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A right inverse of Cauchy–Riemann operator $\bar{\partial}^k + a$ in the weighted Hilbert space $L^2(\mathbb{C}, e^{-|z|^2})$ (*)

Shaoyu Dai (1) and Yifei Pan (2)

Abstract. — Using Hörmander L^2 method for Cauchy–Riemann equations from complex analysis, we study a simple differential operator $\bar{\partial}^k + a$ of any order (densely defined and closed) in the weighted Hilbert space $L^2(\mathbb{C}, e^{-|z|^2})$ and prove the existence of a right inverse that is bounded.

Résumé. — Nous utilisons la méthode des estimées L^2 de Hörmander pour les équations de Cauchy–Riemann pour étudier un opérateur différentiel simple $\bar{\partial}^k + a$ de tout ordre (fermé et densément défini) dans l'espace de Hilbert à poids $L^2(\mathbb{C}, e^{-|z|^2})$. Nous montrons l'existence d'un inverse à droite qui est borné.

1. Introduction

In this paper, using Hörmander L^2 method [2] for Cauchy–Riemann equations from complex analysis, we study the right inverse of the differential operator $\bar{\partial}^k + a$, which is densely defined and closed, in a Hilbert space by proving the following result on the existence of (entire) weak solutions of the equation $\bar{\partial}^k u + au = f$ in the weighted Hilbert space $L^2(\mathbb{C}, e^{-|z|^2})$. Here and throughout, a is a complex constant, k a positive integer, $\bar{\partial}^k := \frac{\partial^k}{\partial \bar{z}^k}$, k^{th}-order Cauchy–Riemann operator, where $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, and $\mathrm{d}\sigma := \frac{1}{2i}\,\mathrm{d}\bar{z} \wedge \mathrm{d}z$, the volume form.

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THEOREM 1.1. — *For each $f \in L^2(\mathbb{C}, e^{-|z|^2})$, there exists a weak solution $u \in L^2(\mathbb{C}, e^{-|z|^2})$ solving the equation*

$$\bar{\partial}^k u + au = f$$

in $\mathbb{C}$ with the norm estimate

$$\int_{\mathbb{C}} |u|^2 e^{-|z|^2}\, \mathrm{d}\sigma \leqslant \frac{1}{k!} \int_{\mathbb{C}} |f|^2 e^{-|z|^2}\, \mathrm{d}\sigma.$$

The novelty of Theorem 1.1 is that the differential operator $\bar{\partial}^k + a$ has a bounded (linear) right inverse

$$\begin{aligned} T_k : L^2(\mathbb{C}, e^{-|z|^2}) &\longrightarrow L^2(\mathbb{C}, e^{-|z|^2}), \\ (\bar{\partial}^k + a)T_k &= I \end{aligned}$$

with the norm estimate $\|T_k\| \leqslant \frac{1}{\sqrt{k!}}$. In particular, the differential operator $\bar{\partial}^k$ has a bounded right inverse $T : L^2(\mathbb{C}, e^{-|z|^2}) \longrightarrow L^2(\mathbb{C}, e^{-|z|^2})$, which, to the best of our knowledge, appears to be new. We also note the fact that the constant a dose not appear in the norm estimate, and it is this fact that we shall use later.

For the first order $\bar{\partial} := \bar{\partial}^1$, the Cauchy–Riemann operator, we have the following slight extension of the simplest case of Hörmander's theorem in the complex plane ([3] and [4]) ($a = 0$; see [1] for a related result). Note that $\Delta = 4\partial\bar{\partial}$.

THEOREM 1.2. — *Let φ be a smooth and nonnegative function on $\mathbb{C}$ with $\Delta\varphi > 0$. For each $f \in L^2(\mathbb{C}, e^{-\varphi})$ such that $\frac{f}{\sqrt{\Delta\varphi}} \in L^2(\mathbb{C}, e^{-\varphi})$, there exists a weak solution $u \in L^2(\mathbb{C}, e^{-\varphi})$ solving the equation*

$$\bar{\partial} u + au = f$$

with the norm estimate

$$\int_{\mathbb{C}} |u|^2 e^{-\varphi}\, \mathrm{d}\sigma \leqslant 4 \int_{\mathbb{C}} \frac{|f|^2}{\Delta\varphi} e^{-\varphi}\, \mathrm{d}\sigma.$$

The organization of the paper is as follows. In Section 2, we will prove several key lemmas based on functional analysis in terms of Hörmander L^2 method, while the proof of Theorem 1.1 and 1.2 will be given in Section 3. In Section 4, we will give some further remarks.

2. Several lemmas

Here, we consider the weighted Hilbert space

$$L^2(\mathbb{C}, e^{-\varphi}) = \left\{ f \,\middle|\, f \in L^2_{loc}(\mathbb{C}); \int_{\mathbb{C}} |f|^2 e^{-\varphi}\, \mathrm{d}\sigma < +\infty \right\},$$

