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Strict convexity of the Mabuchi functional for energy minimizers (*)

Long Li (1)

Abstract. — The aim of this paper is to investigate the strict convexity of the Mabuchi functional up to a holomorphic automorphism. We partially answered this question, and proved this strict convexity when a $C^{1,\bar{1}}$-geodesic connects two non-degenerate energy minimizers.

Résumé. — Le but de cet article est d'étudier la convexité stricte de la fonctionnelle de Mabuchi modulo automorphismes holomorphes. Nous avons partiellement répondu à cette question, et prouvé cette convexité stricte lorsqu'une $C^{1,\bar{1}}$-géodésique relie deux minimisateurs d'énergie non dégénérés.

1. Introduction

Suppose X is an n-dimensional compact complex Kähler manifold, and ω is its associated Kähler form. Let $\mathcal{H}$ be the space of all the smooth Kähler potentials

$$\mathcal{H} := \left\{\varphi \in C^\infty(X)\,;\ \omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0\right\}.$$

Up to a chosen normalization, it can be identified with the space of all Käher metrics in the cohomology class $[\omega]$. The tangent space $T_\varphi\mathcal{H}$ at a point φ can be identified with the space of all real-valued smooth functions on X, namely, we have

$$T_\varphi\mathcal{H} := \left\{\psi \in C^\infty(X)\,;\ \int_X \psi\omega_\varphi^n = 0\right\}$$

Furthermore, we can introduce an L^2 metric on this tangent space as

$$\langle\psi_0, \psi_1\rangle_\varphi := \int_X \psi_0\psi_1\omega_\varphi^n,$$

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(1) Institute Fourier, 100 rue des maths 38610 Gières, Grenoble, France *and* Mathematics Institute, ShanghaiTech University, Pudong, Shanghai, 201210, China — Long.Li1@univ-grenoble-alpes.fr — lilong1@shanghaitech.edu.cn
Article proposé par Vincent Guedj.

for any $\psi_0, \psi_1 \in T_\varphi \mathcal{H}$. Equipped with this metric, the space $\mathcal{H}$ becomes an infinite dimensional Riemannian manifold, and the distance between two points $\varphi_0, \varphi_1 \in \mathcal{H}$ can be computed as

$$d(\varphi_0, \varphi_1) = \inf_{\varphi_t \in \gamma} \int_0^1 \left(\int_X \dot{\varphi_t}^2 \omega_{\varphi_t}^n \right)^{\frac{1}{2}} \mathrm{d}t,$$

where γ denotes the class of all the smooth curves in $\mathcal{H}$ from φ_0 to φ_1. Then we can discuss the existence problem of geodesics on this special Riemannian manifold.

Unfortunately, it is not possible to find such a curve in $\mathcal{H}$ in general, as shown in the example of Darvas–Lempert ([10]). However, a weaker replacement can be found. X.X. Chen [6] proved that the "$C^{1,\bar{1}}$-geodesic" always exists for any two points in $\mathcal{H}$. As a curve in the space of ω-plurisubharmonic functions, it indeed realizes the distance between these two points (see more details in the next section).

Later such $C^{1,\bar{1}}$-geodesics became an important tool in the study of canonical metrics on Kähler manifolds. Especially, people are interested with investigating the behaviour of certain energy functionals when they are restricted to these geodesics. As is well known, the so called *Ding-functional* plays a major role in the study of Kähler–Einstein metrics, and similarly, we study the *Mabuchi functional* for constant scalar curvature Kähler (cscK) metrics.

It is proved by Berndtsson [5] that the Ding -functional is convex along any $C^{1,\bar{1}}$-geodesic (or even bounded geodesic) $\mathcal{G}$ on a Fano manifold. Moreover, this functional is actually *strictly convex*, in the sense that $\mathcal{G}$ will be generated by a holomorphic vector field whenever the Ding-functional is linear along it. These convexity results turned out to be very useful in the study of uniqueness and existence problems of the *Kähler–Einstein* metrics.

In the work of Berman–Berndtsson [4] and Chen–Li–Păun [8], the Mabuchi functional $\mathcal{M}$ is also proved to be convex and continuous along any $C^{1,\bar{1}}$-geodesic $\mathcal{G}$. Now we can further ask the following question.

Conjecture 1.1. — *Suppose the Mabuchi functional $\mathcal{M}$ is linear along a $C^{1,\bar{1}}$-geodesic $\mathcal{G}$. Then the geodesic $\mathcal{G}$ is generated by a holomorphic vector field.*

In other words, we are asking if the Mabuchi functional is strictly convex along any $C^{1,\bar{1}}$-geodesic. It is well known that this conjecture holds on any smooth geodesic. However, there are two difficulties that prevent the generalisation: the degeneracy and the lack of regularities on $C^{1,\bar{1}}$-geodesics.

A closed positive $(1,1)$-current is said to be non-degenerate at a point $p \in X$ if it is a Kähler current near p. Otherwise, it is degenerate at this point. As a solution of the homogenous complex Monge–Ampère (HCMA) equation, a $C^{1,\bar{1}}$-geodesic must have some vanishing directions, and the best hope is that the vanishing only appears in the time direction. In other words, the geodesic is non-degenerate on the fiber direction X_t.

The *key observation* is that the *truncated Mabuchi functional*, introduced by Berman–Berndtsson ([4]), will coincide with the Mabuchi functional, if the latter is linear along a $C^{1,\bar{1}}$-geodesic $\mathcal{G}$. Combined with a $W^{1,2}$-estimate (first discovered in Chen–Tian ([9])), we conclude that $\mathcal{G}$ must be non-degenerate along each fiber X_t, if its boundaries are two non-degenerate energy minimizers of $\mathcal{M}$. (see Corollary 3.5).

Moreover, we prove that such a energy minimizer satisfies the weak cscK equation in the sense of He–Zeng ([13]), and He–Zeng's regularities estimates enable us to improve the regularities of the geodesic on the fiber direction. In fact, the restriction $\mathcal{G}|_{X_t}$ must be a smooth cscK metric for each $t \in [0,1]$.

