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Wandering Fatou component for polynomials

Xavier Buff [(1)]

Abstract. — The filled-in Julia set $\mathcal{K}_f$ of a polynomial $f : \mathbb{C} \to \mathbb{C}$ is the set of points with bounded orbit under iteration of f. The No Wandering Theorem proved by Sullivan in the 1980's asserts that every connected component of the interior of $\mathcal{K}_f$ is eventually periodic. Sullivan's original proof uses Beltrami forms and the straightening of almost complex structures. We present a proof due to Adam Epstein based on a density theorem of Bers for quadratic differentials. This density theorem is proved by duality and requires solving the equation $\bar{\partial}\boldsymbol{\xi} = \boldsymbol{\mu}$ when $\boldsymbol{\mu} \in L^\infty$.

We show that this result does not hold for polynomials $F : \mathbb{C}^2 \to \mathbb{C}^2$. More precisely, we show that if

$$F(z,w) = \left(z + z^2 + az^3 + \frac{\pi^2}{4}w, w - w^2\right)$$

with $a < 1$ sufficiently close to 1, then F admits a wandering Fatou component. The proof uses techniques of parabolic implosion for skew products. The approach was initially suggested by Misha Lyubich and Han Peters.

Résumé. — L'ensemble de Julia $\mathcal{K}_f$ d'un polynôme $f : \mathbb{C} \to \mathbb{C}$ est l'ensemble des points dont l'orbite sous itération de f est bornée. Le théorème de non errance démontré par Sullivan dans les années 1980 affirme que chaque composante connexe de l'intérieur de $\mathcal{K}_f$ est prépériodique. La preuve originale de Sullivan repose sur les formes de Beltrami et le théorème d'intégrabilité des structures presque complexes en dimension 1. Nous présentons une preuve due à Adam Epstein qui repose sur un théorème de densité de Bers concernant les différentielles quadratiques. Ce théorème de densité se montre par dualité et nécessite la résolution de l'équation $\bar{\partial}\boldsymbol{\xi} = \boldsymbol{\mu}$ pour $\boldsymbol{\mu} \in L^\infty$.

Nous montrons ensuite que ce résultat ne se généralise pas aux applications $F : \mathbb{C}^2 \to \mathbb{C}^2$. Plus précisément, nous montrons que si

$$F(z,w) = \left(z + z^2 + az^3 + \frac{\pi^2}{4}w, w - w^2\right)$$

avec $a < 1$ suffisamment proche de 1, alors F possède une composante de Fatou errante. La preuve repose sur des techniques d'implosion parabolique pour des produits fibrés. Cette approche a été initialement suggérée par Misha Lyubich et Han Peters.

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1. Analytic preliminaries

This part is dedicated to the proof of the following density theorem: on a compact Riemann surface X, the set of meromorphic quadratic differentials whose poles are simple and contained in a set $B \subseteq X$ is dense, for the L^1-norm, in the space of integrable quadratic differentials on X which are holomorphic outside $\overline{B}$.

We will first introduce the actors: quasiconformal vector fields, Beltrami differentials and quadratic differentials. We will then solve the equation $\bar\partial\boldsymbol{\xi} = \boldsymbol{\mu}$ when $\boldsymbol{\xi}$ is a vector field on a Riemann surface X and $\boldsymbol{\mu}$ is essentially bounded. We will finally prove the density result.

1.1. The actors

1.1.1. Hyperbolic Riemann surfaces

In this section, X is a Riemann surface. In applications, X will usually be an open subset of $\mathbb{C}$. The Riemann surface X is naturally oriented: if z is a local coordinate, then X is locally oriented by the $(1,1)$-form $\frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}\bar z$.

Definition 1.1. — *The Riemann surface X is* hyperbolic *if there is a universal cover $\boldsymbol{\pi} : \mathbb{D} \to X$ from the unit disk $\mathbb{D} := \{z \in \mathbb{C}\ ;\ |z| < 1\}$ to X.*

If $\boldsymbol{\pi} : \mathbb{D} \to X$ is a universal cover, the deck transformations are Möbius transformations. They preserve the Poincaré metric $\rho_{\mathbb{D}}$ on the unit disk:

$$\rho_{\mathbb{D}} = \frac{2\,|\mathrm{d}z|}{1 - |z|^2}.$$

It follows that the Poincaré metric on $\mathbb{D}$ descends to a *hyperbolic* metric ρ_X on X which satisfies

$$\rho_X \circ \mathrm{D}\boldsymbol{\pi} = \rho_{\mathbb{D}}.$$

Definition 1.2. — *A vector field $\boldsymbol{\xi}$ on a hyperbolic Riemann surface X is* hyperbolically bounded on X *if the function $\rho_X(\boldsymbol{\xi}) : X \to [0, +\infty)$ is bounded. We set*

$$\|\boldsymbol{\xi}\|_X := \sup_X \rho_X(\boldsymbol{\xi}).$$

1.1.2. Beltrami differentials

A vector field $\boldsymbol{\xi}$ on a Riemann surface X is a section of the holomorphic bundle $\mathrm{T}X$. When $\boldsymbol{\xi}$ is a smooth vector field, $\bar\partial\boldsymbol{\xi}$ is a $\mathbb{C}$-antilinear bundle map $\bar\partial\boldsymbol{\xi} : \mathrm{T}X \to \mathrm{T}X$: for all $(v,w) \in \mathrm{T}_xX \times \mathrm{T}_xX$ and all $\lambda \in \mathbb{C}$,

$$\bar\partial\boldsymbol{\xi}(x;\lambda v + w) = \bar\lambda \cdot \bar\partial\boldsymbol{\xi}(x;v) + \bar\partial\boldsymbol{\xi}(x;w).$$

Definition 1.3. — *We denote by* $\overline{\mathrm{Hom}}(\mathrm{T}X,\mathrm{T}X)$ *the space of* $\mathbb{C}$*-antilinear bundle maps.*

When $\boldsymbol{\mu} \in \overline{\mathrm{Hom}}(\mathrm{T}X,\mathrm{T}X)$, the restriction $\boldsymbol{\mu}_x$ of $\boldsymbol{\mu}$ to each fiber T_xX is an endomorphism and since T_xX has dimension 1, its operator norm does not depend on the choice of norm on T_xX. We shall denote by $|\boldsymbol{\mu}|(x)$ this constant, defining a function $|\boldsymbol{\mu}| : X \to [0,+\infty)$.

