Annales de l'institut Fourier

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Annales de l'institut Fourier, tome 35, n° 3 (1985), p. 119-135

<http://www.numdam.org/item?id=AIF_1985__35_3_119_0>

Ann. Inst. Fourier, Grenoble
35, 3 (1985), 119-135

THE RATIONAL HOMOTOPY OF THOM SPACES AND THE SMOOTHING OF ISOLATED SINGULARITIES

by Stefan PAPADIMA

1. Introduction.

Let $Y \subset C^{n+1}$ be an algebraic variety having the origin as an isolated singularity. Suppose that Y is pure-dimensional and denote its codimension by s $(s > 0)$.

The link of the singularity, i.e. the intersection $K = Y \cap \partial B$ with the boundary of a sufficiently small ball centered at the origin, is known to be a differentiable submanifold $K \hookrightarrow S^{2n+1}$ with complex normal bundle of complex dimension s (compare with [8]).

Let F denote C or R and let $MF(s)$ stand for the universal Thom space $MU(s)$ or $MO(2s)$. The Thom map [14] of the link K determines a well-defined element $\Theta_F(Y) \in \pi_{2n+1} MF(s)$.

1.1. Definition. – *The singularity of* Y *is* F-*topologically smoothable if* $\Theta_F(Y) = 0$.

This definition appeared as a topological approximation of the *algebraic smoothability* problem : if the singularity is algebraically smoothable, i.e. there is (locally) an algebraic family of varieties Y_t (t being a small parameter in a neighbourhood of the origin in C) such that $Y_0 = Y$ and the other fibers Y_t are smooth, then the singularity is complex, and consequently real topologically smoothable (see [4], [10]). As we are not going to deal with the algebraic version of the problem, we shall mean by smoothability the topological smoothability.

Mots-clés : Topological smoothing – Nonzero multiple – Thom construction – Formality – Minimal model.

At a first glance the computational difficulties encountered when trying to decide the smoothability might seem enormous, since one knows that the rational homotopy Lie algebras $\pi_* \mathrm{MF}(s) \otimes \mathrm{Q}$ contain free Lie algebras with infinitely many generators, for $s > 1$ ([1]). D. Sullivan suggested in [13] that, when starting with algebraic objects, there must be severe restrictions on the associated homotopy obstructions.

1.2. *Sullivan's conjecture.* – $\mathcal{O}_{\mathrm{C}}(\mathrm{Y}) \otimes \mathrm{Q}$ is expressible as a linear combination of simple Whitehead products, for any Y.

One may find a proof in [11], for Y a conical singularity. The reason is that the general method of proof developed in [11] depends on the order of connectivity of the link K, which is provided for conical Y by the result of [7]. Once one has this result for an arbitrary Y, as generalized in [3], the rational homotopy method of [11] solves the conjecture in full generality.

The first main result of this paper is related to the real form of the above conjecture and asserts that there are no rational obstructions to real smoothability. Since our method is a rational homotopy one, it is quite expectable that it will imply geometric results only up to a nonzero factor :

1.3. DEFINITION. – *We shall say that some nonzero multiple of* Y *is* F*-smoothable if there is an algebraic map* $\mathrm{C}^{n+1} \xrightarrow{f} \mathrm{C}^{n+1}$, *with* $f^{-1}(0) = 0$ *and transverse to* $\mathrm{Y} \setminus \{0\}$, *such that* $f^{-1}\mathrm{Y}$ *is* F*-smoothable.*

THEOREM I. – *Some nonzero multiple of any* Y *is* R*-smoothable.*

Section 2 contains the proof of this result, which is based on the method of formal rational homotopy computations ([6], [13]) and on the general formality properties of Thom spaces derived in [9].

In Section 3 we restrict our attention to the complex smoothability problem for conical singularities. If $\mathrm{V}^r \subset \mathrm{P}^n \mathrm{C}$ is smooth algebraic, of codimension $s = n - r$, there are certain well-known relations among the degrees of the Chern monomials of the normal bundle of V which must hold if the vertex of the cone $\mathrm{Y} = \mathrm{CV}$ is smoothable (in fact, this was the way which enabled Thom to produce the first examples of non-smoothable algebraic singularities, see [4]). It is also shown in [11] that these numerical conditions are the only rational obstruction to the smoothability of the cone.

Our aim is to generalize this kind of results, by formulating these conditions (in fact a larger number of them, including the Thom ones) for an arbitrary differentiable submanifold $V^{2r} \hookrightarrow P^n$ with complex normal bundle of complex dimension s and by pointing out their relationship with the problem of realizing (a multiple of) V as a linear section in a higher-dimensional complex projective space.

