Annales Henri Lebesgue
4 (2021) 283-324

ANNALES
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MARKOV PARTITIONS FOR TORAL $\mathbb{Z}^2$-ROTATIONS FEATURING JEANDEL–RAO WANG SHIFT AND MODEL SETS

PARTITIONS DE MARKOV POUR $\mathbb{Z}^2$-ROTATIONS FAISANT INTERVENIR LE SOUS-DÉCALAGE DE JEANDEL–RAO ET LES ENSEMBLES MODÈLES

Abstract. — We define a partition $\mathcal{P}_0$ and a $\mathbb{Z}^2$-rotation ($\mathbb{Z}^2$-action defined by rotations) on a 2-dimensional torus whose associated symbolic dynamical system is a minimal proper subshift of the Jeandel–Rao aperiodic Wang shift defined by 11 Wang tiles. We define another partition $\mathcal{P}_\mathcal{U}$ and a $\mathbb{Z}^2$-rotation on $\mathbb{T}^2$ whose associated symbolic dynamical system is equal to a minimal and aperiodic Wang shift defined by 19 Wang tiles. This proves that $\mathcal{P}_\mathcal{U}$ is a Markov partition for the $\mathbb{Z}^2$-rotation on $\mathbb{T}^2$. We prove in both cases that the toral $\mathbb{Z}^2$-rotation is the maximal equicontinuous factor of the minimal subshifts and that the set of fiber cardinalities of the factor map is $\{1, 2, 8\}$. The two minimal subshifts are uniquely ergodic and are isomorphic as measure-preserving dynamical systems to the toral $\mathbb{Z}^2$-rotations. It provides a construction

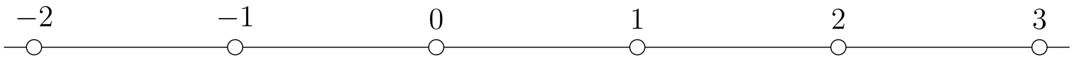
Keywords: Wang tilings, aperiodic, rotation, Markov partition, cut and project.
2020 *Mathematics Subject Classification:* 37B50, 52C23, 28D05, 37B05.
DOI: `https://doi.org/10.5802/ahl.73`
(*) The author acknowledges financial support from the Laboratoire International Franco-Québécois de Recherche en Combinatoire (LIRCO), the Agence Nationale de la Recherche through the project CODYS (ANR-18-CE40-0007) and the Horizon 2020 European Research Infrastructure project OpenDreamKit (676541).

of these Wang shifts as model sets of 4-to-2 cut and project schemes. A do-it-yourself puzzle is available in the appendix to illustrate the results.

Résumé. — Nous définissons une partition $\mathcal{P}_0$ et une $\mathbb{Z}^2$-rotation ($\mathbb{Z}^2$-action définie par des rotations)sur un tore 2-dimensionnel dont le système dynamique symbolique associé est un sous-décalage propre et minimal du sous-décalage apériodique de Jeandel–Rao décrit par un ensemble de 11 tuiles de Wang. Nous définissons une autre partition $\mathcal{P}_\mathcal{U}$ et une $\mathbb{Z}^2$-rotation sur $\mathbb{T}^2$ dont le système dynamique symbolique associé est égal au sous-décalage minimal et apériodique défini par un ensemble de 19 tuiles de Wang. On montre que $\mathcal{P}_\mathcal{U}$ est une partition de Markov pour la $\mathbb{Z}^2$-rotation sur $\mathbb{T}^2$. Nous prouvons dans les deux cas que la $\mathbb{Z}^2$-rotation sur le tore est le facteur équicontinu maximal des sous-décalages minimaux et que l'ensemble des cardinalités des fibres du facteur est $\{1, 2, 8\}$. Les deux sous-décalages minimaux sont uniquement ergodiques et sont isomorphes en tant que systèmes dynamiques mesurés à la $\mathbb{Z}^2$-rotation sur le tore. Les résultats fournissent une construction de ces sous-décalages de Wang en tant qu'ensembles modèles par la méthode de coupe et projection 4 sur 2. Un puzzle à faire soi-même est disponible en annexe pour illustrer les résultats.

1. Introduction

We build a biinfinite necklace by placing beads at integer positions on the real line:

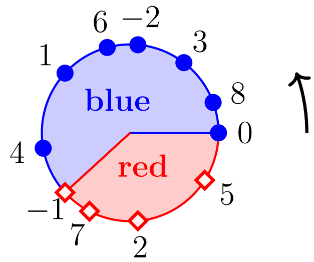


Beads come in two colors: **light red** ◇ and **dark blue** ●. Given $\alpha > 0$, we would like to place the colored beads in such a way that the relative frequency

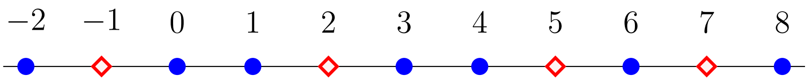
$$\frac{\text{number of blue beads in } \{-n, -n+1, \ldots, n\}}{\text{number of red beads in } \{-n, -n+1, \ldots, n\}}$$

converges to α as n goes to infinity.

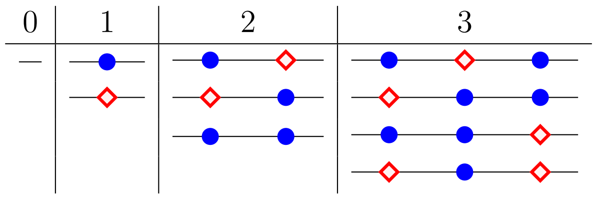
A well-known approach is to use coding of rotations on a circle of circumference $\alpha + 1$ whose radius is $\frac{1}{2\pi}(\alpha + 1)$. The coding is given by the partition of the circle $\mathbb{R}/(\alpha + 1)\mathbb{Z}$ into one arc of length α associated with dark blue beads and another arc of length 1 associated with light red beads. The two end points of the arcs are associated with red and blue beads respectively in one way or the other. Then, we wrap the biinfinite necklace around the circle and each bead is given the color according to which of the two arcs it falls in. For example, when $\alpha = \frac{1+\sqrt{5}}{2}$ and if the zero position is assigned to one of the end points, we get the picture below:



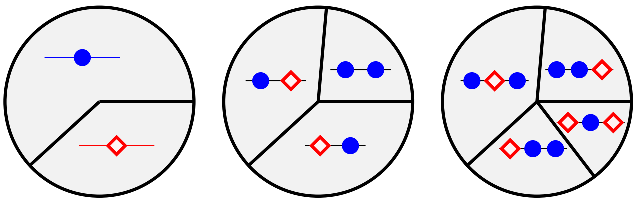
Then, we unwrap the biinfinite necklace and we get an assignment of colored beads to each integer position such that the relative frequency between blue and red beads is α. Here is what we get after zooming out a little:



We observe that this colored necklace has very few distinct patterns. The patterns of size 0, 1, 2 and 3 that we see in the necklace are shown in the table below:



We do not get other patterns of size 1, 2 or 3 in the whole biinfinite necklace since every pattern is uniquely determined by the position of its first bead on the circle. For each $n \in \mathbb{N}$ there exists a partition of the circle according to the pattern associated with the position of its first bead:



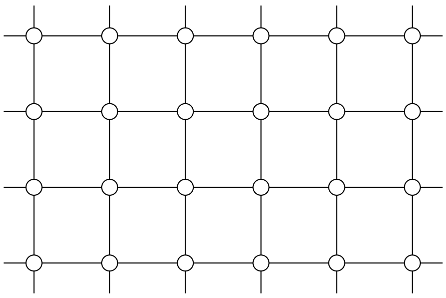
When α is irrational, one can prove that the partition of the circle for patterns of size n is made of $n+1$ parts. The proof follows from the fact that the distance between two consecutive beads on the necklace is equal to the length of one of the original arc (here, the red arc of length 1). So the partition at a given level is obtained from the previous one by adding exactly one separation which increases the number of patterns by 1. This shows that the colored necklace is a Sturmian sequence, that is, a sequence whose pattern complexity is $n+1$, see [Lot02]. When $\alpha = \frac{1+\sqrt{5}}{2}$, this is a construction of the biinfinite Fibonacci word [Ber80]. Note that it is known that sequences having strictly less than $n+1$ patterns of length n, for some $n \in \mathbb{N}$, are eventually periodic [MH38]. Therefore, Sturmian sequences are the simplest aperiodic sequences in terms of pattern (or factor) complexity [CN10].

What Coven and Hedlund proved in [CH73] based on the initial work of Morse and Hedlund [MH40] on Sturmian sequences dating from 1940 is that a biinfinite sequence is Sturmian if and only if it is the coding of an irrational rotation. Proving that the coding of an irrational rotation is a Sturmian sequence is the easy part and corresponds to what we did above. The difficult part is to prove that a Sturmian sequence can be obtained as the coding of an irrational rotation for some starting point. The proof is explained nowadays in terms of S-adic development of Sturmian sequences, Rauzy induction of circle rotations, the continued fraction expansion of real numbers and the Ostrowski numeration system [Fog02]. Rauzy discovered that the connection between Sturmian sequences and rotations can be generalized to sequences using three symbols [Rau82] involving a rotation on a 2-dimensional torus

$\mathbb{T}^2$. This result was extended recently for almost all rotations on $\mathbb{T}^2$ [BST19], see also [Thu19].

1.1. From biinfinite necklaces to 2-dimensional configurations

In this work, we want to extend the behavior of Sturmian sequences beyond the 1-dimensional case by considering d-dimensional configurations. We say that a *configuration* is an assignment of colored beads from a finite set $\mathcal{A}$ to every coordinate of the lattice $\mathbb{Z}^d$. Are there rules describing how to place colored beads in a configuration in such a way that it encodes rotations on a higher dimensional torus?



This is related to a question of Adler: "*how and to what extent can a dynamical system be represented by a symbolic one*" [Adl98]. The kind of dynamical system we consider are toral $\mathbb{Z}^d$-rotations, that is, $\mathbb{Z}^d$-actions by rotations on a torus. When $d = 1$, the answer is given in terms of Sturmian sequences and factor complexity. While Berthé and Vuillon [BV00] considered the coding of $\mathbb{Z}^2$-rotations on the 1-dimensional torus, we consider $\mathbb{Z}^d$-rotations on the d-dimensional torus. We show that an answer to the question when $d = 2$ can be made in terms of sets of configurations avoiding a finite set of forbidden patterns known as *subshifts of finite type* and more precisely in terms of aperiodic tilings by Wang tiles. This contrasts with the one-dimensional case, since Sturmian sequences can not be described by a finite set of forbidden patterns (a one-dimensional shift of finite type is nonempty if and only if it has a periodic point [LM95, Section 13.10]).

1.2. Jeandel-Rao's aperiodic set of 11 Wang tiles

The study of aperiodic order [BG13, GS87] gained a lot of interest since the discovery in 1982 of quasicrystals by Shechtman [SBGC84] for which he was awarded the Nobel Prize in Chemistry in 2011. The first known aperiodic structure was based on the notion of Wang tiles. *Wang tiles* can be represented as unit square with colored edges, see Figure 1.1.

Given a finite set of Wang tiles $\mathcal{T}$, we consider *tilings* of the Euclidean plane using arbitrarily many translated (but not rotated) copies of the tiles in $\mathcal{T}$. Tiles are placed on the integer lattice points of the plane with their edges oriented horizontally and vertically. The tiling is *valid* if every pair of contiguous edges have the same

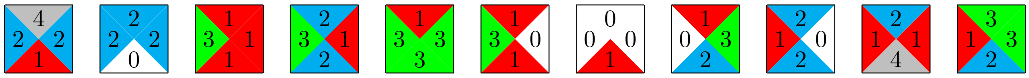


Figure 1.1. The aperiodic set $\mathcal{T}_0$ of 11 Wang tiles discovered by Jeandel and Rao in 2015 [JR15].

color. Deciding if a set of Wang tiles admits a valid tiling of the plane is a difficult question known as the *domino problem*. Answering a question of Wang [Wan61], Berger proved that the domino problem is undecidable [Ber66] using a reduction to the halting problem of Turing machines. As noticed by Wang, if every set of Wang tiles that admits a valid tiling of the plane would also admit a periodic tiling, then the domino problem would be decidable. As a consequence, there exist aperiodic sets of Wang tiles. A set $\mathcal{T}$ of Wang tiles is called *aperiodic* if there exists a valid tiling of the plane with the tiles from $\mathcal{T}$ and none of the valid tilings of the plane with the tiles from $\mathcal{T}$ is invariant under a nonzero translation.

Berger constructed an aperiodic set made of 20426 Wang tiles [Ber66], later reduced to 104 by himself [Ber65] and further reduced by others [Knu68, Rob71]. Small aperiodic sets of Wang tiles include Ammann's 16 tiles [GS87, p. 595], Kari's 14 tiles [Kar96] and Culik's 13 tiles [Cul96]. The search for the smallest aperiodic set of Wang tiles continued until Jeandel and Rao proved the existence of an aperiodic set $\mathcal{T}_0$ of 11 Wang tiles and that no set of Wang tiles of cardinality $\leqslant 10$ is aperiodic [JR15]. Thus their set, shown in Figure 1.1, is a smallest possible set of aperiodic Wang tiles. An equivalent geometric representation of their set of 11 tiles is shown in Figure 1.2.

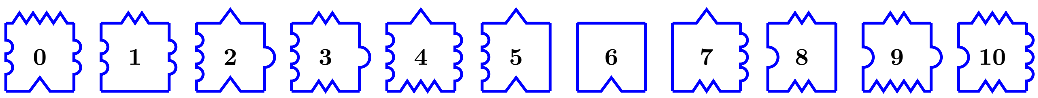


Figure 1.2. Jeandel-Rao tiles can be encoded into a set of equivalent geometrical shapes in the sense that every tiling using Jeandel-Rao tiles can be transformed into a unique tiling with the corresponding geometrical shapes and vice versa.

The aperiodicity of the Jeandel–Rao set of 11 Wang tiles follows from the decomposition of tilings as horizontal strips of height 4 or 5. Using the representation of Wang tiles by transducers, Jeandel and Rao proved that the language of sequences describing the heights of consecutive horizontal strips in the decomposition is exactly the language of the Fibonacci word on the alphabet $\{4, 5\}$. Thus it contains the same patterns as in the necklace we constructed above where 5 corresponds to the dark blue bead ● and 4 corresponds to the light red bead ◇. This proves the absence of any vertical period in every tiling with Jeandel–Rao tiles. This is enough to conclude aperiodicity in all directions, see [BG13, Proposition 5.9]. The presence of the Fibonacci word in the vertical structure of Jeandel–Rao tilings is a first hint that Jeandel–Rao tilings are related to irrational rotations on a torus.

1.3. Results

In this article, we consider Wang tilings from the point of view of symbolic dynamics [Rob04]. While a tiling by a set of Wang tiles $\mathcal{T}$ is a tiling of the plane $\mathbb{R}^2$ whose validity is preserved by translations of $\mathbb{R}^2$ (leading to the notion of *hull*, see [BG13]), we prefer to consider maps $\mathbb{Z}^2 \to \mathcal{T}$, that we call *configurations*, whose validity is preserved by translations of $\mathbb{Z}^2$. The set $\Omega_{\mathcal{T}}$ of all valid configurations $\mathbb{Z}^2 \to \mathcal{T}$ is called a *Wang shift* as it is closed under the shift σ by integer translates. The passage from Wang shifts ($\mathbb{Z}^2$-actions) to Wang tiling dynamical systems ($\mathbb{R}^2$-action) can be made with the 2-dimensional suspension of the former as in the classical construction of a "flow under a function" in Ergodic Theory, see [Rob96].

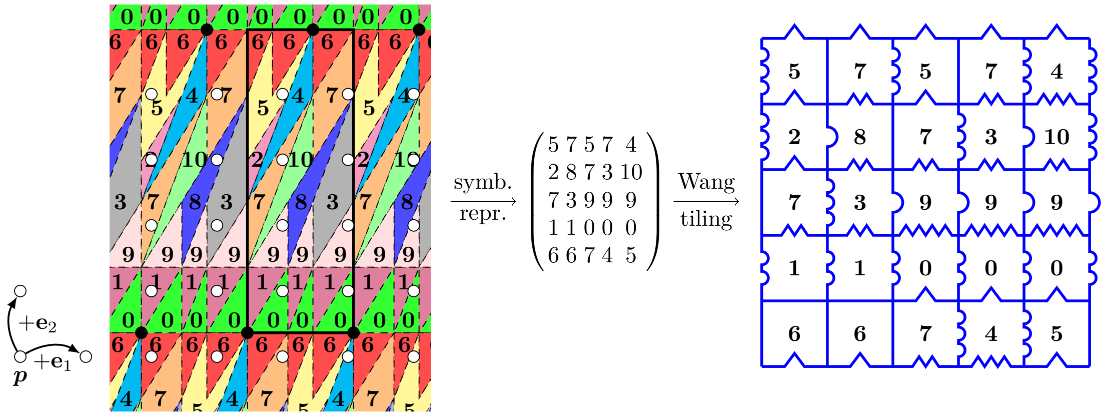


Figure 1.3. On the left, we illustrate the lattice $\Gamma_0 = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$, with black vertices, a rectangular fundamental domain of the flat torus $\mathbb{R}^2/\Gamma_0$ with a black contour and a polygonal partition $\mathcal{P}_0$ of $\mathbb{R}^2/\Gamma_0$ with indices in the set $\{0, 1, \dots, 10\}$. We show that for every starting point $\mathbf{p} \in \mathbb{R}^2$, the coding of the shifted lattice $\mathbf{p} + \mathbb{Z}^2$ under the polygonal partition yields a configuration $w : \mathbb{Z}^2 \to \{0, 1, \dots, 10\}$ which is a symbolic representation of $\mathbf{p}$. The configuration w corresponds to a valid tiling of the plane with Jeandel–Rao's set of 11 Wang tiles.

We may now state the main results of this article together with an illustration. A partition $\mathcal{P}_0$ of the plane into well-chosen polygons indexed by integers from the set $\{0, 1, 2, \dots, 10\}$ is shown in Figure 1.3 (left). The partition $\mathcal{P}_0$ is invariant under the group of translations $\Gamma_0 = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$ where $\varphi = \frac{1+\sqrt{5}}{2}$. Equivalently, it is a partition of the torus $\mathbb{R}^2/\Gamma_0$ given by a partition of the rectangular fundamental domain $[0, \varphi) \times [0, \varphi + 3)$. On the torus $\mathbb{R}^2/\Gamma_0$, we consider the continuous $\mathbb{Z}^2$-action defined by $R_0^{\mathbf{n}}(\mathbf{x}) := R_0(\mathbf{n}, \mathbf{x}) = \mathbf{x} + \mathbf{n}$ for every $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ which defines a dynamical system that we denote $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$. The symbolic dynamical system $\mathcal{X}_{\mathcal{P}_0,R_0}$ corresponding to $\mathcal{P}_0, R_0$ is the topological closure of the set of all configurations $w \in \{0, 1, \dots, 10\}^{\mathbb{Z}^2}$ obtained from the coding by the partition $\mathcal{P}_0$ of the orbit of some starting point in $\mathbb{R}^2/\Gamma_0$ by the $\mathbb{Z}^2$-action of R_0 (see Lemma 4.1). We say that $\mathcal{X}_{\mathcal{P}_0,R_0}$ is a *subshift* as it is also closed under the shift σ by integer translations.

We state the first theorem below. The fact that $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$ is illustrated in Figure 1.3 where $\Omega_0 \subset \{0,1,\dots,10\}^{\mathbb{Z}^2}$ is the Jeandel–Rao Wang shift. The definitions of the terms used in the theorem can be found in Section 2 and Section 3.