where φ is a nonnegative function on $\mathbb{C}$. We denote the weighted inner product for $f, g \in L^2(\mathbb{C}, e^{-\varphi})$ by $\langle f, g\rangle_\varphi = \int_{\mathbb{C}} \bar{f} g e^{-\varphi}\, \mathrm{d}\sigma$, and the weighted norm of $f \in L^2(\mathbb{C}, e^{-\varphi})$ by $\|f\|_\varphi = \sqrt{\langle f, f\rangle_\varphi}$. Let $C_0^\infty(\mathbb{C})$ denote the set of all smooth functions $\phi : \mathbb{C} \to \mathbb{C}$ with compact support. For $u, f \in L^2_{loc}(\mathbb{C})$, we say that f is the k^{th} weak $\bar{\partial}$ partial derivative of u, written $\bar{\partial}^k u = f$, provided $\int_{\mathbb{C}} u \bar{\partial}^k \phi\, \mathrm{d}\sigma = (-1)^k \int_{\mathbb{C}} f\phi\, \mathrm{d}\sigma$ for all test functions $\phi \in C_0^\infty(\mathbb{C})$; we say that f is the k^{th} weak ∂ partial derivative of u, written $\partial^k u = f$, provided $\int_{\mathbb{C}} u \partial^k \phi\, \mathrm{d}\sigma = (-1)^k \int_{\mathbb{C}} f\phi\, \mathrm{d}\sigma$ for all test functions $\phi \in C_0^\infty(\mathbb{C})$.

In the following, let φ be a smooth and nonnegative function on $\mathbb{C}$. For $\forall\, \phi \in C_0^\infty(\mathbb{C})$, we first define the following formal adjoint of $\bar{\partial}^k$ with respect to the weighted inner product in $L^2(\mathbb{C}, e^{-\varphi})$. Let $u \in L^2_{loc}(\mathbb{C})$. We integrate as follows by the definition of the weak partial derivative.

$$\begin{aligned}
\left\langle \phi, \bar{\partial}^k u \right\rangle_\varphi &= \int_{\mathbb{C}} \bar{\phi}\, \left(\bar{\partial}^k u\right) e^{-\varphi}\, \mathrm{d}\sigma \\
&= (-1)^k \int_{\mathbb{C}} \left(\bar{\partial}^k \left(\bar{\phi} e^{-\varphi}\right)\right) u\, \mathrm{d}\sigma \\
&= (-1)^k \int_{\mathbb{C}} e^{\varphi} \left(\bar{\partial}^k \left(\bar{\phi} e^{-\varphi}\right)\right) u e^{-\varphi}\, \mathrm{d}\sigma \\
&= (-1)^k \int_{\mathbb{C}} \overline{e^{\varphi} \partial^k \left(\phi e^{-\varphi}\right)} u e^{-\varphi}\, \mathrm{d}\sigma \\
&= \left\langle (-1)^k e^{\varphi} \partial^k \left(\phi e^{-\varphi}\right), u \right\rangle_\varphi \\
&=: \left\langle \bar{\partial}^{k*}_\varphi \phi, u \right\rangle_\varphi,
\end{aligned}$$

where $\bar{\partial}^{k*}_\varphi \phi = (-1)^k e^{\varphi} \partial^k \left(\phi e^{-\varphi}\right)$ is so called the formal adjoint of $\bar{\partial}^k$ with domain in $C_0^\infty(\mathbb{C})$. Let $\left(\bar{\partial}^k + a\right)^*_\varphi$ be the formal adjoint of $\bar{\partial}^k + a$ with domain in $C_0^\infty(\mathbb{C})$. Note that $I^*_\varphi = I$, where I is the identity operator. Then $\left(\bar{\partial}^k + a\right)^*_\varphi = \bar{\partial}^{k*}_\varphi + \bar{a}$.

Now we give several lemmas for a general weight based on functional analysis, which are the core elements of Hörmander L^2 method.

Lemma 2.1. — *For each $f \in L^2(\mathbb{C}, e^{-\varphi})$, there exists an entire weak solution $u \in L^2(\mathbb{C}, e^{-\varphi})$ solving the equation*

$$\bar{\partial}^k u + au = f$$

in $\mathbb{C}$ with the norm estimate

$$\|u\|_{\varphi}^{2} \leqslant c$$

if and only if

$$|\langle f, \phi\rangle_{\varphi}|^{2} \leqslant c\left\|\left(\bar{\partial}^{k}+a\right)_{\varphi}^{*} \phi\right\|_{\varphi}^{2}, \quad \forall\, \phi \in C_{0}^{\infty}(\mathbb{C}),$$

where c is a constant.

Proof. — Let $\bar{\partial}^{k}+a=H$. Then $\left(\bar{\partial}^{k}+a\right)_{\varphi}^{*}=H_{\varphi}^{*}$.

Necessity. — For $\forall\, \phi \in C_{0}^{\infty}(\mathbb{C})$, from the definition of H_{φ}^{*} and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|\langle f, \phi\rangle_{\varphi}|^{2}=|\langle H u, \phi\rangle_{\varphi}|^{2}=\left|\langle u, H_{\varphi}^{*} \phi\rangle_{\varphi}\right|^{2} \leqslant \|u\|_{\varphi}^{2}\left\|H_{\varphi}^{*} \phi\right\|_{\varphi}^{2} \\
\leqslant c\left\|H_{\varphi}^{*} \phi\right\|_{\varphi}^{2}=c\left\|(\bar{\partial}^{k}+a)_{\varphi}^{*} \phi\right\|_{\varphi}^{2}.
\end{aligned}$$

Sufficiency. — Consider the subspace

$$E=\left\{H_{\varphi}^{*} \phi \,\middle|\, \phi \in C_{0}^{\infty}(\mathbb{C})\right\} \subset L^{2}(\mathbb{C}, e^{-\varphi}).$$