We shall make use of the Thom map of V, $t : P^n \longrightarrow MU(s)$ in order to define normal degrees as follows : if $K \in N^s$, $K = (k_1, \ldots, k_s)$, put : $|K| = k_1 + 2k_2 + \ldots + sk_s$, and define an integer D_K for any such K with $|K| \leqslant n - s$ by the equality :

$$t^* (u_s . c^K) = D_K . a^{s + |K|} \qquad (*)$$

where u_s is the universal Thom class, c^K is the monomial $c_1^{k_1} \ldots c_s^{k_s}$ in the universal Chern classes and $a \in H^2(P^n ; Z)$ is the canonical generator. Notice that, if V is algebraic, then D_K coincides with deg $c_K(N)$, N being the normal bundle of V ([4], [11]). Define next integers R_{KL}, for any K and L such that $|K + L| \leqslant n - s$ by :

$$R_{KL} = D_K D_L - D_0 D_{K+L} \qquad (**)$$

and consider the following conditions on the L-class [14] of V, defined for any $1 \leqslant k \leqslant s$:

(R_k) $R_{KL} = 0$, for any K and L such that :

$$|K + L| \leqslant n - 2s + k$$

(for $k = 1$, one recovers Thom's conditions [4]).

1.4. Definition. – *We shall say that a nonzero multiple of* V *is a linear section in* P^{n+k} *if there is a differentiable map :* $f : P^n \longrightarrow P^n$ *transverse to* V, *which is a rational homotopy equivalence (i.e. has nonzero degree,* $\deg(f) = D$ *being defined by :* $f^*a = D \cdot a$*) such that* $f^{-1}V$ *is* L-*equivalent to a transverse intersection :* $P^n \cap W$, W *being a differentiable submanifold of* P^{n+k} *with complex normal bundle of complex dimension* s.

THEOREM II. – *Suppose* : $1 \leqslant k \leqslant s$.

i) *If some nonzero multiple of* V *is a linear section in* P^{n+k}, *then relations* (R_k) *hold.*

ii) *The converse is also valid, provided* : $n \leqslant 3s + 3$ *or the Thom map of* V *is formal. If in addition* V *is algebraic, then the "nonzero factor"* f *in Definition 1.4 may be chosen to be a regular algebraic map transverse to* V.

Thanks are due to A. Dimca for providing Lemma 2.1.

2. Proof of Theorem I.

2.1. LEMMA. – *Some nonzero multiple of* Y *is* F-*smoothable iff* $\Theta_F(Y) \otimes Q = 0$.

Proof. – If $f : C^{n+1} \longrightarrow C^{n+1}$ is an algebraic map as in Definition 1.3, the transversality assumption implies the equality : $\Theta_F(f^{-1}Y) = \deg(f) \cdot \Theta_F(Y)$, where :

$$\deg(f) = \dim_C C[z_0, \ldots, z_n]/\text{ideal}(f_0, \ldots, f_n)$$

is finite, due to the fact that : $f^{-1}(0) = 0$.

Suppose then that : $m \cdot \Theta_F(Y) = 0$, with $m > 0$. Anyway, we may find a hyperplane in C^{n+1} transverse to $Y \setminus \{0\}$. Assuming that it is given by : $z_0 = 0$, we may define f by :

$$f(z_0, z_1, \ldots, z_n) = (z_0^m, z_1, \ldots, z_n).$$

2.2. *Proof of Theorem I.* – We may well suppose $n \geqslant 2s + 1$, since otherwise $\pi_{2n+1} MU(s) \otimes Q = 0$ ([1]). Set $m = \left[\dfrac{n-2s}{4}\right] + 1$

and start as in [11] by factoring the rationalized classifying map of the normal bundle of K, $\nu_0 : K_0 \longrightarrow BO(2s)_0$, through $\Pi_{i \geqslant m} K(Q, 4i)$; this uses [3]. If $m < s$, this helps to factor $(M\nu)_0$ through $M(m, s) \xrightarrow{\mu} MO(2s)_0$, where $M(m, s)$ appears as a cofibre in the sequence :

$$\prod_{i=m}^{s-1} K(Q, 4i) \longrightarrow \prod_{i=m}^{s} K(Q, 4i) \longrightarrow M(m, s). \qquad (*)$$

Otherwise, a similar construction factors $(M\nu)_0$ through

$M \xrightarrow{\mu} MO(2s)_0$, with $M = K(Q, 4s)$ or $M = S_0^{2s}$ and we are done, since in both these cases $\pi_{2n+1} M = 0$. Write then : $\Theta_R(Y) \otimes Q = \pi_{2n+1} \mu(\alpha)$, with $\alpha \in \pi_{2n+1} M(m, s)$.

Observe next that $\Theta_R(Y) \otimes Q$ comes from $\pi_* MSO(2s) \otimes Q$, hence it is annihilated by the rational Hurewicz map ([1]). On the other hand, since it is easy to see that μ induces an injection in rational homology, it follows that α has the same property. Using this, we are going to show that $\alpha = 0$.

