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m-subharmonic and m-plurisubharmonic functions: on two problems of Sadullaev

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To professor Ahmed Zeriahi

Abstract. — We show that the spaces of A-m-subharmonic and B-m-subharmonic functions differ in sufficiently high dimensions. We also prove that the Monge–Ampère type operator $\mathcal{M}_m$ associated to the space of m-plurisubharmonic functions does not allow an integral comparison principle except in the classical cases $m = 1$ and $m = n$. These answer in the negative two problems posed by A. Sadullaev.

Résumé. — Nous montrons que les fonctions A-m-sousharmoniques et B-m-sousharmoniques diffèrent en dimension suffisamment grande. Nous prouvons que l'opérateur de type Monge–Ampère $\mathcal{M}_m$ associé à l'espace des fonctions m-plurisousharmoniques ne permet pas un principe de comparaison intégral sauf dans les cas classiques $m = 1$ et $m = n$. Cela répond par la négative à deux problèmes posés par A. Sadullaev.

1. Introduction

Given a domain $\Omega \subset \mathbb{C}^n$ an upper semicontinuous function u defined in Ω is *plurisubharmonic* if for any affine complex line L the restriction $u|_{\Omega\cap L}$ is subharmonic. This class of functions plays a prominent role in complex analysis. We refer to [9] for some applications.

It is hence natural to try to generalize this class and investigate weaker positivity notions which should be less rigid. Below we mention two natural generalizations which share many potential theoretic properties with plurisubharmonic functions.

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A function is said to be *p-plurisubharmonic* (or p-psh) if it is upper semicontinuous and is subharmonic whenever it is restricted to any affine complex p-plane. Thus usual plurisubharmonic functions are 1-psh, while n-psh function in $\mathbb{C}^n$ are exactly the subharmonic ones.

When u is additionally $\mathcal{C}^2$ smooth it is easy to see that p-plurisubharmonicity can be formulated in either of the following two equivalent ways:

(1) at each point the sum of the p-smallest eigenvalues of the complex Hessian of u is nonnegative;
(2) the form $i\partial\bar{\partial}u \wedge (i\partial\bar{\partial}\|z\|^2)^{p-1}$ is positive.

Such function classes have been previously investigated by Dieu [10], Verbitsky [20], Abdullaev [1, 2] and Harvey–Lawson [15] and appear naturally in various branches of complex analysis, from the regularity of the Bergman projection (see [16]) to approximation of $\bar{\partial}$-closed differential forms and Andreotti–Grauert theory (see [4, 9, 17]).

A related but different class of functions are the m-subharmonic ones:

A $\mathcal{C}^2$ function u is said to be m-subharmonic if

$$(i\partial\bar{\partial}u)^j \wedge (i\partial\bar{\partial}\|z\|^2)^{n-j}$$

are positve top degree forms for every $j = 1, \ldots, m$. Using this positivity and the theory of positive currents it is possible to extend this definition to merely bounded upper semicontinuous functions (see [6, 7]).

In [18] A. Sadullaev discussed several aspects of the potential theory associated to m-subharmonic and m-plurisubharmonic functions. This nice survey covers in particular numerous results of the Uzbekistani complex analysis group which are otherwise hardly accessible[(1)].

In the case of m-sh functions there is a natural Hessian operator

$$(i\partial\bar{\partial}u)^m \wedge (i\partial\bar{\partial}\|z\|^2)^{n-m},$$

which could be defined for all locally bounded m-sh functions (see [6, 7]) and thus one can recover many analogues of pluripotential theory of Bedford and Taylor (see [2, 3, 6, 12, 13, 14, 18]).

Attempts to build such a theory for m-psh functions have been only partially successful ([1, 2, 18]). The basic reason is the lack of a natural Hessian operator associated to this function class. In fact [18] lists two approaches:

[(1)] In [18] the m-psh functions are called m-subharmonic, while our definition of m-subharmonicity agrees with the notion of B-$(n-m+1)$-subharmonic functions studied there. As the notion of m-subharmonicity provided above is now widely used (see [6, 12, 13, 14]) we prefer to stick to this terminology.

The first one is to use simply the Hessian $(i\partial\bar{\partial}u)^{n-m+1} \wedge (i\partial\bar{\partial}\|z\|^2)^{m-1}$. The problem is that this need not return a *positive* form as the example of the function $u(z) = -|z_1|^2 + |z_2|^2 + |z_3|^2$ shows. Thus one naturally restricts the class of m-psh functions to the set $A - m - sh(\Omega)$ defined by

$$\{u \in \mathcal{C}^2(\Omega)|\ u \text{ is } m - psh, (i\partial\bar{\partial}u)^{n-m+1} \wedge (i\partial\bar{\partial}\|z\|^2)^{m-1} \geqslant 0\}. \tag{1.1}$$

Alternatively one can seek an operator acting on smooth m-psh functions and then try to generalize its action suitably to all locally bounded ones. One possible approach for such an operator is given by

$$\mathcal{M}_m(u) := \Pi_{1\leqslant j_1<j_2<\cdots<j_m\leqslant n}(\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m}), \tag{1.2}$$

with λ_j denoting the eigenvalues of the complex Hessian of u. Obviously this operator is nonnegative on smooth m-psh functions and it can be shown that it is elliptic when restricted to this class. It seems however hard to apply pluripotential techniques to $\mathcal{M}_m$ directly. We remark nevertheless that $\mathcal{M}_m$ has also been investigated on manifolds (see [19]) where it appears naturally in geometric problems.

