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Preface

A thematic semester entitled *Calculus of Variations and Probability* sponsored by the *International Center for Mathematics and Computer Science in Toulouse (CIMI)* was organized from February to June 2019.

Calculus of variations is a branch of mathematical analysis that aims at finding and analyzing extrema of functionals via studying the effects of small perturbations in their argument. Many concrete problems from natural sciences can be recast in terms of calculus of variations. For example, equilibrium configurations of physical systems are often critical points of some energy functional, while evolutions of many physical systems follow some least action principle (optics, Newtonian mechanics, general relativity, superfluidity, magnetic theories, liquid crystals…). Techniques from calculus of variations have found applications several branches of mathematics: partial differential equations, geometry (minimal surfaces, isoperimetric problems), numerical analysis, probability (large deviations, statistical physics), statistics, linear algebra (eigenvalue problems) or dynamical systems (Hamiltonian systems).

The thematic program consisted of three conferences, several intensive courses and lectures, and a winter school aimed at promoting modern and active research areas linked to Calculus of Variations, with a particular focus on topics at the boundary with probability theory. The main topics included the study of singularities in physical problems such as micromagnetics or liquid crystals; optimal transport, Monge-Ampère equation and gradient flow structures for evolution equations; convex integration and fluid mechanics; quantitative stochastic homogenization; geometric flows; minimal surfaces; shape optimization; isoperimetric and functional inequalities.

This volume collects several research and expository articles by some of the plenary lecturers and their coauthors, aiming at presenting an overview of some topics covered during the semester program.

The contribution of Y. Ge and J. Ye investigates the smoothness of the optimal transport map between the smooth densities with respect to the squared Riemannian distance cost. The optimal transport map is characterized by $\exp(\nabla u)$, where the potential function u satisfies a Monge–Ampère type equation. To study the regularity of u, Ma–Trudinger–Wang introduced for the first time the MTW tensor. Later on, Kim–McCann interpreted the MTW tensor as a curvature tensor of some pseudo-Riemannian metric. The authors study the regularity issue on Riemannian manifolds with curvature sufficiently close to curvature of round sphere in C^2 norm in all dimensions. They prove that the MTW tensor on such Riemannian manifolds satisfies an improved Ma–Trudinger–Wang condition. As a consequence, they imply the smoothness of the optimal transport map by the continuity method.

Optimal transport methods have found many applications to geometric functional inequalities, and the extension to a discrete setting has recently seen a lot of activity. D. Halikias, B. Klartag and B. A. Slomka study discrete versions of the classical Brunn–Minkowski inequality on the volume of the Minkowski sum of sets. They give a unified proof of the discrete Brunn–Minkowski inequality of Klartag and Lehec and of the four function theorem of Ahlswede and Daykin, and extend them to finitely-generated abelian groups.

One of the important applications of optimal transport in mathematics is that it can be used to study geometry of metric spaces, including non-smooth situations. F. Cavalletti, N. Gigli and F. Santarcangelo consider synthetic notions of Ricci curvature lower bounds in terms of optimal transport. They show that for non-branching metric-measure spaces, the Lott–Sturm–Villani characterization in terms of convexity of entropy along geodesics for the L^2 optimal transport distance is equivalent to a notion of curvature defined from L^1 optimal transport. The latter is formulated in terms of convexity along one-dimensional curves defined from an L^1 transport map, rather than L^2. In particular, they show that the Lott–Sturm–Villani conditon can be equivalently expressed using L^1 optimal transport, instead of L^2.

G. Canevari and G. Orlandi briefly review the distributional Jacobian and the oriented coarea formula, which are classical tools for the study of vector-valued Sobolev maps, and indicate some applications to variational problems with emphasis on the Ginzburg-Landau energy. Then they propose new tools — flat chains with coefficients in an abelian group — for the study of singularities of manifold-valued Sobolev maps and apply them to variational models arising in material science. The lifting of BV maps with values into manifolds is also discussed.

A. Pisante reviews very recent results obtained with F. Dipasquale and V. Millot on minimizers of a Landau–De Gennes functional appearing in the modeling of three-dimensional nematic liquid crystals. The main goal of that series of results is to provide new insight into the structure of axially symmetric minimizers under a pointwise norm constraint. He describes how minimizers encode topological information on the domain and presents several regimes and scenarios, such as the emergence of torus-like solutions and of split solutions.

P. Mironescu and J. Van Schaftingen consider the problem of characterizing the traces of functions in the Sobolev space $W^{1,p}(\mathbf{B}^{m-1} \times (0,1); \mathcal{N})$, where $\mathbf{B}^{m-1}$ is the unit ball in $\mathbb{R}^{m-1}$ and $\mathcal{N}$ is a compact manifold. The classical trace theory implies that traces must belong to $W^{1-\frac{1}{p},p}(\mathbf{B}^{m-1}; \mathcal{N})$. A natural question is whether *any* function in the latter space is the trace of a function in the former one. If the answer is affirmative, it is also natural to ask whether the trace operator has a continuous right inverse; in other words, whether it is possible to find an extension of any function in $W^{1-\frac{1}{p},p}(\mathbf{B}^{m-1}; \mathcal{N})$ whose norm can be estimated in terms of the norm of the trace. The article provides a review of the current state of the art on these topics, and presents new, original results. In particular, the authors find a new analytical obstruction for the existence of an extension, and generalize a result of R. Hardt and F. H. Lin on the existence of an extension under a topological condition on the target manifold $\mathcal{N}$.

The contribution of S. Dipierro, O. Savin, and E. Valdinoci deals with the notion of *divergent fractional Laplacian* previously introduced by the authors. They show that any function u defined on $\mathbb{R}^N$ with polynomial growth at infinity can be locally approximated by s-harmonic functions: given $\epsilon > 0$, there exist $R_\epsilon > 0$ and an s-harmonic function u_ϵ such that $\|u - u_\epsilon\|_{C^m(B_1)} \leqslant \epsilon$ and $u = u_\epsilon$ on $\mathbb{R}^N \setminus B(0, R_\epsilon)$. Of course, the price to pay is that u_ϵ may oscillate heavily and may be far away from u on $B(0, R_\epsilon) \setminus B(0,1)$. Similarly, it is shown that any function with mild growth at infinity can be locally approximated by solutions of nonlinear equations involving the divergent fractional Laplacian. The solvability of the Dirichlet problem for the divergent fractional Laplacian in a ball is also proven.

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