Since $M(m, s)$ is the cofibre of a formal map, it is a formal space ([2]). Moreover, the cofibration (*) indicates that it should behave homotopically like a localized universal Thom space. This is indeed the case : using the same argument as in [9] (Proposition 2.7 and Corollary 2.8) it can be shown that :

$$\ker h = \sum_{i \geq 1} \mathrm{ad}^i (\pi^{\circ}_{> 4s}) (\pi^{\circ}_{> 4s}),$$

where h is the rational Hurewicz map, $\pi_j M(m, s) = \bigoplus_{p \geq 0} \pi_j^p$ as a consequence of the formality of $M(m, s)$ ([6], [13]), and ad refers to the Whitehead product bracket structure.

The graded space π°_* is isomorphic to

$$H^+ M(m, s) / H^+ M(m, s) \cdot H^+ M(m, s).$$

Therefore $\pi^{\circ}_{> 4s}$ is nonzero only in degrees $\geq 4(s + m)$ (compare with (2.4) from [9]). It follows that h is injective in degree $2n + 1$, and α equals zero.

2.3. *Remark.* – It should be mentioned that all the obvious implications between the various smoothability properties are strict.

In [12] one may find examples of complex topologically smoothable (conical) singularities which are not algebraically smoothable.

Taking Y to be the cone on the Segre embedding : $P^1 \times P^4 \hookrightarrow P^{11}$ we may see, using the stability properties of the homotopy groups of Thom spaces [14] that Y is both real topologically smoothable and complex topologically smoothable up to a nonzero factor (by the result on rational homotopy groups of complex Thom spaces from [1]). The computations in [10] show that Y is not complex topologically smoothable.

At last there are examples, such as the cone on $P^1 \times P^2$ Segre embedded in P^5, of singularities which are neither real topologically smoothable (see [4]) nor complex topologically smoothable up to a nonzero factor (were $\Theta_C(Y) \otimes Q = 0$, we would also have $\Theta_C(Y) = 0$, since $\pi_{11} MU(2) = Z$, see [5]).

3. Conical singularities.

From now on V will denote (the L-class of) a differentiable submanifold V^{2r} embedded in $P^n C$ with complex normal bundle of complex dimension s and will be often identified with the homotopy class of its Thom map : $t : P^n \longrightarrow MU(s)$, [14].

3.1. PROPOSITION. – i) *Suppose* : $1 \leqslant k \leqslant s$. *Then* : *relations* (R_k) *hold iff there is a graded algebra map* F *making the diagram below commutative, where* i *stands for the usual inclusion* :

$$\begin{array}{ccc} H^*(P^{n+k};Z) & \overset{F}{\dashleftarrow} & \\ {\scriptstyle i^*}\downarrow & & \\ H^*(P^n;Z) & \xleftarrow{\ t^*\ } & H^*(MU(s);Z) \end{array}$$

ii) *If* $|K + L| \leqslant n - 2s$, *then* $R_{KL} = 0$.

iii) If k_s *or* l_s *is nonzero, then* $R_{KL} = 0$.

Proof. – i) The following descriptions of the algebra $H^*MU(s)$ make clear the role of the relations R_{KL}. We have an isomorphism :

$$\Lambda\{z_0^I \mid I \in N^{s-1} ; |z_0^I| = 2s + 2|I|\} / \text{ideal}\{r_{IJ} \mid 0 < I \leqslant J\} \longrightarrow H^*MU(s)$$

where :

(*) $r_{IJ} = z_0^I z_0^J - z_0^0 z_0^{I+J}$ and the order is lexicographic, which sends z_0^I to $u_s c^I$ (see [9], [11]), coming from the standard identification :

$$H^{+}MU(s) = \text{ideal } (c_s) \subset H^{*}BU(s) \qquad (**)$$

given by : $u_s c^{K} \longrightarrow c_s \cdot c^{K}$.

If relations (R_k) are satisfied, one obtains an algebra map F by defining : $F(z_0^{I}) = D_I \cdot a^{s+|I|}$ if $|I| \leqslant n-s$, and zero otherwise (where the integers D_I are constructed from the Thom map t as in (*) of Section 1) which plainly lifts t^* to H^*P^{n+k} .

If there is an algebra lifting of t^*, it defines integers F_K, for any $K \in N^s$ such that $|K| \leqslant n-s+k$, by the equality : $F(u_s c^{K}) = F_K \cdot a^{s+|K|}$. Writing that :

$$F[(u_s c^{K})(u_s c^{L}) - (u_s c^{0})(u_s c^{K+L})] = 0$$

one obtains, if $|K+L| \leqslant n-2s+k$, equalities :

$$F_K F_L - F_0 F_{K+L} = 0 .$$

Since : $i^* F = t^*$, one finds out that we have : $F_M = D_M$, if $|M| \leqslant n-s$, which gives, by (**) of Section 1, the relations (R_k).

ii) is rather standard. One has only to write that :

$$t^*[(u_s c^{K})(u_s c^{L}) - (u_s c^{0})(u_s c^{K+L})] = 0, \text{ for } |K+L| \leqslant n-2s .$$

As far as iii) is concerned, one starts by examining, for any K such that $|K| \leqslant n-s$, the equality :

$$t^*[(u_s c^{K}) - (u_s c^{0})^{k_s}(u_s c^{I})] = 0$$

where : $I = (k_1, \ldots, k_{s-1})$ and deducing that : $D_K = (D_0)^{k_s} D_I$. Consequently : $R_{KL} = (D_0)^{k_s + 1_s} \cdot R_{IJ}$, for any K and L such that $|K+L| \leqslant n-s$. If : $k_s + 1_s > 0$, then $|I+J| \leqslant n-2s$ and by ii) : $R_{KL} = 0$.