Theorem 1.1. — *The Jeandel–Rao Wang shift Ω_0 has the following properties:*
- (i) $\mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$ *is a proper minimal and aperiodic subshift of* Ω_0,
- (ii) *the partition* $\mathcal{P}_0$ *gives a symbolic representation of* $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$,
- (iii) *the dynamical system* $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$ *is the maximal equicontinuous factor of* $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma)$,
- (iv) *the set of fiber cardinalities of the factor map* $\mathcal{X}_{\mathcal{P}_0,R_0} \to \mathbb{R}^2/\Gamma_0$ *is* $\{1,2,8\}$,
- (v) *the dynamical system* $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma)$ *is strictly ergodic and the measure-preserving dynamical system* $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma, \nu)$ *is isomorphic to* $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0, \lambda)$ *where* ν *is the unique shift-invariant probability measure on* $\mathcal{X}_{\mathcal{P}_0,R_0}$ *and* λ *is the Haar measure on* $\mathbb{R}^2/\Gamma_0$.

A larger picture of the partition $\mathcal{P}_0$ is illustrated in the appendix together with a DIY puzzle allowing hand made construction of configurations in $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$ as the symbolic representation of starting points in $\mathbb{R}^2/\Gamma_0$.

Theorem 1.1 corresponds to the easy direction in the proof of Morse–Hedlund's theorem, namely that codings of irrational rotations have pattern complexity $n+1$. Proving the converse, i.e., that almost every (for some shift-invariant probability measure) configuration in the Jeandel–Rao Wang shift is obtained as the coding of the shifted lattice $\mathbf{p}+\mathbb{Z}^2$ for some unique $\mathbf{p} \in \mathbb{R}^2/\Gamma_0$ is harder. This has lead to split the proof of the converse [Lab20].

Note that a similar result was obtained for Penrose tilings [Rob96, Theorem A]. In particular, it was shown that the set of fiber cardinalities for Penrose tilings (with the action of $\mathbb{R}^2$) is $\{1,2,10\}$. In [LM13], it was proved that the set of fiber cardinalities is $\{1,2,6,12\}$ for a minimal hull among Taylor–Socolar hexagonal tilings. We show in Lemma 2.2 that the set of fiber cardinalities of the maximal equicontinuous factor of a minimal dynamical system is invariant under topological conjugacy. Therefore, the Jeandel–Rao tilings, the Penrose tilings and the Taylor–Socolar tilings are inherently different.

We also provide a stronger result on another example. We define a polygonal partition $\mathcal{P}_\mathcal{U}$ of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ into 19 atoms. We consider the continuous $\mathbb{Z}^2$-action $R_\mathcal{U}$ defined on $\mathbb{T}^2$ by $R_\mathcal{U}^{\mathbf{n}}(\mathbf{x}) = \mathbf{x} + \varphi^{-2}\mathbf{n}$ for every $\mathbf{n} \in \mathbb{Z}^2$ where $\varphi = \frac{1+\sqrt{5}}{2}$. It defines a dynamical system that we denote $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$. We prove that the symbolic dynamical system $\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}}$ corresponding to $\mathcal{P}_\mathcal{U}, R_\mathcal{U}$ is *equal* to the Wang shift $\Omega_\mathcal{U}$ where $\mathcal{U}$ is the set of 19 Wang tiles introduced by the author in [Lab19a] and discovered from the study of the Jeandel–Rao Wang shift [Lab19b].

Theorem 1.2. — *The Wang shift $\Omega_\mathcal{U}$ has the following properties:*
- (i) *the subshift* $\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}}$ *is minimal, aperiodic and is equal to* $\Omega_\mathcal{U}$,
- (ii) $\mathcal{P}_\mathcal{U}$ *is a Markov partition for the dynamical system* $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$,
- (iii) $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$ *is the maximal equicontinuous factor of* $(\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma)$,
- (iv) *the set of fiber cardinalities of the factor map* $\Omega_\mathcal{U} \to \mathbb{T}^2$ *is* $\{1,2,8\}$,
- (v) *the dynamical system* $(\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma)$ *is strictly ergodic and the measure-preserving dynamical system* $(\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma, \nu)$ *is isomorphic to* $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U}, \lambda)$ *where*

ν is the unique shift-invariant probability measure on $\Omega_{\mathcal{U}}$ and λ is the Haar measure on $\mathbb{T}^2$.

Since a Wang shift is a shift of finite type, the equality $\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}} = \Omega_{\mathcal{U}}$ implies that $\mathcal{P}_{\mathcal{U}}$ is a Markov partition (see Definition 3.2) for the $\mathbb{Z}^2$-action $R_{\mathcal{U}}$. Note that Markov partitions "*remained abstract objects for a long time*" [Fog02, Section 7.1]. Explicit constructions of Markov partitions were originally given for hyperbolic automorphisms of the torus, see [Adl98, AW70]. More recent references relate Markov partitions with arithmetics [Ken99, KV98], algebraic numbers [AFHI11] and numeration systems [Pra99].

The link between aperiodic order and cut and project schemes (Definition 12.1) and model sets (Definition 12.2) is not new. In one dimension, the fact that Sturmian sequences are codings of rotations implies that they can be seen as model sets of cut and project schemes, see [BG13, BMP05]. Since the contribution of N. G. de Bruijn [Bru81], we know that Penrose tilings are obtained as the projection of discrete surfaces in a 5-dimensional space onto a 2-dimensional plane. Other typical examples include Ammann–Beenker tilings [BF13] and Taylor–Socolar aperiodic hexagonal tilings for which Lee and Moody gave a description in terms of model sets [LM13]. Likewise, a consequence of Theorem 1.1 and Theorem 1.2 is a description of the two aperiodic Wang shifts Ω_0 and $\Omega_{\mathcal{U}}$ with cut and project schemes. More precisely, we show that the occurrences of patterns in the two Wang shifts are regular model sets. Definitions of generic and singular configurations is in Section 4 and definitions of regular, generic and singular models sets can be found in Section 12.

Theorem 1.3. — *There exists a cut and project scheme such that for every Jeandel–Rao configuration $w \in \mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$, the set $Q \subseteq \mathbb{Z}^2$ of occurrences of a pattern in w is a regular model set. If w is a generic (resp. singular) configuration, then Q is a generic (resp. singular) model set.*

We prove the same result for the Wang shift $\Omega_{\mathcal{U}}$ (see Theorem 14.1). As opposed to the Kari–Culik Wang shift, for which a minimal subsystem is related to a dynamical system on p-adic numbers [Sie17], windows used for the cut and project schemes are Euclidean.

It was shown that the action of $\mathbb{R}^2$ by translation on the set of Penrose tilings is an almost one-to-one extension of a minimal $\mathbb{R}^2$-action by rotations on $\mathbb{T}^4$ [Rob96] (the fact that it is $\mathbb{T}^4$ instead of $\mathbb{T}^2$ is related to the consideration of tilings instead of shifts). This result can also be seen as a higher dimensional generalization of the Sturmian dynamical systems. Note that a shift of finite type or Wang shift can be explicitly constructed from the Penrose tiling dynamical system, as shown in [SW03]. This calls for a common point of view including Jeandel–Rao aperiodic tilings, Penrose tilings and others. For example, we do not know if Penrose tilings can be seen as a symbolic dynamical system associated to a Markov partition like it is the case for the Jeandel–Rao Wang shift. It is possible that such Markov partitions exist only for tilings associated to some algebraic numbers, see [BF20].

1.4. Structure of the article

This article is divided into three parts. In the first part, we construct symbolic representations of toral $\mathbb{Z}^2$-rotations and a factor map which provides an isomorphism between symbolic dynamical systems and toral $\mathbb{Z}^2$-rotations. In the second part, we construct sets of Wang tiles and Wang shifts as the coding of $\mathbb{Z}^2$-rotations on the 2-torus. We illustrate the method on two examples including Jeandel–Rao aperiodic Wang shift. In the third part, we express occurrences of patterns in these Wang shifts in terms of model sets of cut and project schemes. In the appendix, we propose a do-it-yourself puzzle to explain the construction of valid configurations in the Jeandel–Rao Wang shift as the coding of $\mathbb{Z}^2$-rotations on the 2-torus.

Acknowledgements

I am thankful to Jarkko Kari and Michaël Rao for their presentations at the meeting *Combinatorics on words and tilings* (CRM, Montréal, April 2017). I am grateful to Michaël Rao who provided me a text file of 986×2583 characters in the alphabet $\{\mathtt{a}, \mathtt{b}\}$ describing a rectangular pattern with Jeandel–Rao tiles. This allowed me to start working on aperiodic tilings.

I want to thank Michael Baake, Sebastián Barbieri, Vincent Delecroix, Franz Gähler, Ilya Galanov, Maik Gröger, Jeong-Yup Lee, Jean-François Marckert, Samuel Petite and Mathieu Sablik for profitable discussions during the preparation of this article which allowed me to improve my knowledge on tilings, measure theory and dynamical systems and write the results in terms of existing concepts. I am very thankful to the anonymous referees for their in-depth reading and valuable comments which lead to a great improvement of the presentation.

Part 1. Symbolic dynamics of toral $\mathbb{Z}^2$-rotations

This part is divided into 5 sections. After introducing dynamical systems and subshifts, we define the symbolic representations of toral $\mathbb{Z}^2$-rotations from a topological partition of the 2-torus. We introduce a one-to-one map from the 2-torus to symbolic representations and a factor map from symbolic representations to the 2-torus. We show that the factor map provides an isomorphism between symbolic dynamical systems and toral $\mathbb{Z}^2$-rotations.

2. Dynamical systems, maximal equicontinuous factors and subshifts

In this section, we introduce dynamical systems, maximal equicontinuous factors, set of fiber cardinalities of a factor map, subshifts and shifts of finite type. We let $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ denote the integers and $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the nonnegative integers.

2.1. Topological dynamical systems

Most of the notions introduced here can be found in [Wal82]. A *dynamical system* is a triple (X, G, T), where X is a topological space, G is a topological group and T is a continuous function $G \times X \to X$ defining a left action of G on X: if $x \in X$, e is the identity element of G and $g, h \in G$, then using additive notation for the operation in G we have $T(e, x) = x$ and $T(g + h, x) = T(g, T(h, x))$. In other words, if one denotes the transformation $x \mapsto T(g, x)$ by T^g, then $T^{g+h} = T^g T^h$. In this work, we consider the Abelian group $G = \mathbb{Z} \times \mathbb{Z}$.

If $Y \subset X$, let $\overline{Y}$ denote the topological closure of Y and let $T(Y) := \cup_{g \in G} T^g(Y)$ denote the T-closure of Y. A subset $Y \subset X$ is *T-invariant* if $T(Y) = Y$. A dynamical system (X, G, T) is called *minimal* if X does not contain any nonempty, proper, closed T-invariant subset. The left action of G on X is *free* if $g = e$ whenever there exists $x \in X$ such that $T^g(x) = x$.

Let (X, G, T) and (Y, G, S) be two dynamical systems with the same topological group G. A *homomorphism* $\theta : (X, G, T) \to (Y, G, S)$ is a continuous function $\theta : X \to Y$ satisfying the commuting property that $T^g \circ \theta = \theta \circ S^g$ for every $g \in G$. A homomorphism $\theta : (X, G, T) \to (Y, G, S)$ is called an *embedding* if it is one-to-one, a *factor map* if it is onto, and a *topological conjugacy* if it is both one-to-one and onto and its inverse map is continuous. If $\theta : (X, G, T) \to (Y, G, S)$ is a factor map, then (Y, G, S) is called a *factor* of (X, G, T) and (X, G, T) is called an *extension* of (Y, G, S). Two subshifts are *topologically conjugate* if there is a topological conjugacy between them.

The set of all T-invariant probability measures of a dynamical system (X, G, T) is denoted by $\mathcal{M}^T(X)$. An invariant probability measure on X is called *ergodic* if for every set $B \in \mathcal{B}$ such that $T^g(B) = B$ for all $g \in G$, we have that B has either zero or full measure. A dynamical system (X, G, T) is *uniquely ergodic* if it has only one invariant probability measure, i.e., $|\mathcal{M}^T(X)| = 1$. A dynamical system (X, G, T) is said *strictly ergodic* if it is uniquely ergodic and minimal.

A *measure-preserving dynamical system* is defined as a system $(X, G, T, \mu, \mathcal{B})$, where μ is a probability measure defined on the Borel σ-algebra $\mathcal{B}$ of subsets of X, and $T^g : X \to X$ is a measurable map which preserves the measure μ for all $g \in G$, that is, $\mu(T^g(B)) = \mu(B)$ for all $B \in \mathcal{B}$. The measure μ is said to be *T-invariant*. In what follows, $\mathcal{B}$ is always the Borel σ-algebra of subsets of X, so we omit $\mathcal{B}$ and write (X, G, T, μ) when it is clear from the context.

Let $(X, G, T, \mu, \mathcal{B})$ and $(X', G, T', \mu', \mathcal{B}')$ be two measure-preserving dynamical systems. We say that the two systems are *isomorphic* if there exist measurable sets $X_0 \subset X$ and $X'_0 \subset X'$ of full measure (i.e., $\mu(X \setminus X_0) = 0$ and $\mu'(X' \setminus X'_0) = 0$) with $T^g(X_0) \subset X_0$, $T'^g(X'_0) \subset X'_0$ for all $g \in G$ and there exists a map $\phi : X_0 \to X'_0$, called an *isomorphism*, that is one-to-one and onto and such that for all $A \in \mathcal{B}'(X'_0)$,

- $\phi^{-1}(A) \in \mathcal{B}(X_0)$,
- $\mu(\phi^{-1}(A)) = \mu'(A)$, and
- $\phi \circ T^g(x) = T'^g \circ \phi(x)$ for all $x \in X_0$ and $g \in G$.

The role of the set X_0 is to make precise the fact that the properties of the isomorphism need to hold only on a set of full measure.

2.2. Maximal equicontinuous factor

In this section, we provide the definition of maximal continuous factor and of related notions. We recall a sufficient condition for a factor to be the maximal equicontinuous factor and we prove a result on the set of fiber cardinalities of the maximal equicontinuous factor of a minimal dynamical system.

A metrizable dynamical system (X, G, T) is called *equicontinuous* if the family of homeomorphisms $\{T^g\}_{g\in G}$ is equicontinuous, i.e., if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathrm{dist}(T^g(x), T^g(y)) < \varepsilon$$

for all $g \in G$ and all $x, y \in X$ with $\mathrm{dist}(x, y) < \delta$. According to a well-known theorem [ABKL15, Theorem 3.2], equicontinuous minimal systems defined by the action of an Abelian group are rotations on groups.

We say that $\theta : (X, G, T) \to (Y, G, S)$ is an *equicontinuous factor* if θ is a factor map and (Y, G, S) is equicontinuous. We say that $(X_{\max}, G, T_{\max})$ is the *maximal equicontinuous factor* of (X, G, T) if there exists an equicontinuous factor $\pi_{\max} : (X, G, T) \to (X_{\max}, G, T_{\max})$, such that for any equicontinuous factor $\theta : (X, G, T) \to (Y, G, S)$, there exists a unique factor map $\psi : (X_{\max}, G, T_{\max}) \to (Y, G, S)$ with $\psi \circ \pi_{\max} = \theta$. The maximal equicontinuous factor exists and is unique (up to topological conjugacy), see [ABKL15, Theorem 3.8] and [Kur03, Theorem 2.44].

Let $\theta : (X, G, T) \to (Y, G, S)$ be a factor map. We call the preimage set $\theta^{-1}(y)$ of a point $y \in Y$ the *fiber* of θ over y. The cardinality of the fiber $\theta^{-1}(y)$ for some $y \in Y$ has an important role and is related to the definition of other notions. In particular, the factor map θ is *almost one-to-one* if $\{y \in Y : \mathrm{card}(\theta^{-1}(y)) = 1\}$ is a G_δ-dense set in Y. In that case, (X, G, T) is an *almost one-to-one extension* of (Y, G, S). Moreover, it provides a sufficient condition to prove that an equicontinuous factor of a minimal dynamical system is the maximal one as stated in the next Lemma 2.1 from [ABKL15].

Lemma 2.1. — [ABKL15, Lemma 3.11] *Let (X, G, T) be a minimal dynamical system and (Y, G, S) an equicontinuous dynamical system. If (Y, G, S) is a factor of (X, G, T) with factor map θ and there exists $y \in Y$ such that $\mathrm{card}(\theta^{-1}(y)) = 1$, then (Y, G, S) is the maximal equicontinuous factor.*

The *set of fiber cardinalities* of a factor map $\theta : (X, G, T) \to (Y, G, S)$ is the set $\{\mathrm{card}(\theta^{-1}(y)) : y \in Y\} \subset \mathbb{N} \cup \{\infty\}$, see [Fie01]. Note that different terminology is used in [Rob96] as the set of fiber cardinalities of a factor map is called *thickness spectrum* and its supremum is called *thickness* whereas the supremum is called *maximum rank* in [ABKL15]. As shown in the next Lemma 2.2, the set of fiber cardinalities of the maximal equicontinuous factor of a minimal dynamical system is invariant under topological conjugacy.

Lemma 2.2. — *Let (X, G, T) and (Y, G, S) be a minimal dynamical systems. Let $f : (X, G, T) \to (X_{\max}, G, T_{\max})$ and $g : (Y, G, S) \to (Y_{\max}, G, S_{\max})$ be two maximal equicontinuous factors. If X and Y are topologically conjugate, then f and g have the same set of fiber cardinalities.*

The maximal equicontinuous factor $f : (X, G, T) \to (X_{\max}, G, T_{\max})$ defines an equivalence relation on the elements $a, b \in X$ as $a \equiv b$ if and only if $f(a) = f(b)$. A theorem of Auslander [Aus88, p. 130] on the equivalence relation defined by the maximal equicontinuous factor says that if (X, G, T) is minimal, then $f(a) = f(b)$ if and only if a and b are regionally proximal. Two elements $x, y \in X$ are said to be *regionally proximal* if there are sequences of elements $x_i, y_i \in X$ and a sequence of elements $g_i \in G$ such that $\lim_{i\to\infty} x_i = x$, $\lim_{i\to\infty} y_i = y$ and $\lim_{i\to\infty} \mathrm{dist}(g_i\, x_i,\, g_i\, y_i$ $= 0$.

Proof. — Let $\theta : (X, G, T) \to (Y, G, S)$ be a topological conjugacy. Let us show that the formula $\pi = g \circ \theta \circ f^{-1}$ defines a map $X_{\max} \to Y_{\max}$. Let $x \in X_{\max}$. Since f is onto, there exists $a \in X$ such that $f(a) = x$. Suppose that $a, b \in f^{-1}(x)$. Thus $f(a) = f(b)$ and by Auslander's theorem, a and b are regionally proximal. That property depends only on the distance so it is preserved by the topological conjugacy. Thus $\theta(a)$ and $\theta(b)$ are regionally proximal. Therefore $g(\theta(a)) = g(\theta(b))$ which shows that π is well-defined.

The map π is one-to-one. Let $x, x' \in X_{\max}$ and suppose that $\pi(x) = \pi(x')$. Let $a, a' \in X$ such that $f(a) = x$ and $f(a') = x'$. Then $g(\theta(a)) = \pi(x) = \pi(x') = g(\theta(a'))$. Thus $\theta(a)$ and $\theta(a')$ are regionally proximal from Auslander's theorem. Thus a and a' are regionally proximal and we obtain $x = f(a) = f(a') = x'$.