In the last step, Chen–Feldman–Hu's estimate ([7]) tells us that such a geodesic $\mathcal{G}$ must have at least C^4-regularities along the time direction. Therefore, we conclude the main theorem by a direction computation as the case of smooth geodesics.

Theorem 1.2. — *Suppose $\mathcal{G}$ is a $C^{1,\bar{1}}$-geodesic connecting two non-degenerate energy minimizers of $\mathcal{M}$. Then $\mathcal{G}$ is generated by a holomorphic vector field.*

As an upshot of this theorem, we give a alternative proof of the uniqueness of the cscK metric, namely, any two cscK metrics in the same cohomology class can only differ by a holomorphic automorphism on X.

Finally, we would like to compare our result with a recently result by Berman [3]. He constructed a counter-example of the Conjecture 1.1, when the geodesic has degenerate boundaries. This example shows that the Mabuchi functional is essentially different from the Ding-functinal along $C^{1,\bar{1}}$-geodesics.

On the other hand, Berman's example is constructed on $\mathbb{CP}^1$ and has toric symmetry. If we require that the boundaries of $\mathcal{G}$ are in $\mathcal{H}$, then this kind of examples can never appear, since such $C^{1,\bar{1}}$-geodesics with toric symmetry must be smooth. Therefore, we believe that there is still hope to confirm this conjecture with "correct" boundary conditions.

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2. Preliminary

Suppose Σ is an annuals domain in $\mathbb{C}$ with boundary, and π is the holomorphic projection from the product space $Y := X \times \Sigma$ to X. Therefore, Y is a compact complex Kähler manifold with boundary. Let Φ be a quasi-plurisubharmonic function on Y continuous up to the boundary. Denote $\mathcal{G}$ by the closed positive $(1,1)$ current

$$\pi^*\omega + \mathrm{dd}^c\Phi$$

on Y. We say that $\mathcal{G}$ is a geodesic in the space of Kähler potential, if it is S^1-invariant in the argument direction of Σ, and satisfies the following *Homogeneous complex Monge–Ampère* (HCMA) equation

$$\mathcal{G}^{n+1} = (\pi^*\omega + \mathrm{dd}^c\Phi)^{n+1} = 0, \tag{2.1}$$

in a suitable sense on Y. The boundary value of Φ is required to be in the space of the smooth Kähler potentials. Hence we say that $\mathcal{G}$ is a geodesic connecting two points $\varphi_0, \varphi_1 \in \mathcal{H}$ if $\Phi|_{X\times\{0\}} = \varphi_0$ and $\Phi|_{X\times\{1\}} = \varphi_1$, where we identify the annulus Σ by a cylinder $[0,1]\times S^1$ via the standard diffeomorphism.

It is proved by Chen ([6]) that such a geodesic is unique with fixed boundary value, and has the so called $\mathcal{C}^{1,\bar{1}}$-regularities, namely, writing $\mathcal{G}$ locally as

$$g_{\tau\bar{\tau}}\mathrm{d}\tau\wedge\mathrm{d}\bar{\tau} + \sum_{\alpha,\beta=1}^{n}\big(g_{\tau\bar{\beta}}\mathrm{d}\tau\wedge\mathrm{d}\bar{z}^\beta + g_{\alpha\bar{\tau}}\mathrm{d}z^\alpha\wedge\mathrm{d}\bar{\tau} + g_{\alpha\bar{\beta}}dz^\alpha\wedge\mathrm{d}\bar{z}^\beta\big),$$

we have

$$\|g_{\tau\bar{\tau}}\|_{L^\infty} + \sum_{\alpha,\beta=1}^{n}\big(\|g_{\tau\bar{\beta}}\|_{L^\infty} + \|g_{\alpha\bar{\tau}}\|_{L^\infty} + \|g_{\alpha\bar{\beta}}\|_{L^\infty}\big) < +\infty.$$

Moreover, it is also proved in ([6]) that there exist a uniform constant $C>0$ such that

$$0 \leqslant \mathcal{G} \leqslant C(\pi^*\omega + i\mathrm{d}\tau\wedge\mathrm{d}\bar{\tau})$$

on Y. Therefore the quasi-plurisubharmonic function Φ is of class $C^{1,\alpha}$ for any $\alpha \in (0,1)$, and the wedge product $\mathcal{G}^{n+1}$ can be interpreted in the sense of Bedford and Talyor ([2]).

As a closed positive $(1,1)$ current, the restriction of $\mathcal{G}$ on a fiber X_τ for some $\tau \in \Sigma$ is not necessary to be a Kähler metric, and hence we introduce the following notation.

Definition 2.1. — *A geodesic $\mathcal{G}$ is called non-degenerate at a point $p \in X_\tau, \tau \in \Sigma$, if its restriction $\mathcal{G}|_{X_\tau}$ is a Kähler current near p, namely, there exists an $\varepsilon > 0$ such that*

$$\mathcal{G}|_{X_\tau} \geqslant \varepsilon\omega$$

in an open neighbourhood of p. Otherwise, it is degenerate at p.

Suppose a geodesic $\mathcal{G}$ is non-degenerate at each point of Y. We can consider the following vector field locally defined as

$$V_\tau := \frac{\partial}{\partial\tau} - g^{\bar{\beta}\alpha} g_{\tau\bar{\beta}} \frac{\partial}{\partial z^\alpha}. \tag{2.2}$$

As the horizontal lift of the vector field $\partial/\partial\tau$ by the metric $\mathcal{G}$, it is actually globally defined (see [8]). This vector field plays an important role in the study of geodesics. It is a well known fact that the geodesic $\mathcal{G}$ is generated by a holomorphic automorphism if this vector field V_t is holomorphic.