3.2. *Remark.* – If V is algebraic there are many other relations R_{KL}, besides those in the above Proposition, which are trivially satisfied. For instance, an interesting result in [11] asserts that, if n is even, all relations (R_1) hold.

3.3. LEMMA. - *Let* $(\Lambda Z, d)$ *be the bigraded model of the formal space* MU(s) ([6], [9]).

i) *The set of homotopy classes of maps between rational spaces :* $[P_0^n, MU(s)_0]$ *is in one-to-one correspondence with the set of homotopy classes of d.g.a. maps between minimal models :*

$[(\Lambda Z^{\leqslant 2n}, d), (\Lambda_2(x), 0)]$. *If* $P^n \xrightarrow{i} P^{n+k}$ *is inclusion then the induced map between topological homotopy classes is identified with the map induced between algebraic homotopy classes by restriction ;*

ii) *If* $i > 1$, *then :* $Z_i^P = 0$ *for* $p < (i+1)(2s+1)+3$.

Proof. – The first part is proved by Sullivan's equivalence of rational topological and minimal d.g.a. homotopy categories [13], using standard algebraic obstruction theory, taking into account that : $H^p P^n = 0$ for $p > 2n$ and that the bigraded minimal model of the formal space P^n is given by an elementary extension :

$$(\Lambda_2(x), 0) \underset{d}{\otimes} \Lambda_{2n+1}(y)), \text{ with : } dy = x^{n+1} .$$

The second part uses [9] : by Corollary 2.8. [9] we have for any $i > 0$ an equality of graded vector spaces :

$$(Z_i)^* = \text{span}\,\{[(z_0^{I_0})^*, \ldots, [(z_0^{I_{i-1}})^*, (z_0^{I_i})^*] \ldots] | I_j > 0\}$$

where the star denotes Sullivan duals of indecomposables of the bigraded model as rational homotopy elements, the bracket stands for the Whitehead product, I_j is a multiindex in N^{s-1} and the space Z_0 of indecomposable generators of $H^+MU(s)$ is to be identified (see the proof of Proposition 3.1.i)) with : span $\{z_0^I | I \geqslant 0\}$. We may now evaluate :

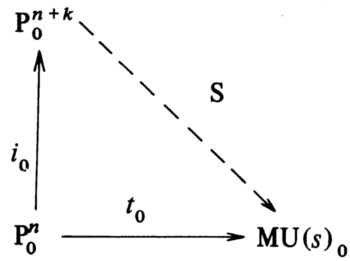
$$|[(z_0^{I_0})^*, \ldots, [(z_0^{I_{i-1}})^*, (z_0^{I_i})^*] \ldots]|$$
$$= 2s(i+1) + 2(|I_0| + \ldots + |I_i|) - i.$$

If $i > 1$ and the iterated bracket above is nonzero we must have :

$\Sigma\, |I_j| \geqslant i + 2$, which proves claim ii).

The next lemma shows that Definition 1.4 is of a rational homotopic nature.

3.4. Lemma. – *If there exists a map* S *making the following diagram homotopy-commutative :*



t being the Thom map of V, *then some nonzero multiple of* V *is a linear section in* P^{n+k}.

Proof. – We first make some general remarks related to Definition 1.4. A complex normal V of complex codimension s in P^n is itself (L-equivalent to) a transverse intersection with W embedded in P^{n+k} as in Definition 1.4 iff the Thom map $t(V)$ is extendible over P^{n+k} ([14]). Since : $t(f^{-1}V) = t(V)\circ f$, for f transverse to V, the property stated in our definition is equivalent to the fact that : there is f with $\deg(f) \neq 0$ and such that $t(V)\circ f$ extends to P^{n+k}.

Denote by 1_n, 1_{n+k} and 1_M the localization maps for P^n, P^{n+k} and $MU(s)$. Given S, standard obstruction theory shows the existence of a self-map of P^{n+k}, F′, with : $\deg(F') > 0$, and of a map : $T' : P^{n+k} \longrightarrow MU(s)$ whose localization is S, with respect to the localization maps $1_{n+k}F'$ and 1_M. Denoting by f' the restriction of F′ to P^n, an easy localization argument gives that : $(T'i)_0 = (tf')_0$ (with respect to 1_n and 1_M), therefore, by obstruction theory again, one finds a self-map of P^n, f'', with : $\deg(f'') > 0$ such that : $T'\,if'' \simeq tf'f''$. If F'' is a self-map of P^{n+k} such that : $\deg(F'') = \deg(f'')$, one may take : $f = f'f''$ and notice that $T'F''$ extends tf over P^{n+k}.