It is sufficient to show that the fiber cardinalities of f is a subset of the fiber cardinalities of g. Let $x \in X_{\max}$ such that $\pi(x) = y$. Then $g \circ \theta \circ f^{-1}(x) = y$ which means that $\theta(f^{-1}(x)) \subseteq g^{-1}(y)$ and $\{x\} \subseteq f(\theta^{-1}(g^{-1}(y)))$. Since π is one-to-one, we deduce $\{x\} = f(\theta^{-1}(g^{-1}(y)))$. Thus $\theta^{-1}(g^{-1}(y)) \subseteq f^{-1}(x)$ and $g^{-1}(y) \subseteq \theta(f^{-1}(x))$. We conclude that $g^{-1}(y) = \theta(f^{-1}(x))$. In particular, $\mathrm{card}(f^{-1}(x)) = \mathrm{card}(g^{-1}(y))$ and

$$\left\{\mathrm{card}(f^{-1}(x)) : x \in X_{\max}\right\} \subseteq \left\{\mathrm{card}(g^{-1}(y)) : y \in Y_{\max}\right\}.$$

The equality follows from the symmetry of the argument. □

2.3. Subshifts and shifts of finite type

We follow the notation of [Sch01]. Let $\mathcal{A}$ be a finite set, $d \geqslant 1$, and let $\mathcal{A}^{\mathbb{Z}^d}$ be the set of all maps $x : \mathbb{Z}^d \to \mathcal{A}$, equipped with the compact product topology. An element $x \in \mathcal{A}^{\mathbb{Z}^d}$ is called *configuration* and we write it as $x = (x_{\mathbf{m}}) = (x_{\mathbf{m}} : \mathbf{m} \in \mathbb{Z}^d)$, where $x_{\mathbf{m}} \in \mathcal{A}$ denotes the value of x at $\mathbf{m}$. The topology on $\mathcal{A}^{\mathbb{Z}^d}$ is compatible with the metric defined for all configurations $x, x' \in \mathcal{A}^{\mathbb{Z}^d}$ by $\mathrm{dist}(x, x') = 2^{-\min\{\|\mathbf{n}\| \,:\, x_{\mathbf{n}} \neq x'_{\mathbf{n}}\}}$ where $\|\mathbf{n}\| = |n_1| + \cdots + |n_d|$. The *shift action* $\sigma : \mathbf{n} \mapsto \sigma^{\mathbf{n}}$ of $\mathbb{Z}^d$ on $\mathcal{A}^{\mathbb{Z}^d}$ is defined by

$$(\sigma^{\mathbf{n}}(x))_{\mathbf{m}} = x_{\mathbf{m}+\mathbf{n}} \tag{2.1}$$

for every $x = (x_{\mathbf{m}}) \in \mathcal{A}^{\mathbb{Z}^d}$ and $\mathbf{n} \in \mathbb{Z}^d$. If $X \subset \mathcal{A}^{\mathbb{Z}^d}$, let $\overline{X}$ denote the topological closure of X and let $\sigma(X) = \{\sigma^{\mathbf{n}}(x) \mid x \in X, \mathbf{n} \in \mathbb{Z}^d\}$ denote the shift-closure of X. A subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is *shift-invariant* if $\sigma(X) = X$ and a closed, shift-invariant subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a *subshift*. If $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift we write $\sigma = \sigma^X$ for the restriction of the shift action (2.1) to X. When X is a subshift, the triple $(X, \mathbb{Z}^d, \sigma)$ is a dynamical system and the notions presented in the previous section hold.

A configuration $x \in X$ is *periodic* if there is a nonzero vector $\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ such that $x = \sigma^{\mathbf{n}}(x)$ and otherwise it is said *nonperiodic*. We say that a nonempty subshift X is *aperiodic* if the shift action σ on X is free.

For any subset $S \subset \mathbb{Z}^d$ let $\pi_S : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^S$ denote the projection map which restricts every $x \in \mathcal{A}^{\mathbb{Z}^d}$ to S. A *pattern* is a function $p \in \mathcal{A}^S$ for some finite subset $S \subset \mathbb{Z}^d$. To every pattern $p \in \mathcal{A}^S$ corresponds a subset $\pi_S^{-1}(p) \subset \mathcal{A}^{\mathbb{Z}^d}$ called *cylinder*. A subshift $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a *shift of finite type* (SFT) if there exists a finite set $\mathcal{F}$ of *forbidden* patterns such that

$$X = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \middle| \pi_S \circ \sigma^{\mathbf{n}}(x) \notin \mathcal{F} \text{ for every } \mathbf{n} \in \mathbb{Z}^d \text{ and } S \subset \mathbb{Z}^d \right\}. \tag{2.2}$$

In this case, we write $X = \mathrm{SFT}(\mathcal{F})$. In this article, we consider shifts of finite type on $\mathbb{Z} \times \mathbb{Z}$, that is, the case $d = 2$.

3. Symbolic representations and Markov partitions for toral $\mathbb{Z}^2$-rotations

We follow the section [LM95, Section 6.5] on Markov partitions where we adapt it to the case of invertible $\mathbb{Z}^2$-actions. A *topological partition* of a metric space M is a finite collection $\{P_0, P_1, \dots, P_{r-1}\}$ of disjoint open sets such that $M = \overline{P_0} \cup \overline{P_1} \cup \dots \cup \overline{P_{r-1}}$.

Suppose that M is a compact metric space, $(M, \mathbb{Z}^2, R)$ is a dynamical system and that $\mathcal{P} = \{P_0, P_1, \dots, P_{r-1}\}$ is a topological partition of M. Let $\mathcal{A} = \{0, 1, \dots, r-1\}$ and $S \subset \mathbb{Z}^2$ be a finite set. We say that a pattern $w \in \mathcal{A}^S$ is *allowed* for $\mathcal{P}, R$ if

$$\bigcap_{\mathbf{k} \in S} R^{-\mathbf{k}}(P_{w_\mathbf{k}}) \neq \varnothing.$$

Let $\mathcal{L}_{\mathcal{P},R}$ be the collection of all allowed patterns for $\mathcal{P}, R$. It can be checked that $\mathcal{L}_{\mathcal{P},R}$ is the language of a subshift. Hence, using standard arguments [LM95, Proposition 1.3.4], there is a unique subshift $\mathcal{X}_{\mathcal{P},R} \subset \mathcal{A}^{\mathbb{Z}^2}$ whose language is $\mathcal{L}_{\mathcal{P},R}$. We call $\mathcal{X}_{\mathcal{P},R}$ the *symbolic dynamical system corresponding to* $\mathcal{P}, R$. For each $w \in \mathcal{X}_{\mathcal{P},R} \subset \mathcal{A}^{\mathbb{Z}^2}$ and $n \geqslant 0$ there is a corresponding nonempty open set

$$D_n(w) = \bigcap_{\|\mathbf{k}\| \leqslant n} R^{-\mathbf{k}}(P_{w_\mathbf{k}}) \subseteq M.$$

The closures $\overline{D}_n(w)$ of these sets are compact and decrease with n, so that $\overline{D}_0(w) \supseteq \overline{D}_1(w) \supseteq \overline{D}_2(w) \supseteq \dots$. It follows that $\cap_{n=0}^{\infty} \overline{D}_n(w) \neq \varnothing$. In order for configurations in $\mathcal{X}_{\mathcal{P},R}$ to correspond to points in M, this intersection should contain only one point. This leads to the following definition.

Definition 3.1. — *Let M be a compact metric space and $(M, \mathbb{Z}^2, R)$ be a dynamical system. A topological partition $\mathcal{P}$ of M gives a* symbolic representation *of $(M, \mathbb{Z}^2, R)$ if for every configuration $w \in \mathcal{X}_{\mathcal{P},R}$ the intersection $\cap_{n=0}^{\infty} \overline{D}_n(w)$ consists of exactly one point $m \in M$. We call w a* symbolic representation of *m.*

Markov partition were originally defined for one-dimensional dynamical systems $(M, \mathbb{Z}, R)$ and were extended to $\mathbb{Z}^d$-actions by automorphisms of compact Abelian group in [ES97]. We allow ourselves to use the same terminology for dynamical systems $(M, \mathbb{Z}^2, R)$ defined by higher-dimensional actions by rotations.

Definition 3.2. — *A topological partition $\mathcal{P}$ of M is a* Markov partition *for $(M,\mathbb{Z}^2,R)$ if*

- *$\mathcal{P}$ gives a symbolic representation of $(M,\mathbb{Z}^2,R)$ and*
- *$\mathcal{X}_{\mathcal{P},R}$ is a shift of finite type (SFT).*

Of course, 2-dimensional SFTs are much different then 1-dimensional SFTs. For example, there exist 2-dimensional aperiodic SFTs with zero entropy. But this is not possible in the one-dimensional case, since one-dimensional infinite SFTs have positive entropy. In this article, we consider partitions associated to 2-dimensional aperiodic Wang shifts with zero entropy.

The partitions we consider are partitions of the 2-dimensional torus. Let Γ be a *lattice* in $\mathbb{R}^2$, i.e., a discrete subgroup of the additive group $\mathbb{R}^2$ with 2 linearly independent generators. This defines a 2-dimensional torus $\mathbf{T}=\mathbb{R}^2/\Gamma$. By analogy with the rotation $x\mapsto x+\alpha$ on the circle $\mathbb{R}/\mathbb{Z}$ for some $\alpha\in\mathbb{R}/\mathbb{Z}$, we use the terminology of *rotation* to denote the following $\mathbb{Z}^2$-action defined on a 2-dimensional torus.

Definition 3.3. — *For some $\alpha,\beta\in\mathbf{T}$, we consider the dynamical system $(\mathbf{T},\mathbb{Z}^2,R)$ where $R:\mathbb{Z}^2\times\mathbf{T}\to\mathbf{T}$ is the continuous $\mathbb{Z}^2$-action on $\mathbf{T}$ defined by*

$$R^{\mathbf{n}}(\mathbf{x}):=R(\mathbf{n},\mathbf{x})=\mathbf{x}+n_1\alpha+n_2\beta$$

for every $\mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2$. We say that the $\mathbb{Z}^2$-action R is a toral $\mathbb{Z}^2$-rotation *or a* $\mathbb{Z}^2$-rotation on $\mathbf{T}$.

From now on, we assume that the compact metric space M is $\mathbf{T}$ and that R is a $\mathbb{Z}^2$-rotation on $\mathbf{T}$ when we consider dynamical systems $(M,\mathbb{Z}^2,R)$.

Lemma 3.4. — *Let $(\mathbf{T},\mathbb{Z}^2,R)$ be a minimal dynamical system and $\mathcal{P}=\{P_0,P_1,\ldots,P_{r-1}\}$ be a topological partition of $\mathbf{T}$. If there exists an atom P_i which is invariant only under the trivial translation in $\mathbf{T}$, then $\mathcal{P}$ gives a symbolic representation of $(\mathbf{T},\mathbb{Z}^2,R)$.*

Proof. — Let $\mathcal{A}=\{0,1,\ldots,r-1\}$. Let $w\in\mathcal{X}_{\mathcal{P},R}\subset\mathcal{A}^{\mathbb{Z}^2}$. As already noticed, the closures $\overline{D}_n(w)$ are compact and decrease with n, so that $\overline{D}_0(w)\supseteq\overline{D}_1(w)\supseteq\overline{D}_2(w)\supseteq\ldots$. It follows that $\cap_{n=0}^{\infty}\overline{D}_n(w)\neq\varnothing$.

We show that $\cap_{n=0}^{\infty}\overline{D}_n(w)$ contains at most one element. Let $\mathbf{x},\mathbf{y}\in\mathbf{T}$. We assume $\mathbf{x}\in\cap_{n=0}^{\infty}\overline{D}_n(w)$ and we want to show that $\mathbf{y}\notin\cap_{n=0}^{\infty}\overline{D}_n(w)$ if $\mathbf{x}\neq\mathbf{y}$. Let $P_i\subset\mathbf{T}$ for some $i\in\mathcal{A}$ be an atom which is invariant only under the trivial translation. Since $\mathbf{x}\neq\mathbf{y}$, $\overline{P_i}\setminus(\overline{P_i}-(\mathbf{y}-\mathbf{x}))$ contains an open set O. Since $(\mathbf{T},\mathbb{Z}^2,R)$ is minimal, any orbit $\{R^{\mathbf{k}}\mathbf{x}\mid\mathbf{k}\in\mathbb{Z}^2\}$ is dense in $\mathbf{T}$. Therefore, there exists $\mathbf{k}\in\mathbb{Z}^2$ such that $R^{\mathbf{k}}\mathbf{x}\in O\subset\mathring{P}_i$. Also $\mathbf{x}\in\cap_{n=0}^{\infty}\overline{D}_n(w)\subset R^{-\mathbf{k}}\overline{P_{w_\mathbf{k}}}$ which implies $R^{\mathbf{k}}\mathbf{x}\in\overline{P_{w_\mathbf{k}}}$. Thus $\overline{P_{w_\mathbf{k}}}\cap\mathring{P}_i\neq\varnothing$ which implies that $P_{w_\mathbf{k}}=P_i$ and $w_\mathbf{k}=i$ since $\mathcal{P}$ is a topological partition. Thus

$$\cap_{n=0}^{\infty}\overline{D}_n(w)\subset R^{-\mathbf{k}}\overline{P_{w_\mathbf{k}}}=R^{-\mathbf{k}}\overline{P_i}.$$

The fact that $R^{\mathbf{k}}\mathbf{x}\in O$ also means that $R^{\mathbf{k}}\mathbf{x}\notin\overline{P_i}-(\mathbf{y}-\mathbf{x})$ which can be rewritten as $R^{\mathbf{k}}\mathbf{y}\notin\overline{P_i}$ or $\mathbf{y}\notin R^{-\mathbf{k}}\overline{P_i}$ and we conclude that $\mathbf{y}\notin\cap_{n=0}^{\infty}\overline{D}_n(w)$. Thus $\mathcal{P}$ gives a symbolic representation of $(\mathbf{T},\mathbb{Z}^2,R)$. □

Remark 3.5. — Note that minimality hypothesis in Lemma 3.4 is not necessary. For example, the partition $\boxtimes$ of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ gives a symbolic representation of the toral $\mathbb{Z}^2$-rotation defined by $R(\mathbf{n}, \mathbf{x}) = \mathbf{x} + n_1(\sqrt{2}, 0) + n_2(\sqrt{3}, 0)$ even if $(\mathbb{T}^2, \mathbb{Z}^2, R)$ is not minimal.

4. A one-to-one map from the 2-torus to symbolic representations

The goal of this section is to express the symbolic dynamical system $\mathcal{X}_{\mathcal{P},R}$ as the closure of the image of a one-to-one map defined on the 2-torus. First we define the map on the points of the torus having a unique symbolic representation. Then, we extend it on all points of the torus by approaching them from some direction $\mathbf{v} \in \mathbb{R}^2$.

The set

$$\Delta_{\mathcal{P},R} := \bigcup_{\mathbf{n}\in\mathbb{Z}^2} R^{\mathbf{n}}\left(\bigcup_{a\in\mathcal{A}} \partial P_a\right) \subset \mathbf{T}$$

is the set of points whose orbits under the toral $\mathbb{Z}^2$-rotation R intersect the boundary of the topological partition $\mathcal{P} = \{P_a\}_{a\in\mathcal{A}}$. From the Baire Category Theorem [LM95, Theorem 6.1.24], the set $\mathbf{T} \setminus \Delta_{\mathcal{P},R}$ is dense in $\mathbf{T}$.

For every starting point $\mathbf{x} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$, the coding of its orbit under the toral $\mathbb{Z}^2$-rotation R is a 2-dimensional configuration:

$$\begin{aligned} \textsc{Config}^{\mathcal{P},R}_{\mathbf{x}} : \mathbb{Z}\times\mathbb{Z} &\to \mathcal{A} \\ \mathbf{n} &\mapsto a \quad \text{if and only if} \quad R^{\mathbf{n}}(\mathbf{x}) \in P_a. \end{aligned}$$

Thus it defines a map

$$\begin{aligned} \textsc{SymbRep} : \mathbf{T} \setminus \Delta_{\mathcal{P},R} &\to \mathcal{A}^{\mathbb{Z}^2} \\ \mathbf{x} &\mapsto \textsc{Config}^{\mathcal{P},R}_{\mathbf{x}}. \end{aligned}$$

The map SymbRep can not be extended continuously on $\mathbf{T}$. Up to some choice to be made, it can still be extended to the whole domain $\mathbf{T}$. Recall that for interval exchange transformations, one way to deal with this issue is to consider two copies x^- and x^+ for each discontinuity point [Kea75]. Here we use this idea in order to extend SymbRep on the whole domain $\mathbf{T}$ by approaching any point from a chosen direction. Not all directions work, so we need some care to formalize this properly. Let $\Theta^{\mathcal{P}}$ with $\{\mathbf{0}\} \subseteq \Theta^{\mathcal{P}} \subset \mathbb{R}^2$ be the set of vectors parallel to a segment included in the boundary of some atom $P_a \in \mathcal{P}$. If all atoms have curved boundaries, then $\Theta^{\mathcal{P}} = \{\mathbf{0}\}$. If the atoms are polygons like in this article, then the set $\Theta^{\mathcal{P}}$ contains nonzero directions. In any case, we assume that $\mathbb{R}\Theta^{\mathcal{P}} = \Theta^{\mathcal{P}}$. For every $\mathbf{v} \in \mathbb{R}^2 \setminus \Theta^{\mathcal{P}}$ we define

$$\begin{aligned} \textsc{SymbRep}^{\mathbf{v}} : \mathbf{T} &\to \mathcal{A}^{\mathbb{Z}^2} \\ \mathbf{x} &\mapsto \lim_{\epsilon\to\mathbf{0}} \textsc{SymbRep}(\mathbf{x} + \epsilon\cdot\mathbf{v}). \end{aligned}$$

We say that the configuration $\textsc{SymbRep}(\mathbf{x}) = \textsc{Config}^{\mathcal{P},R}_{\mathbf{x}}$ is *generic* if $\mathbf{x} \in \mathbf{T}\setminus\Delta_{\mathcal{P},R}$ and that $\textsc{SymbRep}^{\mathbf{v}}(\mathbf{x})$ is *singular* if $\mathbf{x} \in \Delta_{\mathcal{P},R}$ for some $\mathbf{v} \in \mathbb{R}^2 \setminus \Theta^{\mathcal{P}}$. The choice of direction $\mathbf{v}$ is not so important since the topological closure of the range of $\textsc{SymbRep}^{\mathbf{v}}$ does not depend on $\mathbf{v}$ as shown in the next Lemma 4.1. In other words, singular configurations are limits of generic configurations and $\mathcal{X}_{\mathcal{P},R}$ is equal to the topological closure of the range of $\textsc{SymbRep}$.

Lemma 4.1. — *For every $\mathbf{v} \in \mathbb{R}^2 \setminus \Theta^{\mathcal{P}}$, the following equalities hold*

$$\overline{\textsc{SymbRep}^{\mathbf{v}}(\mathbf{T})} = \overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})} = \mathcal{X}_{\mathcal{P},R}$$

where $\mathcal{X}_{\mathcal{P},R}$ is the symbolic dynamical system corresponding to $\mathcal{P}, R$.

Proof. — ($\supseteq$) If $\mathbf{x} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$, then $\textsc{SymbRep}(\mathbf{x}) = \textsc{SymbRep}^{\mathbf{v}}(\mathbf{x})$. Thus $\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R}) = \textsc{SymbRep}^{\mathbf{v}}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})$. Then

$$\overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})} = \overline{\textsc{SymbRep}^{\mathbf{v}}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})} \subseteq \overline{\textsc{SymbRep}^{\mathbf{v}}(\mathbf{T})}.$$

($\subseteq$) Let $w \in \textsc{SymbRep}^{\mathbf{v}}(\Delta_{\mathcal{P},R})$. Then $w = \lim_{\epsilon\to\mathbf{0}} \textsc{SymbRep}(\mathbf{x} + \epsilon\mathbf{v})$ for some $\mathbf{x} \in \Delta_{\mathcal{P},R}$. We may extract a subsequence $(\textsc{SymbRep}(\mathbf{x} + \epsilon_n\mathbf{v}))_{n\in\mathbb{N}}$ with $\epsilon_n \in \mathbb{R}$ such that $\mathbf{x} + \epsilon_n\mathbf{v} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$ for all $n \in \mathbb{N}$. This implies that $w \in \overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})}$. Therefore $\textsc{SymbRep}^{\mathbf{v}}(\Delta_{\mathcal{P},R}) \subseteq \overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})}$. We obtain

$$\begin{aligned}\overline{\textsc{SymbRep}^{\mathbf{v}}(\mathbf{T})} &= \overline{\textsc{SymbRep}^{\mathbf{v}}(\mathbf{T} \setminus \Delta_{\mathcal{P},R}) \cup \textsc{SymbRep}^{\mathbf{v}}(\Delta_{\mathcal{P},R})} \\ &\subseteq \overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})}\end{aligned}$$

which proves the first equality.