Define a linear functional $L_{f}: E \rightarrow \mathbb{C}$ by

$$L_{f}\left(H_{\varphi}^{*} \phi\right)=\langle f, \phi\rangle_{\varphi}=\int_{\mathbb{C}} \bar{f} \phi e^{-\varphi}\, \mathrm{d} \sigma.$$

Since

$$\left|L_{f}\left(H_{\varphi}^{*} \phi\right)\right|=|\langle f, \phi\rangle_{\varphi}| \leqslant \sqrt{c}\left\|H_{\varphi}^{*} \phi\right\|_{\varphi},$$

then L_{f} is a bounded functional on E. Let $\overline{E}$ be the closure of E with respect to the norm $\|\cdot\|_{\varphi}$ of $L^{2}(\mathbb{C}, e^{-\varphi})$. Note that $\overline{E}$ is a Hilbert subspace of $L^{2}(\mathbb{C}, e^{-\varphi})$. So by Hahn–Banach's extension theorem, L_{f} can be extended to a linear functional $\widetilde{L}_{f}$ on $\overline{E}$ such that

$$\left|\widetilde{L}_{f}(g)\right| \leqslant \sqrt{c}\,\|g\|_{\varphi}, \quad \forall\, g \in \overline{E}. \tag{2.1}$$

Using the Riesz representation theorem for $\widetilde{L}_{f}$, there exists a unique $u_{0} \in \overline{E}$ such that

$$\widetilde{L}_{f}(g)=\langle u_{0}, g\rangle_{\varphi}, \quad \forall\, g \in \overline{E}. \tag{2.2}$$

Now we prove $\bar{\partial}^{k} u_{0}+a u_{0}=f$. For $\forall\, \phi \in C_{0}^{\infty}(\mathbb{C})$, apply $g=H_{\varphi}^{*} \phi$ in (2.2). Then

$$\widetilde{L}_{f}\left(H_{\varphi}^{*} \phi\right)=\left\langle u_{0}, H_{\varphi}^{*} \phi\right\rangle_{\varphi}=\langle H u_{0}, \phi\rangle_{\varphi}.$$

Note that

$$\widetilde{L}_{f}\left(H_{\varphi}^{*} \phi\right)=L_{f}\left(H_{\varphi}^{*} \phi\right)=\langle f, \phi\rangle_{\varphi}.$$

Therefore,

$$\langle H u_{0}, \phi\rangle_{\varphi}=\langle f, \phi\rangle_{\varphi}, \quad \forall\, \phi \in C_{0}^{\infty}(\mathbb{C}),$$

i.e.,

$$\int_{\mathbb{C}} \overline{Hu_0} \phi e^{-\varphi} \,\mathrm{d}\sigma = \int_{\mathbb{C}} \bar{f} \phi e^{-\varphi} \,\mathrm{d}\sigma, \quad \forall\, \phi \in C_0^\infty(\mathbb{C}).$$

Thus, $Hu_0 = f$, i.e., $\bar{\partial}^k u_0 + a u_0 = f$.

Next we give a bound for the norm of u_0. Let $g = u_0$ in (2.1) and (2.2). Then we have

$$\|u_0\|_\varphi^2 = |\langle u_0, u_0 \rangle_\varphi| = \left|\widetilde{L}_f(u_0)\right| \leqslant \sqrt{c}\, \|u_0\|_\varphi .$$

Therefore, $\|u_0\|_\varphi^2 \leqslant c$.

Note that $u_0 \in \overline{E}$ and $\overline{E} \subset L^2(\mathbb{C}, e^{-\varphi})$. Then $u_0 \in L^2(\mathbb{C}, e^{-\varphi})$. Let $u = u_0$. So there exists $u \in L^2(\mathbb{C}, e^{-\varphi})$ such that $\bar{\partial}^k u + au = f$ with $\|u\|_\varphi^2 \leqslant c$. The proof is complete. □

Lemma 2.2.

$$\left\|(\bar{\partial}^k + a)_\varphi^* \phi\right\|_\varphi^2 = \left\|(\bar{\partial}^k + a)\, \phi\right\|_\varphi^2 + \left\langle \phi, \bar{\partial}^k(\bar{\partial}_\varphi^{k*}\phi) - \bar{\partial}_\varphi^{k*}(\bar{\partial}^k \phi)\right\rangle_\varphi , \quad \forall\, \phi \in C_0^\infty(\mathbb{C}).$$

Proof. — Let $\bar{\partial}^k + a = H$. Then $\left(\bar{\partial}^k + a\right)_\varphi^* = H_\varphi^*$. For $\forall\, \phi \in C_0^\infty(\mathbb{C})$,

$$\begin{aligned}
\left\|H_\varphi^* \phi\right\|_\varphi^2 &= \left\langle H_\varphi^* \phi, H_\varphi^* \phi \right\rangle_\varphi \\
&= \left\langle \phi, H H_\varphi^* \phi \right\rangle_\varphi \\
&= \left\langle \phi, H_\varphi^* H \phi \right\rangle_\varphi + \left\langle \phi, H H_\varphi^* \phi - H_\varphi^* H \phi \right\rangle_\varphi \\
&= \left\langle H\phi, H\phi \right\rangle_\varphi + \left\langle \phi, H H_\varphi^* \phi - H_\varphi^* H \phi \right\rangle_\varphi \\
&= \|H\phi\|_\varphi^2 + \left\langle \phi, H H_\varphi^* \phi - H_\varphi^* H \phi \right\rangle_\varphi .
\end{aligned} \tag{2.3}$$