Motivated by these two approaches A. Sadullaev in [18] posed the following question:

Question 1.1. — *Let u be a $(n - m + 1)$-subharmonic function (called B-m-subharmonic in* [18]*) It can be shown that u is A-m-subharmonic. Is the converse true i.e. do we have the equality*

$$A - m - sh(\Omega) = B - m - sh(\Omega)?$$

In [18] it is shown that the answer is affirmative for $m = 2$, while for $m = 1, n - 1$ and n the equivalence is trivially true.

A basic tool in pluripotential theory is the *integral comparison principle* of Bedford and Taylor (see [5]). Thus in [18] it was asked whether comparison principle holds for $\mathcal{M}_m$:

Question 1.2. — *Let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be two m-psh functions with $u > v$ on Ω with equality on $\partial\Omega$. Is it true that for some $\alpha > 0$ and all such tuples (u, v) one has*

$$\int_\Omega \mathcal{M}_m^\alpha(u) \leqslant \int_\Omega \mathcal{M}_m^\alpha(v)?$$

The goal of this note is to answer in the negative both questions. We will show that the inclusion $B - m - sh(\Omega) \subset A - m - sh(\Omega)$ is strict for domains $\Omega \subset \mathbb{C}^n$ if $n \geqslant 11$. Interestingly these function classes are indeed the same in dimensions less or equal to 7 (see Theorem 3.1). As for the second question we will show that this inequality holds only when $\alpha = 1$ and furthermore

$m = 1$ or n i.e. when we deal with the complex Monge–Ampère operator or the Laplacian.

In Author's opinion these negative results show that a construction of potential theory for m-psh function is necessarily subtler than in the case of m-subharmonic ones and attempts to apply directly tools from Bedford–Taylor theory are doomed to fail. It seems however possible that a suitable *viscosity* potential theory can be constructed (see [11] for such an approach).

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2. Preliminaries

In this section we shall fix our terminology. Throughout the note we shall work with $\mathcal{C}^2$ functions hence all operators involved will have a *classical* meaning. We refer to [6, 12, 13, 14] for the nonlinear potential theory of weak m-subharmonic functions.

Consider the set $\mathcal{A}_n$ of all Hermitian symmetric $n \times n$ matrices. For a given matrix $M \in \mathcal{A}_n$ let $\lambda(M) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be its eigenvalues arranged in increasing order and let

$$\sigma_k(M) = \sigma_k(\lambda(M)) = \sum_{0<j_1<\cdots<j_k\leqslant n} \lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_k}$$

be the k-th elementary symmetric polynomial applied to the vector $\lambda(M)$. We shall simply write λ and $\sigma_k(\lambda)$ if the matrix M in question is clear from the context. Also we shall use the convention $\sigma_0(\lambda) := 1$ and $\sigma_j(\lambda) = 0$ if λ is a vector of less than j coordinates.

We denote by $\sigma_j(\lambda|\lambda_{i_1}, \ldots, \lambda_{i_r})$ the value of σ_j when the coefficients λ_{i_m} are exchanged by zero. Alternatively this is the j-th elementary symmetric polynomial on the remaining coefficients.

Denote by $S_k(\lambda) := \frac{\sigma_k(\lambda)}{\binom{n}{k}}$ the normalized Hessian operators. The normalization is chosen so that $S_k(t\mathbf{1}) = t^k$ if $\mathbf{1}$ denotes the vector with all coefficients equal to one.

Then one can define the positive cones Γ_m as follows

$$\Gamma_m = \{\lambda \in \mathbb{R}^n \mid S_1(\lambda) > 0, \ \ldots, \ S_m(\lambda) > 0\}. \tag{2.1}$$

Note that the definition of Γ_m is non linear if $m > 1$.

Below we list the properties of these cones that will be used later on.

Proposition 2.1 (Maclaurin's inequality). — *If $\lambda \in \Gamma_m$ then*

$$(S_j(\lambda))^{\frac{1}{j}} \geqslant (S_i(\lambda))^{\frac{1}{i}}$$

for $1 \leqslant j \leqslant i \leqslant m$.

Proposition 2.2 (Newton inequality). — *Let $\lambda \in \mathbb{R}^n$ be any vector. Then for any $k \in \{1, 2, \ldots, n-1\}$ one has*

$$S_{k-1}(\lambda)S_{k+1}(\lambda) \leqslant S_k^2(\lambda).$$

We emphasize that the inequality holds for any vector and not only for those belonging to cones Γ_j.

