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Thermal approximation of the equilibrium measure and obstacle problem [*]

Scott Armstrong [1] and Sylvia Serfaty [2]

Abstract. — We consider the probability measure minimizing a free energy functional equal to the sum of a Coulomb interaction, a confinement potential and an entropy term, which arises in the statistical mechanics of Coulomb gases. In the limit where the inverse temperature β tends to ∞ the entropy term disappears and the measure, which we call the "thermal equilibrium measure" tends to the well-known equilibrium measure, which can also be interpreted as a solution to the classical obstacle problem. We provide quantitative estimates on the convergence of the thermal equilibrium measure to the equilibrium measure in strong norms in the bulk of the latter, with a sequence of explicit correction terms in powers of β^{-1}, as well as an analysis of the tail after the boundary layer of size $\beta^{-1/2}$.

Résumé. — On considère la mesure de probabilité qui minimise une énergie libre égale à la somme d'une interaction coulombienne, d'un potentiel de confinement et d'un terme d'entropie, et qui apparaît en mécanique statistique des gaz de Coulomb. Dans la limite où la température inverse β tend vers l'infini, le terme d'entropie disparaît et la mesure, que l'on appelle "mesure d'équilibre thermique", tend vers la mesure d'équilibre habituelle qui peut également être interprétée comme solution du problème de l'obstacle classique. On obtient des estimées quantitatives de convergence de la mesure d'équilibre thermique vers la mesure d'équilibre dans des normes fortes à l'intérieur du support de cette dernière, avec une série de termes correctifs explicites en puissances inverses de β, de même qu'une analyse des queues apparaissant après une couche limite de taille $\beta^{-1/2}$.

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1. Introduction

1.1. Setting of the problem

The Coulomb gas is a system of points in $\mathbb{R}^{\mathsf{d}}$ with pairwise interaction $\mathbf{g}$ defined by

$$\mathbf{g}(x) := \begin{cases} -\log|x| & \text{if } \mathsf{d}=2, \\ |x|^{2-\mathsf{d}} & \text{if } \mathsf{d}>2, \end{cases}$$

and an external (or confinining) potential (or field) V, so that the total energy of the system of N point at locations $x_1, \dots, x_N$ is given by

$$\mathcal{H}_N(x_1,\dots,x_N) := \frac{1}{2}\sum_{1\leqslant i\neq j\leqslant N} \mathbf{g}(x_i - x_j) + N\sum_{i=1}^N V(x_i). \tag{1.1}$$

Here, the strength of the external potential V has been scaled so that the potential energy is of the same order as the interaction energy. In the limit $N \to \infty$, called the "mean field limit," one is led to minimizing among probability measures the (mean-field) energy

$$\mathcal{E}(\mu) := \frac{1}{2}\int_{\mathbb{R}^{\mathsf{d}}\times\mathbb{R}^{\mathsf{d}}} \mathbf{g}(x-y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) + \int_{\mathbb{R}^{\mathsf{d}}} V(x)\,\mathrm{d}\mu(x). \tag{1.2}$$

Here μ should be thought of as the limit as $N\to\infty$ of the empirical measures $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$.

It is well known that if V grows sufficiently fast at infinity, problem (1.2) has a unique minimizer among probability measures, called the *equilibrium measure*, or the *Frostman equilibrium measure*, see for instance [19] for the two-dimensional case. This measure will be denoted μ_∞. It is well-known that minimizers of (1.1) converge as $N\to\infty$ to μ_∞ in the sense of measures (see [9] or [20, Chapter 2]).

The equilibrium measure μ_∞ is typically compactly supported and characterized by the fact that there exists a constant c_∞ such that letting

$$\zeta(x) := \int_{\mathbb{R}^{\mathsf{d}}} \mathbf{g}(x-y)\,\mathrm{d}\mu_\infty(y) + V(x) - c_\infty, \tag{1.3}$$

we have $\zeta = 0$ q.e. in supp μ_∞ and $\zeta \geqslant 0$ q.e. where q.e. is the abbreviation of "quasi-everywhere" which means except on a set of zero capacity.

This way we can see that μ_∞ can be interpreted in terms of the classical obstacle problem. Using the notation

$$h^\mu(x) := \int_{\mathbb{R}^{\mathsf{d}}} \mathbf{g}(x-y)\mathrm{d}\mu(y) \tag{1.4}$$

the function h^{μ_∞} satisfies $-\Delta h^{\mu_\infty} = \mathsf{c_d}\mu_\infty$, where

$$\mathsf{c_d} := \begin{cases} 2\pi & \text{if } \mathsf{d} = 2, \\ \mathsf{d}(\mathsf{d}-2)|B_1| & \text{if } \mathsf{d} > 2, \end{cases} \tag{1.5}$$

is the constant for which $-\Delta \mathbf{g} = \mathsf{c_d}\delta_0$. By the above properties on ζ it holds that

$$\min\left(h^{\mu_\infty} + V - c_\infty, -\Delta h^{\mu_\infty}\right) = 0 \quad \text{in } \mathbb{R}^{\mathsf{d}}, \tag{1.6}$$

which is precisely the equation for the solution to the classical obstacle problem in whole space with obstacle $c_\infty - V$. For more details about this correspondance between equilibrium measure and obstacle problem, one can see for instance [20, Chapter 2], [3] and references therein. The dependence of μ_∞ in V has been previously examined in this full space context in [22].

The Gibbs measure corresponding to a Coulomb gas at inverse temperature β is

$$\exp\left(-\frac{\beta}{N}\mathcal{H}_N(x_1, \ldots, x_N)\right) \mathrm{d}x_1 \ldots \mathrm{d}x_N. \tag{1.7}$$

Different normalizations of β with respect to N can be chosen, the specific above choice with $1/N$ in front of the energy leads in the mean-fied limit $N \to \infty$ to a minimization problem with an added entropy term of the form:

$$\mathcal{E}_\beta(\mu) := \mathcal{E}(\mu) + \frac{1}{\beta}\int_{\mathbb{R}^{\mathsf{d}}} \mu \log \mu, \tag{1.8}$$

see for instance [6, 8, 14, 16]. Again (1.8) should be minimized among probability measures, and if V grows sufficiently fast, it has a unique solution μ_β which we will call the *thermal equilibrium measure*. The functional (1.8) can also be seen as the free energy associated to the McKean–Vlasov equation which is its Wasserstein gradient flow, see for instance [13] and references therein.

On the other hand, a natural normalization for the energy and temperature in (1.7) is shown in [2, 15] to be

$$\exp\left(-\beta N^{\frac{2}{\mathsf{d}}-1}\mathcal{H}_N(x_1, \ldots, x_N)\right) \mathrm{d}x_1 \ldots \mathrm{d}x_N, \tag{1.9}$$

it is natural as β fixed is then shown to be the temperature choice that leads to a competition at the microsopic scale between interaction energy and entropy. This is in particular the normalization most studied in dimension two where $\beta = 2$ then corresponds to the famous determinantal case of the Ginibre ensemble. This choice, for which β can still be considered to depend on N, then leads in the mean-field limit to minimizing

$$\mathcal{E}_\beta(\mu) := \mathcal{E}(\mu) + \frac{1}{\beta N^{\frac{2}{\mathsf{d}}}}\int_{\mathbb{R}^{\mathsf{d}}} \mu \log \mu \tag{1.10}$$

in place of (1.8). In other words it leads to considering the regime where β in (1.8) tends to ∞ as $N \to \infty$, and thus formally to minimizing just (1.2). In [2], we showed however that, compared to the usual equilibrium measure minimizing (1.2), the thermal equilibrium measure still provides a more precise description of a Coulomb gas, even for the regime with β in (1.10) of order one, equivalently β of order $N^{2/\mathsf{d}}$ in (1.8).

In this paper we thus focus on the regime $\beta \gg 1$ in (1.8), where one expects $\mu_\beta \to \mu_\infty$. This can also be seen as a way to smoothly approximate the obstacle problem solution. The goal of this short paper is to specify how μ_β is close to μ_∞ and h^{μ_β} to h^{μ_∞}, which we will do in C^k norms. The quantitative estimates we provide are crucially used in the papers [2, 21] and allow to treat possibly quite large temperature regimes in (1.9) (note that large temperature regimes for Coulomb or log gases have started to gain interest quite recently, see [2, 12, 18]).

We note that this question, although quite natural, does not seem to have been fully answered in the literature, the only results that we are aware of are less precise, they are those in [14] which consider the two-dimensional case with no external potential, and [18] which provide some results in the particular case $V(x) = |x|^2$, and finally the work [4] motivated by Kähler geometry, which proves an L^∞ bound on the difference of h^{μ_β} and h^{μ_∞} a bit weaker than (1.24) (with an extra $\log \beta$ factor) but in the compact setting of a manifold. There were also explicit formulae for the one-dimensional logarithmic case (related but slightly out of our scope) and quadratic potential in [1].

By contrast with μ_∞, μ_β is not compactly supported, but always positive in $\mathbb{R}^\mathsf{d}$ and regular. In fact h^{μ_β} defined as in (1.4) solves the PDE

$$h^{\mu_\beta} + V + \frac{1}{\beta} \log \mu_\beta = c_\beta, \tag{1.11}$$

for some constant c_β. Taking the Laplacian of that equation leads to a PDE on $\log \mu_\beta$ with notoriously delicate exponential nonlinearity

$$\Delta \log \mu_\beta = \beta(\mathsf{c_d} \mu_\beta - \Delta V). \tag{1.12}$$

Instead of studying this equation directly, we observe for the first time that when subtracting two such equations (with possible error term) with solutions μ and ν respectively, the quotient $u = \mu/\nu - 1$ rewrites nicely as a divergence form equation

$$\operatorname{div} \frac{\nabla u}{1+u} = \beta \mu u + \text{error} \tag{1.13}$$

for which elliptic regularity theory is readily applicable as soon as u is small enough. This allows to obtain corrections to arbitrary order of the approximation $\mu_\beta \simeq \mu_\infty$, see (1.30) below. In fact our proofs only use maximum principle-based arguments and regularity theory, and do not require going through energy estimates.