Note that

$$H H_\varphi^* \phi = \left(\bar{\partial}^k + a\right)\left(\bar{\partial}^k + a\right)_\varphi^* \phi = \bar{\partial}^k \left(\bar{\partial}_\varphi^{k*} \phi\right) + \bar{a} \bar{\partial}^k \phi + a \bar{\partial}_\varphi^{k*} \phi + |a|^2 \phi$$

and

$$H_\varphi^* H \phi = \left(\bar{\partial}^k + a\right)_\varphi^* \left(\bar{\partial}^k + a\right) \phi = \bar{\partial}_\varphi^{k*} \left(\bar{\partial}^k \phi\right) + a \bar{\partial}_\varphi^{k*} \phi + \bar{a} \bar{\partial}^k \phi + |a|^2 \phi.$$

Then

$$H H_\varphi^* \phi - H_\varphi^* H \phi = \bar{\partial}^k \left(\bar{\partial}_\varphi^{k*} \phi\right) - \bar{\partial}_\varphi^{k*} \left(\bar{\partial}^k \phi\right). \tag{2.4}$$

So by (2.3) and (2.4), we have

$$\left\|H_\varphi^* \phi\right\|_\varphi^2 = \|H\phi\|_\varphi^2 + \left\langle \phi, \bar{\partial}^k \left(\bar{\partial}_\varphi^{k*} \phi\right) - \bar{\partial}_\varphi^{k*} \left(\bar{\partial}^k \phi\right) \right\rangle_\varphi .$$

This lemma is proved. □

Lemma 2.3. — *For* $\forall\, \phi \in C_0^\infty(\mathbb{C})$,

$$\bar{\partial}^k\left(\bar{\partial}_\varphi^{k*}\phi\right) - \bar{\partial}_\varphi^{k*}\left(\bar{\partial}^k\phi\right) = (-1)^k \sum_{i=1}^k \sum_{j=1}^k \binom{k}{i}\binom{k}{j} \partial^{k-i}\bar{\partial}^{k-j}\phi\bar{\partial}^j P_i, \qquad (2.5)$$

where

$$P_i = \sum \frac{i!}{m_1!m_2!\cdots m_i!} \prod_{\gamma=1}^{i}\left(\frac{-\partial^\gamma\varphi}{\gamma!}\right)^{m_\gamma}, \qquad (2.6)$$

and the sum is over all i*-tuples of nonnegative integers* $(m_1, m_2, \cdots m_i)$ *satisfying the constraint* $1m_1 + 2m_2 + \cdots + im_i = i$.

Proof.

$$\begin{aligned}
\bar{\partial}_\varphi^{k*}\phi &= (-1)^k e^\varphi \partial^k\left(\phi e^{-\varphi}\right) \\
&= (-1)^k e^\varphi \sum_{i=0}^k \binom{k}{i} \partial^{k-i}\phi\partial^i e^{-\varphi} \\
&= (-1)^k \sum_{i=0}^k \binom{k}{i} \left(\partial^{k-i}\phi\right)\left(e^\varphi\partial^i e^{-\varphi}\right). \qquad (2.7)
\end{aligned}$$

Then from (2.7), we have

$$\begin{aligned}
\bar{\partial}^k\left(\bar{\partial}_\varphi^{k*}\phi\right) &= (-1)^k \sum_{i=0}^k \binom{k}{i} \bar{\partial}^k\left(\left(\partial^{k-i}\phi\right)\left(e^\varphi\partial^i e^{-\varphi}\right)\right) \\
&= (-1)^k \sum_{i=0}^k \binom{k}{i}\left(\sum_{j=0}^k \binom{k}{j} \bar{\partial}^{k-j}\partial^{k-i}\phi\bar{\partial}^j\left(e^\varphi\partial^i e^{-\varphi}\right)\right) \\
&= (-1)^k \sum_{i=0}^k \sum_{j=0}^k \binom{k}{i}\binom{k}{j} \partial^{k-i}\bar{\partial}^{k-j}\phi\bar{\partial}^j\left(e^\varphi\partial^i e^{-\varphi}\right)
\end{aligned}$$

and

$$\bar{\partial}_\varphi^{k*}\left(\bar{\partial}^k\phi\right) = (-1)^k \sum_{i=0}^k \binom{k}{i}\left(\partial^{k-i}\bar{\partial}^k\phi\right)\left(e^\varphi\partial^i e^{-\varphi}\right).$$

Therefore,

$$\begin{aligned}
&\bar{\partial}^k\left(\bar{\partial}_\varphi^{k*}\phi\right) - \bar{\partial}_\varphi^{k*}\left(\bar{\partial}^k\phi\right) \\
&\qquad = (-1)^k \sum_{i=1}^k \sum_{j=1}^k \binom{k}{i}\binom{k}{j} \partial^{k-i}\bar{\partial}^{k-j}\phi\bar{\partial}^j\left(e^\varphi\partial^i e^{-\varphi}\right). \qquad (2.8)
\end{aligned}$$

