are denoted by the same letter ρ, such that

$$\rho(\gamma x) = \rho(\gamma)\, \rho(x)\, {}^t\rho(\gamma) \quad \text{for} \quad \gamma \in \Gamma,\ x \in \Omega.$$

We denote by F_ρ the vector bundle over M associated with the universal covering $\Omega \longrightarrow M$ and ρ. Let U be an evenly covered open set in M. Choosing a section σ on U we set

$$a = (\rho \circ \sigma)^{-1}.$$

Then a is a fiber metric on F_ρ and we have

$$Ha = -a.$$

Therefore

COROLLARY 7.2. – *We have*

$$H^{p,q}(F_\rho) = 0 \quad \textit{for} \quad p > 0 \quad \textit{and} \quad q \geqslant 0.$$

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Manuscrit reçu le 17 juillet 1985
révisé le 14 avril 1986.

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