Finally, we comment that the other extreme regime $\beta \to 0$ is easier to treat. We can formally expect the interaction energy to become negligible and we are then led to minimizing, among probability measures, the quantity

$$\int_{\mathbb{R}^\mathsf{d}} V \,\mathrm{d}\mu + \frac{1}{\beta}\int_{\mathbb{R}^\mathsf{d}} \mu \log \mu,$$

the minimizer of which is $\mu = \frac{e^{-\beta V}}{\int e^{-\beta V}}$, see [18].

1.2. Assumptions and results

We let $\Sigma := \operatorname{supp} \mu_\infty$ and assume that $\partial\Sigma \in C^{1,1}$. Note that it was very recently established in [10] that this holds generically with respect to V. We assume in addition

$$V \in C^2 \tag{1.14}$$

$$\begin{cases} V \to +\infty \quad \text{as } |x| \to \infty & \text{if } \mathsf{d} \geqslant 3 \\ \lim_{|x|\to\infty} (V + \mathbf{g}) = +\infty & \text{if } \mathsf{d} = 2, \end{cases} \tag{1.15}$$

$$\begin{cases} \displaystyle\int_{|x|\geqslant 1} \exp\left(-\frac{\beta}{2}V(x)\right) \mathrm{d}x < \infty, & \text{if } \mathsf{d} \geqslant 3, \\ \displaystyle\int_{|x|\geqslant 1} e^{-\frac{\beta}{2}(V(x)-\log|x|)}\,\mathrm{d}x + \int_{|x|\geqslant 1} e^{-\beta(V(x)-\log|x|)}|x|\log^2|x|\,\mathrm{d}x < \infty & \\ & \text{if } \mathsf{d} = 2, \end{cases} \tag{1.16}$$

and

$$\Delta V \geqslant \alpha > 0 \quad \text{in a neighborhood of } \Sigma. \tag{1.17}$$

Observe that

$$\Sigma \subseteq \{\zeta = 0\}. \tag{1.18}$$

The set $\{\zeta = 0\}$ is called the *contact set* or *coincidence set* of the obstacle problem, and Σ is the set in which the obstacle is *active*, sometimes called the *droplet*. The assumption (1.17) ensures that these coincide. Note that $h^{\mu_\infty} = c_\infty - V$ in $\{\zeta = 0\}$, hence the density satisfies

$$\mu_\infty = \frac{\Delta V}{\mathsf{c_d}} \mathbb{1}_\Sigma.$$

Thanks to this connection, the regularity of μ_V and of Σ can be known by the standard regularity theory for the classical obstacle problem [7] (see also [22] for the formulated in the whole space).

Since we assume $\partial\Sigma \in C^{1,1}$ (which rules out boundary cusps), (1.15) and (1.17), by standard results on the obstacle problem [7], we have

$$\zeta(x) \geqslant \alpha \,\mathrm{dist}(x,\Sigma)^2 \quad \text{in a neighborhood of } \Sigma, \tag{1.19}$$

with ζ the function of (1.3), and a corresponding upper bound also holds. We now assume in addition that

$$\zeta(x) \geqslant \alpha \min(\mathrm{dist}(x,\Sigma)^2, 1), \tag{1.20}$$

which amounts, up to changing to constant $\alpha > 0$ if necessary, to assume that the solution to the obstacle problem never gets very close to the obstacle, outside of Σ. A sufficient condition is for instance that V be strictly convex.

Theorem 1.1. — *Assume* (1.14)–(1.17) *and* (1.20). *Then* (1.10) *has a unique minimizer μ_β. Moreover, there exists $C(V,\mathsf{d}) > 0$ such that, for every $x \in \mathbb{R}^{\mathsf{d}}$ and $\beta \in (2,\infty)$, we have*

$$0 < \mu_\beta(x) \leqslant \begin{cases} \min(C, C\exp\big(-\beta(V(x)-C)\big) & \text{if } \mathsf{d} \geqslant 3 \\ \min(C, C\exp\big(-\beta(V(x)-\log|x|-C)\big) & \text{if } \mathsf{d} = 2 \end{cases} \tag{1.21}$$

$$\mu_\beta(x) > \frac{1}{C} > 0 \quad \text{for } x \in \Sigma, \tag{1.22}$$

$$\exp\left(-\frac{\beta}{C}\,\mathrm{dist}(x,\Sigma)^2 - C\right) \leqslant \mu_\beta(x) \leqslant \exp\left(-\frac{\beta}{C}\,\mathrm{dist}(x,\Sigma)^2 + C\right) \tag{1.23}$$

in a β-independent neighborhood of Σ,

$$\|h^{\mu_\beta} - c_\beta - h^{\mu_\infty} + c_\infty\|_{L^\infty(\mathbb{R}^{\mathsf{d}})} \leqslant \frac{C}{\beta}, \tag{1.24}$$

$$\|\nabla(h^{\mu_\infty} - h^{\mu_\beta})\|_{L^\infty(\mathbb{R}^{\mathsf{d}})} \leqslant \frac{C}{\sqrt{\beta}}, \tag{1.25}$$

$$\mu_\beta(\Sigma^c) \leqslant \frac{C}{\sqrt{\beta}} \tag{1.26}$$

and

$$\left|\int_{\Sigma^c} \mu_\beta \log \mu_\beta\right| \leqslant \frac{C}{\sqrt{\beta}}. \tag{1.27}$$

Let m be an integer $\geqslant 2$ such that $V \in C^{2m,\gamma}$ for some $\gamma \in (0,1]$ and letting f_k be defined iteratively by

$$f_0 = \frac{1}{\mathsf{c_d}}\Delta V, \qquad f_{k+1} = \frac{1}{\mathsf{c_d}}\Delta V + \frac{1}{\beta \mathsf{c_d}}\Delta \log f_k, \tag{1.28}$$

we have $f_k \in C^{2(m-k-1),\gamma}(\Sigma)$ *and for every even integer* $n \leqslant 2m-4$ *and* $0 \leqslant \gamma' \leqslant \gamma$*, if* β *is large enough depending on* m*, for any* $U \subseteq \Sigma$*, we have*

$$\|\mu_\beta - f_{m-2-n/2}\|_{C^{n,\gamma'}(U)} \leqslant C\beta^{\frac{n+\gamma'}{2}} \exp\left(-C\log^2(\beta\,\mathrm{dist}(U,\partial\Sigma)^2)\right) + C\beta^{1+n-m+\frac{\gamma'}{2}}. \quad (1.29)$$

The functions f_k provide a sequence of improving approximations to μ_β defined iteratively. Spelling out the iteration we easily find the expansion in powers of $1/\beta$

$$\mu_\beta \simeq \frac{1}{\mathsf{c_d}}\Delta V + \frac{1}{\mathsf{c_d}\beta}\Delta\log\frac{\Delta V}{\mathsf{c_d}} + \frac{1}{\mathsf{c_d}\beta^2}\Delta\left(\frac{\Delta\log\frac{\Delta V}{\mathsf{c_d}}}{\Delta V}\right) + \cdots \quad \text{inside } \Sigma \quad (1.30)$$

up to an order dictated by the regularity of V and the size of β.

The relation (1.24) improves in particular the equivalent result in [4] (a bound in $\frac{\log\beta}{\beta}$), while (1.25) improves on the energy comparison-based estimate in $1/\sqrt{\beta}$ given in [18]. The estimates reveal the natural lengthscale $1/\sqrt{\beta}$ appearing in the approximation of μ_β by μ_∞.

Remark 1.2. — Since $h^{\mu_\beta} - h^{\mu_\infty}$ vanishes at infinity because μ_β and μ_∞ are both probability measures, (1.24) also implies that

$$|c_\infty - c_\beta| \leqslant \frac{C}{\beta}.$$

It seems difficult to obtain such a precise estimate from energy considerations only.

The rest of the paper is organized as follows: in Section 2 we check the existence of a minimizer to $\mathcal{E}_\beta$ under the assumptions (1.14)–(1.16) and prove a few of its qualitative properties. In Section 3 we obtain a first L^∞ bound on the difference between the solutions to (1.11) and (1.6) via a comparison principle, and a uniform bound on μ_β. This then serves to obtain a lower bound for μ_β inside Σ by a barrier argument in the following section. This in turn leads to the optimal uniform estimates on $h^{\mu_\beta} - h^{\mu_\infty}$ in Section 5. These estimates are then eventually upgraded in Section 6 to C^k spaces via the iterative approximation sequence f_k thanks to DeGiorgi–Schauder elliptic regularity theory applied to (1.13).

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2. Existence of a unique solution and first properties

Throughout the paper, C denotes a positive constant which depends only on V and d and may vary in each occurrence.

Some of the results of this section may be known, but we could not find a reference and therefore we include them here for the convenience of the reader.

LEMMA 2.1. — *If (1.14)–(1.16) hold, then $\mathcal{E}_\beta$ has a unique minimizer.*

Proof. — Let us first consider $\mathsf{d} \geqslant 3$. We may write

$$\mathcal{E}_\beta(\mu) = \int_{\mathbb{R}^\mathsf{d}} \frac{1}{2} V \mathrm{d}\mu + \frac{1}{2} \iint_{\mathbb{R}^\mathsf{d} \times \mathbb{R}^\mathsf{d}} \mathbf{g}(x-y) \mathrm{d}\mu(x) \mathrm{d}\mu(y) + \int_{\mathbb{R}^\mathsf{d}} \frac{1}{2} V \mathrm{d}\mu + \frac{1}{\beta} \mu \log \mu.$$

The function $x \mapsto \frac{V}{2} x + \frac{1}{\beta} x \log x$ achieves its minimum at $x = \exp\left(-\frac{\beta}{2} V - 1\right)$ hence we may bound from below the last integral

$$\int_{\mathbb{R}^\mathsf{d}} \frac{1}{2} V \mathrm{d}\mu + \frac{1}{\beta} \mu \log \mu \geqslant - \int_{\mathbb{R}^\mathsf{d}} \frac{1}{\beta} \exp\left(-\frac{\beta}{2} V - 1\right). \tag{2.1}$$

This is finite by (1.16). On the other hand, since $V \in C^2$ and $\mathbf{g} \geqslant 0$ in the case $\mathsf{d} \geqslant 3$, we have

$$\int_{\mathbb{R}^\mathsf{d}} \frac{1}{2} V \mathrm{d}\mu + \frac{1}{2} \iint_{\mathbb{R}^\mathsf{d} \times \mathbb{R}^\mathsf{d}} \mathbf{g}(x-y) \mathrm{d}\mu(x) \mathrm{d}\mu(y) > -\infty. \tag{2.2}$$

We deduce that $\inf \mathcal{E}_\beta > -\infty$. In view of (1.15) we deduce from the above that minimizing sequences are tight and therefore the existence of a minimizer.