Recall that the collection of all allowed patterns for $\mathcal{P}, R$ is the language $\mathcal{L}_{\mathcal{P},R}$. The set $\overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})}$ is a subshift and contains $\mathcal{L}_{\mathcal{P},R}$. Moreover the language of $\overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})}$ is contained in $\mathcal{L}_{\mathcal{P},R}$. The equality $\overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})} = \mathcal{X}_{\mathcal{P},R}$ follows since the symbolic dynamical system $\mathcal{X}_{\mathcal{P},R}$ is the unique subshift whose language is $\mathcal{L}_{\mathcal{P},R}$. $\square$

Lemma 4.2. — *Let $\mathcal{P}$ give a symbolic representation of the dynamical system $(\mathbf{T}, \mathbb{Z}^2, R)$ and let $\mathbf{v} \in \mathbb{R}^2 \setminus \Theta^{\mathcal{P}}$. Then $\textsc{SymbRep} : \mathbf{T} \setminus \Delta_{\mathcal{P},R} \to \mathcal{X}_{\mathcal{P},R}$ and $\textsc{SymbRep}^{\mathbf{v}} : \mathbf{T} \to \mathcal{X}_{\mathcal{P},R}$ are one-to-one. Moreover, the following diagrams commute:*

$$\begin{array}{ccc} \mathbf{T} \setminus \Delta_{\mathcal{P},R} & \xrightarrow{R^{\mathbf{k}}} & \mathbf{T} \setminus \Delta_{\mathcal{P},R} \\ {\scriptstyle \textsc{SymbRep}}\big\downarrow & & \big\downarrow{\scriptstyle \textsc{SymbRep}} \\ \mathcal{X}_{\mathcal{P},R} & \xrightarrow{\sigma^{\mathbf{k}}} & \mathcal{X}_{\mathcal{P},R} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{T} & \xrightarrow{R^{\mathbf{k}}} & \mathbf{T} \\ {\scriptstyle \textsc{SymbRep}^{\mathbf{v}}}\big\downarrow & & \big\downarrow{\scriptstyle \textsc{SymbRep}^{\mathbf{v}}} \\ \mathcal{X}_{\mathcal{P},R} & \xrightarrow{\sigma^{\mathbf{k}}} & \mathcal{X}_{\mathcal{P},R} \end{array}$$

for every $\mathbf{k} \in \mathbb{Z}^2$.

Proof. — The fact that $w = \textsc{SymbRep}^{\mathbf{v}}(\mathbf{x})$ implies that $\mathbf{x} \in \cap_{n=0}^{\infty}\overline{D}_n(w)$. Therefore, if $\textsc{SymbRep}^{\mathbf{v}}(\mathbf{x}) = \textsc{SymbRep}^{\mathbf{v}}(\mathbf{y}) = w$ then $\mathbf{x}, \mathbf{y} \in \cap_{n=0}^{\infty}\overline{D}_n(w)$. Since $\mathcal{P}$ gives a symbolic representation of the dynamical system $(\mathbf{T}, \mathbb{Z}^2, R)$, the set $\cap_{n=0}^{\infty}\overline{D}_n(w)$ contains at most one element, and it implies that $\mathbf{x} = \mathbf{y}$. Thus, $\textsc{SymbRep}^{\mathbf{v}}$ is one-to-one. As $\textsc{SymbRep}^{\mathbf{v}}$ and $\textsc{SymbRep}$ agree on $\mathbf{T} \setminus \Delta_{\mathcal{P},R}$, we also have that $\textsc{SymbRep}$ is one-to-one.

We now show conjugacy of $\mathbb{Z}^2$-actions. Let $\mathbf{k} \in \mathbb{Z}^2$, $\mathbf{x} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$ and $\mathbf{n} \in \mathbb{Z}^2$. We have

$$\begin{aligned}\left(\sigma^{\mathbf{k}} \circ \textsc{SymbRep}(\mathbf{x})\right)(\mathbf{n}) &= \left(\sigma^{\mathbf{k}} \circ \textsc{Config}^{\mathcal{P},R}_{\mathbf{x}}\right)(\mathbf{n}) = \textsc{Config}^{\mathcal{P},R}_{\mathbf{x}}(\mathbf{n}+\mathbf{k})\\ &= \textsc{Config}^{\mathcal{P},R}_{R^{\mathbf{k}}\mathbf{x}}(\mathbf{n}) = \left(\textsc{SymbRep}\left(R^{\mathbf{k}}\mathbf{x}\right)\right)(\mathbf{n})\\ &= \left(\textsc{SymbRep} \circ R^{\mathbf{k}}(\mathbf{x})\right)(\mathbf{n}).\end{aligned}$$

Therefore $\sigma^{\mathbf{k}} \circ \textsc{SymbRep} = \textsc{SymbRep} \circ R^{\mathbf{k}}$. The conjugacy of $\mathbb{Z}^2$-actions by the map $\textsc{SymbRep}$ also extends to $\textsc{SymbRep}^{\mathbf{v}}$. □

The fact that $\textsc{SymbRep}^{\mathbf{v}}$ is one-to-one means that it admits a left-inverse map $f : \textsc{SymbRep}^{\mathbf{v}}(\mathbf{T}) \to \mathbf{T}$ such that $f \circ \textsc{SymbRep}^{\mathbf{v}} = \mathrm{Id}_{\mathbf{T}}$. But we can say more and define the map f on the closure $\overline{\textsc{SymbRep}^{\mathbf{v}}(\mathbf{T})} = \mathcal{X}_{\mathcal{P},R}$. Indeed, if $\mathcal{P}$ gives a symbolic representation of the dynamical system $(\mathbf{T}, \mathbb{Z}^2, R)$, then there is a well-defined function f from $\mathcal{X}_{\mathcal{P},R}$ to $\mathbf{T}$ which maps a configuration $w \in \mathcal{X}_{\mathcal{P},R} \subset \mathcal{A}^{\mathbb{Z}^2}$ to the unique point $f(w) \in \mathbf{T}$ in the intersection $\cap_{n=0}^{\infty} \overline{D}_n(w)$. We consider the map f in the next section.

5. A factor map from symbolic representations to the 2-torus

In the spirit of [LM95, Proposition 6.5.8], the following result shows that there exists a continuous and onto homomorphism and therefore a factor map from $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$ to $(\mathbf{T}, \mathbb{Z}^2, R)$.

Proposition 5.1. — *Let $\mathcal{P}$ give a symbolic representation of the dynamical system $(\mathbf{T}, \mathbb{Z}^2, R)$. Let $f : \mathcal{X}_{\mathcal{P},R} \to \mathbf{T}$ be defined such that $f(w)$ is the unique point in the intersection $\cap_{n=0}^{\infty} \overline{D}_n(w)$. The map f is a factor map from $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$ to $(\mathbf{T}, \mathbb{Z}^2, R)$ which makes the following diagram commute*

$$\begin{tikzcd} \mathcal{X}_{\mathcal{P},R} \arrow[r, "\sigma^{\mathbf{k}}"] \arrow[d, "f"'] & \mathcal{X}_{\mathcal{P},R} \arrow[d, "f"] \\ \mathbf{T} \arrow[r, "R^{\mathbf{k}}"] & \mathbf{T} \end{tikzcd}$$

for every $\mathbf{k} \in \mathbb{Z}^2$. The map f is one-to-one on $f^{-1}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})$.

Proof. — Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{r-1}\}$. We show that the map f is continuous. Let $\varepsilon > 0$. Let $w \in \mathcal{X}_{\mathcal{P},R}$. Since the partition gives a symbolic representation, there exists $n \in \mathbb{N}$ such that the diameter of

$$\overline{D}_n(w) = \bigcap_{\|\mathbf{k}\| \leqslant n} R^{-\mathbf{k}}\left(\overline{P_{w_{\mathbf{k}}}}\right)$$

is smaller than or equal to ε. That set contains $\left(\cap_{n=0}^{\infty} \overline{D}_n(w)\right) \cup \left(\cap_{n=0}^{\infty} \overline{D}_n(w')\right)$ if $w' \in \mathcal{X}_{\mathcal{P},R}$ is such that $\mathrm{dist}_{\mathcal{X}_{\mathcal{P},R}}(w, w') < \frac{1}{2^n}$. We conclude that if $\mathrm{dist}_{\mathcal{X}_{\mathcal{P},R}}(w, w') < \frac{1}{2^n}$, then $\mathrm{dist}_{\mathbf{T}}(f(w), f(w')) < \varepsilon$ which means that f is continuous.

We show that the map f is onto. Let $\mathbf{x} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$ and $w = \textsc{SymbRep}(\mathbf{x})$. Then $\mathbf{x} \in \cap_{n=0}^{\infty} \overline{D}_n(w)$. Since $\mathcal{P}$ gives a symbolic representation of $(\mathbf{T}, \mathbb{Z}^2, R)$, we have that

$$\{\mathbf{x}\} = \bigcap_{n=0}^{\infty} \overline{D}_n(w) = \bigcap_{n=0}^{\infty} D_n(w),$$

so that $f(w) = \mathbf{x}$. Thus the image of f contains the dense set $\mathbf{T} \setminus \Delta_{\mathcal{P},R}$. Since the image of a compact set via a continuous map is compact and therefore closed, it follows that the image of f is all of $\mathbf{T}$.

An alternate proof that f is onto uses $\textsc{SymbRep}^{\mathbf{v}}$. Let $\mathbf{x} \in \mathbf{T}$ and $w = \textsc{SymbRep}^{\mathbf{v}}(\mathbf{x})$ for some $\mathbf{v} \in \mathbb{R}^2 \setminus \Theta^{\mathcal{P}}$. We have that $\cap_{n=0}^{\infty} \overline{D}_n(w) = \{\mathbf{x}\}$. Therefore, $f(w) = \mathbf{x}$ and f is onto.

We show that the map f is a homomorphism:

$$\begin{aligned} R^{\mathbf{k}}\{f(w)\} &= R^{\mathbf{k}}\left(\bigcap_{n=0}^{\infty} \overline{D}_n(w)\right) = R^{\mathbf{k}} \bigcap_{\mathbf{n}\in\mathbb{Z}^2} R^{-\mathbf{n}}\overline{P_{w_{\mathbf{n}}}} = \bigcap_{\mathbf{n}\in\mathbb{Z}^2} R^{-(\mathbf{n}-\mathbf{k})}\overline{P_{\sigma^{\mathbf{k}}w_{\mathbf{n}-\mathbf{k}}}} \\ &= \bigcap_{\mathbf{m}\in\mathbb{Z}^2} R^{-\mathbf{m}}\overline{P_{\sigma^{\mathbf{k}}w_{\mathbf{m}}}} = \bigcap_{n=0}^{\infty} \overline{D}_n\left(\sigma^{\mathbf{k}}w\right) = \left\{f\left(\sigma^{\mathbf{k}}w\right)\right\} \end{aligned}$$

where $\mathbf{m} = \mathbf{n} - \mathbf{k}$. Therefore $R^{\mathbf{k}} \circ f = f \circ \sigma^{\mathbf{k}}$ for every $\mathbf{k} \in \mathbb{Z}^2$ and $f : \mathcal{X}_{\mathcal{P},R} \to \mathbf{T}$ is a factor map.

We show that f is one-to-one on $f^{-1}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})$. Let $\mathbf{x} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$ and suppose that $w, w' \in f^{-1}(\mathbf{x})$. This means that $\cap_{n=0}^{\infty} \overline{D}_n(w) = \cap_{n=0}^{\infty} \overline{D}_n(w') = \{\mathbf{x}\}$. Therefore for every $\mathbf{n} \in \mathbb{Z}^2$ we have

$$\mathbf{x} \in \left(R^{-\mathbf{n}}\overline{P_{w_{\mathbf{n}}}}\right)^{\circ} \cap \left(R^{-\mathbf{n}}\overline{P_{w'_{\mathbf{n}}}}\right)^{\circ}.$$

Then $w_{\mathbf{n}} = w'_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^2$ and $w = w'$. Therefore for every $\mathbf{x} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$, $f^{-1}(\mathbf{x})$ contains exactly one element. □

As mentioned in Remark 3.5, it is possible that $\mathcal{X}_{\mathcal{P},R}$ is not minimal. But as shown in the next Lemma 5.2, it is minimal if R is minimal.

Lemma 5.2. — *Let $\mathcal{P}$ give a symbolic representation of the dynamical system $(\mathbf{T}, \mathbb{Z}^2, R)$. Then*

(i) *if $(\mathbf{T}, \mathbb{Z}^2, R)$ is minimal, then $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$ is minimal,*
(ii) *if R is a free $\mathbb{Z}^2$-action on $\mathbf{T}$, then $\mathcal{X}_{\mathcal{P},R}$ aperiodic.*

Proof. — Let $f : \mathcal{X}_{\mathcal{P},R} \to \mathbf{T}$ be the factor map from Proposition 5.1.

(i) Let $Y \subseteq \mathcal{X}_{\mathcal{P},R}$ be a nonempty subshift. Thus Y is compact. Continuous map preserve compact sets, thus $f(Y)$ is compact. The set $f(Y)$ is also R-invariant since $R^{\mathbf{k}}f(Y) = f(\sigma^{\mathbf{k}}Y) = f(Y)$ for every $\mathbf{k} \in \mathbb{Z}^2$. Since $(\mathbf{T}, \mathbb{Z}^2, R)$ is minimal, the only nonempty compact subset of $\mathbf{T}$ which is invariant under R is $\mathbf{T}$. Thus $f(Y) = \mathbf{T}$.

For every $x \in \mathbf{T}$, $f^{-1}(x) \cap Y \neq \varnothing$. Then Y contains $\textsc{SymbRep}(x)$ for every $x \in \mathbf{T}$ such that $f^{-1}(x)$ is a singleton. Then Y contains $\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})$. Since Y is closed, it must contain $\overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})}$. From Lemma 4.1, this means that $\mathcal{X}_{\mathcal{P},R} \subseteq Y$. Thus $Y = \mathcal{X}_{\mathcal{P},R}$ and $\mathcal{X}_{\mathcal{P},R}$ is minimal.

(ii) Suppose that there exists $w \in \mathcal{X}_{\mathcal{P},R}$ such that w is periodic, i.e., there exists $\mathbf{k} \in \mathbb{Z}^2$ such that $\sigma^{\mathbf{k}} w = w$. Since f commutes the $\mathbb{Z}^2$-actions, we obtain

$$R^{\mathbf{k}} f(w) = f(\sigma^{\mathbf{k}} w) = f(w).$$

Since we assume that R is a free $\mathbb{Z}^2$-action, this implies that $\mathbf{k} = \mathbf{0}$. Thus $\mathcal{X}_{\mathcal{P},R}$ is aperiodic. □

We can now deduce a corollary of Proposition 5.1.

Corollary 5.3. — *If the dynamical system $(\mathbf{T}, \mathbb{Z}^2, R)$ is minimal and $\mathcal{P}$ gives a symbolic representation of $(\mathbf{T}, \mathbb{Z}^2, R)$, then $(\mathbf{T}, \mathbb{Z}^2, R)$ is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$.*

Proof. — From Lemma 5.2 (i), $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$ is minimal. The dynamical system $(\mathbf{T}, \mathbb{Z}^2, R)$ is equicontinuous. We proved in Proposition 5.1 that the factor map f is one-to-one on $f^{-1}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})$. In particular, there exists at least one element $\mathbf{y} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$ such that $\operatorname{card}(f^{-1}(\mathbf{y})) = 1$. From Lemma 2.1, $(\mathbf{T}, \mathbb{Z}^2, R)$ is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$. □

Remark 5.4. — There are some more consequences. From Proposition 5.1, we deduce that

$$\lambda\left(\left\{\mathbf{x} \in \mathbf{T} : \operatorname{card}\left(f^{-1}(\mathbf{x})\right) > 1\right\}\right) \leqslant \lambda\left(\Delta_{\mathcal{P},R}\right) = 0$$

where λ be the Haar measure on $\mathbf{T}$. From [FGL18], this implies that $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$ is a *regular* extension of $(\mathbf{T}, \mathbb{Z}^2, R)$ and that $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$ is *mean equicontinuous* which has more structural consequences including the fact of having *discrete spectrum with continuous eigenfunctions*. We refer the reader to [FGL18] for the definitions of regular extension and mean equicontinuous.

6. An isomorphism between symbolic dynamical systems and toral $\mathbb{Z}^2$-rotations

Let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be a subshift. Recall that for any subset $S \subset \mathbb{Z}^2$, $\pi_S : X \to \mathcal{A}^S$ is the projection map which restricts every $w \in X$ to S. To every finite pattern $p \in \mathcal{A}^S$ correspond a cylinder $[p] = \pi_S^{-1}(p) \subset X$. The set of all cylinders

$$\left\{[p] \,\middle|\, p \in \mathcal{A}^S \text{ with } S \subset \mathbb{Z}^2 \text{ finite}\right\}$$

generates the Borel σ-algebra on X.

Let $\mathcal{P} = \{P_a\}_{a\in\mathcal{A}}$ give a symbolic representation of the dynamical system $(\mathbf{T}, \mathbb{Z}^2, R)$ and let $f : \mathcal{X}_{\mathcal{P},R} \to \mathbf{T}$ be the factor map from Proposition 5.1. For each $a \in \mathcal{A}$, we have that

$$f([a]) = \overline{P_a} \subset \mathbf{T}$$

is a closed set. Thus the image of a cylinder $[p]$ under f for some finite pattern $p \in \mathcal{A}^S$ is a closed set called *coding region* for the pattern p being the finite intersection of closed sets:

$$f\left([p]\right) = \bigcap_{\mathbf{n}\in S} R^{-\mathbf{n}} \overline{P_{p_{\mathbf{n}}}} \subset \mathbf{T}. \tag{6.1}$$

The following proposition can be seen as an explicit construction of a strictly ergodic symbolic dynamical system isomorphic to the $\mathbb{Z}^2$-rotation R on the torus $\mathbf{T}$ as established by the Theorem of Jewett and Krieger [DGS76] for one-dimensional dynamical systems and generalized to $\mathbb{Z}^2$-actions by Rosenthal [Ros87].

Proposition 6.1. — *Let $\mathcal{P}$ give a symbolic representation of a minimal dynamical system $(\mathbf{T}, \mathbb{Z}^2, R)$. Suppose that $\lambda(\partial P) = 0$ for each atom $P \in \mathcal{P}$ where λ is the Haar measure on $\mathbf{T}$. Then the dynamical system $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$ is strictly ergodic and the measure-preserving dynamical system $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma, \nu)$ is isomorphic to $(\mathbf{T}, \mathbb{Z}^2, R, \lambda)$ where ν is the unique shift-invariant probability measure on $\mathcal{X}_{\mathcal{P},R}$.*

Proof. — We prove that the factor map $f : \mathcal{X}_{\mathcal{P},R} \to \mathbf{T}$ from Proposition 5.1 provides the isomorphism. The map f is measurable as f is continuous and $f^{-1}(K)$ is compact for any compact subset $K \subset \mathbf{T}$. Let λ be the Haar measure on $\mathbf{T}$. By hypothesis, $(\mathbf{T}, \mathbb{Z}^2, R)$ is minimal. It is also strictly ergodic [Wal82] with λ being the only R-invariant probability measure on $\mathbf{T}$.

Since σ is continuous and $\mathcal{X}_{\mathcal{P},R}$ is a compact metric space, the set $\mathcal{M}^\sigma(\mathcal{X}_{\mathcal{P},R})$ of σ-invariant probability measures on $\mathcal{X}_{\mathcal{P},R}$ is nonempty [Wal82, Corollary 6.9.1]. Thus let $\nu \in \mathcal{M}^\sigma(\mathcal{X}_{\mathcal{P},R})$. Let $Z = [p] \subset \mathcal{X}_{\mathcal{P},R}$ be the cylinder corresponding to some pattern $p \in \mathcal{A}^S$ for some finite subset $S \subset \mathbb{Z}^2$. From Equation (6.1) we know that $f(Z)$ is a closed set being the intersection of a finite number of closed sets. Closed sets as well as their interior are both measurable for the Haar measure λ. Continuity of f implies that $f^{-1}(f(Z))$ and $f^{-1}(f(Z)^\circ)$ are both measurable for ν.