Let $h(g) = e^g, g = -\varphi$. By Faà di Bruno's formula [5],

$$\begin{aligned}
\partial^i e^{-\varphi} &= \partial^i (h(g)) \\
&= \sum \frac{i!}{m_1! m_2! \cdots m_i!} h^{(m_1+\cdots+m_i)}(g) \prod_{\gamma=1}^{i} \left(\frac{\partial^\gamma g}{\gamma!} \right)^{m_\gamma} \\
&= \left(\sum \frac{i!}{m_1! m_2! \cdots m_i!} \prod_{\gamma=1}^{i} \left(\frac{-\partial^\gamma \varphi}{\gamma!} \right)^{m_\gamma} \right) e^{-\varphi} \\
&:= P_i e^{-\varphi}, \qquad (2.9)
\end{aligned}$$

where the sum is over all i-tuples of nonnegative integers $(m_1, m_2, \cdots m_i)$ satisfying the constraint $1m_1 + 2m_2 + \cdots + im_i = i$. So (2.5) is proved by (2.8) and (2.9). □

However, unlike in the previous lemmas, here we have to be confined with a special weight $\varphi = |z|^2$ in order to deal with the high order differential operator and we note that the space $L^2(\mathbb{C}, e^{-|z|^2})$ is a well-known space, sometimes, called Fock space.

Lemma 2.4. — *Let $\varphi = |z|^2$. Then*

$$\left\langle \phi, \bar{\partial}^k \left(\bar{\partial}_\varphi^{k*} \phi \right) - \bar{\partial}_\varphi^{k*} \left(\bar{\partial}^k \phi \right) \right\rangle_\varphi = \sum_{j=0}^{k-1} \frac{(k!)^2}{(j!)^2 (k-j)!} \left\| \bar{\partial}^j \phi \right\|_\varphi^2, \quad \forall\, \phi \in C_0^\infty(\mathbb{C}). \qquad (2.10)$$

Proof. — By $\varphi = |z|^2$, we have

$$\partial^\gamma \varphi = \begin{cases} \bar{z}, & \gamma = 1, \\ 0, & \gamma \geqslant 2. \end{cases}$$

Then from (2.6),

$$P_i = (-\partial \varphi)^i = (-\bar{z})^i = (-1)^i (\bar{z})^i . \qquad (2.11)$$

Note that

$$\bar{\partial}^j \bar{z}^i = \begin{cases} 0, & i < j, \\ \frac{i!}{(i-j)!} \bar{z}^{i-j}, & i \geqslant j. \end{cases}$$

Let $s = i - j$. So for $\forall\ \phi \in C_0^\infty(\mathbb{C})$, by (2.5) and (2.11) we have

$$\begin{aligned}
&\bar\partial^k\left(\bar\partial_\varphi^{k*}\phi\right) - \bar\partial_\varphi^{k*}\left(\bar\partial^k\phi\right) \\
&\quad = (-1)^k \sum_{j=1}^{k}\sum_{i=j}^{k}\binom{k}{i}\binom{k}{j}\left(\partial^{k-i}\bar\partial^{k-j}\phi\right)\frac{(-1)^i i!}{(i-j)!}\bar z^{i-j} \\
&\quad = (-1)^k \sum_{j=1}^{k}\sum_{s=0}^{k-j}\binom{k}{j+s}\binom{k}{j}\left(\partial^{k-j-s}\bar\partial^{k-j}\phi\right)\frac{(-1)^{j+s}(j+s)!}{s!}\bar z^{s} \\
&\quad = (-1)^k \sum_{j=1}^{k} A_{k-j}\sum_{s=0}^{k-j}\binom{k-j}{s}\left(\partial^{k-j-s}\bar\partial^{k-j}\phi\right)(-1)^s\bar z^{s} \\
&\quad = (-1)^k \sum_{j=1}^{k} A_{k-j}\sum_{s=0}^{k-j}\binom{k-j}{s}\left(\partial^{k-j-s}\bar\partial^{k-j}\phi\right)P_s, \qquad (2.12)
\end{aligned}$$

where

$$A_{k-j} = \frac{(-1)^j (k!)^2}{((k-j)!)^2 j!}.$$

Then by (2.12) and (2.9), we have

$$\begin{aligned}
&\left(\bar\partial^k\left(\bar\partial_\varphi^{k*}\phi\right) - \bar\partial_\varphi^{k*}\left(\bar\partial^k\phi\right)\right)e^{-\varphi} \\
&\quad = (-1)^k \sum_{j=1}^{k} A_{k-j}\sum_{s=0}^{k-j}\binom{k-j}{s}\left(\partial^{k-j-s}\bar\partial^{k-j}\phi\right)P_s e^{-\varphi} \\
&\quad = (-1)^k \sum_{j=1}^{k} A_{k-j}\sum_{s=0}^{k-j}\binom{k-j}{s}\left(\partial^{k-j-s}\bar\partial^{k-j}\phi\right)\partial^s e^{-\varphi} \\
&\quad = (-1)^k \sum_{j=1}^{k} A_{k-j}\partial^{k-j}\left(\left(\bar\partial^{k-j}\phi\right)e^{-\varphi}\right)
\end{aligned}$$