If all the σ_j's are positive it is easy to derive a slightly weaker inequality:

Proposition 2.3 (weak Newton inequality). — *If $\lambda \in \Gamma_k$ then for any $j \in \{1, 2, \ldots, k-1\}$ one has*

$$\sigma_{j-1}(\lambda)\sigma_{j+1}(\lambda) \leqslant \sigma_j^2(\lambda).$$

Proof. — Newton inequality in terms of σ_j's is simply

$$\sigma_j(\lambda)^2 \geqslant \sigma_{j-1}(\lambda)\sigma_{j+1}(\lambda)\frac{(n-j+1)(j+1)}{(n-j)j}.$$

It remains to observe that the last constant is larger than 1. □

The next proposition is a classical result in vector analysis:

Proposition 2.4. — *If the vector β belongs to Γ_k, then the sum of any $n-k+1$-coeffictients of β is non-negative. In particular any $\mathcal{C}^2$ smooth m-subharmonic function is $n-m+1$-plurisubharmonic.*

The following summation formula is easy to prove:

Proposition 2.5 (Summation formula). — *For a vector $\gamma \in \mathbb{R}^{n-1}$ let α denotes the vector in $\mathbb{R}^p$ formed by the first p-coordinates of γ, and let β denotes the vector formed by the remaining coordinates. Then*

$$\sigma_j(\gamma) = \sum_{i=0}^{j} \sigma_i(\alpha)\sigma_{j-i}(\beta).$$

We refer to [6] or [21] for further properties of these cones.

Recall that the operator $\mathcal{M}_m$ is given by

$$\mathcal{M}_m(u) := \Pi_{1\leqslant j_1<j_2<\cdots<j_m\leqslant n}(\lambda_{j_1}+\lambda_{j_2}+\cdots+\lambda_{j_m}).$$

As $\mathcal{M}_m(u)$ is defined through a symmetric polynomial of the eigenvalues of $\frac{\partial^2 u}{\partial z_j\partial \bar{z}_k}(z)$ it follows from the fundamental theorem of symmetric polynomials that it can be expressed through $\sigma_p(\lambda)=\sigma_p(\frac{\partial^2 u}{\partial z_j\partial \bar{z}_k}(z))$, $p=1,\dots,m$. One observes that $\mathcal{M}_1(u)=\sigma_n(\frac{\partial^2 u}{\partial z_j\partial \bar{z}_k}(z))$ is the complex Monge–Ampère operator, $\mathcal{M}_n(u)=\sigma_1(\frac{\partial^2 u}{\partial z_j\partial \bar{z}_k}(z))$ is simply the Laplacian. The expression of concrete n and m can be complicated. In particular for $m=2$ and $n=3$ it can be computed that $\mathcal{M}_2(u)=\sigma_1(u)\sigma_2(u)-\sigma_3(u)$.

3. A-m-subharmonicity versus B-m-subharmonicity

Recall that a (smooth) function u is m-subharmonic if $\sigma_j(\frac{\partial^2 u}{\partial z_j\partial \bar{z}_k}(z))\geqslant 0$ for every $j=1,\dots,m$ and every point in the domain of definition of u. These are called B-$(n-m+1)$-subharmonic in [18], a terminology that we shall apply in this section.

A smooth function is A-m-sh if it is m-plurisubharmonic and satisfies $\sigma_{n-m+1}(\frac{\partial^2 u}{\partial z_j\partial \bar{z}_k}(z))\geqslant 0$.

Note that $B-m-sh\subset A-m-sh$ thanks to Proposition 2.4 and if there were an equality that would mean that checking m-subharmonicity reduces to checking that the m-Hessian is positive (a thing which in potential theory is usually given a priori) and furthermore

$$i\partial\bar{\partial}u\wedge(i\partial\bar{\partial}\|z\|^2)^{n-m}\geqslant 0$$

which is a linear condition.

In [18] it was shown that if $m=2$ then for every n both notions indeed coincide. More generally they coincide for functions with at most one non-positive eigenvalue.

In this section we solve Sadullaev's problem. More precisely we prove that in a domain $\Omega\subset\mathbb{C}^n$ Blocki's notion of m-subharmonic functions agrees with the one of Abdullaev provided that $n\leqslant 7$. We also show that this fails in large dimensions.

Theorem 3.1. — *Let $u\in\mathcal{C}^2(\Omega)$, where $\Omega\subset\mathbb{C}^n$, $n\leqslant 7$. Then u is $n-k+1$-subharmonic (or B-k-subharmonic) if and only if it satisfies*

$$i\partial\bar{\partial}u\wedge(i\partial\bar{\partial}\|z\|^2)^{k-1}\geqslant 0,\quad (i\partial\bar{\partial}u)^{n-k+1}\wedge(i\partial\bar{\partial}\|z\|^2)^{k-1}\geqslant 0.$$

Proof. — If u is $(n-k+1)$-subharmonic then it satisfies Abdullaev's conditions by Proposition 2.4 (see also [18]). In order to prove the reverse implication we argue at a fixed point $z_0 \in \mathbb{C}^n$. By a complex linear change of coordinates we can assume that the complex Hessian of u is diagonal at z_0.

Observe first that the case $k = n-1$ is trivial and the case $k = 2$ was done in [18].