We now consider the case $\mathsf{d} = 2$. We note that $\mathbf{g}(x-y) \geqslant -C - \log\max(|x|,|y|,1)$ and thus, by symmetry,

$$\begin{aligned} &\mathcal{E}_\beta(\mu) \\ &\geqslant \int_{\mathbb{R}^\mathsf{d}} V \mathrm{d}\mu - \iint_{|x| \geqslant |y|} \log\max(|x|,1) \mathrm{d}\mu(x) \mathrm{d}\mu(y) - C + \frac{1}{\beta} \int_{\mathbb{R}^\mathsf{d}} \mu \log \mu \\ &\geqslant \int_{\mathbb{R}^\mathsf{d}} V \mathrm{d}\mu - \int_{\mathbb{R}^\mathsf{d}} (\log|x|)_+ \mathrm{d}\mu(x) - C + \frac{1}{\beta} \int_{\mathbb{R}^\mathsf{d}} \mu \log \mu. \end{aligned} \tag{2.3}$$

Arguing as in the case $\mathsf{d} \geqslant 3$ but with V replaced by $V - (\log|x|)_+$ and using (1.15) and (1.16), we deduce the existence result for $\mathsf{d} = 2$.

The uniqueness of the minimizing measure is immediate from the strict convexity of the energy functional. □

LEMMA 2.2. — *Under the same assumptions, the minimizer μ_β of $\mathcal{E}_\beta$ is positive almost everywhere in $\mathbb{R}^\mathsf{d}$, bounded above, locally bounded below,* *continuous, and satisfies* (1.11) *almost everywhere in* $\mathbb{R}^\mathsf{d}$*. Moreover, we have the following asymptotics*

$$\begin{cases} \lim_{|x|\to\infty} \left(\frac{h^{\mu_\beta}(x)}{\log|x|}+1\right)=0 & \text{in } \mathsf{d}=2, \\ \lim_{|x|\to\infty} h^{\mu_\beta}(x)=0 & \text{in } \mathsf{d}>2. \end{cases} \tag{2.4}$$

Proof.

Step 1. — We start by showing that $h^{\mu_\beta}\in L^1_{\mathrm{loc}}(\mathbb{R}^\mathsf{d})$. Let B be a bounded set in $\mathbb{R}^\mathsf{d}$, we have

$$\int_B h^{\mu_\beta} = \int_{\mathbb{R}^\mathsf{d}}\int_B \mathbf{g}(x-y)\mathrm{d}x\mathrm{d}\mu_\beta(y). \tag{2.5}$$

If $\mathsf{d}\geqslant 3$ it is then straightforward, by integrability and boundedness of $\mathbf{g}$, that the right-hand side is finite. If $\mathsf{d}=2$, then we first need to show that

$$\int_{\mathbb{R}^2} (\log|x|)_+\mathrm{d}\mu_\beta(x)<\infty. \tag{2.6}$$

To do so, we return to (2.3) and use that for each $x\in\mathbb{R}^2\setminus\{0\}$, the function

$$\phi(\mu) := (V-(\log|x|)_+)\,\mu+\frac{1}{\beta}\mu\log\mu$$

is convex and achieves its minimum at $u_0(x)=\exp(-\beta(V-(\log|x|)_+)-1)$. Its second derivative is the decreasing function $\phi''(\mu)=\frac{1}{\beta\mu}$. Thus

$$\begin{aligned} \phi(\mu) &= \phi(u_0)+\int_{u_0}^{\mu}\frac{1}{\beta}\log\frac{y}{u_0}dy \\ &= \phi(u_0)+\frac{1}{\beta}\left(\mu\log\frac{\mu}{u_0}-\mu+u_0\right) \\ &\geqslant \phi(u_0)+\frac{1}{2\beta}(\mu-u_0)^2\min\left(\mu^{-1},u_0^{-1}\right). \end{aligned} \tag{2.7}$$

Inserting into (2.3) and using (2.7) and (1.15), we obtain

$$\begin{aligned} \mathcal{E}_\beta(\mu) &\geqslant -C+\int_{\mathbb{R}^2}\phi(\mu_\beta(x))\,\mathrm{d}x \\ &\geqslant -C-\frac{1}{\beta}\int_{\mathbb{R}^2}\exp\bigl(-\beta(V-(\log|x|)_+)-1\bigr)\,\mathrm{d}x \\ &\quad+\int_{\mu_\beta(x)\geqslant(1+|x|)u_0(x)}\frac{1}{\beta}\mu_\beta(x)\left(\log(1+|x|)-1\right)\,\mathrm{d}x \\ &\quad+\frac{1}{2\beta}\int_{2u_0(x)\leqslant\mu_\beta(x)\leqslant(1+|x|)u_0(x)}\frac{(\mu_\beta(x)-u_0(x))^2}{(1+|x|)u_0(x)}. \end{aligned} \tag{2.8}$$

We next write

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (\log|x|)_+ \mu_\beta(x)\mathrm{d}x \\
&\leqslant \int_{\mu_\beta(x)\leqslant 2u_0(x)} 2(\log|x|)_+ \exp\big(-\beta(V-(\log|x|)_+)-1\big)\mathrm{d}x \\
&\quad + \int_{\mu_\beta(x)\geqslant(1+|x|)u_0(x)} (\log|x|)_+\mu_\beta(x)\mathrm{d}x \\
&\quad + \int_{2u_0(x)\leqslant\mu_\beta(x)\leqslant(1+|x|)u_0(x)} (\log|x|)_+((1+|x|)u_0(x))^{\frac12}\frac{\mu_\beta(x)}{((1+|x|)u_0(x))^{\frac12}}\mathrm{d}x.
\end{aligned}
$$

The first integral on the right-hand side is finite by (1.16), the second is finite by finiteness of the terms in (2.8) and the third is seen to be finite by applying the Cauchy–Schwarz inequality and using (2.8) and (1.16). This yields (2.6). It then follows from (2.5) that $h^{\mu_\beta}\in L^1_{\mathrm{loc}}(\mathbb{R}^{\mathsf{d}})$ for $\mathsf{d}=2$.

Step 2. — We check that $\mu_\beta>0$ except on a set of measure zero, as in [17, 18]. Assuming by contradiction that $\mu_\beta=0$ in a bounded set S of positive measure, let us consider $\frac{\mu_\beta+\varepsilon\mathbb{1}_S}{1+\varepsilon|S|}$. Let us expand out

$$
\begin{aligned}
&\mathcal{E}_\beta\left(\frac{\mu_\beta+\varepsilon\mathbb{1}_S}{1+\varepsilon|S|}\right) \\
&= \mathcal{E}_\beta(\mu_\beta) - \varepsilon|S|\left(\iint \mathbf{g}(x-y)\mathrm{d}\mu_\beta(x)\mathrm{d}\mu_\beta(y) + \int V\mathrm{d}\mu_\beta + \frac{1}{\beta}\mu_\beta\log\mu_\beta\right) \\
&\qquad\qquad + \varepsilon\int_S (h^{\mu_\beta}+V) + \frac{|S|}{\beta}\varepsilon\log\varepsilon + O(\varepsilon^2).
\end{aligned}
$$

By Step 1 and the fact that S is bounded, we have that $\int_S h^{\mu_\beta}+V<\infty$. We deduce that

$$
\mathcal{E}_\beta\left(\frac{\mu_\beta+\varepsilon\mathbb{1}_S}{1+\varepsilon|S|}\right) \leqslant \mathcal{E}_\beta(\mu_\beta) + C\varepsilon + \frac{|S|}{\beta}\varepsilon\log\varepsilon,
$$

a contradiction with the minimality of μ_β if $|S|>0$ when ε is chosen small enough.

Step 3. — We next check that (1.11) is satisfied. For every smooth compactly supported function f such that $\int f\mathrm{d}\mu_\beta=0$ and $t\in\mathbb{R}$ with $|t|$ sufficiently small, $(1+tf)\mu_\beta$ is a probability measure and we may expand

$$
\mathcal{E}_\beta(\mu_\beta) \leqslant \mathcal{E}_\beta((1+tf)\mu_\beta)
$$

to find

$$
t\int_{\mathbb{R}^{\mathsf{d}}} (h^{\mu_\beta}+V+\frac{1}{\beta}\log\mu_\beta)f\mathrm{d}\mu_\beta + O(t^2) \geqslant 0,
$$

where h^{μ_β} is defined as in (1.4) and may take infinite values. Since this is true for all small enough $|t|$ and any smooth f with $\int f\mathrm{d}\mu_\beta=0$, and since

$\mu_\beta > 0$ almost everywhere, it follows that (1.11) holds almost everywhere, for some constant c_β.