Therefore, as the key step of the proof, we have

$$\begin{aligned}
&\left\langle \phi, \bar\partial^k\left(\bar\partial_\varphi^{k*}\phi\right) - \bar\partial_\varphi^{k*}\left(\bar\partial^k\phi\right)\right\rangle_\varphi \\
&\quad = \int_{\mathbb{C}} \bar\phi\left(\bar\partial^k\left(\bar\partial_\varphi^{k*}\phi\right) - \bar\partial_\varphi^{k*}\left(\bar\partial^k\phi\right)\right)e^{-\varphi}\,\mathrm{d}\sigma \\
&\quad = (-1)^k \sum_{j=1}^{k} A_{k-j}\int_{\mathbb{C}} \bar\phi\partial^{k-j}\left(\left(\bar\partial^{k-j}\phi\right)e^{-\varphi}\right)\mathrm{d}\sigma \\
&\quad = (-1)^k \sum_{j=1}^{k} A_{k-j}(-1)^{k-j}\int_{\mathbb{C}} \left(\partial^{k-j}\bar\phi\right)\left(\left(\bar\partial^{k-j}\phi\right)e^{-\varphi}\right)\mathrm{d}\sigma
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^k \sum_{j=1}^{k} A_{k-j} (-1)^{k-j} \int_{\mathbb{C}} \overline{\left(\bar{\partial}^{k-j}\phi\right)} \left(\bar{\partial}^{k-j}\phi\right) e^{-\varphi}\, \mathrm{d}\sigma \\
&= (-1)^k \sum_{j=1}^{k} A_{k-j} (-1)^{k-j} \left\langle \bar{\partial}^{k-j}\phi, \bar{\partial}^{k-j}\phi \right\rangle_{\varphi} \\
&= (-1)^k \sum_{j=1}^{k} A_{k-j} (-1)^{k-j} \left\| \bar{\partial}^{k-j}\phi \right\|_{\varphi}^2 \\
&= \sum_{j=1}^{k} \frac{(k!)^2}{((k-j)!)^2 j!} \left\| \bar{\partial}^{k-j}\phi \right\|_{\varphi}^2 \\
&= \sum_{j=0}^{k-1} \frac{(k!)^2}{(j!)^2 (k-j)!} \left\| \bar{\partial}^{j}\phi \right\|_{\varphi}^2 .
\end{aligned}
$$

Then (2.10) is proved. □

3. Proof of theorems

First we give the proof of Theorem 1.1.

Proof. — Let $\varphi = |z|^2$. By Lemma 2.2 and Lemma 2.4, we have for $\forall\, \phi \in C_0^\infty(\mathbb{C})$,

$$
\begin{aligned}
\left\| \left(\bar{\partial}^k + a\right)_{\varphi}^{*} \phi \right\|_{\varphi}^2 &\geqslant \left\langle \phi, \bar{\partial}^k \left(\bar{\partial}_{\varphi}^{k*}\phi\right) - \bar{\partial}_{\varphi}^{k*}\left(\bar{\partial}^k\phi\right) \right\rangle_{\varphi} \\
&= \sum_{j=0}^{k-1} \frac{(k!)^2}{(j!)^2 (k-j)!} \left\| \bar{\partial}^{j}\phi \right\|_{\varphi}^2 \\
&\geqslant k! \left\| \phi \right\|_{\varphi}^2 .
\end{aligned} \tag{3.1}
$$

By Cauchy–Schwarz inequality and (3.1), we have for $\forall\, \phi \in C_0^\infty(\mathbb{C})$,

$$
\begin{aligned}
|\langle f, \phi\rangle_{\varphi}|^2 &\leqslant \|f\|_{\varphi}^2 \|\phi\|_{\varphi}^2 \\
&= \left(\frac{1}{k!}\|f\|_{\varphi}^2\right)\left(k!\,\|\phi\|_{\varphi}^2\right) \\
&\leqslant \left(\frac{1}{k!}\|f\|_{\varphi}^2\right) \left\| \left(\bar{\partial}^k + a\right)_{\varphi}^{*} \phi \right\|_{\varphi}^2 .
\end{aligned}
$$

Then by Lemma 2.1, there exists $u \in L^2(\mathbb{C}, e^{-\varphi})$ such that

$$
\bar{\partial}^k u + au = f \quad \text{with} \quad \|u\|_{\varphi}^2 \leqslant \frac{1}{k!}\|f\|_{\varphi}^2 .
$$

The proof is complete. □

Second we prove the following theorem.

THEOREM 3.1. — *There exists a bounded (linear) operator*

$$T_k : L^2(\mathbb{C}, e^{-|z|^2}) \to L^2(\mathbb{C}, e^{-|z|^2})$$

such that

$$\left(\bar{\partial}^k + a\right) T_k = I \quad \textit{with} \quad \|T_k\| \leqslant \frac{1}{\sqrt{k!}},$$

where $\|T_k\|$ *is the norm of* T_k *in* $L^2(\mathbb{C}, e^{-|z|^2})$.

Proof. — Let $\varphi = |z|^2$. For each $f \in L^2(\mathbb{C}, e^{-\varphi})$, from Theorem 1.1, there exists $u \in L^2(\mathbb{C}, e^{-\varphi})$ such that

$$\left(\bar{\partial}^k + a\right) u = f \quad \text{with} \quad \|u\|_\varphi \leqslant \frac{1}{\sqrt{k!}} \|f\|_\varphi .$$

Denote this u by $T_k(f)$. Then $T_k(f)$ satisfies

$$\left(\bar{\partial}^k + a\right) T_k(f) = f \quad \text{with} \quad \|T_k(f)\|_\varphi \leqslant \frac{1}{\sqrt{k!}} \|f\|_\varphi .$$

Note that f is arbitrary in $L^2(\mathbb{C}, e^{-\varphi})$. So $T_k : L^2(\mathbb{C}, e^{-\varphi}) \to L^2(\mathbb{C}, e^{-\varphi})$ is a bounded (linear) operator such that

$$\left(\bar{\partial}^k + a\right) T_k = I \quad \text{with} \quad \|T_k\| \leqslant \frac{1}{\sqrt{k!}}.$$

The proof is complete. □

Lastly we prove Theorem 1.2.