For each letter $a \in \mathcal{A}$, we have $f^{-1}(f([a])^\circ) \subset [a]$. Thus we have

$$f^{-1}(f(Z)^\circ) \subset Z \subset f^{-1}(f(Z))$$

so that

$$\nu\left(f^{-1}(f(Z)^\circ)\right) \leqslant \nu(Z) \leqslant \nu\left(f^{-1}(f(Z))\right).$$

Let $f_\star$ be the pushforward map

$$\begin{aligned} f_\star : \mathcal{M}^\sigma(\mathcal{X}_{\mathcal{P},R}) &\to \mathcal{M}^R(\mathbf{T}) \\ \nu &\mapsto \nu \circ f^{-1} \end{aligned}$$

which maps shift-invariant measures on $\mathcal{X}_{\mathcal{P},R}$ to R-invariant measures on $\mathbf{T}$. But there is only one such measure, so that $f_\star\nu = \lambda$ for every $\nu \in \mathcal{M}^\sigma(\mathcal{X}_{\mathcal{P},R})$. For every $\nu \in \mathcal{M}^\sigma(\mathcal{X}_{\mathcal{P},R})$, we have for the left-hand side

$$\nu\left(f^{-1}(f(Z)^\circ)\right) = f_\star\nu(f(Z)^\circ) = \lambda(f(Z)^\circ)$$

and for the right-hand side

$$\nu(f^{-1}(f(Z))) = f_\star\nu(f(Z)) = \lambda(f(Z)).$$

As the boundary of $f(Z)$ is a λ-null set, we obtain

$$\lambda(f(Z)) = \lambda(f(Z)^\circ) \leqslant \nu(Z) \leqslant \lambda(f(Z))$$

and we conclude that

$$\nu(Z) = \lambda(f(Z)).$$

Since measures are defined from the measure of cylinders which generate the Borel σ-algebra, we conclude that there is a unique shift-invariant probability measure on

$\mathcal{X}_{\mathcal{P},R}$. Thus $\mathcal{X}_{\mathcal{P},R}$ is uniquely ergodic and therefore strictly ergodic since minimality of $\mathcal{X}_{\mathcal{P},R}$ was proved in Lemma 5.2. □

Proposition 6.1 implies uniform pattern frequencies for configurations in $\mathcal{X}_{\mathcal{P},R}$. It also means that the symbolic dynamical system $\mathcal{X}_{\mathcal{P},R}$ is an almost one-to-one extension of a Kronecker dynamical system (a rotation action on a compact Abelian group) and from Von Neumann's Theorem [Que10, Theorem 3.9], it implies that $\mathcal{X}_{\mathcal{P},R}$ has discrete spectrum. See also [Rob07] for a treatment of Von Neumann's Theorem in the context of tiling dynamical systems.

Part 2. Wang shifts as codings of toral $\mathbb{Z}^2$-rotations

This part is divided into 5 sections. After introducing Wang tiles and Wang shifts, we present a generic method for constructing sets of Wang tiles and valid configurations in the associated Wang shift as codings of toral $\mathbb{Z}^2$-rotations. We illustrate the method on Jeandel–Rao's set of 11 Wang tiles and on a self-similar set of 19 Wang tiles. We expose the limitations of the method by presenting two "non-examples".

7. Wang shifts

A *Wang tile*

$$\tau = \boxed{\begin{smallmatrix} & b & \\ c & & a \\ & d & \end{smallmatrix}}$$

is a unit square with colored edges formally represented as a tuple of four colors $(a, b, c, d) \in I \times J \times I \times J$ where I, J are two finite sets (the vertical and horizontal colors respectively). For each Wang tile $\tau = (a, b, c, d)$, let $\textsc{right}(\tau) = a$, $\textsc{top}(\tau) = b$, $\textsc{left}(\tau) = c$, $\textsc{bottom}(\tau) = d$ denote the colors of the right, top, left and bottom edges of τ [Rob71, Wan61].

Let $\mathcal{T}$ be a set of Wang tiles. A configuration $x : \mathbb{Z}^2 \to \mathcal{T}$ is *valid* if it assigns tiles to each position of $\mathbb{Z}^2$ so that contiguous edges have the same color, that is,

$$\textsc{right}(x_{\mathbf{n}}) = \textsc{left}(x_{\mathbf{n}+\mathbf{e}_1}) \tag{7.1}$$

$$\textsc{top}(x_{\mathbf{n}}) = \textsc{bottom}(x_{\mathbf{n}+\mathbf{e}_2}) \tag{7.2}$$

for every $\mathbf{n} \in \mathbb{Z}^2$ where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Let $\Omega_{\mathcal{T}} \subset \mathcal{T}^{\mathbb{Z}^2}$ denote the set of all valid configurations $\mathbb{Z}^2 \to \mathcal{T}$ and we call it the *Wang shift* of $\mathcal{T}$. Together with the shift action σ of $\mathbb{Z}^2$ on $\mathcal{T}^{\mathbb{Z}^2}$, $\Omega_{\mathcal{T}}$ is a SFT of the form (2.2) since there exists a finite set of forbidden patterns made of all horizontal and vertical dominoes of two tiles that do not share an edge of the same color.

A set of Wang tiles $\mathcal{T}$ is *periodic* if there exists a periodic configuration $x \in \Omega_{\mathcal{T}}$. Originally, Wang thought that every set of Wang tiles $\mathcal{T}$ is periodic as soon as $\Omega_{\mathcal{T}}$ is nonempty [Wan61]. This statement is equivalent to the existence of an algorithm solving the *domino problem*, that is, taking as input a set of Wang tiles and returning *yes* or *no* whether there exists a valid configuration with these tiles. Berger, a student of Wang, later proved that the domino problem is undecidable and he also provided a first example of an aperiodic set of Wang tiles [Ber66]. A set of Wang tiles $\mathcal{T}$ is *aperiodic* if the Wang shift $\Omega_\mathcal{T}$ is a nonempty aperiodic subshift. This means that in general one can not decide the emptiness of a Wang shift $\Omega_\mathcal{T}$. This illustrates that the behavior of d-dimensional SFTs when $d \geqslant 2$ is much different than the one-dimensional case where emptiness of a SFT is decidable [LM95]. Note that another important difference between $d = 1$ and $d \geqslant 2$ is expressed in terms of the possible values of entropy of d-dimensional SFTs, see [HM10].

8. From toral partitions and $\mathbb{Z}^2$-rotations to Wang shifts

We consider the 2-torus $\mathbf{T} = \mathbb{R}^2/\Gamma$ where Γ is a lattice in $\mathbb{R}^2$. We suppose that $(\mathbf{T}, \mathbb{Z}^2, R)$ is a dynamical system where R is a toral $\mathbb{Z}^2$-rotation. Let $\mathcal{Y} = \{Y_i\}_{i\in I}$, $\mathcal{Z} = \{Z_j\}_{j\in J}$ be two finite topological partitions of $\mathbf{T}$. For each $(i,j,k,\ell) \in I \times J \times I \times J$ we define the intersection of 4 atoms in the following way

$$P_{(i,j,k,\ell)} = Y_i \cap Z_j \cap R^{\mathbf{e}_1}(Y_k) \cap R^{\mathbf{e}_2}(Z_\ell)$$

where $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$. The quadruples τ for which the intersection P_τ is nonempty define a set

$$\mathcal{T} = \{\tau \in I \times J \times I \times J \mid P_\tau \neq \varnothing\}$$

that we see as a set of Wang tiles. Naturally, this comes with a topological partition

$$\mathcal{P} = \{P_\tau\}_{\tau\in\mathcal{T}}$$

of $\mathbf{T}$ which is the refinement of the four partitions $\mathcal{Y}$ (the right color), $\mathcal{Z}$ (the top color), $R^{\mathbf{e}_1}(\mathcal{Y})$ (the left color) and $R^{\mathbf{e}_2}(\mathcal{Z})$ (the bottom color). Thus to each $\mathbf{x} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$ corresponds a unique Wang tile, that is, a right, a top, a left and a bottom color according to which atom it belongs in each of the four partitions.

Proposition 8.1. — *Let $(\mathbf{T}, \mathbb{Z}^2, R)$ be a dynamical system where R is a $\mathbb{Z}^2$-rotation and let $\mathcal{Y}$ and $\mathcal{Z}$ be two finite topological partitions of $\mathbf{T}$. Let $\mathcal{P} = \mathcal{Y} \wedge \mathcal{Z} \wedge R^{\mathbf{e}_1}(\mathcal{Y}) \wedge R^{\mathbf{e}_2}(\mathcal{Z})$ be the refinement of four partitions. Let $\mathcal{T}$ be the set of Wang tiles defined above as the set of quadruples τ such that P_τ is a nonempty atom of the partition $\mathcal{P}$. Then $\mathcal{X}_{\mathcal{P},R}$ is a subshift of the Wang shift $\Omega_\mathcal{T}$.*

Proof. — Let $\mathbf{x} \in \mathbf{T} \setminus \Delta_{\mathcal{P},R}$ and $w = \textsc{SymbRep}(\mathbf{x})$. Let $\mathbf{n} \in \mathbb{Z}^2$. First we check that Equation (7.1) is satisfied. There exists $i \in I$ such that $R^\mathbf{n}(\mathbf{x}) \in Y_i$. Equivalently, $R^{\mathbf{n}+\mathbf{e}_1}(\mathbf{x}) \in R^{\mathbf{e}_1}(Y_i)$. Thus we have

$$\begin{aligned}\textsc{right}(w_\mathbf{n}) &= \textsc{right}\left(\textsc{Config}^{\mathcal{P},R}_\mathbf{x}(\mathbf{n})\right) = i\\ &= \textsc{left}\left(\textsc{Config}^{\mathcal{P},R}_\mathbf{x}(\mathbf{n}+\mathbf{e}_1)\right) = \textsc{left}(w_{\mathbf{n}+\mathbf{e}_1}).\end{aligned}$$

Similarly we check that Equation (7.2) is satisfied. There exists $j \in J$ such that $R^\mathbf{n}(\mathbf{x}) \in Z_j$. Equivalently, $R^{\mathbf{n}+\mathbf{e}_2}(\mathbf{x}) \in R^{\mathbf{e}_2}(Z_j)$. Thus we have

$$\begin{aligned}\textsc{top}(w_\mathbf{n}) &= \textsc{top}\left(\textsc{Config}^{\mathcal{P},R}_\mathbf{x}(\mathbf{n})\right) = j\\ &= \textsc{bottom}\left(\textsc{Config}^{\mathcal{P},R}_\mathbf{x}(\mathbf{n}+\mathbf{e}_2)\right) = \textsc{bottom}(w_{\mathbf{n}+\mathbf{e}_2}).\end{aligned}$$

Then the configuration w is valid and $w \in \Omega_{\mathcal{T}}$. Thus $\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R}) \subseteq \Omega_{\mathcal{T}}$.

Remark that $\Omega_{\mathcal{T}}$ is closed since it is a subshift. Therefore the topological closure of the image of $\textsc{SymbRep}$ is in the Wang shift $\Omega_{\mathcal{T}}$. Using Lemma 4.1, we conclude that

$$\mathcal{X}_{\mathcal{P},R} = \overline{\textsc{SymbRep}(\mathbf{T} \setminus \Delta_{\mathcal{P},R})} \subseteq \Omega_{\mathcal{T}}. \qquad \square$$

Lemma 8.2. — *If the refined partition $\mathcal{P} = \mathcal{Y} \wedge \mathcal{Z} \wedge R^{\mathbf{e}_1}(\mathcal{Y}) \wedge R^{\mathbf{e}_2}(\mathcal{Z})$ gives a symbolic representation of $(\mathbf{T}, \mathbb{Z}^2, R)$, then for every $\mathbf{v} \in \mathbb{R}^2 \setminus \Theta^{\mathcal{P}}$, $\textsc{SymbRep}^{\mathbf{v}}$ is a one-to-one map $\mathbf{T} \to \Omega_{\mathcal{T}}$.*

Proof. — Follows from Lemma 4.2 and Proposition 8.1. $\square$

9. Example 1: Jeandel-Rao aperiodic Wang shift

Consider the lattice $\Gamma_0 = \langle(\varphi, 0), (1, \varphi+3)\rangle_{\mathbb{Z}}$ where $\varphi = \frac{1+\sqrt{5}}{2}$. On the torus $\mathbb{R}^2/\Gamma_0$, we consider the $\mathbb{Z}^2$-rotation $R_0 : \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_0 \to \mathbb{R}^2/\Gamma_0$ defined by

$$R_0^{\mathbf{n}}(\mathbf{x}) := R_0(\mathbf{n}, \mathbf{x}) = \mathbf{x} + \mathbf{n}$$

for every $\mathbf{n} \in \mathbb{Z}^2$. We consider the fundamental domain $\mathcal{D} = [0, \varphi[\times [0, \varphi + 3[$ of $\mathbb{R}^2$ for the group of translations Γ_0. Let $I = \{0, 1, 2, 3\}$ and $J = \{0, 1, 2, 3, 4\}$ be sets of colors and consider the partitions $\mathcal{Y} = \{Y_i\}_{i \in I}$ and $\mathcal{Z} = \{Z_j\}_{j \in J}$ shown in Figure 9.1.

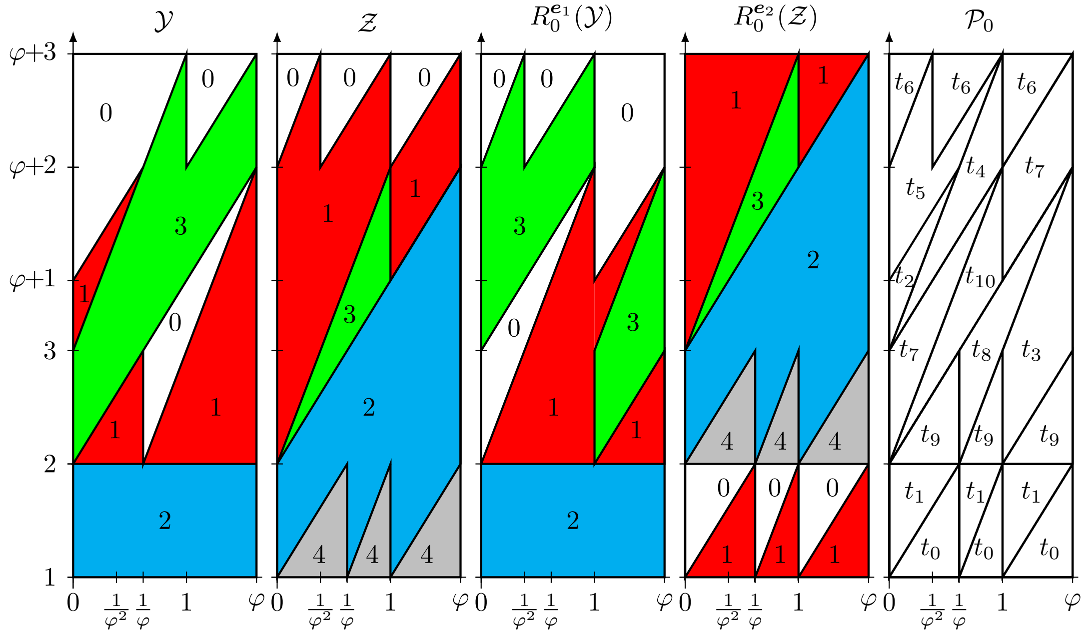


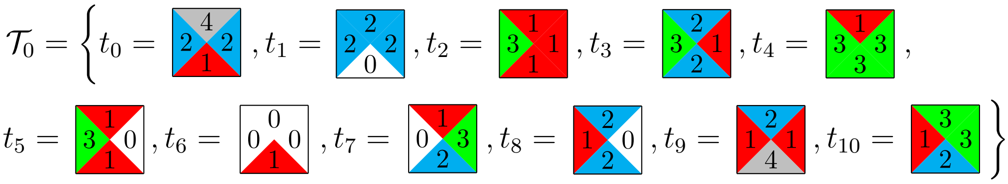
Figure 9.1. Partitions for the 11 Jeandel–Rao Wang tiles. From left to right, the partition $\mathcal{Y}$ for the right color, $\mathcal{Z}$ for the top color, $R_0^{\mathbf{e}_1}(\mathcal{Y})$ for the left color and $R_0^{\mathbf{e}_2}(\mathcal{Z})$ for the bottom color. Their refinement is the partition $\mathcal{P}_0$ where each part is associated with one of the Jeandel–Rao Wang tiles.

The refined partition is $\mathcal{P}_0 = \mathcal{Y} \wedge \mathcal{Z} \wedge R_0^{\mathbf{e}_1}(\mathcal{Y}) \wedge R_0^{\mathbf{e}_2}(\mathcal{Z}) = \{P_t\}_{t \in I \times J \times I \times J}$. The set of quadruples (i, j, k, ℓ) such that $P_{(i,j,k,\ell)} = Y_i \cap Z_j \cap R^{\mathbf{e}_1}(Y_k) \cap R^{\mathbf{e}_2}(Z_\ell)$ is nonempty is

$$\mathcal{T}_0 = \big\{(2,4,2,1), (2,2,2,0), (1,1,3,1), (1,2,3,2), (3,1,3,3), (0,1,3,1), (0,0,0,1), (3,1,0,2), (0,2,1,2), (1,2,1,4), (3,3,1,2)\big\}$$

which can be seen as a set of Wang tiles

(9.1)



We observe that $\mathcal{T}_0$ is Jeandel–Rao's set of 11 tiles [JR15]. Let $\Omega_0 = \Omega_{\mathcal{T}_0}$ be the Jeandel–Rao Wang shift. We may now prove Theorem 1.1 which follows mostly from the work done in the Part 1.

Proof of Theorem 1.1. —

(i) The dynamical system $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$ is minimal. Since $R_0^{\mathbf{e}_1}$ and $R_0^{\mathbf{e}_2}$ are linearly independent irrational rotations on $\mathbb{R}^2/\Gamma_0$, we have that R_0 is a free $\mathbb{Z}^2$-action. Thus from Lemma 5.2, $\mathcal{X}_{\mathcal{P}_0,R_0}$ is minimal and aperiodic. From Proposition 8.1 and from Equation (9.1), we have $\mathcal{X}_{\mathcal{P}_0,R_0} \subseteq \Omega_0$. It was proved in [Lab19b] that the Jeandel–Rao Wang shift Ω_0 is not minimal. Thus $\Omega_0 \setminus \mathcal{X}_{\mathcal{P}_0,R_0}$ is nonempty.

(ii) The atom $P_{t_{10}}$ is invariant only under the trivial translation. Therefore, from Lemma 3.4, $\mathcal{P}_0$ gives a symbolic representation of $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$.

(iii) From Proposition 5.1, there exists a factor map f_0 from $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma)$ to $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$ and from Corollary 5.3, $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$ is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma)$.

(iv) From Proposition 5.1, we have that f_0 is one-to-one on $\mathbb{T}^2 \setminus \Delta_{\mathcal{P}_0,R_0}$. Suppose that $\mathbf{x} \in \Delta_{\mathcal{P}_0,R_0}$. We have $\operatorname{card}(f_0^{-1}(\mathbf{x})) \geqslant 2$. If $\operatorname{card}(f_0^{-1}(\mathbf{x})) > 2$, then we may show that there exists $\mathbf{n} \in \mathbb{Z}^2$ such that $\mathbf{x} = R_0^{\mathbf{n}}(\mathbf{0})$. If $\mathbf{x} = R_0^{\mathbf{n}}(\mathbf{0})$ for some $\mathbf{n} \in \mathbb{Z}^2$, then the set $f_0^{-1}(\mathbf{x})$ contains 8 different configurations of the form $\textsc{SymbRep}_0^{\mathbf{v}}(0)$ for some $\mathbf{v} \in \mathbb{R}^2 \setminus \Theta^{\mathcal{P}_0}$ where $\Theta^{\mathcal{P}_0} = \mathbb{R} \cdot \{(1,0), (0,1), (1,\varphi), (1,\varphi^2)\}$. If $\mathbf{x} \in \Delta_{\mathcal{P}_0,R_0}$ but is not in the orbit of $\mathbf{0}$ under R_0, then $\operatorname{card}(f_0^{-1}(\mathbf{x})) = 2$. We conclude that $\{\operatorname{card}(f_0^{-1}(\mathbf{x})) \mid \mathbf{x} \in \mathbb{R}^2/\Gamma_0\} = \{1, 2, 8\}$.