If z is a local coordinate near a point $x \in X$, then we may locally write

$$\boldsymbol{\mu} = \mu(z)\frac{\partial}{\partial z} \cdot \mathrm{d}\bar z \quad \text{and} \quad |\boldsymbol{\mu}| = \big|\mu(z)\big|,$$

for some function μ defined near $z(x)$.

Definition 1.4. — *A Beltrami differential* $\boldsymbol{\mu}$ *on a Riemann surface* X *is a measurable element of* $\overline{\mathrm{Hom}}(\mathrm{T}X,\mathrm{T}X)$ *which satisfies:*

$$\|\boldsymbol{\mu}\|_{L^\infty(X)} := \sup_X |\boldsymbol{\mu}| < +\infty.$$

We denote by $\mathrm{bel}(X)$ *the space of Beltrami differentials on* X*.*

1.1.3. Quadratic differentials

Definition 1.5. — *A quadratic differential* $\mathbf{q}$ *on a Riemann surface* X *is a field of quadratic forms, i.e. a section of the symmetric square* $S^2(\mathrm{T}^*X)$ *of the cotangent bundle. We denote by* $\mathcal{Q}(X)$ *the space of measurable quadratic differentials on* X*.*

If $\mathbf{q}$ is a quadratic differential on X, then $\mathbf{q}(x)$ is a quadratic form on T_xX. With an abuse of notation, we shall write $\mathbf{q}(x;v)$ for $\mathbf{q}(x)(v)$, and thus, consider $\mathbf{q}$ as a map $\mathrm{T}X \to \mathbb{C}$ which satisfies

$$\mathbf{q}(x;\lambda v) = \lambda^2\mathbf{q}(x;v) \quad \text{for all} \quad (x;v) \in \mathrm{T}X \quad \text{and all} \quad \lambda \in \mathbb{C}.$$

If z is a local coordinate near a point $x \in X$, then we may locally write

$$\mathbf{q} = q(z)\ \mathrm{d}z \otimes \mathrm{d}z$$

for some function q defined near $z(x)$. For simplicity, we will write $\mathrm{d}z^2$ in place of $\mathrm{d}z \otimes \mathrm{d}z$, not to be confused with $\mathrm{d}(z^2)$.

Definition 1.6. — *The polar form* $\mathbf{b}$ *of the quadratic differential* $\mathbf{q}$ *is the field of symmetric and bilinear forms defined by*

$$\mathbf{b}(x;v,w) := \frac{1}{4}\mathbf{q}(x;v+w) - \frac{1}{4}\mathbf{q}(x;v-w).$$

Definition 1.7. — *If* $\mathbf{q}$ *is a quadratic differential on* X*, we denote by* $|\mathbf{q}|$ *the positive* $(1,1)$*-form on* X *which takes the value* $\big|\mathbf{q}(x;v)\big|$ *on the pair* $(v,\mathrm{i}v)$*.*

For $x \in X$ and $(v,w) \in \mathrm{T}_x X \times \mathrm{T}_x X$,

$$|\mathbf{q}|(x;v,w) := \frac{1}{2}\big|\mathbf{q}(x;v-\mathrm{i}w)\big| - \frac{1}{2}\big|\mathbf{q}(x;v+\mathrm{i}w)\big|.$$

If z is a local coordinate and $\mathbf{q} = q(z)\ \mathrm{d}z^2$, then

$$|\mathbf{q}| = |q(z)| \cdot \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Definition 1.8. — *A quadratic differential* $\mathbf{q}$ *is integrable on* X *if*

$$\|\mathbf{q}\|_{L^1(X)} := \int_X |\mathbf{q}| < +\infty.$$

Definition 1.9. — *We denote by*

- $\mathcal{Q}(X)$ *the space of measurable quadratic differentials on* X
- $\mathcal{Q}^1(X) \subset \mathcal{Q}(X)$ *the subspace of integrable quadratic differentials;*
- $\mathcal{Q}_{\mathrm{hol}}(X) \subset \mathcal{Q}(X)$ *the subspace of holomorphic quadratic differentials;*
- $\mathcal{Q}^1_{\mathrm{hol}}(X) := \mathcal{Q}_{\mathrm{hol}}(X) \cap \mathcal{Q}^1(X)$.

If X *is a compact Riemann surface and* $B \subseteq X$*, we denote by*

- $\mathcal{Q}^1_{\mathrm{mero}}(X)$ *the space of meromorphic quadratic differentials on* X *which have at worst simple poles and*
- $\mathcal{Q}^1_{\mathrm{mero}}(X;B) \subseteq \mathcal{Q}^1_{\mathrm{mero}}(X)$ *the subspace of quadratic differentials whose poles are contained in* B*.*

Proposition 1.10. — *If* X *is a compact Riemann surface and* $B \subset X$ *is a finite set, then* $\mathcal{Q}^1_{\mathrm{hol}}(X \smallsetminus B) = \mathcal{Q}^1_{\mathrm{mero}}(X;B)$*.*

Proof. — Assume $\mathbf{q} \in \mathcal{Q}^1_{\mathrm{hol}}(X \smallsetminus B)$. By definition, $\mathbf{q}$ is holomorphic on $X \smallsetminus B$. We must show that the singularity at a point $x \in B$ is at worst a simple pole and that a simple pole is integrable. Choose a local coordinate $z : (X,x) \to (\mathbb{C},0)$ and write

$$\mathbf{q} = q(z)\ \mathrm{d}z^2$$

for some function q which is holomorphic near 0 in $\mathbb{C}$.

If q has a simple pole at 0, then there is a constant C such that

$$\int_{|z|<\varepsilon} |q(z)||\mathrm{d}z^2| \leqslant \int_{|z|<\varepsilon} \frac{C}{|z|}|\mathrm{d}z^2| \leqslant \int_0^{2\pi}\left(\int_0^\varepsilon \frac{C}{r} r\mathrm{d}r\right)\mathrm{d}\theta = 2\pi\varepsilon C.$$

Thus, $\mathbf{q}$ is integrable near x.

