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BJÖRN E. J. DAHLBERG
C. E. KENIG
G. C. VERCHOTA

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THE DIRICHLET PROBLEM FOR THE BIHARMONIC EQUATION IN A LIPSCHITZ DOMAIN

by

B.E.J. DAHLBERG(*) C.E. KENIG(**) G.C. VERCHOTA(***)

Introduction.

The main purpose of this work is to study and give optimal estimates for the Dirichlet problem for the biharmonic operator Δ^2 on an arbitrary bounded Lipschitz domain D in $\mathbf{R}^n$, with the boundary values having first derivatives in $L^2(\partial D)$, and with the normal derivative being in $L^2(\partial D)$.

In recent years considerable attention has been given to the Dirichlet and Neumann problem for Laplace's equation in a Lipschitz domain D, with $L^p(\partial D)$ data, and optimal estimates. This started with the work of B. Dahlberg ([5], [6], [7]) on the Dirichlet problem, and has now reached a very satisfactory level of understanding. We now know optimal estimates for both the Dirichlet and the Neumann problem, in the optimal range of $p's$ and we also have good representation formulas for the solution in terms of classical layer potentials. (see [6], [11], [12], [18], and [8]).

In this work we initiate the corresponding study of the Dirichlet problem for the biharmonic operator Δ^2. One of our main results (Theorem 3.1) states that if D is a bounded Lipschitz domain in $\mathbf{R}^n$, and we are given $f \in L_1^2(\partial D)$, and $g \in L^2(\partial D)$, there exists a unique biharmonic function u in D, which takes the boundary values f, and whose normal

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derivative $\frac{\partial u}{\partial N}$ equals g on ∂D, both in the sense of non-tangential convergence, and such that the non-tangential maximal function of ∇u is in $L^2(\partial D)$. In a forthcoming paper, ([9]) the first two authors will show that u is in the Sobolev space $H^{3/2}(D)$. In [16], Nečas has obtained a very weak solution to the above problem (he shows that $u \in L^2(D)$). The novelty here resides in the much stronger (best possible) estimates for the solution.

The main idea in our work is to reduce the Dirichlet problem for the biharmonic operator, to bilinear estimates for harmonic function in D. These bilinear estimates are Lipschitz domain generalizations of the fact that the paraproduct ([3]) of two L^2 functions is in L^1. We obtain our estimates, using results in [5], [7], by integration by parts, and the deep results of Coifman, McIntosh and Meyer ([2]).

We also show that given a Lipschitz domain D, there is $\epsilon > 0$, such that the above L^2 results extend to L^p results for $2 - \epsilon \leqslant p \leqslant 2 + \epsilon$. This is accomplished by real variable methods, which show that L^p results in the neighborhood of $p = 2$ are consequences of the L^2 results. After the submission of this manuscript, the refereee communicated to us that the results of section 4 in fact follow from a general functional analytic, unpublished theorem of G. David and S. Semmes. (See the body of section 4 for the exact statement of their theorem, and its application in our specific instance).

For C^1 domains in the plane, J. Cohen and J. Gosselin ([1]) have recently established results analogous to our, in L^p, $1 < p < \infty$, by the method of multiple layer potentials. G. Verchota ([19]) in a forthcoming paper has shown how to modify the approach in this paper to obtain L^p results, $1 < p < \infty$ for C^1 domains in $\mathbf{R}^n, n \geqslant 2$. In the last section of our paper we show, by appropriate counterexamples, that for Lipschitz domains, our results are sharp in the range $1 < p < 2$. Whether this is also the case for the range $2 < p$ remains an open problem.

At this point we would like to thank Professor E.B. Fabes for his kind interest in our work, and for many helpful discussions. We are particularly indebted to him for the arguments in Section 3.

We would also like to thank the referee for bringing to our attention the unpublished work of G. David ans S. Semmes.

Before beginning the major part of this work, we will introduce some of the basic notations and definitions, and recall some basic results that will be used throughout the paper.

Capital letters X, Y, Z will denote points of a fixed domain $D \subset \mathbf{R}^n$, while P, Q will be reserved for points in ∂D Lower case letters x, y, z are reserved for points in $\mathbf{R}^{n-1}$. The letters s, t will be reserved for real numbers.

DEFINITION. – *A bounded domain $D \subset \mathbf{R}^n$ is called a Lipschitz domain if corresponding to each point $Q \in \partial D$ there is an open, right circular, doubly truncated cylinder $Z(Q, r)$ centered at Q, with radius equal to r, whose basis is at a positive distance from ∂D, such that there is a rectangular coordinate system for $\mathbf{R}^n, (x, s), x \in \mathbf{R}^{n-1}, s \in \mathbf{R}$, with s-axis containing the axis of Z, and a Lipschitz function $\varphi : \mathbf{R}^{n-1} \longrightarrow \mathbf{R}$*

$$(\text{i.e. } |\varphi(x) - \varphi(z)| \leqslant M\,|x - y|, \ \textit{for all} \ x, y \in \mathbf{R}^{n-1})$$

such that $Z \cap D = Z \cap \{(x, s) : s > \varphi(x)\}$, and $Q = (0, \varphi(0))$. Any such cylinder Z will be called a coordinate cylinder, and the pair (Z, φ) will be called a coordinate pair.

By a cone we mean an open, circular, non-empty truncated cone. Assigning one cone $\Gamma(Q)$ to each $Q \in \partial D$, we call the resulting family $\{\Gamma(Q) : Q \in \partial D\}$ regular if there is a finite covering of ∂D by coordinate cylinders, such that for each $(Z(P, r), \varphi)$ there are three cones α, β and γ, each with vertex at the origin and axis along the axis of Z such that

$$\alpha \subset \beta \,|\, \{0\} \subset \gamma$$

and for all $(x, \varphi(x)) = Q \in \frac{4}{5} Z \cap \partial D$,

$$\alpha + Q \subset \Gamma(Q) \subset \Gamma(Q) \,|\, \{Q\} \subset \beta + Q,$$

$$(\gamma + Q) \subset D \cap Z,$$

and such that $\left\{\frac{4}{5} Z\right\}$ still cover ∂D. Here rZ is the dilation of Z about Q by a factor r.

Given a regular family $\{\Gamma\}$, there is a constant $C > 0$, depending only on $\{\Gamma\}$ such that for any $Q \in \partial D$ and any $X \in \Gamma(P)$, $|X - Q| \geqslant C|X - P|$ and $|X - Q| \geqslant C|P - Q|$.

DEFINITION. — *Given a function* u *in* D *and a regular family of cones* $\{\Gamma\}$, *we define the nontangential maximal function* $N_\Gamma(u)(Q) = N(u)(Q) = \sup_{X \in \Gamma(Q)} |u(X)|$.

We say that $u(X)$ converges non-tangentially a.e. to $f(Q)$ if for any regular family of cones $\{\Gamma\}$, we have $\lim_{\substack{X \to Q \\ X \in \Gamma(Q)}} u(X) = f(Q)$, for a.e. $Q \in \partial D$.

A surface ball $\Delta = \Delta(Q, r)$ will be the intersection of a coordinate cylinder $Z = Z(Q, r)$ with ∂D. We need to recall three theorems of B.E.J. Dahlberg.

THEOREM 0.1 ([5]). — *Let* D *be a bounded Lipschitz domain. Fix* $X^* \in D$, *and let* ω *be harmonic measure for* D, *with pole at* X^*. *Then,* a) ω *is absolutely continuous with respect to surface measure* σ *of* ∂D, b) *Let* $k(Q) = \frac{d\omega}{d\sigma}(Q)$. *Then,* $k \in L^2(d\sigma)$. *Moreover, for any surface ball* $\Delta \subset \partial D$,

$$\left(\frac{1}{\sigma(\Delta)} \int_\Delta k^2 \, d\sigma\right)^{1/2} \leqslant \frac{C}{\sigma(\Delta)} \int_\Delta k \, d\sigma .$$

c) *Surface measure is absolutely continuous with respect to harmonic measure.* d) *If* $k^X(Q) = \frac{d\omega^X}{d\sigma}(Q)$, *then* $k^X(Q) = \frac{\partial G}{\partial N_Q}(Q, X)$ *is the non-tangential limit at* Q *of* $N_Q \cdot \nabla G(-, X)$, *where* N_Q *denotes the unit normal at* $Q \in \partial D$.