(v) We have that $\lambda(\partial P) = 0$ for each atom $P \in \mathcal{P}_0$ where λ is the Haar measure on $\mathbb{R}^2/\Gamma_0$. The result follows from Proposition 6.1. □

The frequency of any pattern p in $\mathcal{X}_{\mathcal{P}_0,R_0}$ is equal to the measure of the associated cylinder $[p]$ in $\mathcal{X}_{\mathcal{P}_0,R_0}$ which is equal to the Haar measure of $f_0([p])$ in $\mathbb{R}^2/\Gamma_0$ and can be computed using Equation (6.1) as the area of the coding region which is the intersection of polygons. Here is what it gives for the frequencies of tiles in $\mathcal{X}_{\mathcal{P}_0,R_0}$.

Proposition 9.1. — *The frequencies of each of the 11 Jeandel-Rao tiles t_i for $i \in \{0, \ldots, 10\}$ in the subshift $\mathcal{X}_{\mathcal{P}_0,R_0}$ is given by the measure of the cylinders below:*

$$\begin{aligned}
\nu([7]) &= 5/(12\varphi+14) &&\approx 0.1496,\\
\nu([0]) = \nu([1]) = \nu([3]) = \nu([6]) = \nu([9]) &= 1/(2\varphi+6) &&\approx 0.1083,\\
\nu([5]) &= 1/(5\varphi+4) &&\approx 0.0827,\\
\nu([4]) = \nu([8]) = \nu([10]) &= 1/(8\varphi+2) &&\approx 0.0669,\\
\nu([2]) &= 1/(18\varphi+10) &&\approx 0.0256.
\end{aligned}$$

Proof. — Thanks to Theorem 1.1 (v), it can be computed from the area of atoms of the partition $\mathcal{P}_0$ shown in Figure 9.1 and dividing by the area of the fundamental domain which is $\varphi(\varphi+3) = 4\varphi+1$. □

We may check that the frequency of each tile in the minimal subshift $\mathcal{X}_{\mathcal{P}_0,R_0} \subset \Omega_0$ match those obtained in [Lab19b] for some minimal subshift $X_0 \subset \Omega_0$ and computed from the substitutive structure of X_0. In fact, $X_0 = \mathcal{X}_{\mathcal{P}_0,R_0}$ but we postpone the proof of this in a later work [Lab20] in which the substitutive structure of $\mathcal{X}_{\mathcal{P}_0,R_0}$ is described using Rauzy induction of toral partitions and $\mathbb{Z}^2$-rotations. In [Lab19b], we also proved that X_0 is a shift of finite type as it can be described by the forbidden patterns coming from Ω_0 plus a finite number of other forbidden patterns. The completion of the proof of the equality $X_0 = \mathcal{X}_{\mathcal{P}_0,R_0}$ will imply that the following statement holds, but we can only state it as a conjecture for now.

Conjecture 9.2. — *$\mathcal{P}_0$ is a Markov partition for $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$.*

10. Example 2: A minimal aperiodic Wang shift defined by 19 tiles

On the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we consider the $\mathbb{Z}^2$-rotation $R_{\mathcal{U}} : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$R_{\mathcal{U}}^{\mathbf{n}}(\mathbf{x}) := R_{\mathcal{U}}(\mathbf{n}, \mathbf{x}) = \mathbf{x} + \varphi^{-2}\mathbf{n}$$

for every $\mathbf{n} \in \mathbb{Z}^2$ where $\varphi = \frac{1+\sqrt{5}}{2}$. Let $I = \{\mathrm{A, B, C, D, E, F, G, H, I, J}\}$ and $J = \{\mathrm{K, L, M, N, O, P}\}$ be sets of colors and consider the partitions $\mathcal{Y} = \{Y_i\}_{i\in I}$ and $\mathcal{Z} = \{Z_j\}_{j\in J}$ shown in Figure 10.1.

The refined partition is $\mathcal{P}_{\mathcal{U}} = \mathcal{Y} \wedge \mathcal{Z} \wedge R_{\mathcal{U}}^{\mathbf{e}_1}(\mathcal{Y}) \wedge R_{\mathcal{U}}^{\mathbf{e}_2}(\mathcal{Z}) = \{P_u\}_{u\in I\times J\times I\times J}$. Let $\mathcal{U}$ be the set of quadruples (i, j, k, ℓ) such that $P_{(i,j,k,\ell)} = Y_i \cap Z_j \cap R_{\mathcal{U}}^{\mathbf{e}_1}(Y_k) \cap R_{\mathcal{U}}^{\mathbf{e}_2}(Z_\ell)$ is nonempty. We represent $\mathcal{U}$ as a set of Wang tiles, see Figure 10.2. It corresponds to the set of 19 Wang tiles $\mathcal{U} = \{u_0, u_1, \ldots, u_{18}\}$ introduced in [Lab19a] which was derived from the substitutive structure of the Jeandel–Rao Wang shift [Lab19b].

Let $\Omega_{\mathcal{U}}$ be the Wang shift associated with the set of Wang tiles $\mathcal{U}$. We now prove the second theorem of the article. Note that the fact that $\Omega_{\mathcal{U}}$ is minimal (proved in [Lab19a]) allows to conclude that $\mathcal{P}_{\mathcal{U}}$ is a Markov partition without having to exploit the substitutive structure of $\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}}$ which is computed in [Lab20].

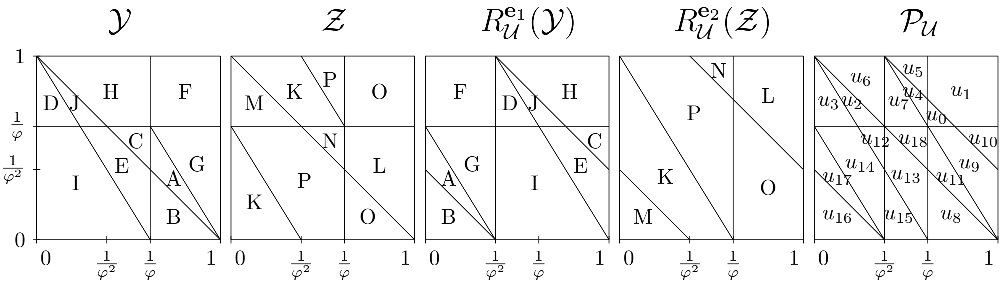


Figure 10.1. Partitions for the set $\mathcal{U}$ of 19 Wang tiles. From left to right, the partition $\mathcal{Y}$ for the right color, $\mathcal{Z}$ for the top color, $R_{\mathcal{U}}^{\mathbf{e}_1}(\mathcal{Y})$ for the left color and $R_{\mathcal{U}}^{\mathbf{e}_2}(\mathcal{Z})$ for the bottom color. Their refinement is the partition $\mathcal{P}_{\mathcal{U}}$ where each part is associated with a Wang tile.

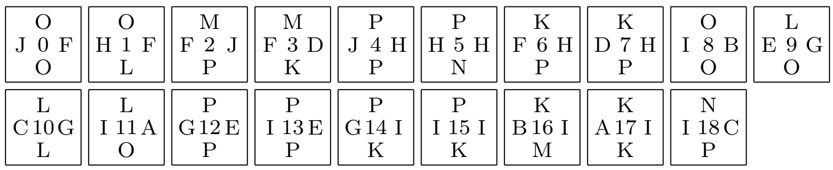


Figure 10.2. The set $\mathcal{U} = \{u_0, \ldots, u_{18}\}$ of 19 Wang tiles. Each index $i \in \{0, \ldots, 18\}$ written in the middle of a tile corresponds to a tile u_i.

Proof of Theorem 1.2. —

(i) The dynamical system $(\mathbb{T}^2, \mathbb{Z}^2, R_{\mathcal{U}})$ is minimal. Since $R_{\mathcal{U}}^{\mathbf{e}_1}$ and $R_{\mathcal{U}}^{\mathbf{e}_2}$ are linearly independent irrational rotations, we have that $R_{\mathcal{U}}$ is a free $\mathbb{Z}^2$-action. Thus from Lemma 5.2, $\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}}$ is minimal and aperiodic. From Proposition 8.1, we have $\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}} \subseteq \Omega_{\mathcal{U}}$. It was proved in [Lab19a] that $\Omega_{\mathcal{U}}$ is minimal. Thus $\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}} = \Omega_{\mathcal{U}}$.

(ii) The atom P_{u_0} is invariant only under the trivial translation. Therefore, from Lemma 3.4, $\mathcal{P}_{\mathcal{U}}$ gives a symbolic representation of $(\mathbb{T}^2, \mathbb{Z}^2, R_{\mathcal{U}})$. Moreover $\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}} = \Omega_{\mathcal{U}}$ is a shift of finite type. Therefore, the conditions of Definition 3.2 are satisfied and $\mathcal{P}_{\mathcal{U}}$ is a Markov partition for the dynamical system $(\mathbb{T}^2, \mathbb{Z}^2, R_{\mathcal{U}})$.

(iii) From Proposition 5.1, there exists a factor map $f_{\mathcal{U}}$ from $(\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}}, \mathbb{Z}^2, \sigma)$ to $(\mathbb{T}^2, \mathbb{Z}^2, R_{\mathcal{U}})$ and from Corollary 5.3, $(\mathbb{T}^2, \mathbb{Z}^2, R_{\mathcal{U}})$ is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}}, \mathbb{Z}^2, \sigma)$.

(iv) In Proposition 5.1, we proved that $f_{\mathcal{U}}$ is one-to-one on $\mathbb{T}^2 \setminus \Delta_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}}$. Suppose that $\mathbf{x} \in \Delta_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}}$. We have $\operatorname{card}(f_{\mathcal{U}}^{-1}(\mathbf{x})) \geqslant 2$. If $\operatorname{card}(f_{\mathcal{U}}^{-1}(\mathbf{x})) > 2$, then we may show that there exists $\mathbf{n} \in \mathbb{Z}^2$ such that $\mathbf{x} = R_{\mathcal{U}}^{\mathbf{n}}(\mathbf{0})$. If $\mathbf{x} = R_{\mathcal{U}}^{\mathbf{n}}(\mathbf{0})$ for some $\mathbf{n} \in \mathbb{Z}^2$, then the set $f_{\mathcal{U}}^{-1}(\mathbf{x})$ contains 8 different configurations of the form $\textsc{SymbRep}_{\mathcal{U}}^{\mathbf{v}}(0)$ for some $\mathbf{v} \in \mathbb{R}^2 \setminus \Theta^{\mathcal{P}_{\mathcal{U}}}$ where $\Theta^{\mathcal{P}_{\mathcal{U}}} = \mathbb{R} \cdot \{(1,0),(0,1),(1,-1),(1,-\varphi)\}$. If $\mathbf{x} \in \Delta_{\mathcal{P}_{\mathcal{U}},R_{\mathcal{U}}}$ but not in the orbit of $\mathbf{0}$ under $R_{\mathcal{U}}$, then $\operatorname{card}(f_{\mathcal{U}}^{-1}(\mathbf{x})) = 2$. We conclude that $\{\operatorname{card}(f_{\mathcal{U}}^{-1}(\mathbf{x})) \mid \mathbf{x} \in \mathbb{T}^2\} = \{1,2,8\}$.

(v) We have that $\lambda(\partial P) = 0$ for each atom $P \in \mathcal{P}_{\mathcal{U}}$ where λ is the Haar measure on $\mathbb{T}^2$. The result follows from Proposition 6.1. $\square$

11. Two non-examples

In this section, we present two "non-examples". The motivation for presenting them is to illustrate that properties of partitions presented in the previous sections are not shared by "randomly" chosen partitions of $\mathbb{T}^2$ and $\mathbb{Z}^2$-rotations on $\mathbb{T}^2$.

Example 3

Let $\alpha, \beta \in \mathbb{R}^2$. On the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we consider the $\mathbb{Z}^2$-rotation $R : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$R^{\mathbf{n}}(\mathbf{x}) := R(\mathbf{n}, \mathbf{x}) = \mathbf{x} + n_1\alpha + n_2\beta$$

for every $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$. Let $\mathcal{Y} = \{Y_A\}$ and $\mathcal{Z} = \{Z_B\}$ be trivial partitions with $Y_A = Z_B = \mathbb{T}^2$. The refined partition is $\mathcal{P} = \mathcal{Y} \wedge \mathcal{Z} \wedge R^{\mathbf{e}_1}(\mathcal{Y}) \wedge R^{\mathbf{e}_2}(\mathcal{Z}) = \{P_{(A,\,B,\,A,\,B)}\}$ where $P_{(A,\,B,\,A,\,B)} = \mathbb{T}^2$. The set of Wang tiles $\mathcal{T} = \{\tau\}$ is a singleton set with $\tau = (A,\, B,\, A,\, B)$. The associated color partitions and the tile coding partition are shown in Figure 11.1.

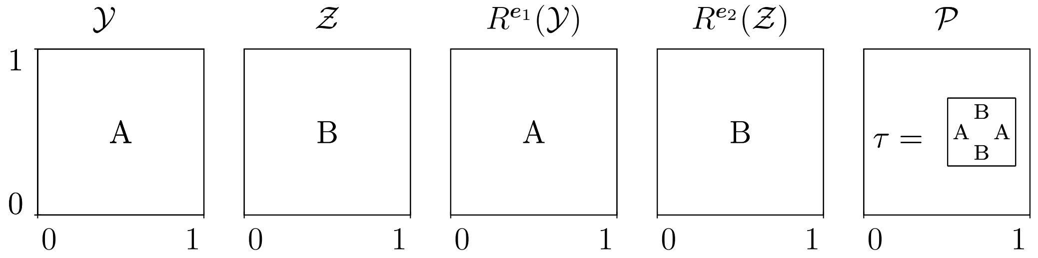


Figure 11.1. Partitions for the Example 3. From left to right, the partition $\mathcal{Y}$ for the right color, $\mathcal{Z}$ for the top color, $R^{\mathbf{e}_1}(\mathcal{Y})$ for the left color and $R^{\mathbf{e}_2}(\mathcal{Z})$ for the bottom color. Their refinement is the trivial partition $\mathcal{P}$ whose single atom is associated with the Wang tile τ.

The map $\textsc{SymbRep} : \mathbb{T}^2 \to \Omega_{\mathcal{T}}$ is clearly not one-to-one, but it is onto.

Lemma 11.1. — *We have $\mathcal{X}_{\mathcal{P},R} = \Omega_{\mathcal{T}}$, but the partition $\mathcal{P}$ does not give a symbolic representation of* $(\mathbb{T}^2, \mathbb{Z}^2, R)$.

Proof. — The set of Wang tiles $\mathcal{T} = \{\tau\}$ is a singleton set with $\tau = (A, B, A, B)$. Therefore $\Omega_{\mathcal{T}}$ contains a unique configuration corresponding to the constant map $(m, n) \mapsto \tau$ for all $m, n \in \mathbb{Z}$. The fact that $\mathcal{X}_{\mathcal{P},R} \subseteq \Omega_{\mathcal{T}}$ follows from Proposition 8.1. The unique constant configuration in $\Omega_{\mathcal{T}}$ can be obtained as $\textsc{SymbRep}(\mathbf{x}) = \textsc{Config}^{\mathcal{P},R}_{\mathbf{x}}$ for any $\mathbf{x} \in \mathbb{T}^2$. Therefore $\textsc{SymbRep}$ is onto.

The partition $\mathcal{P}$ does not give a symbolic representation of $(\mathbb{T}^2, \mathbb{Z}^2, R)$ as every point of $\mathbb{T}^2$ is associated with the same configuration. □

Example 4

Let $\varphi = \frac{1+\sqrt{5}}{2}$. On the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we consider the $\mathbb{Z}^2$-rotation $R : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$R^{\mathbf{n}}(\mathbf{x}) := R(\mathbf{n}, \mathbf{x}) = \mathbf{x} + \varphi\, \mathbf{n}$$

for every $\mathbf{n} \in \mathbb{Z}^2$. Let $I = \{A, B\}$ and $J = \{C, D\}$ be sets of colors. We consider the partitions $\mathcal{Y} = \{Y_A, Y_B\}$ and $\mathcal{Z} = \{Z_C, Z_D\}$ shown in Figure 11.2 involving slopes 1 and -1 in the partition of $\mathbb{T}^2$ into polygons. The refined partition is $\mathcal{P} = \mathcal{Y} \wedge \mathcal{Z} \wedge R^{\mathbf{e}_1}(\mathcal{Y}) \wedge R^{\mathbf{e}_2}(\mathcal{Z}) = \{P_\tau\}_{\tau \in \mathcal{T}}$ where $\mathcal{T}$ is the set of Wang tiles made of 20 tiles shown in Figure 11.3.

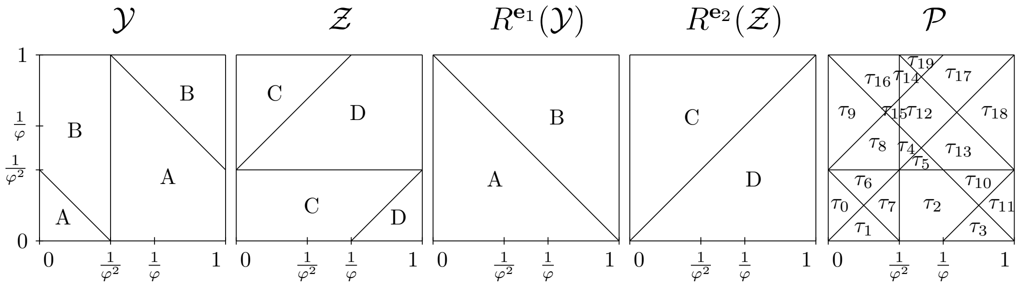


Figure 11.2. Partitions for the Example 4. From left to right, the partition $\mathcal{Y}$ for the right color, $\mathcal{Z}$ for the top color, $R^{\mathbf{e}_1}(\mathcal{Y})$ for the left color and $R^{\mathbf{e}_2}(\mathcal{Z})$ for the bottom color. Their refinement is the partition $\mathcal{P}$ where each part is associated with a Wang tile.

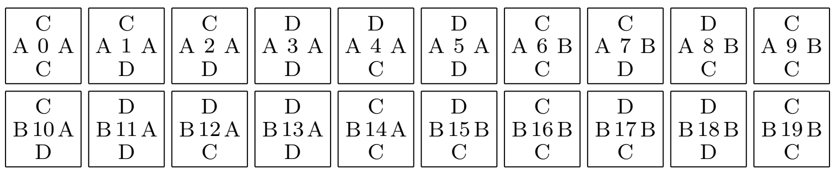


Figure 11.3. The set of 20 tiles $\mathcal{T} = \{\tau_0, \ldots, \tau_{19}\}$. Each index $i \in \{0, \ldots, 19\}$ written in the middle of a tile corresponds to a tile τ_i. The Wang shift $\Omega_{\mathcal{T}}$ contains periodic configurations.

Lemma 11.2. — *The partition $\mathcal{P}$ gives a symbolic representation of $(\mathbb{T}^2, \mathbb{Z}^2, R)$ and $(\mathbb{T}^2, \mathbb{Z}^2, R)$ is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$. We have that $\mathcal{X}_{\mathcal{P},R}$ is a strictly ergodic and aperiodic subshift of $\Omega_{\mathcal{T}}$. But the Wang shift $\Omega_{\mathcal{T}}$ contains a periodic configuration so $\mathcal{X}_{\mathcal{P},R} \subsetneq \Omega_{\mathcal{T}}$.*

Proof. — The dynamical system $(\mathbb{T}^2, \mathbb{Z}^2, R)$ is minimal. The atom P_{τ_0} is invariant only under the trivial translation. Therefore, from Lemma 3.4, $\mathcal{P}$ gives a symbolic representation of $(\mathbb{T}^2, \mathbb{Z}^2, R)$. From Proposition 5.1, there exists a factor map from $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$ to $(\mathbb{T}^2, \mathbb{Z}^2, R)$ and from Corollary 5.3, $(\mathbb{T}^2, \mathbb{Z}^2, R)$ is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma)$.