If all the eigenvalues are non negative they obviously form a vector in Γ_{n-k+1} and there is nothing to prove. Thus we suppose that there is a negative smallest eigenvalue (called α_0) which, afer scaling if necessary we assume to be equal to -1. Let $-1 = \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_p < 0$ denote all the negative eigenvalues. If $p = 0$ then the proof from [18] works, hence we assume $p \geqslant 1$ in what follows. Similarly let $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_{n-1-p}$ denote the nonnegative eigenvalues.

We denote by γ the vector $\gamma := (\alpha, \beta)$ with first p-coordinates equal to $\alpha_j, j = 1, \ldots, p$ and last $(n-p-1)$-coordinates equal to $\beta_j, j = 1, \ldots, n-p-1$. Similarly we define $\eta := (-1, \gamma)$.

Our goal is to prove that for $n \leqslant 7$

$$\eta \in \Gamma_{n-k+1} \quad \text{i.e.} \quad \sigma_j(\eta) \geqslant 0, \quad j = 1, \ldots, n-k-1. \tag{3.1}$$

Note that $(i\partial\bar{\partial}u)^{n-k+1} \wedge (i\partial\bar{\partial}\|z\|^2)^{k-1} \geqslant 0$ at z_0 can be rewritten in the language of eigenvalues as

$$\begin{aligned} 0 \leqslant \sigma_{n-k+1}(\eta) &= \sigma_{n-k+1}(\eta|\eta_1) + \eta_1\sigma_{n-k}(\eta|\eta_1) \\ &= \sigma_{n-k+1}(\gamma) - \sigma_{n-k}(\gamma). \end{aligned} \tag{3.2}$$

Note that for any $j = 1, \ldots, n-k-1$

$$\sigma_j(\eta) = \sigma_j(\gamma) - \sigma_{j-1}(\gamma)$$

thus it suffices to prove

$$\text{For every } j \in 1, \ldots, n-k-1 \text{ the inequality } \sigma_j(\gamma) - \sigma_{j-1}(\gamma) \geqslant 0 \text{ holds.} \tag{3.3}$$

The condition $(i\partial\bar{\partial}u) \wedge (i\partial\bar{\partial}\|z\|^2)^{k-1} \geqslant 0$ means that the sum of any k-tuple of eigenvalues is nonnegative. Note that this in particular implies that $p \leqslant k-2$.

We claim that it suffices to prove that $\sigma_j(\gamma) > 0, j = 1, \ldots, n-k$ i.e. $\gamma \in \Gamma_{n-k}$. Indeed suppose this were true.

Then from (3.2) $\sigma_{n-k+1}(\gamma) \geqslant \sigma_{n-k}(\gamma) \geqslant 0$, and from Proposition 2.3 we have

$$\sigma_{n-k}(\gamma)^2 \geqslant \sigma_{n-k+1}(\gamma)\sigma_{n-k-1}(\gamma) \geqslant \sigma_{n-k}(\gamma)\sigma_{n-k-1}(\gamma).$$

Exploiting the positivity once again we end up with

$$\sigma_{n-k}(\gamma) \geqslant \sigma_{n-k-1}(\gamma).$$

Repeating the argument we obtain $\sigma_{n-k-1}(\gamma) \geqslant \sigma_{n-k-2}(\gamma)$ and so on up until $\sigma_1(\gamma) \geqslant \sigma_0(\gamma)$.

Let us proceed with the proof of the claim.

As $i\partial\bar{\partial}u \wedge (i\partial\bar{\partial}\|z\|^2)^{k-1} \geqslant 0$ we obtain that $i\partial\bar{\partial}u \wedge (i\partial\bar{\partial}\|z\|^2)^{n-1} \geqslant 0$ i.e. $\sigma_1(\gamma) \geqslant 1$ (recall that $\eta_1 = -1$).

From Proposition 2.5 we know that

$$\sigma_j(\gamma) := \sum_{i=0}^{\min\{j,p\}} \sigma_{j-i}(\beta)\sigma_i(\alpha).$$

Recall that the case $k = n-1$ is trivial. Thus we assume from now on that $k \leqslant n-2 \leqslant 5$ and hence $p \leqslant 3$. In fact $\min\{j,p\} \leqslant 2$ (if $p \geqslant 3$ then $n = 7, k = 5$ which yields $j \leqslant 2$). Thus

$$\sigma_j(\gamma) = \sigma_j(\beta) + \sigma_1(\alpha)\sigma_{j-1}(\beta) + \sigma_2(\alpha)\sigma_{j-2}(\beta),$$

where the last term is assumed to be zero if $j = 1$ or $p = 1$. Observe that, whenever defined, this last term is always non negative hence we have the fundamental inequality

$$\sigma_j(\gamma) \geqslant \sigma_j(\beta) + \sigma_1(\alpha)\sigma_{j-1}(\beta). \tag{3.4}$$

Observe that

$$\sigma_j(\beta) = \frac{1}{j}\left(\sum_{1\leqslant l_1<\cdots<l_{j-1}\leqslant n-p-1} \beta_{l_1}\ldots\beta_{l_{j-1}} \left[\sum_{l\notin\{l_1,\ldots l_{j-1}\}} \beta_l \right] \right). \tag{3.5}$$