3.5. *Proof of Theorem II.* – If some nonzero multiple of V is a linear section in P^{n+k}, we just saw that there is a self-map of P^n, with : $\deg(f) = D \neq 0$, and a map : $P^{n+k} \xrightarrow{T} MU(s)$ extending tf. Passing to cohomology, Proposition 3.1 shows that the relations (R_k) are satisfied by tf. An easy computation with definitions (*) and (**) of Section 1 shows that :

$$R_{KL}(tf) = D^{2s+|K+L|} \cdot R_{KL}(t)$$

for any K and L satisfying : $|K + L| \leqslant n - s$, hence the relations (R_k) also hold for t.

For the converse implication, let us notice first that, as soon as relations (R_k) are satisfied, one may find an algebra map F making the diagram below commutative :

$$\begin{array}{ccc} H^*P_0^{n+k} & & \\ {\scriptstyle i_0^*}\downarrow & \nwarrow^{F} & \\ H^*P_0^n & \xleftarrow{t_0^*} & H^*MU(s)_0 \end{array}$$

(without any other assumptions). If t is formal, we may use the previous lemma and take S to be the formal map with : $S^* = F$ (i being formal too).

If $n \leqslant 3s + 3$, it will be enough (by Lemmas 3.3 and 3.4) to construct a d.g.a. map : $\hat{S} : (\Lambda Z^{\leqslant 2(n+k)}, d) \longrightarrow (\Lambda_2(x), 0)$ such that : $\hat{S}|(\Lambda Z^{\leqslant 2n}, d) = \hat{t}$ (the accent indicates the passage to the minimal models). The assumption : $n \leqslant 3s + 3$ gives, taking into account Lemma 3.3 ii) : $\Lambda Z^{\leqslant 2(n+k)} = \Lambda Z_{\leqslant 2}^{\leqslant 2(n+k)}$.

For any $I \in N^{s-1}$ such that : $|I| \leqslant n - s + k$, the algebra map F defines an integer F_I by the equality : $F[z_0^I] = F_I \cdot [x]^{s+|I|}$. Writing, for $|I + J| \leqslant n - 2s + k$, that :

$$F[z_0^I] \cdot F[z_0^J] = F[z_0^0] \cdot F[z_0^{I+J}]$$

one sees that : $F_I F_J - F_0 F_{I+J} = 0$, for any such I and J. The restriction of $\hat{t}$ to $Z_2^{\leqslant 2n}$ is given by a linear function : $C : Z_2^{\leqslant 2n} \longrightarrow Q$, namely : $\hat{t}(z) = C(z) \cdot x^{1/2|z|}$, for $z \in Z_2^{\leqslant 2n}$.

Define then $\hat{S}$ by : $\hat{S}(z_0^I) = F_I \cdot x^{s+|I|}$, for $|I| \leqslant n - s + k$; $\hat{S}|Z_1^{\leqslant 2(n+k)} = 0$ and, for $z \in Z_2^{\leqslant 2(n+k)}$: $\hat{S}(z) = C(z) \cdot x^{1/2|z|}$, if $|z| \leqslant 2n$, and zero otherwise.

Since Z_0 is evenly graded and d is homogenous of degree -1 with respect to the lower graduation [6], one immediately sees that Z_2 is evenly graded too, while Z_1 is oddly graded. Recalling that : $i_0^* F = t_0^*$, one infers that : $\hat{S}|(\Lambda Z^{\leqslant 2n}, d) = \hat{t}$.

It remains to show that $\hat{S}$ is a d.g.a. map indeed, i.e. that : $\hat{S}d|Z^{\leq 2(n+k)} = 0$, which is obvious for Z_0 and Z_2. But one knows ([9]) that : $Z_1 = \text{span}\,\{z_1^{IJ} \mid I, J \in N^{s-1} ; 0 < I \leq J\}$ and : $dz_1^{IJ} = r_{IJ}$, given by (*) in the proof of Proposition 3.1 i) ; then one may use : $F_I F_J - F_0 F_{I+J} = 0, |I + J| \leq n - 2s + k$.

If in addition V is algebraic then, tracing through the proof of Lemma 3.4, one has only to replace f by a regular algebraic map transverse to V of the same degree.

There is a certain overlap between the two additional conditions imposed in Theorem II ii), as shown by the following

3.6. PROPOSITION. – *If $n \leq 3s + 2$, then any map : $t : P^n \longrightarrow MU(s)$ is formal.*

Proof. – It suffices, by algebraic obstruction theory, to show the commutativity of the diagram (see also Lemma 3.3 i)) :

$$\begin{array}{ccc} (\Lambda Z^{\leq 2n}, d) & \xrightarrow{\hat{t}} & (\Lambda_2(x), 0) \\ \downarrow & & \downarrow \\ H^*MU(s) & \xrightarrow{\hat{t}^*} & H^*P^n \end{array}$$

where the vertical arrows are the restrictions of the modelling maps of [6], hence : $z \in Z_0$ goes to $[z]$, x goes to $[x]$, while $Z_{>0}$ goes to zero. But this is immediate, because $n \leq 3s + 2$ implies, via Lemma 3.3 ii), that : $(\Lambda Z^{\leq 2n}, d) = (\Lambda Z_{\leq 1}^{\leq 2n}, d)$ and Z_1 is oddly graded.