Since $R^{\mathbf{e}_1}$ and $R^{\mathbf{e}_2}$ are linearly independent irrational rotations, we have that R is a free $\mathbb{Z}^2$-action. Thus from Lemma 5.2, $\mathcal{X}_{\mathcal{P},R}$ is minimal and aperiodic. From Proposition 6.1, $\mathcal{X}_{\mathcal{P},R}$ is uniquely ergodic thus strictly ergodic. From Proposition 8.1, we have $\mathcal{X}_{\mathcal{P},R} \subseteq \Omega_{\mathcal{T}}$. The set of Wang tiles $\mathcal{T}$ contains the tile $\tau_0 = (A, C, A, C)$. Let w be the constant map $(m, n) \mapsto \tau_0$ for all $m, n \in \mathbb{Z}$. The configuration w is valid and periodic, thus $w \in \Omega_{\mathcal{T}} \setminus \mathcal{X}_{\mathcal{P},R}$. □

The two examples presented in this section show that we can not expect Theorem 1.1 and Theorem 1.2 to hold for any given toral partition and $\mathbb{Z}^2$-rotation. The characterization of toral partitions and $\mathbb{Z}^2$-rotations for which such results hold is an open question.

Part 3. Wang shifts as model sets of cut and project schemes

This part is divided into three sections. Its goal is to show that occurrences of patterns in a minimal subshift of the Jeandel-Rao Wang shift and in the Wang shift $\Omega_{\mathcal{U}}$ are obtained as 4-to-2 cut and project schemes.

12. Cut and project schemes and model sets

In [BHP97], the torus parametrization of three tiling dynamical systems was given. We want to do similarly in the case of symbolic dynamical systems and in particular in the case of Wang shifts. We recall from the more recent book [BG13, Section 7.2] the definition of cut and project scheme and we reuse their notation.

Definition 12.1. — *A* cut and project scheme *(CPS) is a triple $(\mathbb{R}^d, H, \mathcal{L})$ with a (compactly generated) locally compact Abelian group (LCAG) H, a lattice $\mathcal{L}$ in $\mathbb{R}^d \times H$ and the two natural projections $\pi : \mathbb{R}^d \times H \to \mathbb{R}^d$ and $\pi_{\text{int}} : \mathbb{R}^d \times H \to H$, subject to the conditions that $\pi|_{\mathcal{L}}$ is injective and that $\pi_{\text{int}}(\mathcal{L})$ is dense in H.*

A CPS is called *Euclidean* when $H = \mathbb{R}^m$ for some $m \in \mathbb{N}$. A general CPS is summarized in the following diagram.

$$\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\pi} & \mathbb{R}^d \times H & \xrightarrow{\pi_{\text{int}}} & H \\
\cup & & \cup & & \cup \text{ dense} \\
\pi(\mathcal{L}) & \xleftarrow{1-1} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\quad\quad\star\quad\quad} & & L^\star
\end{array}$$

The image is denoted $L = \pi(\mathcal{L})$. Since for a given CPS, π is a bijection between $\mathcal{L}$ and L, there is a well-defined mapping $\star : L \to H$ given by

$$x \mapsto x^\star := \pi_{\text{int}}\left((\pi|_{\mathcal{L}})^{-1}(x)\right)$$

where $(\pi|_{\mathcal{L}})^{-1}(x)$ is the unique point in the set $\mathcal{L} \cap \pi^{-1}(x)$. This mapping is called the *star map* of the CPS. The $\star$-image of L is denoted by $L^\star$. The set $\mathcal{L}$ can be viewed as a diagonal embedding of L as

$$\mathcal{L} = \{(x, x^\star) \mid x \in L\}.$$

For a given CPS $(\mathbb{R}^d, H, \mathcal{L})$ and a (general) set $A \subset H$,

$$\curlywedge(A) := \{x \in L \mid x^\star \in A\}$$

is the projection set within the CPS. The set A is called its *acceptance set*, *window* or *coding set*.

Definition 12.2. — *If $A \subset H$ is a relatively compact set with non-empty interior, the projection set $\curlywedge(A)$, or any translate $t + \curlywedge(A)$ with $t \in \mathbb{R}^d$, is called* a model set.

A model set is termed *regular* when $\mu_H(\partial A) = 0$, where μ_H is the Haar measure of H. If $L^\star \cap \partial A = \varnothing$, the model set is called *generic*. If the window is not in a generic position (meaning that $L^\star \cap \partial A \neq \varnothing$), the corresponding model set is called *singular*.

The shape of the acceptance set A is important and implies structure on the model set $\Lambda = t + \curlywedge(A)$. For example, if A is relatively compact, Λ has finite local complexity and thus also is uniformly discrete; if $A^\circ \neq \varnothing$, Λ is relatively dense. If Λ is a model set, it is also a Meyer set, [BG13, Proposition 7.5]. For regular model set $\Lambda = \curlywedge(A)$ with a compact window $A = \overline{A^\circ}$, it is known [BG13, Theorem 7.2] that the points $\{x^\star \mid x \in \Lambda\}$ are uniformly distributed in A.

Linear repetitivity of model sets is an important notion. Recall that a Delone set $Y \subseteq \mathbb{R}^d$ is called *linearly repetitive* if there exists a constant $C > 0$ such that, for any $r \geqslant 1$, every patch of size r in Y occurs in every ball of diameter Cr in $\mathbb{R}^d$. It was shown by Lagarias and Pleasants in [LP03, Theorem 6.1] that linear repetitivity of a Delone set implies the existence of strict uniform patch frequencies, equivalently the associated dynamical system on the hull of the point set is strictly ergodic (minimal and uniquely ergodic). As a consequence [LP03, Corollary 6.1], a linearly repetitive Delone set X in $\mathbb{R}^n$ has a unique autocorrelation measure γ_X. This measure γ_X is a pure discrete measure supported on $X - X$. In particular X is diffractive. A characterization of linearly repetitive model sets $\curlywedge(A)$ for cubical acceptance set A was recently proved by Haynes, Koivusalo and Walton [HKW18].

Polygon exchange transformations

We end this section with a concept that will be useful for the next two Sections 13 and 14. Suppose that $(\mathbf{T}, \mathbb{Z}^2, R)$ is a dynamical system where R is a $\mathbb{Z}^2$-rotation on $\mathbf{T}$. The rotations $R^{\mathbf{e}_1}$ and $R^{\mathbf{e}_2}$ can be seen as polygon exchange transformations [Sch14] on a fundamental domain of $\mathbf{T}$.

Definition 12.3. — [AKY19] *Let X be a polygon together with two topological partitions of X into polygons*

$$X = \bigcup_{k=0}^{N} P_k = \bigcup_{k=0}^{N} Q_k$$

such that for each k, P_k and Q_k are translation equivalent, i.e., there exists $v_k \in \mathbb{R}^2$ such that $P_k = Q_k + v_k$. A polygon exchange transformation (PET) *is the piecewise translation on X defined for $x \in P_k$ by $T(x) = x + v_k$. The map is not defined for points $x \in \cup_{k=0}^{N} \partial P_k$.*

13. A model set for the Jeandel-Rao Wang shift

We want to describe the positions $Q \subseteq \mathbb{Z}^2$ of patterns in configurations belonging to $\mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$. Because of that, in the construction of a proper cut and project scheme, we need to be careful in the choice of the locally compact Abelian group H so that $\pi|_{\mathcal{L}}$ is an injective map. This is why we introduce the submodule $\Lambda = \langle (1,-1,0,0),(0,0,1,-1)\rangle_{\mathbb{Z}}$ and define the projections π and π_{int} on $\mathbb{R}^4/\Lambda$ as:

$$\begin{array}{rcl} \pi : \mathbb{R}^4/\Lambda & \to & \mathbb{R}^2 \\ (x_1,x_2,x_3,x_4) & \mapsto & (x_1+x_2, x_3+x_4) \end{array}$$

and

$$\begin{array}{rcl} \pi_{\mathrm{int}} : \mathbb{R}^4/\Lambda & \to & \mathbb{R}^2/\Gamma_0 \\ (x_1,x_2,x_3,x_4) & \mapsto & \left(x_1 - \frac{1}{\varphi}x_2 + \frac{1}{\varphi}x_4, x_3 - (\varphi+2)x_4\right) \end{array}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. The product $\pi \times \pi_{\mathrm{int}} : \mathbb{R}^4/\Lambda \to \mathbb{R}^2 \times \mathbb{R}^2/\Gamma_0$ of the projections is one-to-one and onto. Therefore, the projections define a Euclidean cut and project scheme with $d = 2$ and $H = \mathbb{R}^2/\Gamma_0$ on $\mathbb{R}^4/\Lambda \simeq \mathbb{R}^2 \times H$.

Recall that we proved in Theorem 1.1 that $\mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$ and that there exists a factor map f_0 from $(\mathcal{X}_{\mathcal{P}_0,R_0}, \mathbb{Z}^2, \sigma)$ to $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$. Therefore any Jeandel–Rao configuration $w \in \mathcal{X}_{\mathcal{P}_0,R_0} \subsetneq \Omega_0$ can be qualified as a singular or generic according to whether $f_0(w)$ is in the set $\Delta_{\mathcal{P}_0,R_0} \subset \mathbb{R}^2/\Gamma_0$ or not.

Proof of Theorem 1.3. — Let $w \in \mathcal{X}_{\mathcal{P}_0,R_0}$. Let $\mathbf{x} = (r,s) = r'(\varphi,0) + s'(1,\varphi+3) = f_0(w) \in \mathbb{R}^2/\Gamma_0$. We consider the lattice $\mathcal{L} = \mathbb{Z}^4 + (r'+s', -r'-s', s', -s') \subset \mathbb{R}^4/\Lambda$. We have that $\pi|_{\mathcal{L}}$ is injective. Also $L = \pi(\mathcal{L}) = \mathbb{Z}^2$ since $\pi(r'+s', -r'-s', s', -s') = \mathbf{0}$. We also have that $\pi_{\mathrm{int}}(\mathcal{L})$ is dense in $H = \mathbb{R}^2/\Gamma_0$. Also $\pi_{\mathrm{int}}(r'+s', -r'-s', s', -s') = (r,s)$.

Recall that the $\mathbb{Z}^2$-rotation R_0 is defined on the torus $\mathbb{R}^2/\Gamma_0$ by $R_0^{\mathbf{n}}(\mathbf{x}) = \mathbf{x} + \mathbf{n}$ for every $\mathbf{n} \in \mathbb{Z}^2$. The maps $(R_0)^{\mathbf{e}_1}$ and $(R_0)^{\mathbf{e}_2}$ can be seen as polygon exchange transformations on the fundamental domain $W = [0,\varphi) \times [0,\varphi+3)$ of $\mathbb{R}^2/\Gamma_0$ (see Figure 13.1):

$$(R_0)^{\mathbf{e}_1}(\mathbf{x}) = \begin{cases} \mathbf{x} + v_a & \text{if } \mathbf{x} \in P_a, \\ \mathbf{x} + v_b & \text{if } \mathbf{x} \in P_b, \end{cases} \quad \text{and} \quad (R_0)^{\mathbf{e}_2}(\mathbf{x}) = \begin{cases} \mathbf{x} + v_c & \text{if } \mathbf{x} \in P_c, \\ \mathbf{x} + v_d & \text{if } \mathbf{x} \in P_d, \\ \mathbf{x} + v_e & \text{if } \mathbf{x} \in P_e. \end{cases}$$

The translations written in terms of the base of $\mathbb{Z}^2$ and Γ_0 and vice versa are:

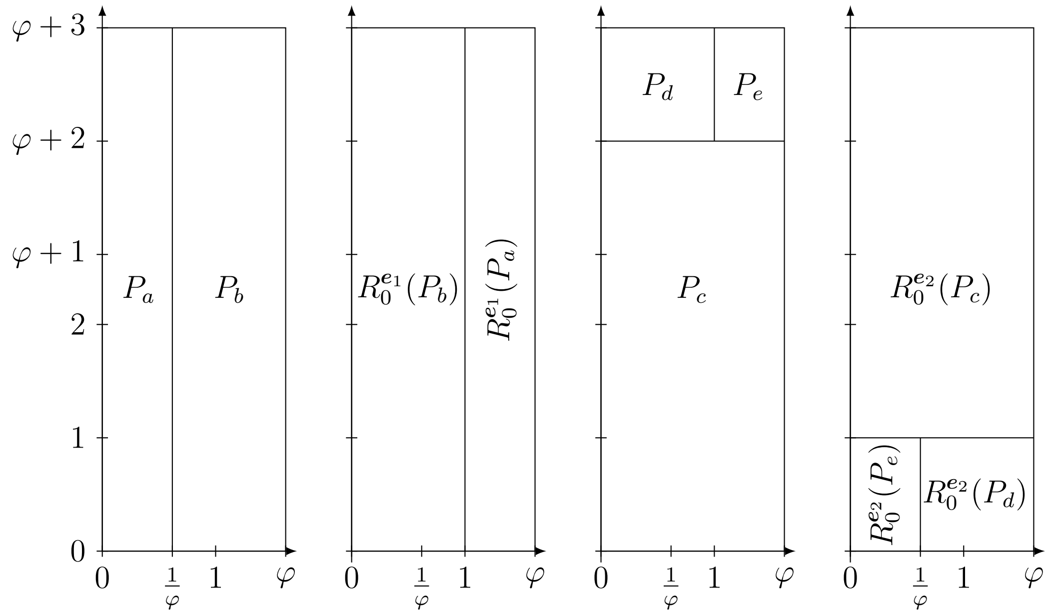


Figure 13.1. The maps $(R_0)^{\mathbf{e}_1}$ and $(R_0)^{\mathbf{e}_2}$ can be seen as polygon exchange transformations on the fundamental domain $W = [0,\varphi) \times [0, \varphi+3)$ of $\mathbb{R}^2/\Gamma_0$.

$$\begin{aligned}
v_a &= \mathbf{e}_1 & \mathbf{e}_1 &= v_a,\\
v_b &= \mathbf{e}_1 - (\varphi,0) & \mathbf{e}_2 &= v_c,\\
v_c &= \mathbf{e}_2 & (\varphi,0) &= v_a - v_b,\\
v_d &= \mathbf{e}_2 - (1,\varphi+3) + (\varphi,0) & (1,\varphi+3) &= v_a - v_b + v_c - v_d,\\
v_e &= \mathbf{e}_2 - (1,\varphi+3).
\end{aligned}$$

Since W is a fundamental domain for $\Gamma_0 = \langle(\varphi,0),(1,\varphi+3)\rangle_{\mathbb{Z}}$, by definition for every $\mathbf{x} \in \mathbb{R}^2$, there exist unique $k,\ell \in \mathbb{Z}$ such that $\mathbf{x} + k(\varphi,0) + \ell(1,\varphi+3) \in W$. Therefore, for every $(m,n) \in \mathbb{Z}^2$ there exist unique $k,\ell \in \mathbb{Z}$ such that the following holds

$$\begin{aligned}
R_0^{(m,n)}(r,s) &= (r,s) + (m,n) \bmod \Gamma_0\\
&= r'(\varphi,0) + s'(1,\varphi+3) + m\mathbf{e}_1 + n\mathbf{e}_2 + k(\varphi,0) + \ell(1,\varphi+3) \in W\\
&= m\mathbf{e}_1 + n\mathbf{e}_2 + (r'+k)(\varphi,0) + (s'+\ell)(1,\varphi+3)\\
&= mv_a + nv_c + (r'+k)(v_a - v_b) + (s'+\ell)(v_a - v_b + v_c - v_d)\\
&= (m + r' + k + s' + \ell)v_a - (r' + k + s' + \ell)v_b\\
&\qquad\qquad + (n + s' + \ell)v_c - (s'+\ell)v_d\\
&= \pi_{\mathrm{int}}\Big((m + r' + k + s' + \ell, -r' - k - s' - \ell,\\
&\qquad\qquad n + s' + \ell, -s' - \ell) + \Lambda\Big) \in \pi_{\mathrm{int}}(\mathcal{L}).
\end{aligned}$$

Notice that the projection into the physical space is

$$\pi\Big((m+r'+k+s'+\ell,-r'-k-s'-\ell,n+s'+\ell,-s'-\ell)+\Lambda\Big)=(m,n).$$

Thus

$$(m+r'+k+s'+\ell,-r'-k-s'-\ell,n+s'+\ell,-s'-\ell)+\Lambda=(\pi|_{\mathcal{L}})^{-1}(m,n)$$

so that

$$\begin{aligned}(m,n)^\star &= \pi_{\text{int}}\left((\pi|_{\mathcal{L}})^{-1}(m,n)\right)\\ &= \pi_{\text{int}}\Big((m+r'+k+s'+\ell,-r'-k-s'-\ell,n+s'+\ell,-s'-\ell)+\Lambda\Big)\\ &= R_0^{(m,n)}(r,s).\end{aligned}$$

Let $p=\pi_S(w)\in\mathcal{T}_0^S$ be a pattern occurring in the configuration w for some subset $S\subset\mathbb{Z}^2$. Let $[p]$ be the cylinder associated with the pattern p and $A=f_0([p])\subset W$ be the acceptance set. The set A is a polygon by construction, see Equation (6.1). Therefore the Lebesgue measure of ∂A is zero. Assume for now that w is a generic configuration. Since $R_0^{(m,n)}(r,s)=(m,n)^\star\notin\partial A$ for every $m,n\in\mathbb{Z}$, the set $Q\subseteq\mathbb{Z}^2$ of occurrences of p in w is

$$Q=\left\{(m,n)\in\mathbb{Z}^2\;\middle|\;R_0^{(m,n)}(r,s)\in A\right\}=\{(m,n)\in L\mid(m,n)^\star\in A\}=\curlywedge(A)$$

which is a regular and generic model set. If w is a singular configuration, then $w=\textsc{SymbRep}_0^{\mathbf{v}}(r,s)$ for some $\mathbf{v}\in\mathbb{R}^2\setminus\Theta^{\mathcal{P}_0}$. If $A=f_0([p])\subset W$, then we take $A'=\lim_{\epsilon\to0}A\cap(A-\epsilon\mathbf{v})$ as acceptance set and we have

$$Q=\left\{(m,n)\in\mathbb{Z}^2\;\middle|\;R_0^{(m,n)}(r,s)\in A'\right\}=\{(m,n)\in L\mid(m,n)^\star\in A'\}=\curlywedge(A')$$

which is a regular and singular model set. □

14. A model set for the Wang shift $\Omega_{\mathcal{U}}$ defined by 19 tiles

As in the previous section we use the submodule $\Lambda=\langle(1,-1,0,0),(0,0,1,-1)\rangle_{\mathbb{Z}}$ and define the projections on $\mathbb{R}^4/\Lambda$ as:

$$\begin{aligned}\pi:\mathbb{R}^4/\Lambda &\to \mathbb{R}^2\\ (x_1,x_2,x_3,x_4)&\mapsto (x_1+x_2,x_3+x_4)\end{aligned}$$

and

$$\begin{aligned}\pi_{\text{int}}:\mathbb{R}^4/\Lambda &\to \mathbb{T}^2\\ (x_1,x_2,x_3,x_4)&\mapsto \left(\frac{1}{\varphi^2}x_1-\frac{1}{\varphi}x_2,\frac{1}{\varphi^2}x_3-\frac{1}{\varphi}x_4\right)\end{aligned}$$

where $\varphi=\frac{1+\sqrt5}{2}$. The product $\pi\times\pi_{\text{int}}:\mathbb{R}^4/\Lambda\to\mathbb{R}^2\times\mathbb{T}^2$ of the projections is one-to-one and onto so we may identify the domain of the projections as $\mathbb{R}^4/\Lambda\simeq\mathbb{R}^2\times\mathbb{T}^2$, in agreement with the definition of cut and project scheme.