Step 4: Proof of (2.4). — For dimension $\mathsf{d} \geqslant 3$ we have that $h^{\mu_\beta} \geqslant 0$ hence from (1.11) and (1.15) we deduce that μ_β is bounded above. To prove (2.4), given $\varepsilon > 0$, we choose R such that $\mu_\beta(B_R^c) < \varepsilon$ (which is possible since μ_β is a probability measure). We then write

$$h^{\mu_\beta}(x) = \int_{B_R} \mathbf{g}(x-y)\mathrm{d}\mu_\beta(y) + \int_{B_R^c \cap \{|y-x| \leqslant \eta\}} \mathbf{g}(x-y)\mathrm{d}\mu_\beta(y) + \int_{B_R^c \cap \{|y-x| \geqslant \eta\}} \mathbf{g}(x-y)\mathrm{d}\mu_\beta(y). \tag{2.9}$$

The first term of the right-hand side tends to zero when $x \to \infty$ because $\mathbf{g}$ does, the second term is bounded by $\|\mu_\beta\|_{L^\infty} \int_{B_\eta} \mathbf{g}$ which tends to zero as η tends to zero by integrability of $\mathbf{g}$ near the origin, and the last term can be bounded by $\mathbf{g}(\eta)\mu_\beta(B_R^c) < \mathbf{g}(\eta)\varepsilon$. We may then choose η appropriately to make all the three terms be at most $\varepsilon^{\frac{1}{2}}$ when $|x|$ is large enough, which proves (2.4) in the case $\mathsf{d} \geqslant 3$.

For $\mathsf{d} = 2$, we use $-\log|x-y| \geqslant -C - \log\max(|x|, |y|, 1)$ and (2.6) to obtain

$$\begin{aligned} h^{\mu_\beta}(x) + (\log|x|)_+ &= \int_{\mathbb{R}^2} \left(-\log|x-y| + (\log|x|)_+\right) \mathrm{d}\mu_\beta(y) \\ &\geqslant -C + \int_{|y| \geqslant |x|} ((\log|x|)_+ - (\log|y|)_+)\mathrm{d}\mu_\beta(y) \\ &\geqslant -C. \end{aligned}$$

Therefore $h^{\mu_\beta} + (\log|x|)_+$ is bounded below. Since $V - (\log|x|)_+$ is also bounded below by (1.15), we deduce from (1.11) that μ_β is bounded above, and then we can finish the proof as in dimension $\mathsf{d} \geqslant 3$ from the decomposition (2.9), using (2.6).

Step 5: Continuity. — The computations of the previous step starting from (2.9) show that h^{μ_β} is locally bounded above, and so is V. It then follows from (1.11) that μ_β is locally bounded below. Once we have shown that μ_β is locally bounded above and below by positive constants, we may rewrite (1.12) as a uniformly elliptic equation for μ_β:

$$\operatorname{div} \frac{\nabla \mu_\beta}{\mu_\beta} = \beta(\mathsf{c}_\mathsf{d}\mu_\beta - \Delta V).$$

By standard elliptic regularity theory (for instance [11]), we thus deduce that μ_β is as regular as V, in particular μ_β is continuous. □

3. The comparison principle and upper bound on μ_β

3.1. A preliminary lemma

We will use the following comparison principle for the obstacle problem in the whole plane.

LEMMA 3.1. — *Suppose that v, w are two continuous function in $\mathbb{R}^2$ which satisfy*

$$\min\{-\Delta v, v - (c_\infty - V)\} \leqslant 0 \leqslant \min\{-\Delta w, w - (c_\infty - V)\} \quad \text{in } \mathbb{R}^2 \tag{3.1}$$

as well as

$$\limsup_{|x|\to\infty} \frac{v(x)}{\log|x|} \leqslant -1 \leqslant \liminf_{|x|\to\infty} \frac{w(x)}{\log|x|}. \tag{3.2}$$

Then $v \leqslant w$ in $\mathbb{R}^2$.

Proof of Lemma 3.1. — Let $\phi = c_\infty - V$ be the obstacle function. We may assume without loss of generality that $\phi \leqslant 0$ (otherwise we may subtract a constant). Then $v \leqslant 0$ by the maximum principle, since the zero function is a harmonic function which, due to (3.2), is larger than v in the complement of a bounded set. Moreover, $\min\{tw, 0\}$, with $0 < t \leqslant 1$, satisfies the same assumptions as w, and thus it suffices to show that $v \leqslant tw$ for every $0 < t < 1$. In light of this, we may assume that

$$\limsup_{|x|\to\infty} \frac{v(x)}{\log|x|} \leqslant -1 < \liminf_{|x|\to\infty} \frac{w(x)}{\log|x|}.$$

In particular, $\{v > w\}$ is bounded. Observe also that $\{v > w\} \subseteq \{v > \phi\}$. Since v is subharmonic in the latter and w is superharmonic in $\mathbb{R}^2$, we deduce that $v - w$ is subharmonic in $\{v - w > 0\}$. Assume that this set is nonempty, to get a contradiction. Let x_0 be the point at which $v - w$ attains its global maximum, say $M := (v - w)(x_0) = \sup_{\mathbb{R}^2}(v - w)$. Then, since $v - w$ is subharmonic at x_0, we deduce that it is constant in a neighborhood of x_0. In fact, this argument shows that the set $\{v - w = M\}$ is open; since $v - w$ is continuous, it is also closed. Since $\{v - w = M\} \neq \emptyset$, we must have that $v - w \equiv M$. Thus v and w are harmonic. Since v is bounded above, it must be constant. This violates the growth condition. □

3.2. Main proof

We now turn to the main comparison result of this section.

Lemma 3.2. — *Let $m_\beta = \sup_{\mathbb{R}^\mathsf{d}} \mu_\beta$. If β is large enough, we have*

$$-\frac{\log m_\beta}{\beta} \leqslant h^{\mu_\beta} - c_\beta - (h^{\mu_\infty} - c_\infty). \tag{3.3}$$

Proof. — To compare $h^{\mu_\beta} - c_\beta$ and $h^{\mu_\infty} - c_\infty$ we recall that h^{μ_β} satisfies (1.11) while h^{μ_∞} satisfies (1.6). We may write from (1.11) that

$$h^{\mu_\beta} + V - c_\beta + \frac{\log m_\beta}{\beta} \geqslant 0 \tag{3.4}$$

It follows that

$$\min\left(h^{\mu_\beta} + V - c_\beta + \frac{\log m_\beta}{\beta}, -\Delta h^{\mu_\beta}\right) \geqslant 0. \tag{3.5}$$

In dimension $\mathsf{d} = 2$, applying the comparison principle of Lemma 3.1 to $h^{\mu_\beta} + c_\infty - c_\beta + \frac{m_\beta}{\beta}$ and h^{μ_∞} we deduce that

$$h^{\mu_\beta} + c_\infty - c_\beta + \frac{\log m_\beta}{\beta} \geqslant h^{\mu_\infty},$$

which is the desired result. For dimension $\mathsf{d} \geqslant 3$, we first show that we have

$$\liminf_{|x|\to\infty} \left(h^{\mu_\beta} + c_\infty - c_\beta + \frac{\log m_\beta}{\beta}\right) \geqslant 0 \tag{3.6}$$

which is equivalent by (2.4) to showing that $c_\infty - c_\beta + \frac{\log m_\beta}{\beta} \geqslant 0$.

To do so, by contradiction assume that $c_\infty - c_\beta + \frac{\log m_\beta}{\beta} < 0$ and let us consider ψ harmonic in $\mathbb{R}^\mathsf{d}\backslash\Sigma$ such that $\psi = 0$ on $\partial\Sigma$ and $\psi = c_\infty - c_\beta + \frac{\log m_\beta}{\beta}$ at ∞. Because $c_\infty - c_\beta + \frac{\log m_\beta}{\beta} < 0$, we have that $\psi - \left(c_\infty - c_\beta + \frac{\log m_\beta}{\beta}\right)$ decays at infinity like the Green's function, i.e. like $|x|^{2-\mathsf{d}}$. On the other hand, setting

$$\varphi := h^{\mu_\beta} - h^{\mu_\infty} + c_\infty - c_\beta + \frac{\log m_\beta}{\beta},$$

by (3.4) and (1.6) we have

$$\begin{cases} \varphi \geqslant 0 & \text{in } \Sigma \\ \Delta\varphi \leqslant 0 & \text{in } \mathbb{R}^\mathsf{d}\backslash\Sigma \end{cases} \tag{3.7}$$

It then follows that $-\Delta(\varphi - \psi) \geqslant 0$ in $\mathbb{R}^\mathsf{d}\backslash\Sigma$ with $\varphi - \psi \to 0$ at ∞ and $\varphi - \psi \geqslant 0$ on $\partial\Sigma$. Thus by the maximum principle $\varphi - \psi \geqslant 0$ in $\mathbb{R}^\mathsf{d}\backslash\Sigma$. On the other hand, since $-\int_{\mathbb{R}^\mathsf{d}} \Delta\varphi = \mathsf{c}_\mathsf{d} \int_{\mathbb{R}^\mathsf{d}} \mu_\beta - \mu_\infty = 0$ we also have that $\varphi - \left(c - c_\beta + \frac{\log m_\beta}{\beta}\right)$ decays at infinity like $|x|^{1-\mathsf{d}}$. This, the fact that $\psi - \left(c - c_\beta + \frac{\log m_\beta}{\beta}\right)$ decays at infinity like $|x|^{2-\mathsf{d}}$, and the fact that $\varphi \geqslant \psi$ bring a contradiction, which shows that $\liminf_{|x|\to\infty} \varphi \geqslant 0$. Since (3.7) holds in any case, we then deduce by the maximum principle that $\varphi \geqslant 0$ in all $\mathbb{R}^{\mathsf{d}}$, which is the desired result. □

We deduce the following bounds on μ_β.