3.7. *Remark.* – Nevertheless, this overlap is far from being total, see Example 4.1 of the next Section.

The next proposition indicates the connection with the complex topological smoothing problem for conical singularities.

3.8. PROPOSITION. – *Suppose $k = 1$. Then :*

i) *Conditions (R_1) are equivalent to : $\pi^{2n+1}\hat{t}|Z_1^{2n+1} = 0$ ($\pi^*\hat{t}$ denoting the map induced by the Thom map t of V between the indecomposables of the minimal models).*

ii) *If $n \leq 4s + 2$, then conditions (R_1) are equivalent to :*

$\pi_{2n+1} t \otimes \mathbf{Q} = 0$ *($\pi_* t \otimes \mathbf{Q}$ being the map induced between rational homotopy groups).*

iii) *If* V *is algebraic, then conditions* (R_1) *are equivalent to the complex topological smoothability of the cone* CV *up to a nonzero factor or, still equivalently, to the existence of a regular algebraic map* $P^n \xrightarrow{f} P^n$, *transverse to* V *and such that the cone on* $f^{-1}V$ *is complex topologically smoothable.*

Proof. – i) Let :

$$\hat{t} : (\Lambda Z, d) \longrightarrow (\Lambda_2(x), 0) \underset{d}{\otimes} \Lambda_{2n+1}(y)\,(dy = x^{n+1})$$

be the minimal model. If $I \in N^{s-1}$ and $|I| \leqslant n - s$, we can easily see that we must have : $\hat{t}(z_0^I) = D_I \cdot x^{s+|I|}$, where D_I is defined by (*) of Section 1. We also have :

$$Z_1^{2n+1} = \text{span}\,\{z_1^{IJ}\,|I, J \in N^{s-1} ; 0 < I \leqslant J ; |I + J| = n - 2s + 1\}$$

with : $dz_1^{IJ} = z_0^I z_0^J - z_0^0 z_0^{I+J}$. For any such I and J we have an equality : $\hat{t}(z_1^{IJ}) = R_{IJ} \cdot y$, due to the fact that $\hat{t}$ is a d.g.a. map. It follows that the condition : $\pi^{2n+1}\hat{t}|Z_1^{2n+1} = 0$ is equivalent to : $R_{IJ} = 0$, any $I, J \in N^{s-1}$ such that : $0 < I \leqslant J$ and $|I + J| = n - 2s + 1$. Recalling Proposition 3.1 ii) and iii), a little analysis shows that this is in turn equivalent to the whole set of relations (R_1).

ii) Since $n \leqslant 4s + 2$, Lemma 3.3 ii) shows that : $Z^{2n+1} = Z_{\leqslant 2}^{2n+1}$, which implies : $Z^{2n+1} = Z_1^{2n+1}$, Z_0 and Z_2 being evenly graded. The result follows now from the previous one, by Sullivan duality.

iii) If V is algebraic, the results of [11] imply that : $\pi^{2n+1}\hat{t}\,|\,Z_{\neq 1}^{2n+1} = 0$, hence conditions (R_1) are equivalent, as above, to : $\pi_{2n+1} t \otimes \mathbf{Q} = 0$. Denoting by : $h : S^{2n+1} \longrightarrow P^n C$ the Hopf map, it is not difficult to see that : $\Theta_C(CV) = [th]$ as elements in $\pi_{2n+1}MU(s)$. By Lemma 2.1, $\pi_{2n+1} t \otimes \mathbf{Q} = 0$ is equivalent to the complex topological smoothability of CV up to a nonzero factor. The equivalent projective version of this property may be immediately deduced, once one notices that : if f is a regular algebraic self-map of $P^n C$ transverse to V, then : $\Theta_C(Cf^{-1}V) = \deg(f)^{n+1} \cdot \Theta_C(CV)$.

3.9. *Remark.* – The example of $P^1 \times P^4$ Segre embedded in P^{11} quoted in Remark 2.3 shows that the conditions of Thom, though being necessary for the complex topological smoothability of the cone, are not in general sufficient.

4. Examples.

The codimension one case is quite uninteresting : every map : $P^n \longrightarrow MU(1)$ is formal, every algebraic hypersurface is a (genuine) linear section.

4.1. *Example.* – On the contrary, for any $s > 1$, there are non-formal maps : $P^{3s+3} \longrightarrow MU(s)$.

We claim that it will be enough to construct a d.g.a. map : $\hat{f} : (\Lambda Z^{<6s+6}, d) \longrightarrow (\Lambda_2(x), 0)$ with the properties : $\hat{f} \mid Z_0^{<6s+6} = 0$ and : $\hat{f} \mid Z_2^{6s+6} \neq 0$.