2.1. The Mabuchi functional

In the study of canonical metrics, Mabuchi ([1]) introduced the following functional on the space $\mathcal{H}$

$$\mathcal{M} := \underline{R}\mathcal{E} - \mathcal{E}^{\mathrm{Ric}\,\omega} + H,$$

where the constant $\underline{R}$ is the average of the scalar curvature

$$\underline{R} = \frac{nc_1(X)\cdot[\omega]^{n-1}}{[\omega]^n}.$$

More precisely, the energy functional $\mathcal{E}$ is defined as

$$\mathcal{E}(\varphi) := \frac{1}{n+1}\sum_{i=0}^{n}\int_X \varphi\omega^i \wedge \omega_\varphi^{n-i}$$

for any $\varphi \in \mathcal{H}$. The twisted energy functional $\mathcal{E}^\alpha$ (by a closed smooth $(1,1)$ form α) is defined as

$$\mathcal{E}^\alpha(\varphi) := \sum_{i=0}^{n-1}\int_X \varphi\omega^i \wedge \omega_\varphi^{n-i-1} \wedge \alpha.$$

Finally, the entropy functional is

$$H(\varphi) := \int_X \left(\log \frac{\omega_\varphi^n}{\omega^n}\right) \omega_\varphi^n.$$

According to the definition, the Mabuchi functional is independent of the normalization of φ. Hence, it is in fact a functional defined on the space of all Kähler metrics cohomologous to ω.

The tangent space of $\mathcal{H}$ at point $\varphi \in \mathcal{H}$ can be identified with the space of all smooth functions on X. Then the first variation of the Mabuchi functional can be found via a standard computation

$$\mathrm{d}\mathcal{M}|_\varphi(\psi) = -\int_X \psi(R_\varphi - \underline{R})\omega_\varphi^n, \tag{2.3}$$

for any $\psi \in C^\infty(X)$. In other words, *the constant scalar curvature Kähler* (cscK) metrics (namely, metrics satisfying the equation $R_\varphi = \underline{R}$ on X), are critical points of the Mabuchi functional.

In fact, the Mabuchi functional can be generalised to the space of all ω-plurisubharmonic functions on X with $C^{1,\bar{1}}$-regularities ([4]). In particular, if Φ is a $\pi^*\omega$-plurisubharmonic function on Y which corresponds to a geodesic $\mathcal{G}$, then its restriction $\varphi_\tau := \Phi|_{X_\tau}$ on a fiber is a ω-plurisubharmonic function on X_τ and has the $\mathcal{C}^{1,\bar{1}}$-regularities. Therefore, we can define the Mabuchi functional along a geodesic as

$$\mathcal{K}(\tau) := \mathcal{M}(\varphi_\tau),$$

Furthermore, we introduce the following modified versions of the Mabuchi functional. Suppose $\Psi(\tau, \cdot\,) = \psi_\tau(\,\cdot\,)$ is a locally bounded singular metric on the relative canonical bundle $K_{Y/\Sigma}$, and then $-\psi_\tau$ is a metric on the anti-canonical line bundle $-K_{X_\tau} := \bigwedge^n TX_\tau$. Therefore, the following is a measure on X

$$\mu := e^{\psi_\tau},$$

which is absolutely continuous with respect to the Lebesgue measure. Define the following functional along the geodesic $\mathcal{G}$ as

$$\mathcal{K}^\Psi(\tau) := \underline{R}\mathcal{E}(\varphi_\tau) - \mathcal{E}^{\mathrm{Ric}\,\omega}(\varphi_\tau) + \int_X \log\left(\frac{e^{\psi_\tau}}{\omega^n}\right)\omega_{\varphi_\tau}^n.$$

Notice that this functional equals to $\mathcal{K}(\tau)$, if Ψ is the (unbounded) metric defined by $\omega_{\varphi_\tau}^n$. In this case, Berman–Berndtsson ([4]) found that its complex Hessian can be computed in the sense of current as

$$\mathrm{dd}^c\mathcal{K}^\Psi(\tau)(v) = \int_{X_\tau} \Psi(\pi^*\omega + \mathrm{dd}^c\Phi)^n \wedge \mathrm{dd}^c v,$$

for any locally supported smooth test function v on Y. In fact, they also proved that this is a positive current.

THEOREM 2.2 (Berman–Berndtsson). — *The Mabuchi functional $\mathcal{M}$ is weakly subharmonic along a geodesic $\mathcal{G}$, namely, we have*

$$dd^c\mathcal{K}(\tau) \geqslant 0,$$

for any τ in the interior of Σ.

In order to prove this theorem, we can consider the following *truncated Mabuchi functional* along the geodesic $\mathcal{G}$ as

$$\mathcal{K}^{\Psi_A}(\tau) := \underline{R}\mathcal{E}(\varphi_\tau) - \mathcal{E}^{\operatorname{Ric}\omega}(\varphi_\tau) + \int_X \log\left(\max\left\{\frac{\omega^n_{\varphi_\tau}}{\omega^n}, \frac{e^{\chi - A}}{\omega^n}\right\}\right)\omega^n_{\varphi_\tau},$$

where χ is a fixed continuous metric on $K_{Y/\Sigma}$ satisfying

$$dd^c\chi \geqslant k_0(\pi^*\omega + dd^c\Phi),$$

for some positive integer k_0. By invoking the dominated convergence theorem, one can easily show that for each $\tau \in \Sigma$ we have

$$\mathcal{K}^{\Psi_A}(\tau) \to \mathcal{K}(\tau),$$

as $A \to +\infty$. Therefore, it is enough to prove that the truncated Mabuchi functionals are weakly subharmonic. This is true, since the complex Hessian of $\mathcal{K}^{\Psi_A}$ can be computed as

$$dd^c\mathcal{K}^{\Psi_A}(\tau) := \int_{X_\tau} T_A, \quad T_A := dd^c\Psi_A \wedge (\pi^*\omega + dd^c\Phi)^n,$$

and the $(n+1, n+1)$-current T_A is in fact positive, because the volume element $\omega^n_{\varphi_\tau}$ can be approximated by a sequence of Bergman kernel locally (see [4, Theorem (2.1)]).

2.2. Convexity and continuity

In the study of cscK metrics, a stronger result than Theorem 2.2 is needed. Before doing this, we will slightly switch our setting on geodesics.