THEOREM 0.2 ([6]). — *Let* D *be a bounded Lipschitz domain. There exists* $\epsilon = \epsilon(D) > 0$ *such that if* $2 - \epsilon \leqslant p \leqslant \infty$, *and* $f \in L^p(\partial D, d\sigma)$ *there exists a unique harmonic function* u *in* D *such that* u *converges nontangentially a.e. to* f *and*

$$\| N_\Gamma(u) \|_{L^p(d\sigma)} \leqslant C_{p,\Gamma} \| f \|_{L^p(d\sigma)} .$$

u will be called the Poisson extension of f.

THEOREM 0.3 ([7]). – *Let* D *be a bounded Lipschitz domain, and* $f \in L^2(\partial D, d\sigma)$ *be such that* $\int_{\partial D} fk\, d\sigma = 0$. *Then, if* u *is the Poisson extension of* f, *we have*

$$C^{-1} \int_{\partial D} f^2 d\sigma \leqslant \int_D \operatorname{dist}(X, \partial D) \,|\nabla u(X)|^2 dX \leqslant C \int_{\partial D} f^2 d\sigma .$$

DEFINITION. – *For a bounded Lipschitz domain* D, *we say that* $f \in L_1^p(\partial D)$ *if* $f \in L^p(\partial D, d\sigma)$ *and if for each coordinate pair* (Z, φ), *there are* $L^p(Z \cap \partial D)$ *functions* $g_1, \ldots, g_{n_1}$ *so that*

$$\int_{\mathbf{R}^{n-1}} h(x)\, g_j(x, \varphi(x))\, dx = \int_{\mathbf{R}^{n-1}} \frac{\partial}{\partial x_j} h(x)\, f(x, \varphi(x))\, dx ,$$

for all $h \in C_0^\infty(Z \cap \mathbf{R}^{n-1})$.

It is easy to see that given $f \in L_1^p(\partial D)$ it is possible to define a unique vector $\nabla_t f \in \mathbf{R}^n$, at almost every $Q \in \partial D$ so that $\|\nabla_t f\|_{L^p(\partial D, d\sigma)}$ is equivalent to the sum over all the coordinate cylinders in a given covering of ∂D of the L^p norms of the locally defined function g_j for f, occurring in the definition of L_1^p. The resulting vector field, $\nabla_t f$ will be called the tangential gradient of f. In local coordinates, $\nabla_t f$ may be realized as

$$(g_1(x, \varphi(x)), \ldots, g_{n-1}(x, \varphi(x)), 0) - \langle (g_1(x, \varphi(x), \ldots, g_{n-1}(x, \varphi(x)), 0), N_{(x,\varphi(x))} \rangle \cdot N_{(x,\varphi(x))}.$$

Thus, $L_1^p(\partial D)$ may be normed by

$$\|f\|_{L_1^p(\partial D)} = \|f\|_{L^p(d\sigma)} + \|\nabla_t f\|_{L^p(d\sigma)} .$$

THEOREM 0.4 ([12], [18], [8]). – *Let* D *be a bounded domain in* $\mathbf{R}^n$, *with connected boundary. There exists* $\epsilon = \epsilon(D) > 0$ *such that if* $g \in L_1^p(\partial D)$, $1 < p < 2 + \epsilon$, *there exists a unique function,* u, *such that*

(i) $\Delta u = 0$ *in* D

(ii) $u \longrightarrow g$ *nontangentially a.e.*

(iii) $\|N(\nabla u)\|_{L^p(\partial D, d\sigma)} \leqslant C \|g\|_{L_1^p(\partial D, d\sigma)}$.

The case $p = 2$ of Theorem 0.4, is due to D. Jerison and C. Kenig ([12]). The general case is due to G. Verchota ([18]). Also, see [8] for another proof.

DEFINITION. – *Let* (r, θ) *be polar coordinates for*

$$\mathbf{R}^n, \; 0 \leqslant r < \infty, \quad \theta \in S^{n-1}$$

the unit sphere in $\mathbf{R}^n$. *A domain* D *in* $\mathbf{R}^n$ *is a starlike Lipschitz domain (with respect to the origin) if there exists* $\varphi : S^{n-1} \longrightarrow \mathbf{R}, \varphi$ *is strictly positive, and* $|\varphi(\theta) - \varphi(\theta')| \leqslant M |\theta - \theta'|$ *for all* $\theta, \theta' \in S^{n-1}$, *so that* $D = \{(r, \theta) : 0 \leqslant r < \varphi(\theta)\}$.

Note that if D is an arbitrary Lipschitz domain, and (Z, φ) is a coordinate pair, $\|\nabla \varphi\|_\infty \leqslant M$, then, for appropriate $\delta > 0, a > 0, b > 0$, which depend only on M, the domain $D \cap U$ is a starlike Lipschitz domain with respect to $X_0 = (0, b\delta)$, where $U = \{(x, t) : |x| < \delta, |t| < a\delta\}$.

In the sequel we will assume for simplicity that $n \geqslant 3$. The results remain valid when $n = 2$ with the obvious modifications.

1. Estimates for Green potentials and bilinear operators involving harmonic functions on starlike Lipschitz domains.

The main results in this section are :

THEOREM 1.1. – *Let* Ω *be a bounded, starlike (with respect to the origin) Lipschitz domain in* $\mathbf{R}^n$. *Let* v *be the Poisson extension in* Ω *of an* $L^2(\partial D, d\sigma)$ *function* f. *Let* $b \in C^1(\bar{\Omega})$, *and consider the Green's potentials*

$$u(X) = \int_\Omega G(X, Y)\, b(Y) \frac{\partial v}{\partial Y_j}(Y)\, dY, j = 1, \ldots, n.$$

Then,

$$\|N(\nabla u)\|_{L^2(d\sigma)} \leqslant C \|b\|_{C^1(\bar{\Omega})} \cdot \|f\|_{L^2(d\sigma)}.$$

THEOREM 1.2. — *Let Ω be a bounded, starlike (with respect to the origin) Lipschitz domain in $\mathbf{R}^n$. Let u, v be harmonic functions in Ω, with $u, v \in \mathrm{Lip}(\overline{\Omega})$.*
Then, for $j = 1, \ldots, n$, we have

$$\left| \int_\Omega u(X) \frac{\partial v}{\partial X_j}(X) dX \right| \leqslant C \| u \|_{L^2(\partial\Omega, d\sigma)} \| v \|_{L^2(\partial\Omega, d\sigma)},$$

where C depends only on Ω.

This theorem can be thought of as the Lipschitz domain analogue of the fact that the paraproduct of two L^2 functions is in L^1 (see [3]).

The main step in the proof of Theorem 1.1 is the following estimate for Newtonian potentials.

LEMMA 1.3. — *Let $\varphi : \mathbf{R}^{n-1} \longrightarrow \mathbf{R}$ have compact support, with $\| \nabla \varphi \|_\infty \leqslant M$. Let Ω be the region above the graph of φ. Let $g \in C_0^1(\mathbf{R}^n)$. Let v be the Poisson extension of $g|_{\partial\Omega}$ in Ω. (Theorem 0.2 easily extends to this situation). Then, the Newtonian potential*

$$u(X) = \int_\Omega \frac{1}{|X - Y|^{n-2}} \frac{\partial}{\partial Y_n} v(Y)\, dY$$

satisfies

$$\| N(\nabla u) \|_{L^2(d\sigma)} \leqslant C \| g \|_{L^2(d\sigma)},$$

where C depends only on M.

Proof. — Let B be fundamental solution for the biharmonic equation Δ^2, i.e. $\Delta_Y B(X - Y) = \dfrac{1}{|X - Y|^{n-2}}, X \neq Y$. (For example, if $n \geqslant 5$, $B(Y) = C_n |Y|^{4-n}$). Let $e_j, j = 1, \ldots, n$ be the standard basis of $\mathbf{R}^n$. We recall the definition of the Riesz transforms, $R_j v$ of v, $j = 1, \ldots, n-1$. They are harmonic functions which together with v satisfy the generalized Cauchy-Riemann equations, i.e. $\dfrac{\partial v}{\partial X_n} = - \sum_{j=1}^{n-1} \dfrac{\partial}{\partial X_j} R_j v$. (See [17]).