On the other hand for any $(k-p-1)$-tuple $1 \leqslant r_1 < \ldots r_{k-p-1} \leqslant n-p-1$ the sum $-1 + \sigma_1(\alpha) + \sum_{s=1}^{k-p-1} \beta_{r_s}$ is nonnegative by assumption. Summing over all $(k-p-1)$-tuples such that

$$\{r_1, \ldots, r_{k-p-1}\} \cap \{l_1, \ldots, l_j\} = \varnothing$$

we obtain

$$\binom{n-p-j}{k-p-1}(-1+\sigma_1(\alpha)) + \binom{n-p-j-1}{k-p-2} \sum_{l\notin\{l_1,\ldots l_{j-1}\}} \beta_l \geqslant 0.$$

Coupling this with the elementary inequlity $\sigma_1(\alpha) \geqslant -p$ (since all $\alpha_j \geqslant -1$) we obtain

$$\sum_{l\notin\{l_1,\ldots l_{j-1}\}} \beta_l \geqslant \frac{(n-p-j)(p+1)}{(k-p-1)p}(-\sigma_1(\alpha)). \tag{3.6}$$

Summing over in equation (3.5) we get

$$\sigma_j(\beta) \geqslant \sigma_{j-1}(\beta)(-\sigma_1(\alpha))\frac{(n-p-j)(p+1)}{j(k-p-1)p}$$

Thus the fundamental inequality (3.4) yields

$$\sigma_j(\gamma) \geqslant \sigma_{j-1}(\beta)(-\sigma_1(\alpha))\left[\frac{(n-p-j)(p+1)}{j(k-p-1)p} - 1\right]. \tag{3.7}$$

Hence $\sigma_j(\gamma) \geqslant 0$ provided

$$\frac{(n-p-j)(p+1)}{j(k-p-1)p} \geqslant 1.$$

The quantity on the left is clearly decreasing in j, hence it is smallest for $j = n-k$ and then reads

$$\frac{(k-p)(p+1)}{(n-k)(k-p-1)p}.$$

It is then straightforward to check that the latter quantity is indeed at least 1 for all triples $(p,k,n) \in \mathbb{N}^3$ such that $1 \leqslant p \leqslant k-2, k \leqslant n-2, n \leqslant 7$. □

The following example shows that Blocki's and Abdullaev's notions are different in large dimensions even for $k = 3$:

Example 3.2. — Consider the function

$$u(z_1,\ldots,z_{11}) = -\sum_{j=1}^{2}|z_j|^2 + 2\sum_{j=3}^{11}|z_j|^2.$$

Then $(i\partial\bar{\partial}u)\wedge(i\partial\bar{\partial}\|z\|^2)^2 \geqslant 0, (i\partial\bar{\partial}u)^9\wedge(i\partial\bar{\partial}\|z\|^2)^2 \geqslant 0$, but $(i\partial\bar{\partial}u)^8\wedge(i\partial\bar{\partial}\|z\|^2)^3 < 0$. i.e. u is m-sh in the sense of Abdullaev but not in the sense of Blocki.

Proof. — By computation

$$\sigma_9\left(\lambda\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)\right)(z) = \sigma_2(-1,-1,2,2,\ldots,2) = 2^9 - 2.9.2^8 + 36.2^7 > 0$$

but

$$\sigma_8(-1,-1,2,\ldots,2) = 9.2^8 - 2.36.2^7 + 84.2^6 = -24.2^6,$$

as claimed. □

4. Failure of the integral comparison principle

Recall that an elliptic operator $F(\frac{\partial^2 u}{\partial z_p \partial \bar{z}_q})$ is said to satisfy the integral comparison principle if for any two $\mathcal{C}^2$ admissible functions u and v defined in a domain $\Omega \subset \mathbb{C}^n$ one has

$$\int_{\{u<v\}} F\left(\frac{\partial^2 v}{\partial z_p \partial \bar{z}_q}\right) \leqslant \int_{\{u<v\}} F\left(\frac{\partial^2 v}{\partial z_p \partial \bar{z}_q}\right)$$

provided $u \geqslant v$ on $\partial\Omega$. Classical examples include the Laplacian and the complex Monge–Ampère operator restricted to the class of plurisubharmonic functions. Recall that in [5] the validity of the comparison principle has been extended to all locally bounded plurisubharmonic functions.

Such an inequality would have been very helpful in developing a version pluripotential theory associated to $\mathcal{M}_m$. On the other hand if one wants Chern–Levine–Nirenberg inequalities to hold (see [5]), which is again a basic property in pluripotential theory allowing in particular to define relative capacities, it is more natural to consider the operator $\mathcal{M}_m^\alpha$ with the exponent α chosen properly.