Indeed, having such an $\hat{f}$ at hand, we may use Lemma 3.3 i) and then obstruction theory as in the proof of Lemma 3.4 in order to find a nonzero integer D and a map : $t : P^{3s+3} \longrightarrow MU(s)$ such that : $\hat{t} \mid (\Lambda Z^{<6s+6}, d) = D^* \cdot \hat{f}$. Since : $\hat{t}^* = 0$, the formality of t would imply that $\hat{t}$ is nullhomotopic. Supposing this, we shall derive a contradiction.

Let : $H : (\Lambda Z^{<6s+6}, d) \longrightarrow (\Lambda_2(x), 0) \otimes (u, du)$ be a homotopy between the trivial d.g.a. map and $\hat{t}$. For any $z \in Z_0^{<6s+6}$ we must have : $H(z) = p_z(u) \cdot x^{1/2|z|}$. H being a d.g.a. map the polynomial $p_z(u)$ must be constant. Moreover, since : $H(z) \mid (u = 0) = 0$, we conclude that : $H \mid Z_0^{<6s+6} = 0$.

If $z \in Z_1^{<6s+6}$ then : $H(z) = q_z(u) \cdot x^{1/2(|z|-1)} du$,

and if $z \in Z_2^{<6s+6}$, then : $H(z) = r_z(u) \cdot x^{1/2|z|}$.

We have :

$$\hat{t}(z) = H(z) \Big|_{u=0}^{u=1} = \int_0^1 H(dz), \quad \text{for any } z \in Z_2^{<6s+6},$$

which is zero since : $dZ_2 \subset$ ideal (Z_0) for any bigraded model.
zero since : $dZ_2 \subset$ ideal (Z_0) for any bigraded model.

By Lemma 3.3 ii) : $(\Lambda Z^{<6s+6}, d) = (\Lambda Z_{<2}^{<6s+6}, d)$ and : $Z_2^{<6s+6} = Z_2^{6s+6}$. Showing that, for any $s > 1$, $Z_2^{6s+6} \neq 0$, we are done, since we may then take : $\hat{f} \mid Z_{<1}^{<6s+6} = 0$ and : $\hat{f} \mid Z_2^{6s+6} \neq 0$

arbitrary. Consider the rational homotopy element (see the proof of Lemma 3.3 ii)) : $[(z_0^{\mathrm{I}})^*, [(z_0^{\mathrm{I}})^*, (z_0^{\mathrm{J}})^*]]$ with $\mathrm{I}, \mathrm{J} \in \mathrm{N}^{s-1}$ given by : $\mathrm{I} = (1, 0, \ldots, 0)$ and $\mathrm{J} = (2, 0, \ldots, 0)$.

It is nonzero and Sullivan dual to Z_2^{6s+6}. Our proof is complete.

4.2. *Example.* – We are going to show that the range : $n \leqslant 4s + 2$ in Proposition 3.8 ii) is sharp, by constructing a map $\mathrm{T} : \mathrm{P}^{11} \xrightarrow{\mathrm{T}} \mathrm{MU}(2)$ which satisfies relations (R_1) and has the property : $\pi_{23} \mathrm{T} \otimes \mathrm{Q} \neq 0$. (Note that, by Proposition 3.8 i), if $\pi_{2n+1} \mathrm{T} \otimes \mathrm{Q} = 0$, then relations (R_1) hold, for any n).

Suppose that we have constructed :

$$\hat{t} : (\Lambda \mathrm{Z}^{\leqslant 22}, d) \longrightarrow (\Lambda_2(a), 0)$$

with the properties :

$$(*) \quad \hat{t}d \,|\, \mathrm{Z}_1^{23} = 0$$

$$(**) \quad \hat{t}d \,|\, \mathrm{Z}^{23} \neq 0 .$$

As in the previous example, we may find a map T whose minimal model $\hat{\mathrm{T}}$ still satisfies (*) and (**). (*) shows that : $\pi^{23} \hat{\mathrm{T}} | \mathrm{Z}_1^{23} = 0$ while (**) shows that : $\pi^{23} \hat{\mathrm{T}} \neq 0$.