Using the summation convention, the integrand for the Newtonian potential we are considering, is

$$\frac{\partial}{\partial Y_j}\frac{\partial}{\partial Y_j} B \frac{\partial}{\partial Y_n} v + \frac{\partial^2}{\partial Y_n^2} B \frac{\partial v}{\partial Y_n}$$

$$= \frac{\partial}{\partial Y_j}\frac{\partial}{\partial Y_j} B \frac{\partial}{\partial Y_n} v - \frac{\partial}{\partial Y_j}\frac{\partial}{\partial Y_n} B \frac{\partial}{\partial Y_j} v$$

$$+ \frac{\partial}{\partial Y_j}\frac{\partial}{\partial Y_n} B \frac{\partial}{\partial Y_j} v - \frac{\partial^2}{\partial Y_n^2} B \frac{\partial}{\partial Y_j} R_j v$$

$$= \langle\Big(-\frac{\partial}{\partial Y_1}\frac{\partial}{\partial Y_n} B, \dots, -\frac{\partial}{\partial Y_{n-1}}\frac{\partial}{\partial Y_{n-1}} B, \frac{\partial}{\partial Y_j}\frac{\partial}{\partial Y_j} B\Big), \nabla v\rangle$$

$$+ \langle\frac{\partial}{\partial Y_j}\frac{\partial}{\partial Y_n} B\, e_n, \nabla R_j v\rangle - \langle\frac{\partial^2}{\partial Y_n^2} B\, e_j, \nabla R_j v\rangle$$

$$= \langle\vec{\alpha}, \nabla v\rangle + \langle\vec{\beta}_j, \nabla R_j v\rangle,$$

where $\langle\,,\rangle$ is the inner product in $\mathbf{R}^n$, and

$$\vec{\alpha} = \Big(-\frac{\partial}{\partial Y_1}\frac{\partial}{\partial Y_n} B, \dots, -\frac{\partial}{\partial Y_{n-1}}\frac{\partial}{\partial Y_{n-1}} B, \frac{\partial}{\partial Y_j}\frac{\partial}{\partial Y_j} B\Big),$$

$$\vec{\beta}_j = \frac{\partial}{\partial Y_j}\frac{\partial}{\partial Y_n} B\, e_n - \frac{\partial^2}{\partial Y_n^2} B e_j\,.$$

Note that $\vec{\alpha}, \vec{\beta}_j$, $j = 1, \dots, n-1$ are divergence free vectors. Given the conditions on v it is easy to see that integration by parts is allowed inside the Newtonian potential, and thus,

$$u(X) = \int_{\delta\Omega} \Bigg[\Big(- N_j(Q)\frac{\partial}{\partial Q_j}\frac{\partial}{\partial Q_n} B\,(Q - X)$$

$$+ N_n(Q)\frac{\partial}{\partial Q_j}\frac{\partial}{\partial Q_j} B(X - Q)\quad g(Q)\Big)$$

$$+ \Big(N_n(Q)\frac{\partial}{\partial Q_n}\frac{\partial}{\partial Q_j} B(Q - X) - N_j(Q)\frac{\partial^2}{\partial Q_n^2} B(Q - X)\Big)$$

$$R_j v(Q)\Bigg]\, d\sigma(Q)\,.$$

Because of Theorem 0.3, and classical arguments (see [18] for the details in a similar situation), it is easy to see that

$$\| R_j v \|_{L^2(d\sigma)} \leqslant C \| g \|_{L^2(d\sigma)}$$

with C depending only on M.

Thus, $u(X)$ is simply a sum of boundary potentials of the form $\int_{\partial\Omega} \frac{\partial}{\partial Q_j} \frac{\partial}{\partial Q_n} B(X - Q) f(Q) \, d\sigma(Q)$, where

$$\| f \|_{L^2(d\sigma)} \leqslant C \| g \|_{L^2(d\sigma)} .$$

The fact that $N(\nabla u)$ is in $L^2(d\sigma)$ now follows from the theorem of Coifman-McIntosh Meyer on the boundedness of the Cauchy integral on Lipschitz curves, ([2]), by standards arguments. (See [18] for the details in similar circumstances).

Lemma 1.3 localizes in a fairly easy fashion. We have :

LEMMA 1.4. – *Let* D *be a bounded Lipschitz domain in* $\mathbf{R}^n$. *Let* v *be a harmonic function in* $D, v \in C^1(\bar{D})$. *Let* $b \in C^1(\bar{D})$, *and consider the Newtonian potentials*

$$u(X) = \int_D \frac{1}{|X - Y|^{n-2}} b(Y) \frac{\partial}{\partial Y_j} v(Y) \, dY , j = 1, \ldots, n.$$

Then,

$$\| N(\nabla u) \|_{L^2(d\sigma)} \leqslant C \| b \|_{C^1(\bar{D})} \| v \|_{L^2(d\sigma)} .$$

Proof. – If suffices to examine the L^2 norm of $N(\nabla u)$ on the intersection of ∂D with coordinate cylinders Z, with the property that 3Z is still a coordinate cylinder. Fix such a cylinder, and let $\Psi \in C_0^\infty(3Z)$ be such that $\Psi = 1$ on 2Z. For

$$X \in \Gamma(Q), Q \in Z \cap \partial D,$$

consider

$$\nabla_X \int_D \frac{1}{|X - Y|^{n-2}} (1 - \Psi(Y)) \cdot b(Y) \cdot \frac{\partial}{\partial Y_j} v(Y) \, dY .$$

An integration by parts, together with the support porperties of Ψ and Theorem 0.2, yield a non-tangential maximal function with the correct bound on its norm. Thus, it suffices to consider u with

$b \in C_0^1(3Z)$. Without loss of generality, the axis of Z may be taken to be in the X_n direction. Denote the $n-1$ dimensional ball that forms the top of 3Z by B, and let $t_0 > 0$ be such that

$$B = \{X \in \partial\, 3Z : X_n = t_0\}.$$

Let $Y, X \in B$ be denoted by (y, t_0) and (x, t_0) respectively. Let $g(x, y)$ be the Green's function for Laplace's equation in B. Then Riesz transforms of v may be defined in $3Z \cap D$ by

$$R_j v(X) = -\int_{X_n}^{t_0} \frac{\partial}{\partial X_j} v(x, t)\, dt - \frac{\partial}{\partial X_j} \int_B g(x, y) \frac{\partial}{\partial t} v(y, t_0)\, dy$$

(see [18]). Note that the values of $R_j v(X)$, for X in $K \cap 3Z$, where K is a compact subset of D are bounded by $\sup_K |v|$, which, by Theorem 0.2 is bounded by $C \|v\|_{L^2(d\sigma)}$. The same is true for $\int_{K \cap 3Z} |\nabla R_j v(X)|^2\, dX$. Therefore, by Lemma 5.2 in [18], and Theorem 0.3, we see that $\int_{\partial(3Z \cap D)} N(R_j v)^2 \leq C \int_{\partial D} v^2\, d\sigma$.

Consider first the Newtonian potential u, with $j = n$. The arguments of Lemma 1.3 go through unchanged, except for the appearance of integrals over D, in the representation of u, of the form $\int_D \frac{\partial}{\partial Y_i} \frac{\partial}{\partial Y_k} B(X - Y) \frac{\partial b}{\partial Y_2}(Y)\, w(Y)\, dY$, where w is either v or a Riesz transform $R_j v$. These integrals can be thought of as integrals of integrals of the same type that we had before, and hence, they can be handled. The lemma is then proved for $j = n$. For $j \neq n$, we merely replace v by $R_j v$ and the lemma follows.

In order to pass from Lemma 1.4 to Theorem 1.1, we need a couple of simple lemmas.