Proof. — From Lemma 2.3, we have

$$\bar{\partial}\left(\bar{\partial}^*_\varphi \phi\right) - \bar{\partial}^*_\varphi \left(\bar{\partial}\phi\right) = \phi\bar{\partial}\partial\varphi, \quad \forall\, \phi \in C_0^\infty(\mathbb{C}).$$

Then by Lemma 2.2, for $\forall\, \phi \in C_0^\infty(\mathbb{C})$,

$$\begin{aligned} \left\|\left(\bar{\partial} + a\right)^*_\varphi \phi\right\|^2_\varphi &\geqslant \left\langle \phi, \bar{\partial}\left(\bar{\partial}^*_\varphi \phi\right) - \bar{\partial}^*_\varphi \left(\bar{\partial}\phi\right)\right\rangle_\varphi \\ &= \left\langle \phi, \phi\bar{\partial}\partial\varphi\right\rangle_\varphi = \left\|\phi\sqrt{\bar{\partial}\partial\varphi}\right\|^2_\varphi . \end{aligned} \tag{3.2}$$

By Cauchy–Schwarz inequality and (3.2), we have for $\forall\ \phi \in C_0^\infty(\mathbb{C})$,

$$\begin{aligned}
|\langle f, \phi\rangle_\varphi|^2 &= \left|\left\langle \frac{f}{\sqrt{\bar{\partial}\partial\varphi}}, \phi\sqrt{\bar{\partial}\partial\varphi}\right\rangle_\varphi\right|^2 \\
&\leqslant \left\|\frac{f}{\sqrt{\bar{\partial}\partial\varphi}}\right\|_\varphi^2 \left\|\phi\sqrt{\bar{\partial}\partial\varphi}\right\|_\varphi^2 \\
&\leqslant \left\|\frac{f}{\sqrt{\bar{\partial}\partial\varphi}}\right\|_\varphi^2 \left\|\left(\bar{\partial}+a\right)_\varphi^* \phi\right\|_\varphi^2.
\end{aligned}$$

Then by Lemma 2.1, there exists $u \in L^2(\mathbb{C}, e^{-\varphi})$ such that

$$\bar{\partial}u + au = f \quad \text{with} \quad \|u\|_\varphi^2 \leqslant \left\|\frac{f}{\sqrt{\bar{\partial}\partial\varphi}}\right\|_\varphi^2.$$

The proof is complete. □

4. Further remarks

Remark 4.1. — Given $\lambda > 0$ and $z_0 \in \mathbb{C}$, for the weight $\varphi = \lambda|z - z_0|^2$, we obtain the following corollary from Theorem 1.1. Here we stress that the proof is not simply a straightforward scaling, instead it will scale to a different equation.

Corollary 4.2. — *For each $f \in L^2(\mathbb{C}, e^{-\lambda|z-z_0|^2})$, there exists a weak solution $u \in L^2(\mathbb{C}, e^{-\lambda|z-z_0|^2})$ solving the equation*

$$\bar{\partial}^k u + au = f$$

with the norm estimate

$$\int_{\mathbb{C}} |u|^2 e^{-\lambda|z-z_0|^2}\,\mathrm{d}\sigma \leqslant \frac{1}{\lambda^k k!}\int_{\mathbb{C}} |f|^2 e^{-\lambda|z-z_0|^2}\,\mathrm{d}\sigma.$$

Proof. — From $f \in L^2(\mathbb{C}, e^{-\lambda|z-z_0|^2})$, we have

$$\int_{\mathbb{C}} |f(z)|^2 e^{-\lambda|z-z_0|^2}\,\mathrm{d}\sigma < +\infty. \tag{4.1}$$

Let $z = \frac{\omega}{\sqrt{\lambda}} + z_0$ and $g(\omega) = f(z) = f\left(\frac{\omega}{\sqrt{\lambda}} + z_0\right)$. Then by (4.1), we have

$$\frac{1}{\lambda}\int_{\mathbb{C}} |g(\omega)|^2 e^{-|\omega|^2}\,\frac{1}{2i}\,\mathrm{d}\overline{\omega}\wedge \mathrm{d}\omega < +\infty,$$

which implies that $g \in L^2(\mathbb{C}, e^{-|\omega|^2})$. For g, applying Theorem 1.1 with a replacing by $\frac{a}{(\sqrt{\lambda})^k}$, there exists a weak solution $v \in L^2(\mathbb{C}, e^{-|\omega|^2})$ solving the equation

$$\bar{\partial}^k v(\omega) + \frac{a}{(\sqrt{\lambda})^k} v(\omega) = g(\omega) \tag{4.2}$$

in $\mathbb{C}$ with the norm estimate

$$\int_{\mathbb{C}} |v(\omega)|^2 e^{-|\omega|^2} \frac{1}{2i} \,\mathrm{d}\overline{\omega} \wedge \mathrm{d}\omega \leqslant \frac{1}{k!} \int_{\mathbb{C}} |g(\omega)|^2 e^{-|\omega|^2} \frac{1}{2i} \,\mathrm{d}\overline{\omega} \wedge \mathrm{d}\omega. \tag{4.3}$$