Let $\Gamma := \{z \in \mathbb{C}\,;\ 0 \leqslant \operatorname{Re} z \leqslant 1\}$ be a strip domain in $\mathbb{C}$. Then there is a holomorphic map from Γ to Σ as

$$\tau(z) := e^z.$$

This is a branched cover of Σ, and the inverse map is not well defined on the whole annulus. The difficulty arises when one selects a typical branch.

Fortunately this does no harm to us, since a geodesic $\mathcal{G}$ is S^1-invariant in the argument direction of Σ. Therefore, the pull back of geodesic $\mathcal{G}$ is in fact a solution of HCMA equation on $\Gamma \times X$, which is independent of the imaginary part of z. For this reason, we will identify the complex variable

$$z := t + is$$

with its real part t. For the same reason, the z-variable on Γ can be taken as the complex coordinate of the cylinder

$$\mathcal{R} := [0,1] \times S^1,$$

and we think that $\mathcal{G}$ is actually defined on $\mathcal{R} \times X$ in this sense.

Thanks to Theorem 2.2, the Mabuchi functional $\mathcal{K}(t)$ along a geodesic $\mathcal{G}$ is a convex curve in the open interval $(0,1)$. However, the boundary behaviour is not clear up to this stage. In fact, Berman–Berndtsson ([4]) proved the following "strong convexity" of the Mabuchi functional, and the same result was also obtained by Chen–Li–Păun ([8]) via a different method.

Theorem 2.3. — *The Mabuchi functional $\mathcal{M}$ along a geodesic $\mathcal{G}$ is convex and continuous up to the boundary, namely, $\mathcal{K}(t)$ is a convex and continuous function on $[0,1]$.*

In fact, the truncated Mabuchi functional $\mathcal{K}^{\Psi_A}$ also plays a major role in Berman–Berndtsson's proof, and they actually first proved the following fact ([4, Theorem 3.4]).

Theorem 2.4. — *For each positive number A, the truncated Mabuchi functional $\mathcal{K}^{\Psi_A}(t)$ is convex along any geodesic $\mathcal{G}$.*

We emphasis that $\mathcal{K}^{\Psi_A}(t)$ may not be continuous up to the boundaries of the interval $[0,1]$. Nevertheless, it is still upper semi-continuous near the boundaries as a convex function.

Taking the limit of $\mathcal{K}^{\Psi_A}(t)$ as $A \to +\infty$, we conclude that $\mathcal{K}(t)$ is upper semi-continuous near the boundaries, but it is also lower semi-continuous since the entropy functional $H(\varphi)$ is lower semi-continuous in the weak topology of plurisubharmonic functions. Hence Theorem 2.3 follows.

3. Non-degeneracy of the geodesic

As we discussed before, there are usually two difficulties while dealing with problems concerning with a geodesic $\mathcal{G}$. One is the possible degeneracy; the other is the lack of regularities. In this section, we prove that the first difficulty will never happen when the Mabuchi functional is linear along the geodesic.

3.1. Gap phenomenon

Put a function $f_\varphi := \omega_\varphi^n/\omega^n$, and the entropy can be re-written as

$$H(\varphi) = \int_X f_\varphi \log f_\varphi \cdot \omega^n.$$

Moreover, put

$$f_A(\varphi) := \max\left\{\frac{\omega_\varphi^n}{\omega^n}, \frac{e^{\chi-A}}{\omega^n}\right\},$$

and then the truncated entropy is

$$H_A(\varphi) = \int_X f_\varphi \log f_A \cdot \omega^n.$$

If we consider these functions on the fiber of the geodesic, then they are exactly the fiberwise volume form ratio of $\mathcal{G}$. Hence the truncated Mabuchi functional along a geodesic $\mathcal{G}$ can be written as

$$\mathcal{K}^{\Psi_A}(t) = \underline{R}\mathcal{E}(\varphi_t) - \mathcal{E}^{\mathrm{Ric}\,\omega}(\varphi_t) + H_A(\varphi_t).$$

As we have seen in the last section, these functionals converge to $\mathcal{K}(t)$ as $A \to +\infty$. Moreover, a simple but important observation is that they actually build a decreasing sequence, namely, we have

$$\mathcal{K}^{\Psi_A}(t) \searrow \mathcal{K}(t), \tag{3.1}$$

for each $t \in [0,1]$ as $A \to +\infty$. The reason is just because

$$H_{A'}(\varphi) \leqslant H_A(\varphi),$$

for any $A' \geqslant A$, since we have $f_{A'}(\varphi) \leqslant f_A(\varphi)$ and $f_\varphi \geqslant 0$ in this case. Bear this in mind, we can prove the following fact.

Lemma 3.1. — *Suppose the Mabuchi functional $\mathcal{M}$ is linear along a geodesic $\mathcal{G}$. Then there exists a positive number A_0, such that for each $A \geqslant A_0$, the truncated Mabuchi functional is also linear and coincides with $\mathcal{M}$ along the geodesic $\mathcal{G}$, namely, we have*

$$\mathcal{K}^{\Psi_A}(t) = \mathcal{K}(t),$$

for all $t \in [0,1]$.

Proof. — Up to a linear function on $[0,1]$, we can assume that the Mabuchi functional is identically zero along the geodesic, i.e. $\mathcal{K}(t) = 0$ for each $t \in [0,1]$. Let φ_0, φ_1 be the boundary value of the potentials of the geodesic $\mathcal{G}$, and then we can pick up a constant as

$$A_0 := \sup_X(\chi - \log f_{\varphi_i}),$$

for $i = 0, 1$. Observe that on the boundary, we have

$$\mathcal{K}^{\Psi_A}(0) = \mathcal{K}(0), \quad \mathcal{K}^{\Psi_A}(1) = \mathcal{K}(1), \tag{3.2}$$

for all constant $A \geqslant A_0$. Then we have $\mathcal{K}^{\Psi_A}(0) = \mathcal{K}^{\Psi_A}(1) = 0$.