LEMMA 1.5. – *Let* D *be a bounded Lipschitz domain in* $\mathbf{R}^n$. *Let* v *be the Poisson extension in* D *of an* $L^2(\partial D, d\sigma)$ *function* f. *Then, for* $X \in D$,

$$\int_D |G(X, Y)|\, |\nabla v(Y)|\, dY \leq C_X \|f\|_{L^2(d\sigma)},$$

where C_X *depends only on* D *and* $\operatorname{dist}(X, \partial D)$.

Proof. – It suffices to consider only the integration over a strip S, of width comparable to $\mathrm{dist}(X, \partial D)$, up to the boundary. For Y in this strip, projections Q(Y) onto the boundary may be uniquely defined. By [5] (also see [13]),

$$|G(X, Y)| \leqslant C \,\mathrm{dist}(Y, \partial D) \frac{\omega^X(\Delta(Q(Y)))}{\sigma(\Delta(Q(Y)))},$$

where $\Delta(Q(Y))$ is a surface ball centered at $Q(Y)$ of radius comparable to $\mathrm{dist}(Y, \partial D)$. Thus, the integral over S is bounded by

$$C \int_S \mathrm{dist}(Y, \partial D)\, M(k^X)(Q(Y))\, |\nabla v(Y)|\, dY$$
$$\leqslant C \left(\int_S \mathrm{dist}(Y, \partial D)\, |\nabla v(Y)|^2 \, dy \right)^{1/2} \left(\int_S \mathrm{dist}(Y, \partial D)\, M(k^X)(Q(Y))\, dY \right)^{1/2},$$

where $k^X = \dfrac{d\omega^X}{d\sigma}$ (see Theorem 0.1), and M denotes the Hardy-Littlewood maximal operator on ∂D. The first integral is bounded by $\left(\int_{\partial D} f^2(Q)\, d\sigma(Q) \right)^{1/2}$ by Theorem 0.3. The second integral is bounded by C_X by the maximal theorem and Theorem 0.1.

LEMMA 1.6. – *Let* D *be a bounded Lipschitz domain in* $\mathbf{R}^n$. *Let* v *be the Poisson extension in* D *of an* $L^2(\partial D, d\sigma)$ *function* f, *and let* $b \in L^\infty(D)$. *Then, for* $X \in D$,

$$\frac{\partial}{\partial X_j} \int_D G(X, Y)\, b(Y)\, \nabla v(Y)\, dY = \int_D \frac{\partial}{\partial X_j} G(X, Y)\, b(Y)\, \nabla v(Y)\, dY.$$

Moreover,

$$\int_D |\nabla_X G(X, Y)|\, |\nabla v(Y)|\, dY \leqslant C_X \|b\|_\infty \|f\|_{L^2(d\sigma)},$$

where C_X *depends only on* D *and* $\mathrm{dist}(X, \partial D)$.

Proof. – Let $r > 0$ be comparable to $\mathrm{dist}(X, \partial D)$ and so that $\mathrm{dist}(B_r(X), \partial D)$ is comparable to r. For

$$Y \in B_r(X), \quad |\nabla_X G(X, Y)| \leqslant C/|X - Y|^{n-1},$$

and thus, it suffices to consider

$$\lim_{|\epsilon|\to 0} \frac{1}{|\epsilon|} \int_{D\backslash B_r(X)} [G(X+\epsilon, Y) - G(X,Y)]\, b(Y) \cdot \nabla v(Y)\, dY,$$

where $\epsilon \in \mathbf{R}^n$. For ϵ small, $Y \notin B_r(X)$, Harnack's principle and the mean value theorem for harmonic function show that the difference quotient is bounded by $\dfrac{C\,|G(X,Y)|}{r}$. The lemma now follows from Lemma 1.5 and dominated convergence.

Proof of Theorem 1.1. – For $0 < r < 1$, let $v_r(Y) = v(rY)$, and define u_r to be the Green's potential of $b \cdot \dfrac{\partial v_r}{\partial Y_j}$. Let

$$\Gamma_\epsilon(Q) = \{X \in \Gamma(Q) : \operatorname{dist}(X, \partial r) > \epsilon\}.$$

Define, for $Q \in \partial\Omega$, $N_\epsilon(\nabla u)(Q) = \sup_{X \in \Gamma_\epsilon(Q)} |\nabla u(X)|$. Since by Theorem 0.2, $f_r(Q) = v(rQ) \longrightarrow f$ in $L^2(d\sigma)$, as $r \longrightarrow 1$, Lemma 1.6 shows that for all $Q \in \partial\Omega$,

$$N_\epsilon(\nabla u)(Q) = \lim_{r \uparrow 1} N_\epsilon(\nabla u_r)(Q), \epsilon > 0.$$

Each u_r is the sum of a harmonic function, H_r and a Newtonian potential by Theorem 0.4,

$$\| N(\nabla H_r) \|_{L^2(d\sigma)} \leqslant C \| H_r \|_{L^2_1(\partial D, d\sigma)}.$$

But, since Green potentials vanish at the boundary, the $L^2_1(\partial D, d\sigma)$ norm of H_r is identical to the one of the corresponding Newtonian potential. This, by Lemma 1.4 is dominated by $\| v_r \|_{L^2(d\sigma)}$. Thus, $\| N(\nabla u_r) \|_{L^2(d\sigma)} \leqslant C \| b \|_{C^1(\bar{D})} \| v_r \|_{L^2(d\sigma)}$. By Fatou's lemma,

$$\| N_\epsilon(\nabla u) \|_{L^2(d\sigma)} \leqslant C \| b \|_{C^1(\bar{D})} \| f \|_{L^2(d\sigma)}.$$

The theorem now follows by monotone convergence.

In order to deduce Theorem 1.2 from Theorem 1.1, we need one more lemma.

LEMMA 1.7. – *Let* $f \in L^\infty(D)$. *Consider the Green's potential* $u(X) = \int_D G(X,Y)\, f(Y)\, dY$. *Then,*

$$\lim_{\substack{X \to Q \\ X \in \Gamma(Q)}} N_Q \cdot \nabla u(X) = \int_D k^Y(Q) f(Y)\, dY,$$

for a.e. $Q(d\sigma)$. *Here,* $k^Y(Q) = \dfrac{\omega^Y(Q)}{d\sigma(Q)}$ *(see Theorem 0.1).*

Proof. – It suffices to consider the case when $f \equiv 1$. Let $r = \operatorname{dist}(X, \partial D)$. For $Y \in B_{Mr}(Q) \cap D$, where M is a large constant to be chosen depending only on the Lipschitz character of D, $|\nabla_X G(X,Y)| \leqslant C/|X-Y|^{n-1}$. Thus,

$$\lim_{\substack{X \to Q \\ X \in \Gamma(Q)}} \int_{B_{Mr}(Q) \cap D} |\nabla_X G(X,Y)|\, dY = 0.$$

It suffices then to justify the use of the dominated convergence theorem for $\lim_{\substack{X \to Q \\ X \in \Gamma(Q)}} \int_{D \setminus B_{Mr}(Q)} N_Q \cdot \nabla_X G(X,Y)\, dY$. Let $\Delta_r(Q)$ denote the surface ball of radius r about Q. Write

$$k^Y(Q) = K(Y,Q)\, k(Q)$$

where $k(Q)$ is the density of harmonic measure at some fixed $X^* \in D$. (See Theorem 0.1). $K(Y,Q)$ is the so called kernel function of D. Since Y is away from X, $|\nabla_X G(X,Y)| \leqslant \dfrac{C}{r} |G(X,Y)|$. By Lemma 5.8 of [13] (see also [5],

$$|G(X,Y)| \leqslant \frac{C}{r^{n-2}} \int_{\Delta_r(Q)} K(Y,P)\, k(P)\, d\sigma(P).$$

By theorem 5.20 of [13], $K(Y,P) \leqslant CK(Y,Q)$ for $P \in \Delta_r(Q)$, if M is chosen large enough. Thus, for $Y \notin B_{Mr}(Q)$,

$$\begin{aligned} |\nabla_X G(X,Y)| &\leqslant C\Big(\frac{1}{r^{n-1}} \int_{\Delta_r(Q)} k(P)\, d\sigma(P)\Big) \cdot K(Y,Q) \\ &\leqslant C\, M(k)(Q) \cdot K(Y,Q), \end{aligned}$$

where C depends only on D, and M is the Hardy-Littlewood maximal operator. Since $k \in L^2(\partial D, d\sigma)$, $M(k)$ is finite a.e. on ∂D. It suffices to show that $\int_D K(Y,Q)\, dY < +\infty$ a.e. $(d\sigma)$.