Note that $\omega = \sqrt{\lambda}(z - z_0)$ and $g(\omega) = f(z)$. Let $u(z) = \frac{1}{(\sqrt{\lambda})^k} v(\omega) = \frac{1}{(\sqrt{\lambda})^k} v\big(\sqrt{\lambda}(z - z_0)\big)$. Then (4.2) and (4.3) can be rewritten by

$$\bar{\partial}^k u(z) + a u(z) = f(z) \tag{4.4}$$

$$\int_{\mathbb{C}} |u(z)|^2 e^{-\lambda|z-z_0|^2} \,\mathrm{d}\sigma \leqslant \frac{1}{\lambda^k k!} \int_{\mathbb{C}} |f(z)|^2 e^{-\lambda|z-z_0|^2} \,\mathrm{d}\sigma. \tag{4.5}$$

(4.5) implies that $u \in L^2(\mathbb{C}, e^{-\lambda|z-z_0|^2})$. Then by (4.4) and (4.5), the proof is complete. □

Remark 4.3. — From Corollary 4.2, we can obtain the following corollary, which shows that for any choice of a, the differential operator $\bar{\partial}^k + a$ has a bounded right inverse in $L^2(U)$, provided U is a bounded open set.

Corollary 4.4. — *Let $U \subset \mathbb{C}$ be any bounded open set. For each $f \in L^2(U)$, there exists a weak solution $u \in L^2(U)$ solving the equation*

$$\bar{\partial}^k u + a u = f$$

with the norm estimate $\|u\|_{L^2(U)} \leqslant c\|f\|_{L^2(U)}$, where the constant c depends only on the diameter of U.

Proof. — Let $z_0 \in U$. For given $f \in L^2(U)$, extending f to zero on $\mathbb{C} \setminus U$, we have

$$\widetilde{f} = \begin{cases} f, & x \in U \\ 0, & x \in \mathbb{C} \setminus U. \end{cases}$$

Then $\widetilde{f} \in L^2(\mathbb{C}) \subset L^2(\mathbb{C}, e^{-|z-z_0|^2})$. From Corollary 4.2, there exists $\widetilde{u} \in L^2(\mathbb{C}, e^{-|z-z_0|^2})$ such that

$$\bar{\partial}^k \widetilde{u} + a\widetilde{u} = \widetilde{f} \quad \text{with} \quad \int_{\mathbb{C}} |\widetilde{u}|^2 e^{-|z-z_0|^2} \,\mathrm{d}\sigma \leqslant \frac{1}{k!} \int_{\mathbb{C}} |\widetilde{f}|^2 e^{-|z-z_0|^2} \,\mathrm{d}\sigma.$$

Then

$$\int_{\mathbb{C}} |\widetilde{u}|^2 e^{-|z-z_0|^2} \,\mathrm{d}\sigma \leqslant \frac{1}{k!} \int_{\mathbb{C}} |\widetilde{f}|^2 \,\mathrm{d}\sigma = \frac{1}{k!} \int_U |f|^2 \,\mathrm{d}\sigma.$$

Note that

$$\int_{\mathbb{C}} |\widetilde{u}|^2 e^{-|z-z_0|^2}\,\mathrm{d}\sigma \geqslant \int_U |\widetilde{u}|^2 e^{-|z-z_0|^2}\,\mathrm{d}\sigma \geqslant \int_U |\widetilde{u}|^2 e^{-|U|^2}\,\mathrm{d}\sigma = e^{-|U|^2}\int_U |\widetilde{u}|^2\,\mathrm{d}\sigma,$$

where $|U|$ is the diameter of U. Therefore,

$$e^{-|U|^2}\int_U |\widetilde{u}|^2\,\mathrm{d}\sigma \leqslant \frac{1}{k!}\int_U |f|^2\,\mathrm{d}\sigma,$$

i.e.,

$$\int_U |\widetilde{u}|^2\,\mathrm{d}\sigma \leqslant \frac{e^{|U|^2}}{k!}\int_U |f|^2\,\mathrm{d}\sigma.$$

Restricting $\widetilde{u}$ on U to get u, then

$$\bar{\partial}^k u + au = f \quad \text{with} \quad \int_U |u|^2\,\mathrm{d}\sigma \leqslant \frac{e^{|U|^2}}{k!}\int_U |f|^2\,\mathrm{d}\sigma.$$

Note that $u \in L^2(U)$ and let $c = \sqrt{\frac{e^{|U|^2}}{k!}}$. Then the proof is complete. □

Remark 4.5. — As a simple consequence of Theorem 1.1, we can obtain the following result on the existence of entire weak solutions of the equation $\bar{\partial}^k u + au = f$ for square integrable functions and almost everywhere bounded functions.

Corollary 4.6. — *For each $f \in L^2(\mathbb{C})$ or $f \in L^\infty(\mathbb{C})$, there exists a weak solution $u \in L^2_{loc}(\mathbb{C})$ solving the equation*

$$\bar{\partial}^k u + au = f.$$

In particular, the equation $\bar{\partial}^k u = f$ has a weak solution $u \in L^2_{loc}(\mathbb{C})$ for $f \in L^2(\mathbb{C})$ or $f \in L^\infty(\mathbb{C})$.

The proof of Corollary 4.6 follows from the observation that $L^2(\mathbb{C}) \subset L^2(\mathbb{C}, e^{-|z|^2})$, $L^\infty(\mathbb{C}) \subset L^2(\mathbb{C}, e^{-|z|^2})$ and $L^2(\mathbb{C}, e^{-|z|^2}) \subset L^2_{loc}(\mathbb{C})$.

Remark 4.7. — It would be a natural question whether other weights would work by Hörmander L^2 method, but so far we don't know how to do.

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