Unfortunately our next result shows that this is impossible (for every choice of α) unless $p = 1$ or $p = n$:

Theorem 4.1. — *Suppose that the operator $\mathcal{M}_m^\alpha(u)$ satisfies the integral comparison principle. Then $\alpha = 1$ and furthermore $m = 1$ or $m = n$.*

Before starting the proof we need a lemma that is a minor generalization of Lemma 1.2 in [8]:

Lemma 4.2. — *Consider the real valued function*

$$\rho(z) := \chi(z) + a|z_n|^2 + \sum_{j=1}^{n-1} a_j |z_j|^2,$$

where χ is any $\mathcal{C}^2$ function $a_j > 0, a_1 < a_2 < \cdots < a_{n-1} < a$ and $a_j \leqslant C$ for $j = 1, \ldots, n-1$ for some constant C. Then, assuming $a \to \infty$, $(\frac{\partial^2 \rho}{\partial z_p \partial \bar{z}_q})_{p,q=1,\ldots,n}$ has eigenvalues $\lambda_j((\frac{\partial^2 \rho}{\partial z_p \partial \bar{z}_q})_{p,q=1}^n)$, $j = 1, \ldots, n$ at a fixed point z satisfying asymptotically

$$\lambda_j\left(\left(\frac{\partial^2 \rho}{\partial z_p \partial \bar{z}_q}\right)_{p,q=1}^{n}\right) = \widetilde{\lambda}_j\left(\left(\frac{\partial^2 \rho}{\partial z_p \partial \bar{z}_q}\right)_{p,q=1}^{n-1}\right) + o(1)$$

for $1 \leqslant j \leqslant n-1$, while

$$\lambda_n = a + \chi_{n\bar{n}} + o(1).$$

All $o(1)$ terms are uniform and depend on C and the $\mathcal{C}^2$ bound on χ.

Proof. — Dividing the last row of the characteristic equation

$$\det\left(\frac{\partial^2\rho}{\partial z_p\partial\bar{z}_q}-tI_n\right)=0$$

by a_n and then passing with a_n to infinity we obtain that the all but one of the roots satisfy the characteristic equation for the matrix $(\frac{\partial^2\rho}{\partial z_p\partial\bar{z}_q})_{p,q=1}^{n-1}$ and the first part of the claim follows from the continuous dependence of eigenvalues with respect to the matrix coefficients. The equality of λ_n follows simply from taking the traces of $(\frac{\partial^2\rho}{\partial z_p\partial\bar{z}_q})_{p,q=1}^{n}$ and $(\frac{\partial^2\rho}{\partial z_p\partial\bar{z}_q})_{p,q=1}^{n-1}$. $\square$

Proof of Theorem 4.1. — We begin with the following claim providing a lower bound for α:

Claim 4.3. — *Let Ω be a bounded domain with $\mathcal{C}^2$ smooth boundary. If for any m-psh functions $u,v\in\mathcal{C}^2(\overline{\Omega})$ satisfying $u\geqslant v$ in Ω, $u=v$ on $\partial\Omega$, one has*

$$\int_\Omega\mathcal{M}_m^\alpha(u)\leqslant\int_\Omega\mathcal{M}_m^\alpha(v),$$

for some $\alpha>0$, then $\alpha\geqslant 1$.

Fix any strictly negative smooth function χ on the unit ball $B_1(0)$ which vanishes together with its gradient on $\partial B_1(0)$. As a concrete example we may take $\chi(z):=-(1-\|z\|^2)^2$. Take $u(z):=\sum_{j=1}^n a_j|z_j|^2$, $v(z):=\sum_{j=1}^n a_j|z_j|^2+\chi$, for some sufficiently large positive constants a_j, so that both u and v are m-psh. If the integral comparison principle were true then

$$\int_{B_1(0)}\mathcal{M}_m^\alpha(u)\leqslant\int_{B_1(0)}\mathcal{M}_m^\alpha(v). \tag{4.1}$$

Let now a_n to infinity, while keeping other a_j's fixed. Using lemma 4.2 in Equation (4.1), after dividing both sides by $a_n^{\binom{n-1}{m-1}\alpha}$ we end up with

$$\int_{B_1(0)}\widetilde{\mathcal{M}}_m^\alpha(u)\leqslant\int_{B_1(0)}\widetilde{\mathcal{M}}_m^\alpha(v),$$

where the $\widetilde{}$ sign denotes computation of $\mathcal{M}_m$ in the first $(n-1)$-coordinates. Letting now a_{n-1} to infinity we can repeat the process. After the $(n-m)$-th iteration it is easy to see that we end up with

$$\int_{B_1(0)}\left(\sum_{j=1}^m a_j\right)^\alpha\leqslant\int_{B_1(0)}\left(\sum_{j=1}^m a_j+\chi_{j\bar{j}}(z)\right)^\alpha. \tag{4.2}$$

Note that inequality (4.2) holds for *all* $\mathcal{C}^2$ smooth functions χ assuming that a_j's, $j=1,\dots,m$ are large enough. In particular taking the path $v_\varepsilon:=$

$\sum_{j=1}^{n} a_j|z_j|^2 + \varepsilon\chi$ for $a_1, \dots a_m$ fixed and applying the whole process we end up with

$$\int_{B_1(0)} \left(\sum_{j=1}^{m} a_j\right)^{\alpha} \leqslant \int_{B_1(0)} \left(\sum_{j=1}^{m} a_j + \varepsilon\chi_{j\bar{j}}(z)\right)^{\alpha}.$$