Moreover, $\mathcal{K}^{\Psi_A}(t)$ is a convex curve on $[0, 1]$, and hence it is upper semi-continuous near the boundaries, namely, we have

$$\limsup_{t \to 0,1} \mathcal{K}^{\Psi_A}(t) \leqslant 0. \tag{3.3}$$

Therefore, this convex curve must be below the line segment joining its two boundaries. Hence we conclude that

$$\mathcal{K}^{\Psi_A}(t) \leqslant 0$$

for each $t \in [0, 1]$. On the other hand, thanks to the inequality (3.1), we also know that

$$\mathcal{K}^{\Psi_A}(t) \geqslant \mathcal{K}(t) = 0.$$

Therefore, the truncated Mabuchi functional is also identically zero along the geodesic $\mathcal{G}$, i.e. $\mathcal{K}^{\Psi_A}(t) \equiv 0$, for each $t \in [0, 1]$. Then our result follows. □

An immediate consequence of Lemma 3.1 is that the truncated entropy $H_A(\varphi_t)$ also coincides with the entropy $H(\varphi_t)$ along the geodesic $\mathcal{G}$ for each $A \geqslant A_0$. Therefore, it gives a way to describe the degenerate locus of the geodesic along each fiber X_t.

Proposition 3.2. — *Suppose the Mabuchi functional is linear along a geodesic $\mathcal{G}$. Then on each fiber X_t, there exists a non-empty measurable subset $\mathcal{Z}_t$ and a uniform constant A_0, such that the following holds up to a set of measure zero:*

$$\omega^n_{\varphi_t} \geqslant e^{\chi - A_0},$$

on $\mathcal{Z}_t$, and $\omega^n_{\varphi_t} = 0$ on $X_t - \mathcal{Z}_t$.

Proof. — Denote the set $\mathcal{Z}_t$ by the non-zero locus of the function f_φ on the fiber X_t. It is a non-empty set since the integral of $\omega^n_{\varphi_t}$ is the fixed volume of this Kähler metric. Consider the sub-level sets of the function f_φ on a fiber X_t as

$$E_A := \left\{ p \in X_t \, ; \, f_\varphi(p) \geqslant e^{\chi(p) - A} / \omega^n \right\}.$$

We claim that $E_A - E_{A_0}$ has measure zero for each $A > A_0$. Then we have

$$\mathcal{Z}_t = \bigcup_{A \geqslant A_0} E_A = E_{A_0},$$

up to a set of measure zero, and our result follows.

Now on each fiber X_t, we compute as

$$\begin{aligned} 0 = H_{A_0}(\varphi_t) - H(\varphi_t) &= \int_X (\log f_{A_0} - \log f) f \\ &\geqslant \int_{E_A} (\log f_{A_0} - \log f) f \\ &\geqslant \varepsilon \int_{E_A} (\log f_{A_0} - \log f). \end{aligned} \tag{3.4}$$

This inequality implies that $f_{A_0} = f$ almost everywhere on E_A since $\log f_{A_0} \geqslant \log f$ on it. Moreover, we know that

$$\int_{E_A} (\log f_{A_0} - \log f) = \int_{E_A - E_{A_0}} (\log f_{A_0} - \log f). \tag{3.5}$$

However, it is clear that $f_{A_0} = e^{\chi - A_0}/\omega^n$ on $E_A - E_{A_0}$ and we have

$$\frac{e^{\chi - A}}{\omega^n} \leqslant f < \frac{e^{\chi - A_0}}{\omega^n}$$

on this set. Therefore, the set $E_A - E_{A_0}$ must have measure zero, and we complete the proof. □

We emphasis that this constant A_0 is independent of t. In fact, the value of the volume form ratio f_φ on the geodesic is either larger than a positive constant κ or equal to zero almost everywhere, provided the Mabuchi functional is linear. In other words, there exists a "*gap*" for the volume form ratio of the geodesic on $\Gamma \times X$.

3.2. $W^{1,2}$ estimate

The next step is to investigate the regularities of the geodesic $\mathcal{G}$. Recall that the volume form ratio of a Kähler potential φ is defined as

$$f_\varphi := \omega_\varphi^n / \omega^n.$$

Here we will invoke the following estimate from Chen–Tian (see [9, Theorem 7.3.1]) to estimate the $W^{1,2}$ norm of f_φ. However, an extra condition is needed.

Let φ be a ω-plurisubharmonic functions on X with $C^{1,\bar{1}}$-regularities. Suppose the infimum of the Mabuchi functional in the class $[\omega]$ is finite. Denote ϱ by this infimum as

$$\varrho := \min_{u \in \mathcal{H}} \mathcal{M}(u), \tag{3.6}$$

and then we can state the following result.

Proposition 3.3 (Chen–Tian). — *Suppose $\mathcal{M}(\varphi) = \varrho$. Then we have*

$$\|f_\varphi^{1/2}\|_{W^{1,2}} < +\infty.$$

This regularity theorem is proved via the so called "*weak Kähler Ricci flow* ", and a crucial step is to observe that the first derivative of the energy $\frac{\partial \mathcal{M}}{\partial t}|_{t=0}$ is bounded from below uniformly along the flow direction. However, this is unlikely to be true if we merely assume the Mabuchi functional is linear.

Combining with the "gap phenomenon", this $W^{1,2}$ estimate enable us to conclude the following uniform non-degeneracy of the geodesic, provided that the two boundaries of $\mathcal{G}$ are non-degenerate.

Proposition 3.4. — *Suppose the Mabuchi functional $\mathcal{M}$ is linear along a $C^{1,\bar{1}}$-geodesic $\mathcal{G}$. Assume that the fiberwise volume form ratio of $\mathcal{G}$ satisfies*

$$f^\alpha_{\varphi_t} \in W^{1,2}$$

for some $\alpha > 0$ and each $t \in [0,1]$. Then there exists a uniform constant $\kappa > 0$, such that we have

$$f_{\varphi_t} > \kappa,$$

almost everywhere on each X_t.