But this is clear since $\int_{\partial D} \left(\int_D K(Y,Q)\, dY\right) d\omega(Q) = |D|$, using the fact that ω and σ are mutually absolutely continuous (Theorem 0.1).

Proof of Theorem 1.2. – $u(X) = \int_{\partial\Omega} k^X(Q)u(Q)\, d\sigma(Q)$. Hence,

$$\int_\Omega u(X)\frac{\partial v}{\partial X_j}(X)\, dx = \int_\Omega \left(\int_{\partial\Omega} k^X(Q)\, u(Q)\, d\sigma(Q)\right)\cdot\frac{\partial v}{\partial X_j}\, dX$$

$$= \int_{\partial\Omega} u(Q)\left(\int_\Omega k^X(Q)\frac{\partial v}{\partial X_j}(X)\, dX\right) d\sigma(Q).$$

By Lemma 1.7,

$$\int_\Omega k^X(Q)\ \frac{\partial v}{\partial X_j}(X)\, dX = \lim_{\substack{Y\to Q\\ Y\in\Gamma(Q)}} N_Q\cdot\int_\Omega \nabla_X G(X,Y)\frac{\partial v}{\partial X_j}(X)dX.$$

Theorem 1.2 now follows from Theorem 1.1.

2. The Dirichlet problem for Δ^2 with L^2 data in starlike Lipschitz domains.

The main result in this section is :

THEOREM 2.1. – *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, starlike with respect to the origin. Let $g\in L^2(\partial\Omega, d\sigma)$. Then, there exists a unique function in u in Ω, such that*

(a) $\Delta^2 u = 0$ *in* Ω

(b) $\lim\limits_{\substack{X\to Q\\ X\in\Gamma(Q)}} u(X) = 0$ *for a.e.* $Q\in\partial\Omega$,

(c) $\lim\limits_{\substack{X\to Q\\ X\in\Gamma(Q)}} N_Q\cdot\nabla u(X) = g(Q)$ *for a.e.* $Q\in\partial\Omega$

(d) $\|N(\nabla u)\|_{L^2(\partial\Omega, d\sigma)} < +\infty$.

In fact, $\|N(u)\|_{L^2(d\sigma)} + \|N(\nabla u)\|_{L^2(d\sigma)} \leq C\|g\|_{L^2(d\sigma)}$, *where* C *depends only on* Ω.

Moreover, there exists a harmonic function v, that is the Poisson extension of a unique $L^2(\partial\Omega, d\sigma)$ function so that

$$u(X) = \int_\Omega G(X,Y)\,[nv(Y) + 2Y \cdot \nabla v(Y)]\,dY,$$

where n is the dimension.

Proof. – We begin with the existence part of Theorem 2.1. Let $f \in L^2(\partial\Omega, d\sigma)$, and let v be the Poisson extension of f in Ω. Form the Green's potential

$$u(X) = \int_\Omega G(X,Y)\,[nv(Y) + 2Y \cdot \nabla v(Y)]\,dY.$$

By Theorem 1.1, $\|N(\nabla u)\|_{L^2(\partial\Omega, d\sigma)} \leq C\,\|f\|_{L^2(\partial\Omega, d\sigma)}$, where C depends only on Ω. Using the functions u_r and v_r as in the proof of Theorem 1.1, Lemma 1.7 and standard arguments we may conclude that $\lim\limits_{\substack{X \to Q \\ X \in \Gamma(Q)}} N_Q \cdot \nabla u(X) = \dfrac{\partial v}{\partial N}(Q)$ exists a.e. $(d\sigma)$, and in $L^2(\partial\Omega, d\sigma)$. Thus, we can define a bounded linear operator $T : L^2(\partial\Omega, d\sigma) \longrightarrow L^2(\partial\Omega, d\sigma)$ by $Tf = \dfrac{\partial u}{\partial N}$. We claim that T is invertible. In fact, let $f_r = v_r|_{\partial\Omega}$. By Lemma 1.7 and Fubini's theorem, we have

$$\int_{\partial\Omega} f_r \frac{\partial u_r}{\partial N}\,d\sigma$$

$$= \int_{\partial\Omega} f_r(Q)\left(\int_\Omega k^Y(Q)\,\{nv_r(Y) + Y \cdot \nabla v_r(Y)\}\,dY\right) d\sigma(Q)$$

$$= \int_\Omega \{nv_r(Y) + 2Y \cdot \nabla v_r(Y)\} \cdot v_r(Y)\,dY = \int_\Omega \operatorname{div}\,\{Y \cdot (v_r(Y))^2\}\,dY$$

$$= \int_{\partial\Omega} Q \cdot N_Q\, f_r(Q)^2\,d\sigma(Q)$$

by the divergence theorem. Letting $r \longrightarrow 1$, and using the fact that for a bounded Lipschitz domain which is starlike with respect to the origin, $Q \cdot N_Q \geq C$, for $Q \in \partial\Omega$, where C depends only on the Lipschitz character of $\partial\Omega$, we see that

$$\int_{\partial\Omega} f \cdot Tf\,d\partial \geq C\,\|f\|^2_{L^2(\partial\Omega, d\sigma)}.$$

This shows that T is invertible in $L^2(\partial\Omega, d\sigma)$. Therefore, given $g \in L^2(\partial\Omega, d\sigma)$, if we let v be the Poisson extension of $f = T^{-1}(g)$, it is clear that u will satisfy (c) and (d). To verify (b), note that u_r verifies (b) by the proof of Lemma 1.7, and that $N(u) \in L^2(\partial\Omega, d\sigma)$ because of (d) and Lemma 1.5. (b) now follows from standard arguments. That u verifies (a) follows from the fact that $v(Y)$ and $Y \cdot \nabla v(Y)$ are harmonic. This concludes the existence part of Theorem 2.1. In order to establish uniqueness, introduce the smooth, starlike domains Ω_j of Lemma 3.5 of [13], which are contained in Ω and increase to Ω. Note that in Ω_j we have uniqueness of the Dirichlet problem for Δ^2, and so, using the existence part of the Theorem, and the case $p = 2$ of Theorem 0.4, we see that for all $u \in C^\infty(\bar{\Omega}_j)$, with $\Delta^2 u = 0$ in Ω_j,

$$\| N_{\Omega_j}(u) \|_{L^2(\partial\Omega_j, d\sigma)} + \| N_{\Omega_j}(\nabla u) \|_{L^2(\partial\Omega_j, d\sigma)} \leq C \| u \|_{L^2(\partial\Omega_j, d\sigma)} + \left\| \frac{\partial u}{\partial N_j} \right\|_{L^2(\partial\Omega_j, d\sigma)},$$

where C is independent of j, N_{Ω_j} denotes the non-tangential maximal function associated to Ω_j, and $\frac{\partial u}{\partial N_j}$ the normal derivative on $\partial\Omega_j$. If $\Delta^2 u = 0$ in Ω, $u(X) \xrightarrow[X \in \Gamma(Q)]{X \to Q} 0$ for a.e. Q, and $\| N(\nabla u) \|_{L^2(\partial\Omega, d\sigma)} < +\infty$, dominated convergence shows that the right hand side of the above inequality tends to 0 with j, while the left hand side goes to $\| N(u) \|_{L^2(\partial\Omega, d\sigma)} + \| N(\nabla u) \|_{L^2(\partial\Omega, d\sigma)}$. Thus $u \equiv 0$ in Ω, and the proof is complete.

3. The Dirichlet problem for Δ^2 with L^2 data in an arbitrary bounded Lipschitz domain with connected boundary.

In this section we extend Theorem 2.1 to arbitrary bounded Lipschitz domain with connected boundary.