Expanding the right hand side in ε we obtain

$$\begin{aligned}
&\int_{B_1(0)} \left(\sum_{j=1}^{m} a_j + \varepsilon\chi_{j\bar{j}}(z)\right)^{\alpha} \\
&\quad = \int_{B_1(0)} \left(\sum_{j=1}^{m} a_j\right)^{\alpha} \\
&\qquad + \int_{B_1(0)} \alpha\left(\sum_{j=1}^{m} a_j\right)^{\alpha-1} \varepsilon\sum_{k=1}^{m} \chi_{k\bar{k}}(z) \\
&\qquad + \int_{B_1(0)} \alpha(\alpha-1)\left(\sum_{j=1}^{m} a_j\right)^{\alpha-2} \frac{\varepsilon^2}{2}\left(\sum_{k=1}^{m} \chi_{k\bar{k}}(z)\right)^{2} + o(\varepsilon^2) \\
&\quad = I + II + III + IV.
\end{aligned}$$

The first term clearly matches the left hand side in (4.2). The second term is zero as it can be seen from integration by parts (we use the vanishing of the gradient of χ at the boundary). But the third term is strictly negative if $\alpha < 1$ and it dominates the fourth one for small ε. Hence we must have $\alpha \geqslant 1$, which yields the claim.

The next claim in turn provides an upper bound for α:

Claim 4.4. — *Let $B_1(0)$ be the unit ball in $\mathbb{C}^n$. If for any rotationally invariant m-psh functions $u, v \in \mathcal{C}^2(\overline{B_1(0)})$ satisfying $u \geqslant v$ in Ω, $u = v$ on $\partial\Omega$, one has*

$$\int_{B_1(0)} \mathcal{M}_m^{\alpha}(u) \leqslant \int_{B_1(0)} \mathcal{M}_m^{\alpha}(v),$$

for some $\alpha > 0$, then $\alpha \leqslant \frac{1}{\binom{n-1}{m-1}}$.

It is straightforward to compute that if $u(z) := \chi(\|z\|^2)$ for some $\mathcal{C}^2$ smooth function χ, then the eigenvalues of the complex Hessian of u satisfy

$$\lambda_1(z) = \dots = \lambda_{n-1}(z) = \chi'(\|z\|^2),\ \lambda_n(z) = \chi'(\|z\|^2) + \|z\|^2\chi''(\|z\|^2).$$

Thus

$$\mathcal{M}_m(u)(z) = \left[(m\chi'(\|z\|^2))^{\binom{n-1}{m}}(m\chi'(\|z\|^2)) + \|z\|^2\chi''(\|z\|^2))^{\binom{n-1}{m-1}}\right]. \quad (4.3)$$

We apply this for the family of $\chi_A(t) := \frac{1}{2}(\frac{(t+A)^2}{1+A} - 1 + A)$, $A \geqslant 0$. It is easy to see that the corresponding functions $u_A(z) := \chi_A(\|z\|^2)$ are plurisubharmonic, hence m-psh, and they all vanish on the unit sphere. Also u_A is a decreasing sequence as A increases.

Supposing that the comparison principle holds for some α we obtain then

$$\int_{B_1(0)} \mathcal{M}_m^\alpha(u_0) \leqslant \int_{B_1(0)} \mathcal{M}_m^\alpha(u_A)$$

for any $A > 0$. But then, denoting by c_{2n-1} the area of the unit sphere, the left hand side is simply

$$\begin{aligned}
\int_{B_1(0)} \mathcal{M}_m(u_0)^\alpha &= \int_{B_1(0)} \left[(m\chi')^{\binom{n-1}{m}}(m\chi' + \|z\|^2\chi'')^{\binom{n-1}{m-1}}\right]^\alpha \\
&= \int_{B_1(0)} \left[(m\|z\|^2)^{\binom{n-1}{m}}((m+1)\|z\|^2)^{\binom{n-1}{m-1}}\right]^\alpha \\
&= c_{2n-1} m^{\binom{n-1}{m}\alpha}(m+1)^{\binom{n-1}{m-1}\alpha} \int_0^1 r^{2n-1+2\binom{n-1}{m-1}\alpha+2\binom{n-1}{m}\alpha} \mathrm{d}r \\
&= c_{2n-1} \frac{m^{\binom{n-1}{m}\alpha}(m+1)^{\binom{n-1}{m-1}\alpha}}{2n + 2\binom{n}{m}\alpha}.
\end{aligned}$$

On the other hand after taking the limit as $A \to \infty$ the right hand side becomes

$$\begin{aligned}
\lim_{A\to\infty} \int_{B_1(0)} \mathcal{M}_m(u_A)^\alpha &= \int_{B_1(0)} m^{\binom{n}{m}\alpha} \\
&= c_{2n-1} m^{\binom{n}{m}\alpha} \int_0^1 r^{2n-1} \mathrm{d}r \\
&= c_{2n-1} \frac{m^{\binom{n}{m}\alpha}}{2n}.
\end{aligned}$$

Comparing both sides we obtain the numerical inequality

$$c_{2n-1} \frac{m^{\binom{n-1}{m}\alpha}(m+1)^{\binom{n-1}{m-1}\alpha}}{2n + 2\binom{n}{m}\alpha} \leqslant c_{2n-1} \frac{m^{\binom{n}{m}\alpha}}{2n}, \tag{4.4}$$

which reduces to

$$\left(1 + \frac{1}{m}\right)^{\binom{n-1}{m-1}\alpha} \leqslant 1 + \frac{1}{m}\binom{n-1}{m-1}\alpha.$$

If now $\alpha > \frac{1}{\binom{n-1}{m-1}}$ we get the contradiction with the elementary inequality $(1+x)^\beta > 1 + \beta x$, valid for all $x > 0$ and $\beta > 1$.