Proof. — For any fiber X_t, put $u = f^\alpha_\varphi$. Thanks to Proposition 3.2, there exists a uniform constant $\kappa > 0$, such that the gap phenomenon occurs for the value of the function u almost everywhere on X_t. Since the function u itself is in L^∞, we can further assume that the gap exists for each point $p \in X_t$, i.e. either $u(p) > \kappa$ or $u(p) = 0$. Moreover, the function u is also non-trivial since $\int_{X_t} u^2 = 1$.

Let $U_i \subset V_i \subset U'_i$ be open coverings of X_t, such that U_i corresponds to a ball B_1 with radius $1/2$, U'_i corresponds to a ball B_2 with radius 2, and V_i corresponds to the n-interval $[0,1]^n$ in each local coordinate. Then a standard regularity argument (see Lemma A.1) implies that each restriction $u_i := u|_{V_i}$ is either trivial or greater than κ everywhere on V_i. Since U_i forms an open converging of X_t, only the latter situation can occurs for each u_i. Then our result follows. □

With this gap κ, the restriction of the geodesic $\mathcal{G}|_{X_t}$ becomes a (possibly non-smooth) Kähler metric on each fiber $X_t, t \in [0,1]$. Moreover, these Kähler metrics have a uniform lower bound since they already have a uniform upper bound thanks to their $C^{1,\bar{1}}$-regularities. Combining Chen–Tian's $W^{1,2}$-estimate with Propositions 3.2 and 3.4, we proved the following fact.

Corollary 3.5. — *Any $C^{1,\bar{1}}$-geodesic $\mathcal{G}$ connecting two non-degenerate energy minimizers of the Mabuchi functional is uniformly non-degenerate on $\Gamma \times X$, namely, there exists a uniform constant $C > 0$ such that*

$$C^{-1}\omega \leqslant \mathcal{G}|_{X_t} \leqslant C\omega$$

almost everywhere on each $X_t, t \in [0,1]$.

Finally, we emphasis the following fact. To prove Lemma 3.1 and Proposition 3.2, it is enough to assume the volume form ratios $f_{\varphi_0}, f_{\varphi_1}$ on the boundaries are bounded below from zero. Therefore, it is not necessary to require the boundaries of $\mathcal{G}$ are smooth Kähler metrics in Corollary 3.5.

4. Regularities of the energy minimizers

The goal of this section is to deal with the regularities of the geodesic $\mathcal{G}$ when it connects to two minimums of $\mathcal{M}$. Thanks to the convexity of the Mabuchi functional, this implies that the Kähler metric ω_{φ_t} reaches the minimum of $\mathcal{M}$ for each $t \in [0,1]$, where the $\varphi_t := \Phi|_{X_t}$ is the fiberwise potential of the geodesic $\mathcal{G}$.

4.1. Fiber direction

The first step is to prove that any energy minimizer of $\mathcal{M}$ satisfies the cscK equation in some weak sense.

Lemma 4.1. — *Suppose ω_φ is a non-degenerate $C^{1,\bar{1}}$-energy minimizer of the Mabuchi functional. Then for any $C^{1,\bar{1}}$-test function χ supported locally, it satisfies*

$$\int_X \log f_\varphi i\partial\bar{\partial}\chi \wedge \omega_\varphi^{n-1} = \int_X \chi(\mathrm{Ric}(\omega) - \omega_\varphi) \wedge \omega_\varphi^{n-1}. \tag{4.1}$$

Proof. — By the standard approximation method, it is enough to prove that equation (4.1) holds for all smooth compactly supported test function χ.

First notice that the potential φ is a strictly ω-plurisubharmonic function on X, and we actually can gain a bit more than this from the non-degeneracy. The non-degenerate condition on the volume form $f_\varphi > \kappa$ and the upper bound of the coefficients of the metric ω_φ together implies the lower bound of the metric, i.e. $\omega_\varphi > \varepsilon'\omega$ for some small $\varepsilon' > 0$. Therefore, the potential φ is actually a $(1-\varepsilon')\omega$-plurisubharmonic function on X.

Denote $PSH^\infty(X,\omega)$ by the space of all smooth ω-plurisubharmonic functions on X. According to Demailly's regluarization theorem [11], there exists a sequence $\varphi(s) \in PSH^\infty(X,(1-\varepsilon')\omega)$, such that $\varphi(0)=\varphi$, $\|\varphi(s)\|_{C^{1,\bar{1}}} < C$, and $\varphi(s)$ converges to φ in $W^{2,p}$ for any p large.

For any small $s>0$, we want to construct a smooth curve $\varphi(s,t) \in PSH^\infty(X,\omega)$ initiated from $\varphi(s)$, such that it satisfies

$$\left.\frac{\partial\varphi(s,t)}{\partial t}\right|_{t=0} = \chi, \tag{4.2}$$

and it exists for $t\in[0,\delta)$ for some uniform small constant δ. In fact, one can check that a linear combination $\varphi(s,t) := \varphi(s)+t\chi$ would work, since we have

$$\omega + i\partial\bar\partial\varphi(s,t) > \varepsilon'\omega + ti\partial\bar\partial\chi \geqslant 0,$$

for all t small enough.