THEOREM 3.1. – *Let* D *be a bounded Lipschitz domain in* $\mathbf{R}^n$ *with connected boundary. Let* $f \in L_1^2(\partial D)$, $g \in L^2(\partial D)$. *Then, there exists a unique function,* u, *in* D *such that*

(a) $\Delta^2 u = 0$ *in* D

(b) $\lim\limits_{\substack{X \to Q \\ X \in \Gamma(Q)}} u(X) = f(Q)$ *a.e.* $(d\sigma)$

(c) $\lim\limits_{\substack{X \to Q \\ X \in \Gamma(Q)}} N_Q \cdot \nabla u(X) = g(Q)$ *a.e.* $d\sigma$

(d) $\| N(\nabla u) \|_{L^2(\partial D, d\sigma)} < +\infty$.

In fact,

$$\| N(u) \|_{L^2(\partial D, d\sigma)} + \| N(\nabla u) \|_{L^2(\partial D, d\sigma)} \leqslant C\{\| f \|_{L^2_1(\partial D)} + \| g \|_{L^2(\partial D)}\}$$

where C *depends only on the Lipschitz character of* D.

In order to prove existence, we first utilize some results of Nečas ([16]) to see that a very weak solution exists.

DEFINITION. – $M = \left\{ (g_0, g_1) : g_0 = g|_{\partial D}, g_1 = \dfrac{\partial g}{\partial N} \right.$ *for some* $\left. g \in W^{2,2}(D) \right\}$. *Here* $W^{2,2}(D) = \left\{ g \in L^2(D) : \dfrac{\partial g}{\partial X_i}, \dfrac{\partial^2 g}{\partial X_i \partial X_j} \in L^2(D) \right\}$.

THEOREM 3.2 ([16]). – *Let* D *be a bounded Lipschitz domain in* $\mathbf{R}^n$ *with connected boundary. Then,*

(a) M *is a dense subspace of* $L^2_1(\partial D) \times L^2(\partial D)$.

(b) *Given* $g \in W^{2,2}(D)$ *there is a unique* $u \in W^{2,2}(D)$ *so that*

(i) $\int_D \Delta u \, \Delta \varphi = 0$ *for all* $\varphi \in C_0^\infty(D)$

(ii) $u - g \in W_0^{2,2}(D)$ = *the closure of* $C_0^\infty(D)$ *in* $W^{2,2}(D)$.

(c) *There exists a constant* $C > 0$ *depending only on* D *such that for all* u *as in* (b),

$$\| u \|_{L^2(D)} \leqslant C\{\| g_0 \|_{L^2_1(\partial D)} + \| g_1 \|_{L^2(\partial D)}\},$$

where C *depends only on the Lipschitz character of* D.

LEMMA 3.3. – *Let* D *be a bounded Lipschitz domain in* $\mathbf{R}^n$ *with connected boundary. Then, if* u *is a weak solution of the biharmonic Dirichlet problem in the sense of Theorem 3.2, we have*

$$\|N(\nabla u)\|_{L^2(\partial D, d\sigma)} \leqslant C\left\{\|u\|_{L^2_1(\partial D)} + \left\|\frac{\partial u}{\partial N}\right\|_{L^2(\partial D, d\sigma)}\right\},$$

where C *depends only on the Lipschitz character of* D.

Proof. – For a fixed $P_0 \in \partial D$, consider a continuum of coordinate cylinders $\theta Z = \theta Z(P_0), 1 \leqslant \theta \leqslant 2$, all of which have the property that $\theta Z \cap D$ is a starlike Lipschitz domain. Since $u \in W^{2,2}(D)$, by Fubini's theorem, for a.e. θ, u is a weak solution in $\theta Z \cap D$ with data in $L^2_1(\partial(\theta Z \cap D)) \times L^2(\partial(\theta Z \cap D))$. For any such θ, let $\Omega = \theta Z \cap D$. Let now $\widetilde{u}$ be the unique solution of the biharmonic Dirichlet problem with the same data as u on $\partial\Omega$, so that $\|N(\nabla\widetilde{u})\|_{L^2(\partial\Omega)} < \infty$. (Such a $\widetilde{u}$ is provided by Theorem 2.1 and the case $p = 2$ of Theorem 0.4). We can also assume that Ω is starlike with respect to the origin. Let $\widetilde{u}_r(X) = \widetilde{u}(rX), 0 < r < 1$. By the results in Theorem 2.1,

$$\|u - \widetilde{u}_r\|_{L^2_1(\partial\Omega)} + \left\|\frac{\partial u}{\partial N} - \frac{\partial \widetilde{u}_r}{\partial N}\right\|_{L^2(\partial\Omega)} \longrightarrow 0$$

as $r \longrightarrow 1$. Then, as $\widetilde{u}_r \in W^{2,2}(\Omega)$, part (c) of Theorem 3.2 shows that $\|u - \widetilde{u}_r\|_{L^2(\Omega)} \longrightarrow 0$, and hence $u = \widetilde{u}$ in Ω. Therefore, using Theorem 2.1 once more, we see that,

$$\int_{Z\cap\partial D} N(\nabla u)^2 d\sigma \leqslant C\int_1^2 d\sigma \left[\|u\|^2_{L^2(\partial(\theta Z\cap D))} + \left\|\frac{\partial u}{\partial N}\right\|^2_{L^2(\partial(\theta Z\cap D))}\right].$$

The right hand side is clearly bounded by

$$C\left(\|u\|_{L^2_1(\partial D)} + \left\|\frac{\partial u}{\partial N}\right\|_{L^2(\partial D)} + \int_D u^2 + |\nabla u|^2\right).$$

Covering ∂D with finitely many such $Z's$, we obtain

$$\int_{\partial D} N(\nabla u)^2 d\sigma \leqslant C\left\{\|u\|_{L^2_1(\partial D)} + \left\|\frac{\partial u}{\partial N}\right\|_{L^2(\partial D)} + \int_D |\nabla u|^2 + u^2\right\}.$$

Choose now a smooth domain D', compactly contained in D, so that $C \int_{D \setminus D'} |\nabla u|^2 \leqslant \frac{1}{2} \int_{\partial D} N(\nabla u)^2 \, d\sigma$. By standard interior elliptic estimates, $\int_{D'} |\nabla u|^2 \, dx \leqslant C \int_D u^2 \, dx$. The lemma now follows from part (c) of Theorem 3.2.

Proof of Theorem 3.1. – Existence follows easily by (a), (b) of Theorem 3.2, and Lemma 3.3, by standard arguments. To prove uniqueness, we only need to introduce smooth domains $D_j \subset D$, which approximate D in a similar manner as in the proof of uniqueness in Theorem 2.1 (see for example [18] for the existence of D_j). The uniqueness proof is then the same as the corresponding one in Theorem 2.1.

4. L^2 booster theorems.

The purpose of this section is to show that the L^2 results established in section 3 for all Lipschitz domains have an automatic real variable improvement of themselves. In this section we show that given a bounded Lipschitz domain D in $\mathbf{R}^n$, with connected boundary, there exists $\epsilon > 0$, which depends only on the Lipschitz character of D so that we can uniquely solve the Dirichlet problem for the biLaplacian in D, with data in $L_1^p(\partial D)$ and normal derivative in $L^p(\partial D, d\sigma)$, for $2 - \epsilon \leqslant p \leqslant 2 + \epsilon$. This fact does not depend on the particular boundary value problem that we are treating and it is proved from purely real variable considerations. In fact after the submission of this manuscript, the referee communicated to us the following general functional analytic results of G. David and S. Semmes (unpublished), which easily yields our L^p results for $2 - \epsilon < p < 2 + \epsilon$.

THEOREM (G. David and S. Semmes). – *Let Ω be a measure space with a positive measure μ. Let $\sigma_0 > 0, \sigma_0 < 1$, and let $T : L^p(\Omega, \mu) \longrightarrow L^p(\Omega, \mu)$ be a bounded linear operator, with norm smaller than or equal to B for $\left|\frac{1}{2} - \frac{1}{p}\right| \leqslant \sigma_0$. Also assume that*

$T : L^2 \longrightarrow L^2$ *is an isomorphism, and that* $\|T^{-1}\|_{(2,2)} \leqslant A$. *Then,* $T : L^p \longrightarrow L^p$ *is an isomorphism for* $\left|\frac{1}{2} - \frac{1}{p}\right| < \delta\sigma_0$, *with* $\delta = \frac{1}{2AB}$.