Using [9] we may completely describe $(\Lambda \mathrm{Z}_{\leqslant 1}, d)$:

$$\mathrm{Z}_0 = \mathrm{span}\{z_k \,|\, k \geqslant 0 ; |z_k| = 2(k+2) \text{ and } dz_k = 0\}$$

$$\mathrm{Z}_1 = \mathrm{span}\{x_{pq} \,|\, 0 < p \leqslant q ; dx_{pq} = r_{pq} = z_p z_q - z_0 z_{p+q}\} .$$

Lemma 3.3 ii) shows that : $\mathrm{Z}^{\leqslant 23} = \mathrm{Z}_{\leqslant 1}^{\leqslant 23} \oplus \mathrm{Z}_2^{\leqslant 23} \oplus \mathrm{Z}_3^{23}$. By direct computation : $\mathrm{Z}_2^{\leqslant 23} = \mathrm{span}\{u_1, \ldots, u_6\}$ and $\mathrm{Z}_3^{23} = \mathrm{span}\{v\}$,

$$du_1 = z_0(x_{13} - x_{22}) + z_1 x_{12} - z_2 x_{11}$$

$$du_2 = z_0(x_{14} - x_{23}) + z_1 x_{13} - z_3 x_{11}$$

$$du_3 = z_0(x_{14} - x_{23}) + z_1 x_{22} - z_2 x_{12}$$

$$dv = z_0(u_2 - u_3) - z_1 u_1 + x_{11} \Lambda x_{12} .$$

Pick any nonzero integers t_1 and t_2 ; defining $\hat{t}(z_k) = t_1^k t_2 \cdot a^{k+2}$ for $0 \leqslant k \leqslant 9$, $\hat{t} | \mathrm{Z}_1^{\leqslant 22} = 0$, $\hat{t}(u_1) = a^9$ and $\hat{t}(u_i) = 0$ for $1 < i \leqslant 6$, the d.g.a. map $\hat{t}$ is easily seen to have all the required properties.

The last example shows that the condition : $n \leqslant 3s + 3$ in Theorem II ii) is the best one in general.

4.3. *Example.* – There is a map : $T : P^{10} \longrightarrow MU(2)$ for which conditions (R_2) are satisfied but no nonzero multiple of the associated complex normal submanifold V is a linear section in P^{12}.

We claim that, using the d.g.a. map $\hat{t} : (\Lambda Z^{\leqslant 22}, d) \longrightarrow (\Lambda_2(a), 0)$ of the previous example, it is enough to know that :

(*) $\hat{t}d \mid Z_1^{23} = 0$

(**) for any other $\hat{t}' : (\Lambda Z^{\leqslant 22}, d) \longrightarrow (\Lambda_2(a), 0)$ such that $\hat{t}'$ and $\hat{t}$ become homotopic when restricted to $(\Lambda Z^{\leqslant 20}, d)$ we still have : $\hat{t}'d \mid Z^{23} \neq 0$.

Indeed we may find a nonzero integer D and a map T with the property : $\hat{T} \mid (\Lambda Z^{\leqslant 20}, d) = D^* \cdot \hat{t} \mid (\Lambda Z^{\leqslant 20}, d)$. Using definitions (*) and (**) of Section 1, it is easy to see that : $D_k = (Dt_1)^k (D^2 t_2)$, for any $0 \leqslant k \leqslant 8$, which shows (R_2) to be true.

Were some nonzero multiple of V a linear section in P^{12}, we would be able to find a map : $T' : P^{12} \longrightarrow MU(2)$ and a self-map f' of P^{10} with $\deg(f') = D' \neq 0$ such that : $T' | P^{10} \simeq Tf'$. But then we may take : $\hat{t}' = \left(\frac{1}{DD'}\right)^* \hat{T}' \mid (\Lambda Z^{\leqslant 22}, d)$ and see that : $\hat{t}' \simeq \hat{t}$ on $(\Lambda Z^{\leqslant 20}, d)$, but : $\hat{t}'d \mid Z^{23} = 0$, since $\hat{t}'$ extends to $(\Lambda Z^{\leqslant 24}, d)$.

In order to verify the property (**), start with a homotopy : $H : (\Lambda Z^{\leqslant 20}_{\leqslant 2}, d) \longrightarrow (\Lambda_2(a), 0) \otimes (u, du)$ between the restrictions of $\hat{t}$ and $\hat{t}'$. Using the same method as in Example 4.1, one sees that : $H(z_k) = \hat{t}'(z_k) = \hat{t}(z_k)$, for any $0 \leqslant k \leqslant 8$ and that we have : coefficient of

$$\{t_1[\hat{t}'(u_1) - \hat{t}(u_1)] - [\hat{t}'(u_2) - \hat{t}(u_2)] + [\hat{t}'(u_3) - \hat{t}(u_3)]\} = 0.$$

Putting : $\hat{t}'(u_i) = U_i' \cdot a^{1/2|u_i|}$, for $i = 1, 2, 3$, we may compute :

$$\hat{t}'dv = [t_2(U_2' - U_3') - t_1 t_2 U_1'] a^{12} = -t_1 t_2 a^{12}$$

by the previous equality.

4.4. *Remarks.* – Using for example the result of [11] quoted in Remark 3.2 it is easy to produce formal maps $t : P^n \longrightarrow MU(s)$ which do not come from algebraic varieties. Nevertheless it would be interesting to know if the Thom maps of smooth projective varieties are always formal (in connection with Theorem II ii) for instance). It can be shown that this is indeed the case for complete intersections (and, of course, in the range : $n \leqslant 3s + 2$, by Proposition 3.6).

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Manuscrit reçu le 3 juin 1983.

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