Let us now prove that when $\mathbf{q}$ is integrable, it has at worst a simple pole. Consider the Laurent series

$$q(z) = \sum_{k\in\mathbb{Z}} q_k z^k \quad \text{with} \quad q_k = \frac{1}{2\pi\mathrm{i}}\int_{|z|=r}\frac{q(z)}{z^{k+1}}\mathrm{d}z \quad \text{for } r\in(0,1).$$

Writing $z = r\mathrm{e}^{2\pi\mathrm{i}t}$ yields

$$|q_k| = \left|\int_{t\in(0,1)} \frac{q\big(r\mathrm{e}^{2\pi\mathrm{i}t}\big)}{r^k\mathrm{e}^{2\pi\mathrm{i}kt}}\mathrm{d}t\right| \leqslant \frac{1}{r^k}\int_{t\in(0,1)}\left|q\big(r\mathrm{e}^{2\pi\mathrm{i}t}\big)\right|\mathrm{d}t.$$

As a consequence,

$$\|q\|_{L^1(X)} \geqslant \int_{(r,t)\in(0,\varepsilon)\times(0,1)}\left|q\big(r\mathrm{e}^{2\pi\mathrm{i}t}\big)\right| r\mathrm{d}r\mathrm{d}t \geqslant |q_k|\int_{r\in(0,\varepsilon)} r^{k+1}\mathrm{d}r.$$

It follows that $q_k = 0$ for $k \leqslant -2$ and so, q has at worst a simple pole at 0. $\square$

The ratio of any two holomorphic quadratic differentials is a meromorphic function. On a compact Riemann surface, a meromorphic function has as many zeros as poles, counting multiplicities. On a genus g compact Riemann surface, the number of zeros minus the number of poles of a 1-form is $2g-2$. It follows that the number of zeros minus the number of poles of a meromorphic quadratic differential is $4g-4$. In particular, on the Riemann sphere for which $g=0$, the number of poles minus the number of zeros of a non zero meromorphic quadratic differential is equal to 4. As a consequence, if X is a thrice-punctured sphere, then $\mathcal{Q}^1_{\mathrm{hol}}(X) = \{0\}$.

It is natural to pull back quadratic differentials: if $f : X \to Y$ is a holomorphic map and $\mathbf{q}$ is a quadratic differential on Y, then we can define a quadratic differential $f^*\mathbf{q}$ on X by

$$f^*\mathbf{q} := \mathbf{q}\circ \mathrm{D}f.$$

In appropriate situations, we can also push forward quadratic differentials.

Proposition 1.11. — *If $f : X \to Y$ is a covering of Riemann surfaces and $\mathbf{q} \in \mathcal{Q}^1_{\mathrm{hol}}(X)$, the following series converges and defines a quadratic differential $f_*\mathbf{q} \in \mathcal{Q}^1_{\mathrm{hol}}(Y)$:*

$$f_*\mathbf{q} := \sum_{\substack{g_i \text{ inverse} \\ \text{branch of } f}} g_i^*\mathbf{q}.$$

In addition,

$$\|f_*\mathbf{q}\|_{L^1(Y)} \leqslant \|\mathbf{q}\|_{L^1(X)}.$$

Proof. — It is enough to show that the series converges in $L^1(Y)$ and this immediately follows from the triangle inequality. Indeed, if $U \subseteq Y$ is an evenly covered open set, and $g_i : U \to X$ are inverse branches of f, then

$$\int_U \left|\sum g_i^*\mathbf{q}\right| \leqslant \int_U \sum |g_i^*\mathbf{q}| = \int_{f^{-1}(U)} |\mathbf{q}| \leqslant \|\mathbf{q}\|_{L^1(X)} < +\infty. \qquad \square$$

1.1.4. Quasiconformal vector fields

Definition 1.12. — *A vector field $\boldsymbol{\xi}$ on a Riemann surface X is* quasiconformal *if $\bar\partial\boldsymbol{\xi}$ is a Beltrami differential.*

In this definition, the $\bar\partial$-derivative is taken in the sense of distributions. What it means is that there is a Beltrami differential $\boldsymbol{\mu}$ on X such that for any smooth quadratic differential $\mathbf{q}$ with compact support in X, we have

$$\int_X \bar\partial\mathbf{q}\cdot\boldsymbol{\xi} = -\int_X \mathbf{q}\cdot\boldsymbol{\mu}.$$

We need to define what $\bar\partial\mathbf{q}\cdot\boldsymbol{\xi}$ and $\mathbf{q}\cdot\boldsymbol{\mu}$ mean. Let us first define $\mathbf{q}\cdot\boldsymbol{\xi}$.

Definition 1.13. — *If $\mathbf{q}$ is a quadratic differential on X with polar form $\mathbf{b}$, and $\boldsymbol{\xi}$ is a vector field on X, then $\mathbf{q}\cdot\boldsymbol{\xi}$ is the $(1,0)$-form defined by*

$$\mathrm{T}X \ni (x;v) \mapsto \frac{1}{2\mathrm{i}}\mathbf{b}(x;\boldsymbol{\xi}(x),v) \in \mathbb{C}.$$

In local coordinates, if

$$\mathbf{q} = q(z)\,\mathrm{d}z^2 \quad\text{and}\quad \boldsymbol{\xi} = \xi(z)\,\frac{\partial}{\partial z},$$

then

$$\mathbf{q}\cdot\boldsymbol{\xi} = \frac{1}{2\mathrm{i}}q(z)\xi(z)\,\mathrm{d}z.$$

Since $\mathbf{q}\cdot\boldsymbol{\xi}$ is a $(1,0)$-form, its derivative $\bar\partial(\mathbf{q}\cdot\boldsymbol{\xi})$ is a $(1,1)$-form and we wish to have the equality

$$\bar\partial(\mathbf{q}\cdot\boldsymbol{\xi}) = \bar\partial\mathbf{q}\cdot\boldsymbol{\xi} + \mathbf{q}\cdot\bar\partial\boldsymbol{\xi}$$

which corresponds, in a local coordinate z, to

$$\bar\partial\left(q(z)\xi(z)\cdot\frac{1}{2\mathrm{i}}\mathrm{d}z\right) = \left(\frac{\partial q(z)}{\partial\bar z}\xi(z) + q(z)\frac{\partial\xi(z)}{\partial\bar z}\right)\cdot\frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar z.$$

This yields the following definitions.