The argument given here for the range $2 \leqslant p \leqslant 2 + \epsilon$ was first discovered by the first two authors in [8]. In section 5 we will show that our L^p results in the range $1 < p \leqslant 2$ are sharp. Whether they are also sharp in the range $2 \leqslant p < \infty$ remains an open problem.

THEOREM 4.1. – *Let* D *be a bounded Lipschitz domain in* $\mathbf{R}^n$, *with connected boundary. Then, there exists* $\epsilon > 0$, *which depends only on the Lipschitz character of* D, *so that for any* $2 - \epsilon \leqslant p \leqslant 2 + \epsilon$, *and* $f \in L_1^p(\partial D, d\sigma), g \in L^p(\partial D, d\sigma)$, *there exists a unique function* u *in* D, *such that*

(a) $\Delta^2 u = 0$ *in* D

(b) $\lim\limits_{\substack{X \to Q \\ X \in \Gamma(Q)}} u(X) = f(Q)$ *a.e.* $(d\sigma)$

(c) $\lim\limits_{\substack{X \to Q \\ X \in \Gamma(Q)}} N_Q \cdot \nabla u(X) = g(Q)$ *a.e.* $(d\sigma)$

(d) $\|N(\nabla u)\|_{L^p(\partial D, d\sigma)} < +\infty$.

In fact,

$$(*) \quad \|N(u)\|_{L^p(\partial D, d\sigma)} + \|N(\nabla u)\|_{L^p(\partial D, d\sigma)} \leqslant C\,\{\|f\|_{L_1^p(\partial D)} + \|g\|_{L^p(\partial D)}\}$$

where C *depends only on* p *and the Lipschitz character of* D.

Proof. – We will first treat the case $2 - \epsilon < p < 2$. As in the proof of Theorem 3.1, it is enough to treat the existence part of the Theorem, together with the a priori inequality (*). We will first show (*), when $f \in L_1^2(\partial D), g \in L^2(\partial D, d\sigma)$, for the solution u constructed in Theorem 3.1. By standard arguments, this will suffice. Let $\{\Gamma(Q)\}$ be a regular family of cones on ∂D. Pick another regular family of cones $\{\Gamma'(Q)\}$ with $\overline{\Gamma(Q)} \backslash Q \subset \Gamma'(Q)$. Let

$$m(Q) = N_\Gamma(u)(Q) + N_\Gamma(\nabla u)(Q), \bar{m}(Q) = N_{\Gamma'}(u)(Q) + N_{\Gamma'}(\nabla u)(Q).$$

For $\lambda > 0$, let $F_\lambda = \{Q \in \partial D : \tilde{m}(Q) \leqslant \lambda\}$. Let $\widetilde{F}_\lambda = \bigcup_{Q \in F_\lambda} \Gamma'(Q)$. Then, there exists a compact subset $D' \subset D$ such that if $D_\lambda = \widetilde{F}_\lambda \cup D'$, then D_λ is a bounded Lipschitz domain, with connected boundary, whose Lipschitz constants depend only on the ones of D. Moreover, $\partial D_\lambda \cap \partial D = F_\lambda$, $\sigma(\partial D_\lambda \backslash F_\lambda) \leqslant C\,\sigma(\partial D \backslash F_\lambda)$, $|u(Q)| + |\nabla u(Q)| \leqslant \lambda$ for $Q \in \partial D_\lambda$, and $\{\Gamma(Q)\}, Q \in F_\lambda$ extends to a regular family of cones in D_λ, which we will denote by $\{\Gamma_\lambda(Q)\}$. By the L^2 theory in D_λ,

$$\int_{F_\lambda} m^2(Q)\, d\sigma(Q)$$

$$\leqslant \int_{\partial D_\lambda} [N_{\Gamma_\lambda}(u(\,(Q) + N_{\Gamma_\lambda}(\nabla u)(Q)]^2\, d\sigma(Q)$$

$$\leqslant C \int_{\partial D_\lambda} u^2 + |\nabla_t u|^2 + \left(\frac{\partial u}{\partial N}\right)^2 d\sigma,$$

where C depends only on the Lipschitz character of D. The last term is bounded by $C \int_{F_\lambda} (f^2 + |\nabla_t f|^2 + g^2)\, d\sigma + C\lambda^2 \sigma(\partial D \backslash F_\lambda)$.

We next remark that for $0 < \epsilon < 1$ we have

$$\int_{\partial D} m^{2-\epsilon}\, d\sigma \leqslant C \int_{\partial D} m^2 \tilde{m}^{-\epsilon}\, d\sigma.$$

In order to show this, note that a classical argument (see [10], for example) shows that $\tilde{m}(Q) \leqslant C\,M(m)(Q)$, where M denotes the Hardy-Littlewood maximal operator on ∂D. A well known result from the theory of weights (see [4] for example) is that, for any $0 < \epsilon < 1$, $M(m)^\epsilon$ is a weight in the Muckenhoupt class A_1. Thus, $M(m)^{-\epsilon}$ is in the Muckenhoupt class A_2, and hence, by a theorem of Muckenhoupt ([15]), the maximal operator is bounded on $L^2(\partial D, M(m)^{-\epsilon}\, d\sigma)$. Therefore,

$$\int_{\partial D} M(m)^2\, M(m)^{-\epsilon}\, d\sigma \leqslant C \int_{\partial D} m^2\, M(m)^{-\epsilon}\, d\sigma$$

$$\leqslant C \int_{\partial D} m^2\, \tilde{m}^{-\epsilon}\, d\sigma,$$

and our remark follows from the maximal theorem.

Thus,

$$\int_{\partial D} m^{2-\epsilon}\, d\sigma$$

$$\leqslant C \int_{\partial D} m^2 \bar{m}^{-\epsilon}\, d\sigma = \epsilon \int_0^\infty \lambda^{-\epsilon-1} \left(\int_{\{Q \in \partial D : \bar{m} \leqslant \lambda\}} m^2\, d\sigma \right) d\lambda$$

$$\leqslant C \epsilon \int_0^\infty \lambda^{-\epsilon-1} \left(\int_{\{Q \in \partial D : \bar{m} < \lambda\}} f^2 + |\nabla_t f|^2 + g^2 \right) d\sigma$$

$$+ C \epsilon \int_0^\infty \lambda^{-\epsilon+1} \sigma\, \{Q \in \partial D : \bar{m} > \lambda\}\, d\lambda$$

$$\leqslant C \int_{\partial D} (f^2 + |\nabla_t f|^2 + g^2)\, \bar{m}^{-\epsilon}\, d\sigma + C \epsilon \int_{\partial D} \bar{m}^{-\epsilon}\, d\sigma .$$

If we now observe that, by classical arguments (see [10] for example)

$$\int_{\partial D} \bar{m}^{2-\epsilon}\, d\sigma \leqslant C \int_{\partial D} m^{2-\epsilon}\, d\sigma ,$$

and that $(f + |\nabla_t f| + g) \leqslant \bar{m}$ for a.e. Q, we obtain the desired estimate for ϵ small enough.

The case $2 < p < 2 + \epsilon$ follows from an argument of the first two authors (see [8] or [14]).

The results in this section also easily follow from the general functional analytic result of G. David and S. Semmes mentioned above, once we observe that the operator T used in the proof of Theorem 2.1 is also L^p bounded for p sufficiently near 2. This follows from an examination of our L^2 proof.

5. Some L^p counterexamples.

In sections 3 and 4 we have shown that given a bounded Lipschitz domain D in $\mathbf{R}^n$, with connected boundary, there exists $\epsilon > 0$, which depends only on the Lipschitz character of D so that, if $2 - \epsilon \leqslant p \leqslant 2 + \epsilon$, given any $f \in L_1^p(\partial D)$, and $g \in L^p(\partial D)$ there exists a unique biharmonic function u, in D, such that $u\big|_{\partial D} = f$ and $\dfrac{\partial u}{\partial N}\Big|_{\partial D} = g$ in the non-tangential sense, and

$$\| N(\nabla u) \|_{L^p(\partial D, d\sigma)} < \infty .$$

In fact, we showed that, for this range of $p's$,

$$(*)\quad \|N(u)\|_{L^p(\partial D, d\sigma)} + \|N(\nabla u)\|_{L^p(\partial D, d\sigma)} \leqslant C\,\{\|f\|_{L^p_1(\partial D)} + \|g\|_{L^p(\partial D, d\sigma)}\}$$

where C depends only on p and the Lipschitz character of D.