Since the potential φ is an energy minimizer, we have

$$\mathcal{M}(\varphi) = \lim_{s\to 0}\mathcal{M}(\varphi(s,0)) \leqslant \mathcal{M}(s,t),$$

for any $s>0, t\geqslant 0$. Therefore, for any small constant $c>0$, there exists a sequence of points $s_i,t_i\to 0$ such that we have

$$\begin{aligned}-c \leqslant &\left.\frac{\partial\mathcal{M}(s,t)}{\partial t}\right|_{s_i,t_i}\\ &= \int_X \log f_\varphi(\Delta_\varphi\chi)\omega_\varphi^n|_{s_i,t_i} - \int_X \chi(\mathrm{Ric}(\omega)-\omega_\varphi)\wedge\omega_\varphi^{n-1}|_{s_i,t_i}.\end{aligned} \tag{4.3}$$

Since each coefficient of $\omega_{\varphi(s_i,t_i)}$ converges strongly to ω_φ in L^p, we conclude the following one side inequality by letting $s_i,t_i\to 0$

$$\int_X \log f_\varphi i\partial\bar\partial\chi\wedge\omega_\varphi^{n-1} \leqslant \int_X \chi(\mathrm{Ric}(\omega)-\omega_\varphi)\wedge\omega_\varphi^{n-1}. \tag{4.4}$$

Observe that both sides of our equation (4.3) are linear in χ, and then we obtain an inequality with reversed sign of (4.4) by putting $\widetilde{\chi}=-\chi$. Hence the desired equation follows. □

In the language of He–Zeng (see [13, Definition 2.5]), the function f_φ is a Δ-weak solution if it satisfies equation (4.1), and the L^∞ metric ω_φ is called a weak solution of the cscK equation. In the same work, they proved the following regularity estimate (and a apriori estimate).

Theorem 4.2 (He–Zeng). — *Let ω_φ be an L^∞-Kähler metric in the class $[\omega]$ such that*

$$\varepsilon\omega \leqslant \omega_\varphi \leqslant \Lambda\omega$$

holds in L^∞ sense with constants $\Lambda > \varepsilon > 0$. If ω_φ is a weak solution of the equation $R_\varphi = \underline{R}$, then ω_φ is smooth. Moreover, there are uniform positive constants $c_1 := c_1(n,\Lambda)$ and $C_k := C(n,k,\Lambda)$ such that

$$c_1\omega \leqslant \omega_\varphi \leqslant \Lambda\omega\ ;\ \|\varphi\|_{C^k} \leqslant C(n,k,\Lambda). \tag{4.5}$$

We note that the constants c_1 and C_k are independent of ε. Combined with He–Zeng's estimate and our Corollary 3.5, we obtain that the restriction of such geodesic on each fiber is a smooth Kähler metric with uniform lower and upper bound.

Theorem 4.3. — *Suppose $\mathcal{G}$ is a $C^{1,\bar{1}}$-geodesic connecting two non-degenerate energy minimizers of the Mabuchi functional. Then the fiberwise restriction of the geodesic $\omega_{\varphi_t} := \mathcal{G}|_{X_t}$ is a smooth cscK metric for all $t \in [0,1]$. Moreover, there are uniform constants $\Lambda > \kappa' > 0$ such that*

$$\kappa'\omega \leqslant \omega_{\varphi_t} \leqslant \Lambda\omega.$$

4.2. Time direction

We continue to understand the regularities in time direction of the geodesic $\mathcal{G}$. It is a well known question that whether we can perturb a $C^{1,\bar{1}}$-geodesic a bit to get a smooth geodesic. Recently Chen–Feldman–Hu [7] proved the following theorem, and partially answered this question in a local version.

Theorem 4.4 (Chen–Feldman–Hu). — *For any $\varphi \in \mathcal{H}$, and any real number $\alpha \in (0,1)$, there exists a small number $\varepsilon > 0$, such that for any Kähler potential $\varphi_1 \in C^5$ satisfying $|\varphi_1 - \varphi|_{C^5} < \varepsilon$, the $C^{1,\bar{1}}$-geodesic $\mathcal{G}$ connecting φ and φ_1 is non-degenerate and has $C^{5-\alpha}$ regularities on $\Gamma \times X$.*

In our case, the time direction regularities of the geodesic will be improved to $C^{5-\alpha}$ by invoking this theorem, provided that we can prove that the geodesic potentials φ_t and $\varphi_{t'}$ is close in C^5 norm if t and t' is close enough. This is the place where we will use the cscK equation again.

Proposition 4.5. — *Suppose $\mathcal{G}$ is a $C^{1,\bar{1}}$-geodesic connecting two non-degenerate energy minimizers of the Mabuchi functional. Then the geodesic is at least $C^{5-\alpha'}$ continuous on $\Gamma \times X$ for some $0 < \alpha' < 1$.*

Proof. — As we have seen in Theorem 4.3, the metric $\omega_{\varphi_t} := \mathcal{G}|_{X_t}$ is smooth cscK with uniform lower and upper bound for each $t \in [0,1]$. Then He–Zeng's a priori estimate (equation (4.5)) implies that the higher order regularities of ω_{φ_t} are uniformly controlled. In particular, there exists a uniform constant $C := C(n, 6, \Lambda)$ satisfying

$$\|\varphi_t\|_{C^6} \leqslant C(n, 6, \Lambda).$$

Fix a time $t_0 \in [0,1]$, and consider a sequence of points t_i such that $t_i \to t_0$ as $i \to \infty$. Then the potential φ_{t_i} converges to a $C^{5,\alpha}$ potential φ_∞ in C^5 norm by possibly passing to a subsequence. However, since φ_t is continuous in time direction along the geodesic, the limit φ_∞ must coincide with the potential φ_{t_0}. Therefore, for any $\varepsilon > 0$ small, we have

$$|\varphi_{t_i} - \varphi_{t_0}|_{C^5} < \varepsilon \tag{4.6}$$

for all i large. It follows from the uniqueness of the HCMA equation and Theorem 4.4 that in an open neighbourhood of t_0, the geodesic $\mathcal{G}|_{[t_0-\delta, t_0+\delta]\times X}$ is at least $C^{5-\alpha}$-continuous for some $0 < \alpha < 1$. Finally, our result follows from the compactness of the closed interval $[0,1]$. □

Once we have enough regularities of the geodesic $\mathcal{G}$, all calculations on the complex Hessian of the Mabuchi functional along $\mathcal{G}$ will go through, and our main theorem follows.