Definition 1.14. — *If* $\mathbf{q}$ *is a quadratic differential on* X *with polar form* $\mathbf{b}$ *and if* $\boldsymbol{\mu}$ *is a Beltrami differential on* X*, then* $\mathbf{q}\cdot\boldsymbol{\mu}$ *is the* $(1,1)$*-form defined on* X *by*

$$\mathbf{q}\cdot\boldsymbol{\mu}\ (x;v,w)=\frac{\mathrm{i}}{2}\mathbf{b}\big(x;v,\boldsymbol{\mu}(x;w)\big)-\frac{\mathrm{i}}{2}\mathbf{b}\big(x;\boldsymbol{\mu}(x;v),w\big).$$

If z is a local coordinate, if

$$\mathbf{q}=q(z)\,\mathrm{d}z^2\quad\text{and}\quad\boldsymbol{\mu}=\mu(z)\frac{\partial}{\partial z}\cdot\mathrm{d}\bar z,$$

then

$$\mathbf{q}\cdot\boldsymbol{\mu}=q(z)\mu(z)\cdot\frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar z.$$

We now define $\bar\partial\mathbf{q}\cdot\boldsymbol{\xi}$ when $\mathbf{q}$ is a smooth quadratic differential. If $\mathbf{b}$ is the polar form of $\mathbf{q}$, then $\bar\partial\mathbf{b}$ is a $\mathbb{C}$-antilinear bundle map $\mathrm{T}X\to S^2(\mathrm{T}^*X)$.

Definition 1.15. — *If* $\boldsymbol{\xi}$ *is a vector field on* X *and* $\mathbf{q}$ *is a smooth quadratic differential on* X *with polar form* $\mathbf{b}$*, then* $\bar\partial\mathbf{q}\cdot\xi$ *is the* $(1,1)$*-form defined on* X *by*

$$\bar\partial\mathbf{q}\cdot\boldsymbol{\xi}\ (v,w)=\frac{\mathrm{i}}{2}\bar\partial\mathbf{b}(v)(\boldsymbol{\xi},w)-\frac{\mathrm{i}}{2}\bar\partial\mathbf{b}(w)(\boldsymbol{\xi},v).$$

If z is a local coordinate, if

$$\mathbf{q}=q(z)\ \mathrm{d}z^2\quad\text{and}\quad\boldsymbol{\xi}=\xi(z)\frac{\partial}{\partial z},$$

then

$$\bar\partial\mathbf{q}\cdot\boldsymbol{\xi}=\frac{\partial q(z)}{\partial\bar z}\xi(z)\cdot\frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar z.$$

1.2. Solving the equation $\bar\partial\boldsymbol{\xi}=\boldsymbol{\mu}$

First, observe that the equation $\bar\partial\boldsymbol{\xi}=\boldsymbol{\mu}$ always has local solutions. Indeed, if z is a local coordinate sending $x\in X$ to $0\in\mathbb{C}$ and if

$$\boldsymbol{\mu}=\mu(z)\mathrm{d}\bar z\cdot\frac{\partial}{\partial z},$$

then we can define

$$\boldsymbol{\xi}:=\xi(z)\ \frac{\partial}{\partial z}\quad\text{with}\quad\xi(z):=\frac{1}{\pi}\int_{|w|<\varepsilon}\frac{\mu(w)}{z-w}\ |\mathrm{d}w^2|.$$

Then,

$$\frac{\partial\xi}{\partial\bar z}=\begin{cases}\mu & \text{if } |z|<\varepsilon\\ 0 & \text{if } |z|\geqslant\varepsilon.\end{cases}$$

So, $\bar\partial\boldsymbol{\xi}=\boldsymbol{\mu}$ when $|z|<\varepsilon$.

Second, observe that the vector field $\boldsymbol{\xi}$ defined in this way is continuous at $x\in X$. More precisely,

$$\xi(z)=\xi(0)+\mathrm{O}\left(|z|\log\frac{1}{|z|}\right)\quad\text{as}\quad z\to 0.$$

Indeed,

$$\begin{aligned}\bigl|\xi(z)-\xi(0)\bigr| &\leqslant \frac{1}{\pi}\cdot\|\mu\|_{L^\infty(D_\varepsilon)}\cdot\int_{|w|<\varepsilon}\left|\frac{1}{w-z}-\frac{1}{w}\right||\mathrm{d}w^2|\\ &=\frac{|z|}{\pi}\cdot\|\mu\|_{L^\infty(D_\varepsilon)}\int_{|w|<\varepsilon}\left|\frac{1}{w(w-z)}\right||\mathrm{d}w^2|.\end{aligned}$$

Make the change of variables $w=zu$ with $|\mathrm{d}w^2|=|z|^2\cdot|\mathrm{d}u^2|$, to obtain

$$\begin{aligned}\int_{|w|<\varepsilon}\left|\frac{1}{w(w-z)}\right||\mathrm{d}w^2| &= \int_{|u|<\varepsilon/|z|}\left|\frac{1}{u(u-1)}\right||\mathrm{d}u^2|\\ &=\mathrm{O}\left(\int_{1<|u|<\varepsilon/|z|}\frac{1}{|u|^2}|\mathrm{d}u^2|\right)=\mathrm{O}\left(\log\frac{1}{|z|}\right).\end{aligned}$$

Third, observe that if $\boldsymbol{\zeta}$ is another quasiconformal vector field with $\bar\partial\boldsymbol{\zeta}=\bar\partial\boldsymbol{\xi}=\boldsymbol{\mu}$, then the difference $\boldsymbol{\zeta}-\boldsymbol{\xi}$ is a holomorphic vector field, so that $\boldsymbol{\zeta}$ is also continuous at x and

$$\boldsymbol{\zeta}=\zeta(z)\,\frac{\partial}{\partial z}\quad\text{with}\quad\zeta(z)=\zeta(0)+\mathrm{O}\left(|z|\log\frac{1}{|z|}\right)\quad\text{as}\quad z\to 0.$$

We just proved the following result.

Lemma 1.16. — *If $\boldsymbol{\xi}$ is a quasiconformal vector field on a Riemann surface X, then $\boldsymbol{\xi}$ is continuous. More precisely, if z is a local coordinate sending $x\in X$ to $0\in\mathbb{C}$ and*

$$\textit{if}\quad\boldsymbol{\xi}=\xi(z)\,\frac{\partial}{\partial z}\quad\textit{then}\quad\xi(z)=\xi(0)+\mathrm{O}\left(|z|\log\frac{1}{|z|}\right)\quad\textit{as}\quad z\to 0.$$

The following result states that under appropriate circumstances, the equation $\bar\partial\boldsymbol{\xi}=\boldsymbol{\mu}$ has a global solution.