Finally coupling Claim 4.3 with Claim 4.4 it is obvious that the comparison principle can hold iff $\alpha = 1$ and $\binom{n-1}{m-1} = 1$ i.e. if $m = 1$ (the case of the complex Monge–Ampère operator) or $m = n$ (the Laplacian). □

Remark 4.5. — It is interesting to note that for *radial* m-psh functions the comparison principle does hold true for the operator $\mathcal{M}_m$ raised to power $\frac{1}{\binom{n-1}{m-1}}$. We leave the elementary proof to the Reader.

Bibliography

[1] B. Abdullaev, "Subharmonic functions on complex hyperplanes of $\mathbb{C}^n$", *J. Sib. Fed. Univ., Math. Phys.* **6** (2013), no. 4, p. 409-416.

[2] ———, "$\mathcal{P}$-measure in the class of m-wsh functions", *J. Sib. Fed. Univ., Math. Phys.* **7** (2014), no. 1, p. 3-9.

[3] B. Abdullaev & A. Sadullaev, "Potential theory in the class of m-sh functions", *Proc. Steklov Inst. Math.* **279** (2012), p. 155-180.

[4] A. Andreotti & H. Grauert, "Théorèmes de finitude pour la cohomologie des espaces complexes", *Bull. Soc. Math. Fr.* **90** (1962), p. 193-259.

[5] E. Bedford & B. A. Taylor, "A New Capacity for Plurisubharmonic Functions", *Acta Math.* **149** (1982), p. 1-40.

[6] Z. Blocki, "Weak solutions to the complex Hessian equation", *Ann. Inst. Fourier* **55** (2005), no. 5, p. 1735-1756.

[7] ———, "Defining nonlinear elliptic operators for non-smooth functions", in *Complex analysis and digital geometry*, Univ. Uppsala, 2009, p. 111-124.

[8] L. Caffarelli, L. Nirenberg & J. Spruck, "The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian", *Acta Math.* **155** (1985), no. 3-4, p. 261-301.

[9] J.-P. Demailly, "Complex Analytic and Differential Geometry", 1997, `http://www-fourier.ujf-grenoble.fr/~demailly/documents.html`.

[10] N. Q. Dieu, "q-plurisubharmonicity and q-pseudoconvexity in $\mathbb{C}^n$", *Publ. Mat., Barc.* **50** (2006), no. 2, p. 349-369.

[11] S. Dinew, H.-S. Do & T. D. Tô, "A viscosity approach to the Dirichlet problem for degenerate complex Hessian type equations", *Anal. PDE* **12** (2019), no. 2, p. 505-535.

[12] S. Dinew & S. Kołodziej, "A priori estimates for complex Hessian equations", *Anal. PDE* **7** (2014), no. 1, p. 227-244.

[13] ———, "Liouville and Calabi-Yau type theorems for complex Hessian equations", *Am. J. Math.* **139** (2017), no. 2, p. 403-415.

[14] ———, "Non standard properties of m-subharmonic functions", *Dolomites Res. Notes Approx.* **11** (2018), no. 4, p. 35-50.

[15] F. R. Harvey & H. B. j. Lawson, "p-convexity, p-plurisubharmonicity and the Levi problem", *Indiana Univ. Math. J.* **62** (2013), no. 1, p. 149-169.

[16] A.-K. Herbig & J. D. McNeal, "Regularity of the Bergman projection on forms and plurisubharmonicity conditions", *Math. Ann.* **336** (2006), no. 2, p. 335-359.

[17] L.-H. Ho, "$\bar{\partial}$-problem on weakly q-convex domains", *Math. Ann.* **290** (1991), no. 1, p. 3-18.

[18] A. Sadullaev, "Further developments of the pluripotential theory (survey)", in *Algebra, complex analysis, and pluripotential theory*, Springer Proceedings in Mathematics & Statistics, vol. 264, Springer, 2018, p. 167-182.

[19] V. Tosatti & B. Weinkove, "The Monge–Ampère equation for $(n-1)$-plurisubharmonic functions on a compact Kähler manifold", *J. Am. Math. Soc.* **30** (2017), no. 2, p. 311-346.

[20] M. Verbitsky, "Plurisubharmonic functions in calibrated geometry and q-convexity", *Math. Z.* **264** (2010), no. 4, p. 939-957.

[21] X.-J. Wang, *The k-Hessian equation*, Lecture Notes in Mathematics, vol. 1977, Springer, 2009, 177-252 pages.