Proof of Theorem 1.2. — Suppose $\mathcal{G}$ is a $C^{1,\bar{1}}$-geodesic connecting two non-degenerate energy minimizers of $\mathcal{M}$. Thanks to the convexity of $\mathcal{M}$, we have on $[0,1]$

$$\mathcal{K}(t) \equiv \varrho,$$

where the constant ϱ is defined in equation (3.6). By Corollary 3.5 and Proposition 4.5, the geodesic $\mathcal{G}$ is in fact non-degenerate and has at least C^4-regularities on $\Gamma \times X$. Therefore, we can compute the complex Hessian of the functional $\mathcal{K}(t)$ as

$$0 = \mathrm{dd}^c\mathcal{K}(t) = \int_X \|\bar{\partial} V_t\|^2 \omega_{\varphi_t}^n, \tag{4.7}$$

where the vector field V_t is defined in equation (2.2). Finally, we conclude our result since V_t is holomorphic along the geodesic $\mathcal{G}$. □

As we have seen from Lemma 4.1 and He–Zeng's estimate, a non-degenerate $C^{1,\bar{1}}$-energy minimizer of $\mathcal{M}$ is actually a smooth cscK metric. Then the following corollary naturally follows.

Corollary 4.6. — *Suppose ω_0 and ω_1 are two constant scalar curvature Kähler metrics on X in the same cohomology class $[\omega]$. Then there exists a holomorphic automorphism F of X such that*

$$F^*\omega_1 = \omega_0.$$

4.3. Yet another proof

There is another way to figure out the regularities of the geodesic along the time direction, by using the *cscK* equation. Write the equation along the geodesic as follows

$$g_t^{\bar{\beta}\alpha}\partial_\alpha\partial_{\bar{\beta}}\log\det g_t = \underline{R}. \tag{4.8}$$

Then take the first time variation of the family of the *cscK* equations yielding as

$$\Delta_\phi^2(\delta_t\phi) - R^{\bar{\beta}\alpha}(\delta_t\phi)_{,\alpha\bar{\beta}} = 0. \tag{4.9}$$

This is a one parameter family of the fourth order (strict) elliptic equations, with uniformly bounded coefficients. Hence we can lift the regularities of the function $\delta_t\phi$ in the space direction by the standard elliptic estimates. However, this equation (4.9) can not be derived in the usual sense, since $\delta_t\phi$ is merely a Lipschitz continuous function on $X \times \Sigma$.

What we can do is to take the difference quotient along the time direction in the *cscK* equation (4.8), i.e.

$$\delta_t\phi := \frac{\phi(t_0+t,\cdot) - \phi(t_0,\cdot)}{t}.$$

The difference quotient also satisfies the Leibniz rule, and then equation (4.9) indeed holds for it. Finally, observe that all the elliptic estimates coming from equation (4.9) holds uniformly for t. Therefore, we actually improved the regularities of the function $\dot{\phi}$ in the space, and proved $\dot{\phi}$ is in fact a smooth function on X_t.

We circumvent using the Chen–Feldman–Hu Theorem, but still the cscK equation plays an essential role in this argument. When the two boundary of $\mathcal{G}$ are no longer energy minimizers, we lost the cscK condition in general, and it still remains a question to prove the holomorphicity of the vector field V_t.

Appendix

Let Ω denote the domain nth. unit interval $(0,1)^n$ in $\mathbb{R}^n$, and u is a non-negative function on Ω. We say the function u is trivial if it is identically zero outside a set of measure zero.

Lemma A.1. — *Suppose a non-trivial function u belongs to the intersection of the spaces $W^{1,2}(\Omega)$ and $L^\infty(\Omega)$, and satisfies the following gap condition:*

$$u \geqslant 1,$$

whenever $u \neq 0$. Then $u \geqslant 1$ everywhere on Ω.

Proof. — First we can assume the function u is either equal to 1 or 0, otherwise replace u by $(1-u)_+$, and the energy $\int_\Omega |\nabla u|^2$ decreases under this change by the property of maximum operator [12].

Assume $n = 2$, and Ω is the unit square $(0,1) \times (0,1)$. We will first illustrate our idea in this case. The energy can be decomposed as follows

$$\int_\Omega |\nabla u|^2 \mathrm{d}x\mathrm{d}y = \int_\Omega |D_x u|^2 + \int_\Omega |D_y u|^2. \tag{A.1}$$

Hence we have

$$\int_0^1 \left(\int_0^1 |D_x u|^2 \mathrm{d}x \right) \mathrm{d}y < +\infty,$$

and then Fubini's Theorem implies that for almost everywhere $y \in (0,1)$, we have

$$\int_0^1 |D_x u|^2(x,y)\mathrm{d}x < +\infty.$$

Therefore, the restriction of u on the interval $(0,1)\times\{y\}$ is $W^{1,2}$, and then u is continuous along this interval, thanks to the Sobolev embedding theorem. That is to say, for almost everywhere $y \in (0,1)$, the function $u(x,y)$ is identically 1 or 0 on the slice $(0,1) \times \{y\}$.

On the other hand, we have from equation (A.1)

$$\int_0^1 \left(\int_0^1 |D_y u|^2 \mathrm{d}y \right) \mathrm{d}x < +\infty.$$

As we have proved, for any $x_1, x_2 \in (0,1)$, the function $u(x_1, y)$ equals to $u(x_2, y)$ for almost everywhere $y \in (0,1)$. Therefore, the partial derivatives satisfy

$$\int_0^1 |D_y u|^2(x_1,y)\mathrm{d}y = \int_0^1 |D_y u|^2(x_2,y)\mathrm{d}y,$$

and then we have for each $x_0 \in (0,1)$

$$\int_0^1 |D_y u|^2(x_0,y)\mathrm{d}y < +\infty, \tag{A.2}$$

The same argument implies that the restriction $u_{\{x_0\}\times(0,1)}$ is also continuous, and therefore u is identically equal to 1 on the square since it is non-trivial.

For $n > 2$, we can also decompose the energy as

$$\int_\Omega |\nabla u|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_0^1 \int_{\Omega^{n-1}} |\nabla_{n-1} u|^2 + \int_{\Omega^{n-1}} \int_0^1 |D_{x_n} u|^2,$$

and then the result follows in a similar way by induction on the dimension of Ω. □

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