Definition 1.17. — *A Beltrami differential $\boldsymbol{\mu}$ on a Riemann surface X is* trivial *if*

$$\forall\,\mathbf{q}\in\mathcal{Q}^1_{\mathrm{hol}}(X),\quad\int_X\mathbf{q}\cdot\boldsymbol{\mu}=0.$$

Proposition 1.18. — *Let X be a hyperbolic Riemann surface and $\boldsymbol{\mu}$ be a trivial Beltrami differential on X. Then, there exists a unique hyperbolically bounded quasiconformal vector field $\boldsymbol{\xi}$ on X such that $\bar\partial\boldsymbol{\xi} = \boldsymbol{\mu}$. In addition,*

$$\|\boldsymbol{\xi}\|_X \leqslant 4\|\boldsymbol{\mu}\|_{L^\infty(X)}.$$

Proof. — Let us first prove uniqueness. If $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are two hyperbolically bounded quasiconformal vector fields on X with $\bar\partial\boldsymbol{\xi} = \bar\partial\boldsymbol{\zeta}$, then the difference $\boldsymbol{\xi} - \boldsymbol{\zeta}$ is a hyperbolically bounded holomorphic vector field on X. Let $\boldsymbol{\pi} : \mathbb{D} \to X$ be a universal cover. Then, $\boldsymbol{\pi}^*(\boldsymbol{\xi} - \boldsymbol{\zeta})$ is a holomorphic vector field on $\mathbb{D}$, which extends continuously by 0 on the unit circle. It follows that $\boldsymbol{\xi} - \boldsymbol{\zeta} = 0$.

Let us now prove the existence of a vector field $\boldsymbol{\xi}$ such that $\bar\partial\boldsymbol{\xi} = \boldsymbol{\mu}$. Given a universal cover $\boldsymbol{\pi} : \mathbb{D} \to X$, the Beltrami differential $\boldsymbol{\pi}^*\boldsymbol{\mu}$ can be expressed as

$$\boldsymbol{\pi}^*\boldsymbol{\mu} = \mu(z)\ \mathrm{d}\bar z \cdot \frac{\partial}{\partial z}$$

for some function $\mu \in L^\infty(\mathbb{D})$. Let $\boldsymbol{\zeta}$ be the vector field on $\mathbb{C}$ defined by

$$\boldsymbol{\zeta} := \zeta(z)\frac{\partial}{\partial z} \quad \text{with} \quad \zeta(z) := \frac{1}{\pi}\int_{\mathbb{D}} \frac{\mu(w)}{z-w}\ |\mathrm{d}w^2|.$$

Since

$$\frac{\partial\zeta}{\partial\bar z} = \begin{cases} \mu & \text{on } \mathbb{D} \\ 0 & \text{outside } \mathbb{D}, \end{cases}$$

$\boldsymbol{\zeta}$ is quasiconformal on $\mathbb{C}$, $\bar\partial\boldsymbol{\zeta} = \boldsymbol{\pi}^*\boldsymbol{\mu}$ on $\mathbb{D}$ and $\boldsymbol{\zeta}$ is holomorphic outside $\mathbb{D}$. In fact, if $z \notin \mathbb{D}$, then

$$\mathbf{q}_z := \frac{\mathrm{d}w^2}{w-z} \in \mathcal{Q}^1_{\mathrm{hol}}(\mathbb{D}) \quad \text{so} \quad \boldsymbol{\pi}_*(\mathbf{q}_z) \in \mathcal{Q}^1_{\mathrm{hol}}(X).$$

In that case

$$\zeta(z) := \frac{1}{\pi}\int_{\mathbb{D}} \frac{\mu(w)}{w-z}\ |\mathrm{d}w^2| = \frac{1}{\pi}\int_{\mathbb{D}} \mathbf{q}_z \cdot (\boldsymbol{\pi}^*\boldsymbol{\mu}) = \frac{1}{\pi}\int_X (\boldsymbol{\pi}_*\mathbf{q}_z)\cdot\boldsymbol{\mu} = 0$$

since $\boldsymbol{\mu}$ is trivial. Thus, $\boldsymbol{\zeta}$ vanishes outside $\mathbb{D}$.

Now, if $\boldsymbol{\pi}_1 : \mathbb{D} \to X$ and $\boldsymbol{\pi}_2 : \mathbb{D} \to X$ are two universal covers and if $\boldsymbol{\pi}_1(z_1) = \boldsymbol{\pi}_2(z_2)$, there is a Möbius transformation γ fixing $\mathbb{D}$ and sending z_2 to z_1 such that $\boldsymbol{\pi}_2 = \boldsymbol{\pi}_1 \circ \gamma$. Let $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ be the corresponding quasiconformal vector fields defined above. They vanish outside $\mathbb{D}$ and satisfy

$$\bar\partial\boldsymbol{\zeta}_1 = \boldsymbol{\pi}_1^*\boldsymbol{\mu} \quad \text{and} \quad \bar\partial\boldsymbol{\zeta}_2 = \boldsymbol{\pi}_2^*\boldsymbol{\mu} = \gamma^*(\boldsymbol{\pi}_1^*\boldsymbol{\mu}) = \bar\partial(\gamma^*\boldsymbol{\zeta}_1).$$

It follows that $\boldsymbol{\zeta}_2 - \gamma^*\boldsymbol{\zeta}_1$ is holomorphic on $\mathbb{C}$, and since both vector fields vanish outside $\mathbb{D}$, we necessarily have $\boldsymbol{\zeta}_2 = \gamma^*\boldsymbol{\zeta}_1$. This shows that

$$\mathrm{D}_{z_2}\boldsymbol{\pi}_2\big(\boldsymbol{\zeta}_2(z_2)\big) = \mathrm{D}_{z_1}\boldsymbol{\pi}_1\big(\boldsymbol{\zeta}_1(z_1)\big).$$

Taking $\boldsymbol{\pi}_1 = \boldsymbol{\pi}_2 = \boldsymbol{\pi}$, this shows that $\boldsymbol{\zeta}$ descends to a vector field $\boldsymbol{\xi}$ on X such that $\boldsymbol{\xi} = \mathrm{D}\boldsymbol{\pi}(\boldsymbol{\zeta})$. Taking $\boldsymbol{\pi}_1 \neq \boldsymbol{\pi}_2$ shows that $\boldsymbol{\xi}$ does not depend on the choice of universal cover. Since $\bar\partial\boldsymbol{\zeta} = \boldsymbol{\pi}^*\boldsymbol{\mu}$ on $\mathbb{D}$, we have $\bar\partial\boldsymbol{\xi} = \boldsymbol{\mu}$ on X.