Lemma 3.3. — *For every $x \in \mathbb{R}^{\mathsf{d}}$ and $\beta \geqslant 1$, we have*

$$0 < \mu_\beta(x) \leqslant \begin{cases} \min(C, C\exp\left(-\beta(V(x) - C)\right) & \text{for } \mathsf{d} \geqslant 3 \\ \min(C, C\exp\left(-\beta(V(x) - \log|x| - C)\right) & \text{for } \mathsf{d} = 2. \end{cases} \tag{3.8}$$

Proof. — Let us now turn to the upper bound. With the result of (3.3) and bounds on h^{μ_∞}, we have

$$h^{\mu_\beta}(x) - c_\beta \geqslant -\max(1, \log|x|)\mathbb{1}_{\mathsf{d}=2} - C - \frac{\log m_\beta}{\beta}.$$

Inserting into (1.11) we deduce that

$$\log \mu_\beta = \beta(c_\beta - h^{\mu_\beta}) - \beta V \leqslant \beta \max(1, \log|x|)\mathbb{1}_{\mathsf{d}=2} + \beta C + \log m_\beta - \beta V. \tag{3.9}$$

In view of (1.15), there thus exists $R > 0$ independent of β such that if $x \in \mathbb{R}^{\mathsf{d}} \backslash B_R$ we have $\log \mu_\beta < \log m_\beta - 1$ (and recall that $m_\beta < \infty$ by Lemma 2.2). This, with the fact that μ_β is continuous, implies that $\sup_{\mathbb{R}^{\mathsf{d}}} \mu_\beta$ must be a maximum, and it must be achieved at some point x_β in B_R. Then we must have $\Delta \log \mu_\beta(x_\beta) \leqslant 0$, hence by (1.12)

$$\mathsf{c_d} \mu_\beta(x_\beta) - \Delta V(x_\beta) \leqslant 0.$$

We may then deduce that

$$m_\beta = \mu_\beta(x_\beta) \leqslant \frac{1}{\mathsf{c_d}} \max_{B_R} \Delta V$$

i.e. that m_β is bounded independently of β. The first bounds in the right-hand side of (3.8) follow. The bound in (3.9) then gets improved to

$$\log \mu_\beta \leqslant \beta \max(1, \log|x|)\mathbb{1}_{\mathsf{d}=2} + \beta C - \beta V$$

which yields the second set of bounds in (3.8). □

Proposition 3.4. — *There exists $C > 0$ (depending only on V and d) such that if β is large enough, we have*

$$-\frac{C}{\beta} \leqslant h^{\mu_\beta} - c_\beta - (h^{\mu_\infty} - c_\infty) \leqslant \frac{C \log \beta}{\beta}. \tag{3.10}$$

Proof. — The lower bound is an immediate consequence of (3.3) and (3.8). Let us turn to the upper bound.

We know that

$$\min(h^{\mu_\beta} - c_\beta + V, -\Delta h^{\mu_\beta}) = \min\left(-\frac{1}{\beta}\log \mu_\beta, \mu_\beta\right).$$

If the right-hand side were $\leqslant 0$ we could directly conclude by comparison principle. Instead, we need to modify our test function slightly. To that end, let us define

$$E := \{x \in \mathbb{R}^{\mathsf{d}} : \mu_\beta(x) < \beta^{-2}\}.$$

Let us estimate $\mu_\beta(E)$: using (3.8) and (1.16), we find that

$$\mu_\beta(E) \leqslant C \int_E \beta^{-1} \left(\exp\left(-\frac{\beta}{2}(V - C)\right) \wedge 1 \right) \leqslant C\beta^{-1} \tag{3.11}$$

or respectively using $V - \log|x| - C$ in dimension 2. Since $\mu_\beta(\mathbb{R}^{\mathsf{d}}) = 1$ and $\mu_\beta \leqslant C$, it also follows that if β is large enough,

$$|\mathbb{R}^{\mathsf{d}} \backslash E| \geqslant \frac{1}{C}. \tag{3.12}$$

Let now w be

$$w := \mathbf{g} * \left(\mu_\beta \mathbb{1}_E - \frac{\mu_\beta(E)}{|\mathbb{R}^{\mathsf{d}} \setminus E|} \mathbb{1}_{\mathbb{R}^{\mathsf{d}} \backslash E} \right) \tag{3.13}$$

This way w decays like $|x|^{1-\mathsf{d}}$ in all dimensions $\mathsf{d} \geqslant 2$, and in view of (3.11) we have

$$\forall\, x \in \mathbb{R}^{\mathsf{d}}, \quad |w(x)| \leqslant C\beta^{-1}. \tag{3.14}$$

Let us then set

$$v := h^{\mu_\beta} - c_\beta + c_\infty - \frac{2}{\beta}\log\beta - w - C\beta^{-1}$$

for the C of (3.14). Observe that

$$-\frac{1}{\mathsf{c_d}}\Delta v = \mu_\beta \mathbb{1}_{\mathbb{R}^{\mathsf{d}} \backslash E} + \frac{\mu_\beta(E)}{|\mathbb{R}^{\mathsf{d}} \setminus E|} \mathbb{1}_{\mathbb{R}^{\mathsf{d}} \backslash E} \quad \text{in } \mathbb{R}^{\mathsf{d}}.$$

By choice of E, (1.11) and (3.14), we have in $\mathbb{R}^{\mathsf{d}} \setminus E$,

$$\begin{aligned} v + V - c_\infty &= h^{\mu_\beta} + V - c_\beta - \frac{2}{\beta}\log\beta - w - C\beta^{-1} \\ &= -\frac{1}{\beta}\log\mu_\beta - \frac{2}{\beta}\log\beta - w - C\beta^{-1} \leqslant 0. \end{aligned} \tag{3.15}$$

It follows that

$$\min(v + V - c_\infty, -\Delta v) \leqslant 0 \quad \text{in } \mathbb{R}^{\mathsf{d}}. \tag{3.16}$$

In dimension $\mathsf{d} = 2$ the comparison principle of Lemma 3.1 allows to conclude that $v \leqslant h^{\mu_\infty}$ which yields the desired upper bound for h^{μ_β}. Let us now turn to dimension $\mathsf{d} \geqslant 3$. Setting

$$\varphi := h^{\mu_\infty} - v,$$

by (3.15) and (1.6) we have

$$\begin{cases} \varphi \geqslant 0 & \text{in } \mathbb{R}^{\mathsf{d}} \setminus E \\ -\Delta\varphi \geqslant 0 & \text{in } E. \end{cases} \tag{3.17}$$

We also have $\varphi \to c_\beta - c_\infty + \frac{2}{\beta}\log\beta + C\beta^{-1}$ at ∞. Arguing as in the proof of Lemma 3.2, let ψ be a harmonic function equal to zero on ∂E and $c_\beta - c_\infty + \frac{2}{\beta}\log\beta + C\beta^{-1}$ at infinity, we have $\varphi \geqslant \psi$ in E and if $c_\beta - c_\infty + \frac{2}{\beta}\log\beta + C\beta^{-1} < 0$, ψ tends to its limit from above at speed $|x|^{2-\mathsf{d}}$. On the other hand $\int_{\mathbb{R}^{\mathsf{d}}} \Delta\varphi = 0$. As in the proof of Lemma 3.2, we get a contradiction and conclude that $c_\beta - c_\infty + \frac{2}{\beta}\log\beta + C\beta^{-1} \geqslant 0$. We then conclude from (3.17) and the maximum principle that $\varphi \geqslant 0$ everywhere, which yields the desired result. □

We deduce some corollaries.

Lemma 3.5. — *There exists $C > 0$ (depending only on V and d) such that*

$$\mu_\beta(x) \leqslant \exp\big(-\beta\alpha \min(1, \operatorname{dist}(x,\Sigma)^2) + C\big), \tag{3.18}$$

$$\mu_\beta(x) \geqslant \exp(-C\log\beta) \quad \text{for } x \in \Sigma, \tag{3.19}$$

$$\mu_\beta(\Sigma^c) \leqslant \frac{C}{\sqrt{\beta}}, \tag{3.20}$$

and

$$\left| \int_{\Sigma^c} \mu_\beta \log \mu_\beta \right| \leqslant \frac{C}{\sqrt{\beta}}. \tag{3.21}$$

Proof. — Taking the exponential of (1.11) and using (1.3), we find

$$\mu_\beta = \exp\big(\beta(c_\beta - h^{\mu_\beta} - V)\big) = \exp\big(\beta(c_\beta - c_\infty + h^{\mu_\infty} - h^{\mu_\beta} - \zeta)\big). \tag{3.22}$$

Inserting (3.10) and (1.20), we find

$$\begin{aligned} \exp\left(-\beta\zeta(x) - C\log\beta\right) &\leqslant \mu_\beta(x) \\ &\leqslant \exp\big(-\beta\alpha\min(1, \operatorname{dist}(x,\Sigma)^2) + C\big). \end{aligned} \tag{3.23}$$

The upper bound in (3.18) follows, and (3.19) as well since $\zeta = 0$ in Σ.

For (3.20), using (3.23) and (3.8) as well as (1.16) and the coarea formula, we may write

$$\begin{aligned} \mu_\beta(\Sigma^c) \leqslant C\int_0^\alpha \exp\big(-\beta\alpha s^2\big)\,\mathrm{d}s + C\exp\left(-\frac{\beta}{2}\alpha\right)\int_{\mathbb{R}^{\mathsf{d}}} \exp\left(-\frac{\beta}{2}(V - C)\wedge 1\right) \\ \leqslant \frac{C}{\sqrt{\beta}}, \end{aligned} \tag{3.24}$$

or respectively using $V - \log|x| - C$ in dimension $\mathsf{d} = 2$.

Arguing in the same way, and using the behavior of the function $x \log x$ we have

$$\left|\int_{\Sigma^c} \mu_\beta \log \mu_\beta\right| \leqslant C \int_0^\alpha \beta s^2 \exp(-\beta\alpha s^2)\,\mathrm{d}s + C\beta\alpha \exp\left(-\frac{\beta}{2}\alpha\right)\int_{\mathbb{R}^\mathsf{d}} \exp\left(-\frac{\beta}{2}(V-C)\wedge 1\right) \leqslant \frac{C}{\sqrt{\beta}}$$

(respectively with $V - \log|x|$ in dimension 2) which proves (3.21). □

4. Study of the radial case and barrier argument

Here we first specialize to $V(x) = \frac{\lambda}{2}|x|^2$, which will provide a barrier function for the general case. The problem is then radial and the solution $\mu_\beta(x) = e^{u_\beta(|x|)}$ with u_β solving in place of (1.11) the ODE

$$\frac{1}{r^{\mathsf{d}-1}}(r^{\mathsf{d}-1}u_\beta')' = \beta(\mathsf{c}_\mathsf{d} e^{u_\beta} - \lambda). \tag{4.1}$$

By scaling, the coincidence set Σ is then a ball of radius $R_\mathsf{d}\lambda^{-1/\mathsf{d}}$, where R_d only depends on d, more precisely $\mu_\infty = \frac{1}{\mathsf{c}_\mathsf{d}}\mathbb{1}_{B(0,R_\mathsf{d}\lambda^{-1/\mathsf{d}})}$. We first note that at a point of local maximum of μ_β we have $\Delta \log \mu_\beta \leqslant 0$ hence $\mu_\beta \leqslant \frac{\Delta V}{\mathsf{c}_\mathsf{d}} = \frac{\lambda}{\mathsf{c}_\mathsf{d}}$. We thus know that $\mathsf{c}_\mathsf{d}\mu_\beta \leqslant \lambda$ everywhere and thus $(r^{\mathsf{d}-1}u_\beta')' \leqslant 0$ and $r^{\mathsf{d}-1}u_\beta' \leqslant 0$ hence u_β is nonincreasing.