Note that if $\mathcal{L}=\mathbb{Z}^4\subset\mathbb{R}^4/\Lambda$, then $\pi|_{\mathcal{L}}$ is injective and $L=\pi(\mathcal{L})=\mathbb{Z}^2$. If the acceptance set is the whole cubical window $A=\mathbb{T}^2$, we obtain a description of the positions of patterns in a configuration as a model set, that is, $\mathbb{Z}^2 = \curlywedge(A)$. In the result below, noncubical acceptance sets $A \subset \mathbb{T}^2$ are used to describe the positions of patterns occurring in configurations.

Recall that we proved among other things in Theorem 1.2 that $\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}} = \Omega_\mathcal{U}$ and that there exists a factor map $f_\mathcal{U}$ from $(\mathcal{X}_{\mathcal{P}_\mathcal{U},R_\mathcal{U}}, \mathbb{Z}^2, \sigma)$ to $(\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U})$. Therefore any configuration $w \in \Omega_\mathcal{U}$ can be qualified as singular or generic according to whether $f_\mathcal{U}(w)$ is in the set $\Delta_{\mathcal{P}_\mathcal{U},R_\mathcal{U}} \subset \mathbb{T}^2$ or not.

Theorem 14.1. — *Let $\mathcal{U}$ be the self-similar set of Wang tiles shown in Figure 10.2. There exists a cut and project scheme such that for every configuration $w \in \Omega_\mathcal{U}$, the set $Q \subseteq \mathbb{Z}^2$ of occurrences of a pattern in w is a regular model set. If w is a generic (resp. singular) configuration, then Q is a generic (resp. singular) model set.*

Proof. — Let $w \in \Omega_\mathcal{U}$ be a valid configuration and let $\mathbf{x} = (r, s) = f_\mathcal{U}(w) \in \mathbb{T}^2$. We consider $\mathcal{L} = \mathbb{Z}^4 + (r, -r, s, -s) \subset \mathbb{R}^4/\Lambda$. We have that $\pi|_\mathcal{L}$ is injective and $L = \pi(\mathcal{L}) = \mathbb{Z}^2$. We also have that $\pi_{\mathrm{int}}(\mathcal{L})$ is dense in $H = \mathbb{T}^2$. Also $\pi_{\mathrm{int}}(r, -r, s, -s) = (r, s)$.

Since π is a bijection between $\mathcal{L}$ and L, there is a well-defined mapping $\star : L \to H$ given by

$$x \mapsto x^\star := \pi_{\mathrm{int}}\left((\pi|_\mathcal{L})^{-1}(x)\right)$$

where $(\pi|_\mathcal{L})^{-1}(x)$ is the unique point in the set $\mathcal{L} \cap \pi^{-1}(x)$.

Recall that the $\mathbb{Z}^2$-rotation $R_\mathcal{U}$ is defined on the torus $\mathbb{T}^2$ by $R_\mathcal{U}^\mathbf{n}(\mathbf{x}) = \mathbf{x} + \varphi^{-2}\mathbf{n}$ for every $\mathbf{n} \in \mathbb{Z}^2$. The maps $(R_\mathcal{U})^{\mathbf{e}_1}$ and $(R_\mathcal{U})^{\mathbf{e}_2}$ can be seen as polygon exchange transformations on the fundamental domain $W = [0, 1)^2$ of $\mathbb{T}^2$:

$$(R_\mathcal{U})^{\mathbf{e}_1}(\mathbf{x}) = \begin{cases} \mathbf{x} + v_a & \text{if } \mathbf{x} \in P_a, \\ \mathbf{x} + v_b & \text{if } \mathbf{x} \in P_b, \end{cases} \quad \text{and} \quad (R_\mathcal{U})^{\mathbf{e}_2}(\mathbf{x}) = \begin{cases} \mathbf{x} + v_c & \text{if } \mathbf{x} \in P_c, \\ \mathbf{x} + v_d & \text{if } \mathbf{x} \in P_d. \end{cases}$$

with $v_a = (\varphi^{-2}, 0)$, $v_b = (-\varphi^{-1}, 0)$, $v_c = (0, \varphi^{-2})$ and $v_d = (0, -\varphi^{-1})$, see Figure 14.1.

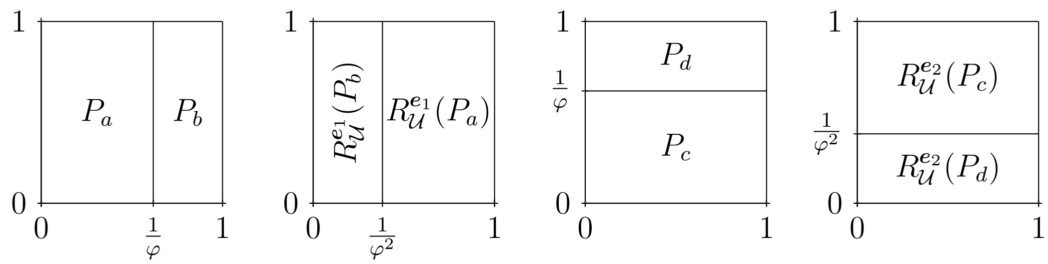


Figure 14.1. The maps $(R_\mathcal{U})^{\mathbf{e}_1}$ and $(R_\mathcal{U})^{\mathbf{e}_2}$ can be seen as polygon exchange transformations on the fundamental domain $W = [0, 1)^2$ of $\mathbb{T}^2$.

Notice that the base of $\mathbb{Z}^2$ can be written in terms of the translations as

$$\mathbf{e}_1 = v_a - v_b \quad \text{and} \quad \mathbf{e}_2 = v_c - v_d.$$

Since W is a fundamental domain for $\mathbb{Z}^2 = \langle \mathbf{e}_1, \mathbf{e}_2\rangle_\mathbb{Z}$, for every $\mathbf{x} \in \mathbb{R}^2$ there exist unique $k, \ell \in \mathbb{Z}$ such that $\mathbf{x} + k\mathbf{e}_1 + \ell\mathbf{e}_2 \in W$. Therefore, for every $(m, n) \in \mathbb{Z}^2$ there exist unique $k, \ell \in \mathbb{Z}$ such that the following holds

$$\begin{aligned}
R_\mathcal{U}^{(m,n)}(r,s) &= (r,s) + \frac{1}{\varphi^2}(m,n) \bmod \mathbb{Z}^2 \\
&= r\mathbf{e}_1 + s\mathbf{e}_2 + mv_a + nv_c \bmod \mathbb{Z}^2 \\
&= r\mathbf{e}_1 + s\mathbf{e}_2 + mv_a + nv_c + k\mathbf{e}_1 + \ell\mathbf{e}_2 \in W \\
&= mv_a + nv_c + (r+k)(v_a - v_b) + (s+\ell)(v_c - v_d) \\
&= (m+r+k)v_a - (r+k)v_b + (n+s+\ell)v_c - (s+\ell)v_d \\
&= \pi_{\text{int}}\Big((m+r+k, -r-k, n+s+\ell, -s-\ell) + \Lambda\Big) \in \pi_{\text{int}}(\mathcal{L}).
\end{aligned}$$

Notice that the projection into the physical space is

$$\pi\Big((m+r+k, -r-k, n+s+\ell, -s-\ell) + \Lambda\Big) = (m,n).$$

Thus

$$(m+r+k, -r-k, n+s+\ell, -s-\ell) + \Lambda = (\pi|_\mathcal{L})^{-1}(m,n)$$

so that

$$\begin{aligned}
(m,n)^\star &= \pi_{\text{int}}\left((\pi|_\mathcal{L})^{-1}(m,n)\right) \\
&= \pi_{\text{int}}\Big((m+r+k, -r-k, n+s+\ell, -s-\ell) + \Lambda\Big) \\
&= R_\mathcal{U}^{(m,n)}(r,s) = \Big(\{r+\varphi m\}, \{s+\varphi n\}\Big)
\end{aligned}$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

Let $p = \pi_S(w) \in \mathcal{U}^S$ be a pattern occurring in the configuration w for some subset $S \subset \mathbb{Z}^2$. Let $[p]$ be the cylinder associated with the pattern p and $A = f_\mathcal{U}([p]) \subset W$ be the acceptance set. The set A is a polygon by construction, see Equation (6.1). Therefore the Lebesgue measure of ∂A is zero. Assume for now that w is a generic configuration. Since $R_\mathcal{U}^{(m,n)}(r,s) = (m,n)^\star \notin \partial A$ for every $m, n \in \mathbb{Z}$, the set $Q \subseteq \mathbb{Z}^2$ of occurrences of p in w is

$$Q = \left\{(m,n) \in \mathbb{Z}^2 \,\middle|\, R_\mathcal{U}^{(m,n)}(r,s) \in A\right\} = \{(m,n) \in L \mid (m,n)^\star \in A\} = \curlywedge(A)$$

which is a regular and generic model set. If w is a singular configuration, then $w = \textsc{SymbRep}_\mathcal{U}^\mathbf{v}(r,s)$ for some $\mathbf{v} \in \mathbb{R}^2 \setminus \Theta^{\mathcal{P}_\mathcal{U}}$. If $A = f_\mathcal{U}([p]) \subset W$, then we take $A' = \lim_{\epsilon\to 0} A \cap (A - \epsilon\mathbf{v})$ as acceptance set and we have

$$Q = \left\{(m,n) \in \mathbb{Z}^2 \,\middle|\, R_\mathcal{U}^{(m,n)}(r,s) \in A'\right\} = \{(m,n) \in L \mid (m,n)^\star \in A'\} = \curlywedge(A')$$

which is a regular and singular model set. □

In [Lab19a], $\Omega_\mathcal{U}$ was proved to be self-similar being invariant under the application of an expansive and primitive substitution. It follows that $\Omega_\mathcal{U}$ is linearly repetitive. Based on [HKW18], an alternate proof of linear repetitivity of $\Omega_\mathcal{U}$ could be obtained now that $\Omega_\mathcal{U}$ is described as a model set. Some more work has to be done as the characterization of linearly repetitive model sets provided in [HKW18] is stated for cubical windows only.

In the present work, we made the choice of uniform 1×1 size for Wang tiles but we can make the following remark on the use of other rectangular shapes and stone inflations.

Remark 14.2. — To use the natural size for Wang tiles in $\mathcal{U}$ as stone inflation deduced from its self-similarity, see [Lab19a, Section 7], one must use

$$\begin{aligned} \pi' : \mathbb{R}^4 &\to \mathbb{R}^2 \\ (x_1, x_2, x_3, x_4) &\mapsto (x_1 + \frac{1}{\varphi} x_2, x_3 + \frac{1}{\varphi} x_4). \end{aligned}$$

as projection into the physical space. In this case, $\pi'|_{\mathcal{L}}$ is injective making it a proper cut and project scheme. Another way to construct the cut and project scheme is to use the Minkowski embedding of $\mathbb{Z}[\sqrt{5}] \times \mathbb{Z}[\sqrt{5}]$

$$\mathcal{L} = \left\{ (x, y, x^\star, y^\star) \,\middle|\, x, y \in \mathbb{Z}\left[\sqrt{5}\right] \right\}$$

where the star map $\star$ corresponds to the algebraic conjugation $(\sqrt{5})^\star = -\sqrt{5}$ in the quadratic field $\mathbb{Q}(\sqrt{5})$, see [BG13, Section 7]. In this setup, the natural window to be used should be $W = [-1, \varphi - 1) \times [-1, \varphi - 1)$ instead of $[0, 1) \times [0, 1)$ following known construction in the Fibonacci case. We do not provide this construction here.

Appendix. A DIY Puzzle to illustrate the results

We encode the 11 Jeandel–Rao tiles into geometrical shapes, see Figure 1.2, where each integer color in $\{0, 1, 2, 3, 4\}$ is replaced by an equal number of triangular or circular bumps. Print one or more copies of this page and cut each of the 25 tiles shown in Figure A.1 with scissors. Use the tiles and the *Universal solver for Jeandel–Rao Wang shift* shown in Figure A.2 to construct every pattern seen in the proper minimal subshift $\mathcal{X}_{\mathcal{P}_0, R_0} \subsetneq \Omega_0$ of the Jeandel–Rao Wang shift.

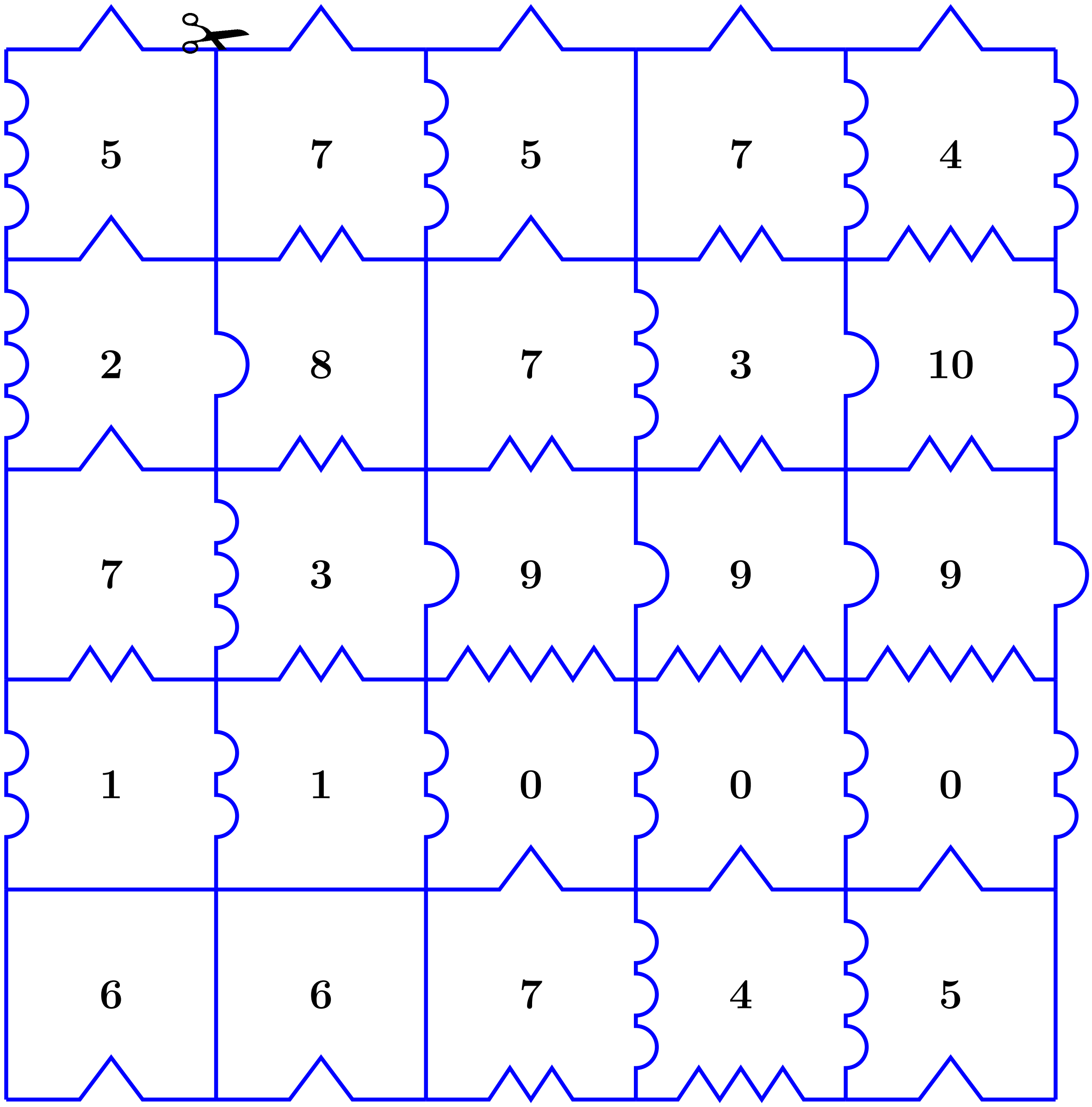


Figure A.1. A 5×5 pattern with Jeandel–Rao tiles ready to laser cut. Tiles should have 3cm size when printed in A4 format.

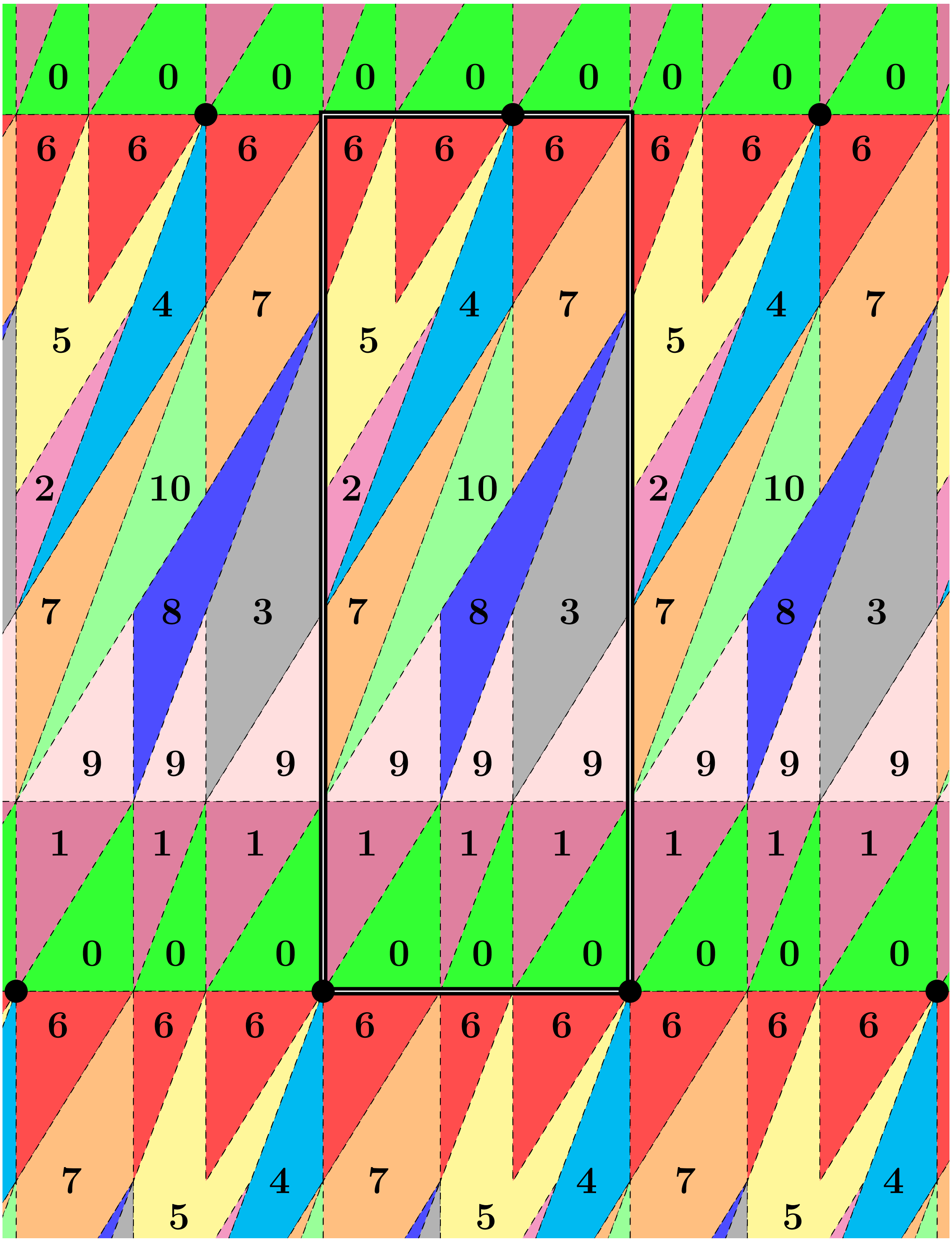


Figure A.2. The Universal solver for Jeandel–Rao Wang shift. Any pattern in the minimal subshift of Jeandel–Rao Wang shift is the coding of the orbit of some starting point by the action of horizontal and vertical translations by 1 unit (3 cm when printed in A4 format).

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Manuscript received on 15th March 2019,
revised on 19th December 2019,
accepted on 30th March 2020.

Recommended by Editor S. Gouëzel.

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