We finally prove that $\boldsymbol{\xi}$ is hyperbolically bounded. Given $x \in X$, let $\boldsymbol{\pi} : \mathbb{D} \to X$ be a universal cover sending 0 to x. By construction

$$\boldsymbol{\xi}(x) = \mathrm{D}_0\boldsymbol{\pi}\left(\zeta(0)\frac{\partial}{\partial z}\right) \quad \text{with} \quad \zeta(0) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{\mu(w)}{w}|\mathrm{d}w^2|.$$

As a consequence,

$$\rho_X\big(\boldsymbol{\xi}(x)\big) = \rho_{\mathbb{D}}\left(\zeta(0)\frac{\partial}{\partial w}\right) = 2\big|\zeta(0)\big| \leqslant \frac{2\|\mu\|_{L^\infty(\mathbb{D})}}{\pi}\int_{\mathbb{D}} \frac{|\mathrm{d}w^2|}{|w|} = 4\|\boldsymbol{\mu}\|_{L^\infty(X)}.$$

□

The next result shows that the converse to Proposition 1.18 holds.

Proposition 1.19. — *Assume $\boldsymbol{\xi}$ is a hyperbolically bounded quasiconformal vector field on X. Then, $\bar\partial\boldsymbol{\xi}$ is a trivial Beltrami differential on X.*

Proof. — Fix a point $x \in X$, let $\delta : X \to (0, +\infty)$ be the hyperbolic distance to x. Assume $\mathbf{q} \in \mathcal{Q}^1_{\mathrm{hol}}(X)$. On the one hand, $\mathbf{q}\cdot\boldsymbol{\mu}$ is integrable on X and $\bar\partial\mathbf{q} = 0$ on X. On the other hand, $\{\delta = R\}$ is smooth by part, so that, according to the Stokes Theorem,

$$\int_{\{\delta=R\}} \mathbf{q}\cdot\boldsymbol{\xi} = \int_{\{\delta<R\}} \bar\partial\mathbf{q}\cdot\boldsymbol{\xi} + \int_{\{\delta<R\}} \mathbf{q}\cdot\boldsymbol{\mu} = \int_{\{\delta<R\}} \mathbf{q}\cdot\boldsymbol{\mu}.$$

Working in local coordinates with

$$\mathbf{q} = q(z)\ \mathrm{d}z^2, \quad \boldsymbol{\xi} = \xi(z)\ \frac{\partial}{\partial z} \quad \text{and} \quad \rho_X = \rho(z)\ |\mathrm{d}z|,$$

we have

$$\big|(\mathbf{q}\cdot\boldsymbol{\xi})\wedge \mathrm{d}\delta\big| \leqslant \left|\mathbf{q}(z)\xi(z)\rho(z)\cdot\frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar z\right| = \rho_X(\boldsymbol{\xi})\cdot|\mathbf{q}| \leqslant \|\boldsymbol{\xi}\|_X\cdot|\mathbf{q}|.$$

Thus, according to Fubini's theorem,

$$\int_{R=0}^{+\infty}\left|\int_{\{\delta=R\}} \mathbf{q}\cdot\boldsymbol{\xi}\right|\mathrm{d}\delta \leqslant \int_X \big|(\mathbf{q}\cdot\boldsymbol{\xi})\wedge\mathrm{d}\delta\big| \leqslant \|\boldsymbol{\xi}\|_X\cdot\|\mathbf{q}\|_{L^1(X)} < +\infty.$$

Since the following limit exists

$$\lim_{R\to+\infty}\int_{\{\delta=R\}} \mathbf{q}\cdot\boldsymbol{\xi} = \lim_{R\to+\infty}\int_{\{\delta<R\}} \mathbf{q}\cdot\boldsymbol{\mu} = \int_X \mathbf{q}\cdot\boldsymbol{\mu},$$

it is necessarily 0, thus $\boldsymbol{\mu}$ is trivial. □

In fact, a similar argument yields the following result.

Proposition 1.20. — *Let Y be a Riemann surface and $X \subset Y$ be a hyperbolic Riemann surface. Let $\boldsymbol{\xi}$ be a vector field on Y which vanishes outside X and whose restriction to X is hyperbolically bounded and quasiconformal. Then, $\boldsymbol{\xi}$ is globally quasiconformal on Y and $\bar{\partial}\boldsymbol{\xi} = 0$ outside X.*

Proof. — Let $\boldsymbol{\mu}$ be the Beltrami differential on Y which coincides with $\bar{\partial}\boldsymbol{\xi}$ on X and vanishes outside X. Let $\mathbf{q}$ be a smooth quadratic differential with compact support in Y. We must prove that

$$\int_Y \bar{\partial}\mathbf{q} \cdot \xi + \int_Y \mathbf{q} \cdot \boldsymbol{\mu} = 0.$$

Since $\boldsymbol{\xi}$ and $\boldsymbol{\mu}$ vanish outside X, this really amounts to proving that

$$\int_X \bar{\partial}\mathbf{q} \cdot \xi + \int_X \mathbf{q} \cdot \boldsymbol{\mu} = 0.$$

As in the previous proof, fix a point $x \in X$, let $\delta : X \to (0, +\infty)$ be the hyperbolic distance to x. According to the Stokes theorem

$$\int_{\{\delta<R\}} \bar{\partial}\mathbf{q} \cdot \boldsymbol{\xi} + \int_{\{\delta<R\}} \mathbf{q} \cdot \boldsymbol{\mu} = \int_{\{\delta=R\}} \mathbf{q} \cdot \boldsymbol{\xi}.$$

Letting R tend to $+\infty$, we have

$$\int_{\{\delta<R\}} \bar{\partial}\mathbf{q} \cdot \boldsymbol{\xi} + \int_{\{\delta<R\}} \mathbf{q} \cdot \boldsymbol{\mu} \underset{R\to+\infty}{\longrightarrow} \int_X \bar{\partial}\mathbf{q} \cdot \boldsymbol{\xi} + \int_X \mathbf{q} \cdot \boldsymbol{\mu}.$$

And as in the previous proof,

$$\int_{R=0}^{+\infty} \left| \int_{\{\delta=R\}} \mathbf{q} \cdot \boldsymbol{\xi} \right| \mathrm{d}\delta \leqslant \|\boldsymbol{\xi}\|_X \cdot \|\mathbf{q}\|_{L^1(X)} < +\infty,$$

so that the only possible limit is

$$\int_{\{\delta=R\}} \mathbf{q} \cdot \boldsymbol{\xi} \underset{R\to+\infty}{\longrightarrow} 0. \qquad \square$$

1.3. The density theorem

We will now prove the following density result.