In view of the exponential decay proved for the general problem in (3.18), for $1 \ll K_\beta \leqslant C\log\beta$, there exists an r_2 (depending on β) and bounded above by $2R_\mathsf{d}\lambda^{-1/\mathsf{d}}$ such that $u_\beta(r_2) = -K_\beta$.

Lemma 4.1. — *Let η be such that $e^{-K_\beta} \leqslant \eta \leqslant \frac{\lambda}{2\mathsf{c}_\mathsf{d}}$, and let r_2 be as above such that $u_\beta(r_2) = -K_\beta$. There exists $r_1 \geqslant r_2 - C\sqrt{\frac{K_\beta + \log\eta}{\beta}}$ (depending on β) such that*

$$u_\beta(r_1) = \log\eta,$$

with C depending only on d and λ.

Proof. — Integrating (4.1), we may write

$$r^{\mathsf{d}-1}u_\beta'(r) = \int_0^r \beta s^{\mathsf{d}-1}(\mathsf{c}_\mathsf{d} e^{u_\beta(s)} - \lambda)\,\mathrm{d}s. \tag{4.2}$$

Let r_1 be the largest $r \leqslant r_2$ such that $u_\beta(r_1) = \log\eta$. For $r \geqslant r_1$ we have $u_\beta(r) \leqslant \log\eta \leqslant \log\frac{\lambda}{2\mathsf{c_d}}$ hence

$$\begin{aligned}\log\eta + K_\beta = u_\beta(r_1) - u_\beta(r_2) &= \beta\int_{r_1}^{r_2}\frac{1}{t^{\mathsf{d}-1}}\int_0^t s^{\mathsf{d}-1}(\lambda - \mathsf{c_d}e^{u_\beta(s)})\,\mathrm{d}s\,\mathrm{d}t\\ &\geqslant \beta\frac{\lambda}{2}\int_{r_1}^{r_2}\frac{1}{t^{\mathsf{d}-1}}\int_{r_1}^t s^{\mathsf{d}-1}\,\mathrm{d}s\,\mathrm{d}t\\ &\geqslant \beta\frac{\lambda}{2\mathsf{d}}\int_{r_1}^{r_2}\frac{t^{\mathsf{d}} - r_1^{\mathsf{d}}}{t^{\mathsf{d}-1}}\,\mathrm{d}t\\ &\geqslant \beta\frac{\lambda}{2\mathsf{d}}\int_{r_1}^{r_2}(t - r_1)\,\mathrm{d}t\\ &\geqslant \beta\frac{\lambda}{4\mathsf{d}}(r_2 - r_1)^2.\end{aligned}$$

The result follows. □

We may now use the radial solution as a barrier for the solution in the general case.

Proposition 4.2. — *Let*

$$M_\beta := \beta\max_{\mathbb{R}^{\mathsf{d}}}(h^{\mu_\beta} - c_\beta - h^{\mu_\infty} + c_\infty), \tag{4.3}$$

and $1 \geqslant \eta \geqslant e^{-M_\beta}$. *There exists* $C > 0$ *depending only on* V *and* d *such that for* $x \in \Sigma$ *satisfying* $\operatorname{dist}(x, \partial\Sigma) \geqslant C\sqrt{\frac{M_\beta}{\beta}}$, *we have*

$$\mu_\beta(x) \geqslant \eta. \tag{4.4}$$

Proof. — We know from (3.10) that $M_\beta \leqslant C\log\beta$. Taking the exponential of (1.11) and using the definition (1.3) and the definition of M_β, we find

$$\begin{aligned}\mu_\beta &= \exp(\beta(c_\beta - h^{\mu_\beta} - V))\\ &= \exp\left(\beta(c_\beta - c_\infty + h^{\mu_\infty} - h^{\mu_\beta} - \zeta)\right) \geqslant e^{-M_\beta}\end{aligned} \tag{4.5}$$

in Σ, since $\zeta = 0$ in Σ. Since $\partial\Sigma \in C^{1,1}$, it satisfies an interior ball condition, with a ball of radius which can be chosen independently of the point, say of radius ε. We then choose $\lambda \geqslant 2\mathsf{c_d}$ large enough that $\lambda \geqslant \alpha$ and $2R_{\mathsf{d}}\lambda^{-1/\mathsf{d}} \leqslant \varepsilon$. Given this λ, we consider ν to be α/λ times the radial $\mu_{\frac{\alpha\beta}{\lambda}}$ of Lemma 4.1, which satisfies

$$\Delta\log\nu = \beta(\mathsf{c_d}\nu - \alpha). \tag{4.6}$$

We also let $K_{\alpha\beta/\lambda} = M_\beta$ in Lemma 4.1 applied at the inverse temperature $\beta\alpha/\lambda$, and we let r be the r_2 given there. Since $r_2 \leqslant 2R_{\mathsf{d}}\lambda^{-\frac{1}{\mathsf{d}}} \leqslant \varepsilon$, a ball B_r tangent to $\partial\Sigma$ at any point can be included in Σ. In view of (4.5), the monotonicity of ν and the definition of r and $K_{\alpha\beta/\lambda}$, we check that $\nu \leqslant \mu_\beta$

on ∂B_r. We now substract (1.12) and (4.6) and test the resulting relation against $(\log\nu - \log\mu_\beta)_+$ which vanishes on ∂B_r. We obtain

$$\int_{B_r} (\Delta \log \nu - \Delta \log \mu_\beta)(\log \nu - \log \mu_\beta)_+ \\ = \beta \int_{B_r} (\mathsf{c_d}\nu - \mathsf{c_d}\mu_\beta + \Delta V - \alpha)(\log \nu - \log \mu_\beta)_+.$$

Using that $\Delta V \geqslant \alpha$ in B_r by (1.17) and an integration by parts, we are led to

$$-\int_{B_r \cap \{\nu \geqslant \mu_\beta\}} |\nabla(\log \nu - \log \mu_\beta)|^2 \geqslant \beta \mathsf{c_d} \int_{B_r} (\nu - \mu_\beta)(\log \nu - \log \mu_\beta)_+ \geqslant 0.$$

It follows that $\nu \leqslant \mu_\beta$ a.e. in B_r, thus ν is a barrier for μ_β. In view of the result of Lemma 4.1, we deduce that $\mu_\beta \geqslant \eta$ as soon as $x \in B_r$ and $\operatorname{dist}(x, \partial B_r) \geqslant C\sqrt{\frac{M_\beta + \log \eta}{\beta}}$ for some C depending only on V and d. The result follows. □

5. Optimal estimates and lower bound on μ_β

We may now conclude

Proposition 5.1. — *There exists $C > 0$ (depending only on V and d) such that if β is large enough, we have*

$$h^{\mu_\beta} - c_\beta - (h^{\mu_\infty} - c_\infty) \leqslant \frac{C}{\beta}, \tag{5.1}$$

$$\exp\left(-\frac{\beta}{C}\operatorname{dist}(x, \Sigma)^2 - C\right) \leqslant \mu_\beta(x) \tag{5.2}$$

for x in a neighborhood of Σ, and

$$\mu_\beta(x) \geqslant \frac{1}{C} > 0 \quad \text{for } x \in \Sigma. \tag{5.3}$$

Proof of Proposition 5.1. — We iterate and improve on the proof of Proposition 3.4. Let M_β be as in (4.3). We know from (3.10) that $M_\beta \leqslant C \log \beta$. If M_β is bounded above independently of β, then there is nothing to prove. If $M_\beta \to +\infty$ as $\beta \to \infty$, let η be a constant in $[e^{-M_\beta}, \min(\frac{\lambda}{2\mathsf{c_d}}, 1)]$, to be determined later. Let then

$$\widehat{\Sigma} = \left\{ x \in \Sigma, \operatorname{dist}(x, \partial\Sigma) \geqslant C\sqrt{\frac{M_\beta}{\beta}} := \tau_\beta \right\}$$

with C as in Proposition 4.2. In view of that proposition we have that $\mu_\beta \geqslant \eta$ in $\widehat{\Sigma}$.

Since $\partial\Sigma \in C^{1,1}$ so is $\partial\widehat{\Sigma}$ for β large enough, and we may consider $R(x)$ to be the reflexion with respect to $\partial\widehat{\Sigma}$, defined in a tubular neighborhood. We next let $\widehat{\mu_\beta} = \mu_\beta \mathbb{1}_E$ where

$$E := \left(\{0 \leqslant \mathrm{dist}(x,\Sigma) < \beta^{-\frac{1}{4}}\} \cup (\Sigma \backslash \widehat{\Sigma})\right) \cap \{\mu_\beta(x) \leqslant \eta\}.$$

Arguing as in (3.24) we have that

$$|\mu_\beta((\Sigma\backslash E)^c) - \widehat{\mu_\beta}(E)| < \frac{C}{\beta}. \tag{5.4}$$

We note that E is included in a $\tau_\beta + \beta^{-1/4}$ neighborhood of $\partial\Sigma$.