The purpose of this section is to show that, at least in the range $p \leqslant 2$ this is sharp. In fact, we will show that given $p < 2$, there exists a bounded Lipschitz domain $D \subset \mathbf{R}^2$, with connected boundary, and a biharmonic function u in D, with

$$N(u) \in L^p(\partial D, d\sigma), N(\nabla u) \in L^p(\partial D, d\sigma), u \not\equiv 0,$$

and such that $u = 0$ on $\partial D, \dfrac{\partial u}{\partial N} = 0$ on ∂D, in the sense of non-tangential convergence. By the uniqueness argument used in the proof of Theorem 3.1, it is clear that (*) cannot then hold as on a priori inequality for all smooth domains. We will also show that if D is as above, and $f \in L^2_1(\partial D), g \in L^2(\partial D)$, and u is the solution given in Theorem 3.1, the estimate (*) cannot hold. We emphasize that our counterexamples work for the range $p < 2$. Whether our results are sharp or not in the range $p > 2$ remains an open problem.

Our counterexamples are based on the following lemma:

LEMMA 5.1. – *Given $q < 2$, but sufficiently close to 2, there exists a real number γ_q and an angle $\theta_q < \pi$ such that if S_q is the sector $S_q = \{re^{i\theta} : 0 < r < \infty, -\theta_q < \theta < \theta_q\}$, the functions*

$$u_q(r, \theta) = r^{1-1/q}\left[\sin\left(1 - \frac{1}{q}\right)\theta + \gamma_q \sin\left(1 + \frac{1}{q}\right)\theta\right]$$

and

$$v_q(r, \theta) = r^{1+1/q}\left[\sin\left(1 - \frac{1}{q}\right)\theta + \gamma_q \sin\left(1 + \frac{1}{q}\right)\theta\right]$$

are biharmonic in S_q, and satisfy

$$u_q(r, \pm\theta_q) = v_q(r, \pm\theta_q) = \frac{1}{r}\frac{\partial}{\partial\theta}u_q(r, \pm\theta_q) = \frac{1}{r}\frac{\partial}{\partial\theta}v_q(r, \pm\theta_q) = 0,$$

for $0 < r < \infty$.

Before proving the lemma, let us discuss its consequences. Let $s\Delta = \{re^{\pm i\theta_q} : 0 \leqslant r < s\}$, $s > 0$. Let $D_q \subset S_q$ be a bounded Lipschitz domain with connected boundary such that $2\Delta \subset \partial D$, and $\partial D \backslash 1\Delta$ is C^∞. To show non-uniqueness for $p < 2$, fix such a p, and choose $p < q < 2$. Consider u as in Lemma 5.1, but restricted to D_q. Using the L^2 theory in D_q, a biharmonic function in D_q, with the same data as u_q, may be subtracted from u_q, so that the resulting function w_q is biharmonic in

$$D_q \,, N(w_q) \,, N(\nabla w_q) \in L^p(\partial D_q \,, d\sigma)\,,$$

but w_q and $\dfrac{\partial w_q}{\partial N}$ are 0, non-tangentially a.e. on $\partial D_q \cdot w_q$ cannot be identically 0, for then $N(\nabla u_q)$ would be in $L^2(\partial D_q \,, d\sigma)$ which is easily seen to be false. To see that (*) cannot hold in D_q as an a priori estimate for the L^2 solutions, first note that $\Delta v_q(r\,,\theta) = \dfrac{4}{q} r^{1/q-1} \sin(1 - 1/q)\,\theta$ does not have boundary values in the dual of $L^p(\partial D_q)$ although it is in $L^2(\partial D_q)$. Let u be an L^2 solution with $u|_{\partial D_q} = 0$, and $\dfrac{\partial u}{\partial N}$ supported in 1Δ. Under these conditions, with v_q as in Lemma 5.1, we can apply the divergence theorem on D_q, to yield

$$\int_{1\Delta} \frac{\partial u}{\partial N}\, \Delta v_q \, d\sigma = \int_{\partial D_q \backslash 2\Delta} \left[(\Delta u) \cdot \frac{\partial v_q}{\partial N} - \frac{\partial \Delta u}{\partial N}\, v_q \right] d\sigma\,.$$

By standard results on C^∞ domains, higher derivatives of u on $\partial D_q \backslash 2\Delta$ may be controlled by the L^p norm of its Dirichlet data on the boundary of a C^∞ domain $D' \subset D_q$, where say,

$$\partial D' \cap \partial D_q = \partial D_q \backslash 3/2\Delta\,.$$

This data is in turn controlled by $N(\nabla u)$, which if (*) holds, is controlled by $\left\| \dfrac{\partial u}{\partial N} \right\|_{L^p(1\Delta)}$. We would then conclude that Δv_q is in the dual of $L^p(\partial D_q)$, which is false.

Proof of 5.1. – In the complex plane, consider the complex valued biharmonic functions

$$z^\alpha + \gamma\, \bar{z} z^{\alpha-1} \,, \alpha\,, \gamma \in \mathbf{R} \,, z \in \mathbf{C}\,.$$

Let $u(r,\theta) = r^\alpha \sin \alpha\theta + \gamma r^\alpha \sin(\alpha - 2)\theta$ be their imaginary parts. Consider the sectors $S_{\theta_0}, 0 < \theta_0 < \pi$. Finding the desired biharmonic functions u with $u = \dfrac{\partial u}{\partial N} = 0$ on ∂S_{θ_0} leads us to finding $\alpha(\theta), \gamma(\theta)$, so that at $\theta = \theta_0$,

$$
(5.2) \quad \begin{cases} f(\alpha,\gamma,\theta) \equiv \sin \alpha\theta + \gamma \sin(\alpha - 2)\theta = 0 \\ \\ g(\alpha,\gamma,\theta) \equiv \alpha \cos \alpha\theta + \gamma(\alpha - 2)\cos(\alpha - 2)\theta = 0. \end{cases}
$$

In the limiting case, when $\theta = \pi$, the interesting solutions are $\alpha(\pi) = \dfrac{1}{2}$, $\gamma(\pi) = -1$. If

$$
\left.\begin{bmatrix} \dfrac{\partial f}{\partial \alpha} & \dfrac{\partial f}{\partial \gamma} \\ \\ \dfrac{\partial g}{\partial \alpha} & \dfrac{\partial g}{\partial \gamma} \end{bmatrix}\right|_{\substack{\theta = \pi \\ \alpha = 1/2 \\ \gamma = -1}}
$$

is an invertible matrix, then, by the implicit function theorem we will be able to solve (5.2) for α and γ as C^1 function of θ, θ in an interval around π, with $\alpha(\pi) = 1/2$, $\gamma(\pi) = -1$. A calculation shows that the above matrix is $\begin{bmatrix} 0 & 1 \\ -2\pi & 0 \end{bmatrix}$, which is clearly invertible. Differentiating (5.2) implicitly with respect to θ, and evaluating at $\theta = \pi$ gives $\alpha'(\pi) = \dfrac{1}{\pi} > 0$. Thus, if $\theta_0 < \pi$ is sufficiently close to $\pi, \alpha(\theta_0) < 1/2$, and $\alpha(\theta_0) \nearrow 1/2$ as $\theta_0 \nearrow \pi$. This establishes the existence of u_q as in 5.1. To obtain v_q all we need to observe is that is $u(z)$ is biharmonic, so is $|z|^2 u(1/z) = v(z)$.

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B.E.J. DAHLBERG,
Dept. of Mathematics
Univ. of Göteborg & Chalmers Inst of Techn.
Göteborg (Sweden).

C.E. KENIG,
Dept. of Mathematics
Univ. of Chicago
5734 Univ. Avenue
Chicago, Ill. 60637 (USA).

G.C. VERCHOTA,
Dept. of Mathematics
California Inst. of Technology
Pasadena, CA 91125 (USA).