Definition 1.21. — *If X is a compact Riemann surface and B is a subset of X, we denote by*

- *$\mathcal{Q}^1(X; \overline{B}) \subseteq \mathcal{Q}^1(X)$ the space of integrable quadratic differentials on X which are holomorphic outside $\overline{B}$ and*
- *$\mathcal{Q}^1_{\mathrm{mero}}(X; B) \subseteq \mathcal{Q}^1(X, \overline{B})$ the space of meromorphic quadratic differentials on X which are holomorphic outside B and have at worst simple poles in B.*

Theorem 1.22 (Bers–Lakic). — *Let X be a compact Riemann surface and B be a subset of X. Then $\mathcal{Q}^1_{\mathrm{mero}}(X;B)$ is dense in $\mathcal{Q}^1(X,\overline{B})$ for the L^1-norm.*

Proof. — By the Hahn–Banach Theorem, it is enough to prove that any linear form $L:\mathcal{Q}^1(X)\to\mathbb{C}$ that vanishes on $\mathcal{Q}^1_{\mathrm{mero}}(X;B)$ also vanishes on $\mathcal{Q}^1(X,\overline{B})$. According to the Riesz Representation Theorem, there exists a Beltrami differential $\boldsymbol{\mu}\in\mathrm{bel}(X)$ such that

$$L(\mathbf{q})=\int_X \mathbf{q}\cdot\boldsymbol{\mu}.$$

By assumption, $\int_X \mathbf{q}\cdot\boldsymbol{\mu}=0$ for all $\mathbf{q}\in\mathcal{Q}^1_{\mathrm{mero}}(X;B)$. We need to prove that $\int_X \mathbf{q}\cdot\boldsymbol{\mu}=0$ for all $\mathbf{q}\in\mathcal{Q}^1(X;\overline{B})$.

If B is finite, then $\mathcal{Q}^1_{\mathrm{mero}}(X;B)=\mathcal{Q}^1(X;\overline{B})$ and the result is obvious. So, without loss of generality, we may assume that B is not finite and we let

$$B_0\subset B_1\subset\ldots\subset B_n\subset\ldots\subset B$$

be an increasing sequence of finite sets such that $X\smallsetminus B_0$ is hyperbolic. Then, $X_n:=X\smallsetminus B_n$ is a hyperbolic Riemann surface and $\mathcal{Q}^1_{\mathrm{hol}}(X_n)\subset\mathcal{Q}^1_{\mathrm{mero}}(X;B)$. In particular, $\boldsymbol{\mu}$ is a trivial Beltrami differential on X_n.

According to Proposition 1.18, there exists a unique hyperbolically bounded quasiconformal vector field $\boldsymbol{\xi}_n$ on X_n such that $\bar\partial\boldsymbol{\xi}_n=\boldsymbol{\mu}$ on X_n. Note that $\boldsymbol{\xi}_n-\boldsymbol{\xi}_0$ is holomorphic on X_n and extends continuously at points in $B_n\smallsetminus B_0$, so that $\boldsymbol{\xi}_n-\boldsymbol{\xi}_0$ is holomorphic on X_0 and $\bar\partial\boldsymbol{\xi}_n=\boldsymbol{\mu}$ on X_0. In addition, $X_n\subset X_0$, thus, $\rho_{X_0}<\rho_{X_n}$ and $\boldsymbol{\xi}_n$ is hyperbolically bounded on X_0. This shows that $\boldsymbol{\xi}_n=\boldsymbol{\xi}_0$ for all $n\geqslant 0$.

Note that ρ_{X_n} tends to $\rho_{X\smallsetminus\overline{B}}$ pointwise on $X\smallsetminus\overline{B}$. Since $\rho_{X_n}(\boldsymbol{\xi}_0)$ is uniformly bounded by $4\|\boldsymbol{\mu}\|_{L^\infty(X)}$, $\boldsymbol{\xi}_0$ is hyperbolically bounded on $X\smallsetminus\overline{B}$. In addition, $\boldsymbol{\xi}_0$ vanishes at all points in B_n, thus on $\overline{B}$ by continuity. According to Proposition 1.20, $\boldsymbol{\xi}_0$ is globally quasiconformal on X and $\boldsymbol{\mu}=\bar\partial\boldsymbol{\xi}_0$ vanishes on $\overline{B}$. According to Proposition 1.19, $\boldsymbol{\mu}$ is trivial on $X\smallsetminus\overline{B}$. Thus,

$$\int_X \mathbf{q}\cdot\boldsymbol{\mu}=\int_{X\smallsetminus\overline{B}}\mathbf{q}\cdot\boldsymbol{\mu}=0. \qquad \square$$

2. Holomorphic dynamics

2.1. Orbits

Discrete dynamics is the study of the long term behavior of sequences $(x_n)_{n\geqslant 0}$ defined by induction:

$$x_0 \in X \quad \text{and} \quad x_{n+1} = f(x_n)$$

where $f : X \to X$ is a map from a set X to itself. Usually, one requires some regularity. Typically, X is a topological space and f is continuous. We shall use the notation $f^{\circ n}$ to denote the n-th iterate of f.

Definition 2.1. — *Given any point $x \in X$, we denote by $\mathcal{O}^+(x)$ the forward orbit of x, by $\mathcal{O}^-(x)$ its backward orbit, and by $\mathcal{GO}(x)$ its grand-orbit, i.e. the sets:*

$$\mathcal{O}^+(x) := \bigcup_{n\geqslant 0} \{f^{\circ n}(x)\}, \quad \mathcal{O}^-(x) := \bigcup_{n\geqslant 0} f^{-n}\{x\} \quad \textit{and} \quad \mathcal{GO}(x) := \bigcup_{y\in\mathcal{O}^+(x)} \mathcal{O}^-(y).$$

Those sets and their closures are the basic objects of study in dynamical systems. When X is compact, we may consider the ω-limit set $\omega(x)$, i.e., the set of accumulation points of the sequence $\big(f^{\circ n}(x)\big)$:

$$\omega(x) := \bigcap_{n\geqslant 0} \overline{\mathcal{O}^+\big(f^{\circ n}(x)\big)}.$$

This set is compact and invariant: $f\big(\omega(x)\big) = \omega(x)$.