Let w be

$$w := \mathbf{g} * \left(\widehat{\mu_\beta} - R\#\widehat{\mu_\beta} + (\mu_\beta \mathbb{1}_{(\Sigma\backslash E)^c} - \widehat{\mu_\beta}) - \frac{\mu_\beta((\Sigma\backslash E)^c) - \widehat{\mu_\beta}(E)}{|\widehat{\Sigma}|}\mathbb{1}_{\widehat{\Sigma}}\right), \tag{5.5}$$

where # denotes the push-forward of measures. We claim that

$$\forall\, x \in \mathbb{R}^{\mathsf{d}}, \quad |w(x)| \leqslant C\frac{\eta}{\beta}(M_\beta + |\log \eta|), \tag{5.6}$$

with C depending only of V and d. We note that w decays like $|x|^{1-\mathsf{d}}$ in all dimensions $\mathsf{d} \geqslant 2$ because its Laplacian integrates to 0, and in view of (5.4) we have

$$\left|\mathbf{g} * \left(\mu_\beta \mathbb{1}_{(\Sigma\backslash E)^c} - \widehat{\mu_\beta} - \frac{\mu_\beta((\Sigma\backslash E)^c) - \widehat{\mu_\beta}(E)}{|\widehat{\Sigma}|}\mathbb{1}_{\widehat{\Sigma}}\right)\right| \leqslant C\beta^{-1}.$$

Hence there remains to show that

$$|\mathbf{g} * (\widehat{\mu_\beta} - R\#\widehat{\mu_\beta})\,| \leqslant C\eta\frac{M_\beta}{\beta}. \tag{5.7}$$

By definition of the push-forward, we have

$$\begin{aligned} &|\mathbf{g} * (\widehat{\mu_\beta} - R\#\widehat{\mu_\beta})\,(x)| \\ &\quad = \left|\int (\mathbf{g}(x-y) - \mathbf{g}(x-R(y)))\,\mathrm{d}\widehat{\mu_\beta}(y)\right| \\ &\quad \leqslant C\eta \int_{\min(|y-x|,|R(y)-x|)\leqslant\tau_\beta} |\mathbf{g}(x-y) - \mathbf{g}(x-R(y))|\mathrm{d}y \\ &\qquad + C\int_{|y-x|\geqslant\tau_\beta,|R(y)-x|\geqslant\tau_\beta} \frac{|\langle y-R(y), y-x\rangle|}{|y-x|^{\mathsf{d}}}\mathrm{d}\widehat{\mu_\beta}(y) \\ &\qquad + C\int_{|y-x|\geqslant\tau_\beta,|R(y)-x|\geqslant\tau_\beta} \frac{|y-R(y)|^2}{|y-x|^{\mathsf{d}}}\mathrm{d}\widehat{\mu_\beta}(y). \end{aligned} \tag{5.8}$$

The first time in the right-hand side is bounded by $C\eta\tau_\beta^2$ in all dimensions, while the second is bounded by

$$C\int_{\substack{|y-x|\geqslant\tau_\beta\\ 0<\mathrm{dist}(y,\Sigma)<\beta^{-1/4}}}\frac{\mathrm{dist}(y,\widehat{\Sigma})|\langle y-x,n(y)\rangle|}{|y-x|^{\mathsf{d}}}\min\bigl(\eta,\exp(-\beta\alpha\ \mathrm{dist}^2(y,\Sigma))\bigr)\mathrm{d}y \\ +C\eta\int_{|y-x|\geqslant\tau_\beta,y\in\Sigma\setminus\widehat{\Sigma}}\frac{\mathrm{dist}(y,\widehat{\Sigma})|\langle y-x,n(y)\rangle|}{|y-x|^{\mathsf{d}}}\,\mathrm{d}y \quad (5.9)$$

where $n(y)$ is the normal vector to $\partial\widehat{\Sigma}$ at y. We claim that these terms are bounded by $C\eta\tau_\beta^2$. To see this, let us use local coordinates adapted to $\widehat{\Sigma}$ of the form $(s,t)\in\partial\widehat{\Sigma}\times\mathbb{R}$ such that each vector y can be decomposed into $y=s+tn(y)$, with $s\in\partial\widehat{\Sigma}$ and $t=\mathrm{dist}(y,\widehat{\Sigma})$. We choose the origin so that x has coordinates $(0,t_0)$, and assume $|t_0|<\varepsilon$ (for otherwise the result is clearly true). The Jacobian of the change of variables is bounded, and $\langle y-x,n(y)\rangle=O(|s|^2)+t-t_0$ as $|s|\to 0$, using that $\partial\widehat{\Sigma}\in C^{1,1}$, so we may locally on $\partial\widehat{\Sigma}$ bound this integral by

$$C\int_{\substack{|t-t_0|\geqslant\tau_\beta\\ \tau_\beta<t<\tau_\beta+\beta^{-1/4}\\ s\in\partial\widehat{\Sigma}}}\frac{t\min\left(\eta,\exp(-\beta\alpha(t-\tau_\beta)^2)\right)\left((t-t_0)+O(s^2)\right)}{\left(|t-t_0|^2+|s|^2+O(|s|^2|t-t_0|)\right)^{\frac{\mathsf{d}}{2}}}\mathrm{d}s\,\mathrm{d}t$$
$$\leqslant C\eta\int_{\substack{\tau_\beta<t<\tau_\beta+\beta^{-1/4}\\ s'\in\partial\frac{\widehat{\Sigma}}{|t-t_0|}}}\frac{t\min\left(\eta,\exp(-\beta\alpha(t-\tau_\beta)^2)\right)\left(O(1+|s'|^2|t-t_0|)\right)}{\frac{1}{2}\left(1+|s'|^2\right)^{\frac{\mathsf{d}}{2}}}\mathrm{d}s'\mathrm{d}t$$
$$\leqslant C\eta\tau_\beta\sqrt{\frac{|\log\eta|}{\beta}}\leqslant C\left(\eta\tau_\beta^2+\frac{\eta|\log\eta|}{\beta}\right)$$

where used the change of variables $s=|t-t_0|s'$, that $\partial\widehat{\Sigma}\in C^{1,1}$ and that ε can be chosen small enough.

The second term in (5.9) and the third term in (5.8) are bounded by $C\eta\tau_\beta^2$ by similar computations, which concludes the proof of (5.7) hence of (5.6). Let us then set

$$v:=h^{\mu_\beta}-c_\beta+c_\infty-w-C\frac{\eta}{\beta}(M_\beta+|\log\eta|)+\frac{\log\eta}{\beta}$$

for the C of (5.6). Observe that

$$-\frac{1}{\mathsf{c}_{\mathsf{d}}}\Delta v=\mu_\beta\mathbb{1}_{\Sigma\setminus E}+R\#\widehat{\mu_\beta}+\frac{\mu_\beta(\Sigma^c)-\widehat{\mu_\beta}(\Sigma^c)}{|\widehat{\Sigma}|}\mathbb{1}_{\widehat{\Sigma}}\quad\text{in }\mathbb{R}^{\mathsf{d}},$$

hence Δv is supported in $\Sigma\backslash E$. By (1.11), (5.6) and $\mu_\beta \geqslant \eta$ in $\Sigma\backslash E$, we have in $\Sigma\backslash E$,

$$\begin{aligned} v + V - c_\infty &= h^{\mu_\beta} + V - c_\beta - w - C\frac{\eta}{\beta}(M_\beta + |\log\eta|) + \frac{\log\eta}{\beta} \\ &= -\frac{1}{\beta}\log\mu_\beta + \frac{\log\eta}{\beta} - C\eta\frac{M_\beta + |\log\eta|}{\beta} - w \leqslant 0. \end{aligned} \tag{5.10}$$

It follows that

$$\min(v + V - c_\infty, -\Delta v) \leqslant 0 \quad \text{in } \mathbb{R}^{\mathsf{d}}. \tag{5.11}$$

In dimension $\mathsf{d} = 2$ the comparison principle of Lemma 3.1 allows to conclude that $v \leqslant h^{\mu_\infty}$ which yields the desired upper bound for h^{μ_β}. Let us now turn to dimension $\mathsf{d} \geqslant 3$. Setting

$$\varphi := h^{\mu_\infty} - v,$$

by (5.10) and (1.6) we have

$$\begin{cases} \varphi \geqslant 0 & \text{in } \Sigma\backslash E \\ -\Delta\varphi \geqslant 0 & \text{in } (\Sigma\backslash E)^c. \end{cases} \tag{5.12}$$

We also have $\varphi \to c_\beta - c_\infty - C\eta\frac{M_\beta+|\log\eta|}{\beta} + \frac{\log\eta}{\beta}$ at ∞. Arguing as in the proof of Lemma 3.2, let ψ be a harmonic function equal vanishing on $\partial(\Sigma\backslash E)$ and $c_\beta - c_\infty - C\eta\frac{M_\beta+|\log\eta|}{\beta} + \frac{\log\eta}{\beta}$ at infinity, we have $\varphi \geqslant \psi$ in $\Sigma\backslash E$ and if $c_\beta - c_\infty - C\eta\frac{M_\beta+|\log\eta|}{\beta} + \frac{\log\eta}{\beta} < 0$, ψ tends to its limit from above at speed $|x|^{2-\mathsf{d}}$. On the other hand $\int_{\mathbb{R}^{\mathsf{d}}} \Delta\varphi = 0$. As in the proof of Lemma 3.2, we get a contradiction and conclude that $c_\beta - c_\infty - C\eta\frac{M_\beta+|\log\eta|}{\beta} + \frac{\log\eta}{\beta} \geqslant 0$. We then conclude from (5.12) and the maximum principle that $\varphi \geqslant 0$ everywhere, which yields

$$h^{\mu_\infty} - c_\infty - h^{\mu_\beta} + c_\beta \geqslant \frac{\log\eta}{\beta} - C\eta\frac{M_\beta + |\log\eta|}{\beta}$$

hence by definition of M_β

$$\frac{M_\beta}{\beta} \leqslant C\eta\frac{M_\beta + |\log\eta|}{\beta} - \frac{\log\eta}{\beta}.$$

Choosing η a small enough constant, we obtain that $M_\beta \leqslant C$, which concludes the proof of (5.1). The result (5.2) follows from combining (3.22) and (5.1), and (5.3) follows from combining (3.22), (5.1) and the fact that $\zeta = 0$ in Σ. ☐

Corollary 5.2. — *We have*

$$\|\nabla(h^{\mu_\beta} - h^{\mu_\infty})\|_{L^\infty(\mathbb{R}^{\mathsf{d}})} \leqslant C\beta^{-\frac{1}{2}}. \tag{5.13}$$

Proof. — This follows from (3.10), the fact that $\|\mu_\beta - \mu_\infty\|_{L^\infty} \leqslant C$ and interpolation (see for instance the appendix in [5]). ☐

6. Regularity theory and iterative approximation

Once μ_β is bounded below, the PDE (1.13) becomes uniformly elliptic and we may apply regularity theory tools to compare μ_β to the expected solution. In the case that ΔV is constant, then we can show that μ_β is very close to the constant μ_∞ inside Σ, however in the case where ΔV is not constant, there are corrections to arbitrary order that need to be added to μ_∞.

Assuming that $V \in C^{2m,\gamma}$ for some $m \in \mathbb{N}$ and exponent $\gamma > 0$, we recursively define f_k by (1.28). We note that, for β sufficiently large depending on the norms of V and on k, and by (1.17),

$$\|f_k\|_{C^{2(m-k-1),\gamma}(\Sigma)} \leqslant C \quad \text{and} \quad f_k \geqslant \frac{\alpha}{4\mathsf{c}_\mathsf{d}} \quad \text{in } \Sigma. \tag{6.1}$$

We also define

$$\varepsilon_k := \Delta \log f_k - \beta(\mathsf{c}_\mathsf{d} f_k - \Delta V) = \beta \mathsf{c}_\mathsf{d}(f_{k+1} - f_k) \tag{6.2}$$

and check that

$$\varepsilon_{k+1} = \Delta \log\left(1 + \frac{\varepsilon_k}{\beta \mathsf{c}_\mathsf{d} f_k}\right)$$

and thus

$$\|\varepsilon_k\|_{C^{2(m-k-2),\gamma}(\Sigma)} \leqslant C\beta^{-k}. \tag{6.3}$$

Thus since ε_k gets small as β gets large, f_k is a good approximate solution to (1.12) for $k \geqslant 1$. In view of (6.2) and (6.3), if $V \in C^\infty$ then f_k converges as $k \to \infty$ in all C^m spaces to f_∞, an exact solution of (1.12).

Proposition 6.1. — *Assume $m \in \mathbb{N}$, $m \geqslant 2$, and $\gamma \in (0,1]$ are such that $V \in C^{2m,\gamma}$. Then for every n even integer with $n \leqslant 2(m-2)$ and every $0 \leqslant \gamma' \leqslant \gamma$, there exists $C > 0$ depending only on V, d, n such that if β is large enough depending on m, for any $U \subseteq \Sigma$, we have*

$$\begin{aligned} &\left\|\mu_\beta(x) - f_{m-2-\frac{n}{2}}(x)\right\|_{C^{n,\gamma'}(U)} \\ &\qquad \leqslant C\beta^{\frac{n+\gamma'}{2}} \exp\left(-C \log^2\left(\beta \operatorname{dist}^2\left(U, \partial\Sigma\right)\right)\right) + C\beta^{n-m+1+\frac{\gamma'}{2}}. \end{aligned} \tag{6.4}$$

Proof. — Define $u_\beta := \frac{\mu_\beta}{f_k} - 1$. By (1.12), we have

$$\Delta \log(f_k(u_\beta + 1)) = \beta(\mathsf{c}_\mathsf{d} f_k u_\beta + \mathsf{c}_\mathsf{d} f_k - \Delta V).$$

In view of (6.2), we get

$$\Delta \log(1 + u_\beta) = \beta \mathsf{c}_\mathsf{d} f_k u_\beta - \varepsilon_k$$

which can be rewritten as

$$-\operatorname{div}\left(\frac{\nabla u_\beta}{1 + u_\beta}\right) + \beta \mathsf{c}_\mathsf{d} f_k u_\beta = \varepsilon_k. \tag{6.5}$$

This equation is uniformly elliptic in Σ since, by (3.8),(5.3) and (6.1),

$$\frac{\alpha}{C} \leqslant u_\beta + 1 \leqslant \frac{C}{\alpha} \quad \text{in } \Sigma. \tag{6.6}$$

We next seek a local L^2 estimate for u_β in Σ. Select $x_0 \in \Sigma$, $r \in \big(0, \operatorname{dist}\big(x_0, \partial\Sigma\big)\big)$ and a cutoff function $\chi \in C_c^\infty(B_r)$. Testing (6.5) with $\chi^2 u_\beta$, we obtain

$$\int_{B_r(x_0)} \chi^2 \frac{|\nabla u_\beta|^2}{1+u_\beta} + \int_{B_r(x_0)} \frac{2u_\beta \chi \nabla\chi \cdot \nabla u_\beta}{1+u_\beta} + \beta \mathsf{c_d} \int_{B_r(x_0)} \chi^2 f_k u_\beta^2 = \int_{B_r(x_0)} \chi^2 \varepsilon_k u_\beta.$$

Using Young's inequality and (6.1), we obtain after rearrangement that

$$\begin{aligned}
\frac{1}{2}\int_{B_r(x_0)} \chi^2 \frac{|\nabla u_\beta|^2}{1+u_\beta} + \frac{\beta \mathsf{c_d}}{2} \int_{B_r(x_0)} \chi^2 f_k u_\beta^2 &\leqslant 4 \int_{B_r(x_0)} \frac{u_\beta^2\, |\nabla \chi|^2}{1+u_\beta} + \frac{1}{2\beta\mathsf{c_d}} \int_{B_r(x_0)} \chi^2 \varepsilon_k^2 \\
&\leqslant C \int_{B_r(x_0)} |\nabla\chi|^2 u_\beta^2 + \frac{C}{\beta}\int_{B_r(x_0)} \varepsilon_k^2.
\end{aligned}$$

Choosing χ such that $\mathbb{1}_{B_{r/2}(x_0)} \leqslant \chi \leqslant \mathbb{1}_{B_r(x_0)}$ and $|\nabla\chi| \leqslant 4r^{-1}$ and using (6.1), (6.3) and (6.6), we find that, if $k \leqslant m-2$, then

$$\fint_{B_{r/2}(x_0)} |\nabla u_\beta|^2 + \beta \fint_{B_{r/2}(x_0)} u_\beta^2 \leqslant \frac{C}{r^2} \fint_{B_r(x_0)} u_\beta^2 + C\beta^{-(2k+1)} \tag{6.7}$$

In particular, keeping only the second term on the left side, we obtain

$$\fint_{B_{r/2}} u_\beta^2 \leqslant \frac{C}{\beta r^2} \fint_{B_r} u_\beta^2 + C\beta^{-2(k+1)}. \tag{6.8}$$

After an iteration of the previous inequality, we obtain, for $s := C\beta^{-\frac{1}{2}}$,

$$\fint_{B_s(x_0)} u_\beta^2 \leqslant \exp\big(-c\log^2\big(\beta r^2\big)\big) + C\beta^{-2(k+1)}.$$

Let us now rescale the equation (6.5) by defining

$$\widehat{u}_\beta(x) := u_\beta(x_0 + sx), \tag{6.9}$$

and similarly $\widehat{f}_k, \widehat{\varepsilon}_k$. In terms of $\widehat{u}_\beta$, the equation becomes

$$-\operatorname{div}\left(\frac{\nabla \widehat{u}_\beta}{1+\widehat{u}_\beta}\right) + \mathsf{c_d} \widehat{f}_k \widehat{u}_\beta = \beta^{-1}\widehat{\varepsilon}_k \quad \text{in } B_1. \tag{6.10}$$

Note that the function $\mathsf{c_d}\widehat{f}_k$ is bounded. Applying the De Giorgi–Nash H́older estimate (see [11, Theorem 8.24]) for uniformly elliptic equations, we obtain, for some $\sigma > 0$ and again for $k \leqslant m-2$,

$$\begin{aligned}\|\widehat{u}_\beta\|_{L^\infty(B_{1/2})} + [\widehat{u}_\beta]_{C^{0,\sigma}(B_{1/2})} &\leqslant C\left(\int_{B_1}\widehat{u}_\beta^2\right)^{\frac{1}{2}} + C\beta^{-1}\|\widehat{\varepsilon}_k\|_{L^\infty(B_s)} \\ &\leqslant C\exp\bigl(-c(\log^2\left(\beta r^2\right)\bigr) + C\beta^{-(k+1)}.\end{aligned}$$

Repeatedly applying Schauder estimates yields, for every $n \leqslant 2(m-k-2)$ and $0 \leqslant \gamma' \leqslant \gamma$,

$$[\nabla^n \widehat{u}_\beta]_{C^{0,\gamma'}(B_{1/2})} \leqslant C\exp\bigl(-c(\log^2\left(\beta r^2\right)\bigr) + C\beta^{-(k+1)}, \tag{6.11}$$

with constants C which now depend on m. Taking $k = m - 2 - \frac{n}{2}$, after rescaling back, by definition of u_β this implies (6.4). □

Remark 6.2. — The estimates above apply in the same way to all solutions of relations of the form (6.5). This allows to handle questions of stability of the solutions with respect to V: if V is changed into $V + t\xi$ with ξ supported in Σ, then letting μ_β^t be the corresponding thermal equilibrium measure, the function $u_t = \frac{\mu_\beta^t}{\mu_\beta} - 1$ satisfies

$$\operatorname{div}\left(\frac{\nabla u_t}{1+u_t}\right) = \beta\left(\mathsf{c_d}\mu_\beta^0 u_t - t\Delta\xi\right). \tag{6.12}$$

which is of the same form as (6.5). The same method then allows to estimate u_t hence μ_β^t/μ_β.

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