Holomorphic dynamics deals with the case where X is a complex manifold and f is holomorphic. From now on, we assume that $X = \mathbb{P}^k(\mathbb{C})$ for some $k \geqslant 1$ and $f : X \to X$ is a holomorphic endomorphism. We will mainly focus on the case where f restricts to a polynomial map $f : \mathbb{C}^k \to \mathbb{C}^k$.

For some exceptional maps $f : X \to X$, there may be exceptional points $x \in X$ with a finite grand-orbit. In dimension 1, i.e. for rational maps $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, those cases are well understood:

- either there are two points with finite grand orbit
  - if the grand orbit are disjoint, up to Möbius change of coordinates, the map is $z \mapsto z^d$ with $d \geqslant 2$ and the points are 0 and ∞;
  - if the two points are in the same orbit, up to Möbius change of coordinates, the map is $z \mapsto 1/z^d$ with $d \geqslant 2$ and the points are 0 and ∞;
- or there is a single point with finite grand orbit and up to Möbius change of coordinates, the map is a polynomial and the point if ∞.

For endomorphisms $f : \mathbb{P}^k(\mathbb{C}) \to \mathbb{P}^k(\mathbb{C})$, the set of exceptional points is much less understood.

2.2. Fatou and Julia sets

Definition 2.2. — *The Fatou set of f is the largest open set $\mathcal{F}_f$ on which the family of iterates $(f^{\circ n})$ is normal. The Julia set of f is its complement $\mathcal{J}_f := X \smallsetminus \mathcal{F}_f$.*

The Fatou set and the Julia set are completely invariant:

$$f^{-1}(\mathcal{J}_f) = f(\mathcal{J}_f) = \mathcal{J}_f \quad \text{and} \quad f^{-1}(\mathcal{F}_f) = f(\mathcal{F}_f) = \mathcal{F}_f.$$

In particular, each component of the Fatou set is mapped to a component of the Fatou set. A famous result due to Sullivan asserts that in dimension $k = 1$, every Fatou component is eventually mapped to a periodic Fatou component. We shall give a proof of this result due to Adam Epstein and based on the density Theorem 1.22. We shall also prove that this result is not valid in dimension $k > 1$.

A particular case is the one of a rational map $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ which restricts to a polynomial $f : \mathbb{C} \to \mathbb{C}$. It is rather elementary to check that when the degree of the polynomial f is at least 2, $\omega(x) = \{\infty\}$ for x large enough. In fact, there is an open set of points x for which $\omega(x) = \{\infty\}$ and its complement is called the filled-in Julia set of f:

$$\mathcal{K}_f := \bigl\{x_0 \in \mathbb{C}\ ;\ \bigl(f^{\circ n}(x)\bigr)_{n \geqslant 0} \text{ is bounded}\bigr\}.$$

It follows from the Maximum modulus principle that $\mathbb{C} \setminus \mathcal{K}_f$ is connected. In particular, the component of the interior of $\mathcal{K}_f$ are simply connected. In addition, the Julia set $\mathcal{J}_f$ is the topological boundary of the filled-in Julia set $\mathcal{K}_f$. Indeed, outside $\mathcal{K}_f$, the sequence $(f^{\circ n})$ converges locally uniformly to ∞ and on the interior of $\mathcal{K}_f$, the sequence $(f^{\circ n})$ is bounded, thus normal.

The Fatou set and the Julia set are completely invariant:

$$f^{-1}(\mathcal{J}_f) = f(\mathcal{J}_f) = \mathcal{J}_f \quad \text{and} \quad f^{-1}(\mathcal{F}_f) = f(\mathcal{F}_f) = \mathcal{F}_f.$$

For rational maps $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, the following properties of the Julia set are easily derived from the Montel Theorem which asserts that a family of maps avoiding three points in $\mathbb{P}^1(\mathbb{C})$ is normal.

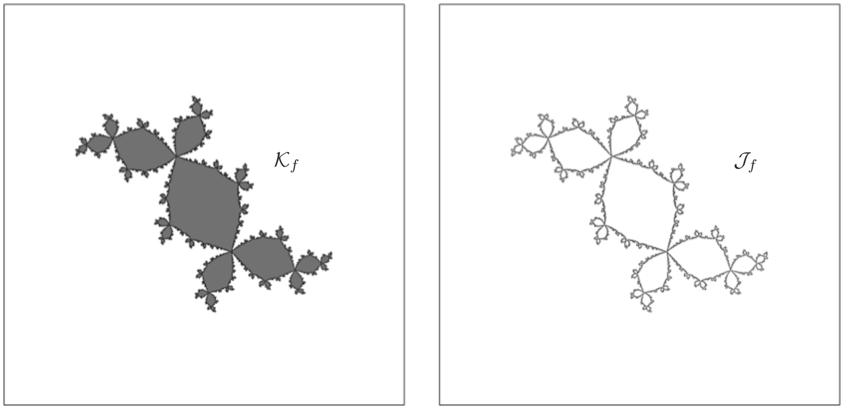

Figure 2.1. Left: the filled-in Julia set of a quadratic polynomial. It is known as the Douady Rabbit. Right: the Julia set for the same quadratic polynomial. It is the topological boundary of the filled-in Julia set.

Proposition 2.3. — *Assume $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational map. Then, the Julia set $\mathcal{J}_f$ is*

- *the smallest completely invariant set containing at least three points;*
- *contained in the closure of the set of periodic points;*
- *contained in the closure of the backward orbit of any point $x \in \mathbb{P}^1(\mathbb{C})$ which is not exceptional.*

Proof. — See [3] for example. □

2.3. Fatou components

Again, we assume that $f : \mathbb{P}^k(\mathbb{C}) \to \mathbb{P}^k(\mathbb{C})$ is a holomorphic endomorphism.

Definition 2.4. — *A Fatou component is a connected component of the Fatou set.*

Since the Fatou set is completely invariant, every Fatou component is mapped to a Fatou component.

Definition 2.5. — *A Fatou component U is eventually periodic if there are integers $n_1 > n_2 \geqslant 0$ such that $f^{\circ n_1}(U) = f^{\circ n_2}(U)$. A* wandering *Fatou component is a Fatou component which is not eventually periodic.*

In the case of 1-dimensional rational maps $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, Fatou gave a classification of periodic Fatou components: every periodic Fatou component is a (super)attracting basin, a parabolic basin, a Siegel disk or a Herman ring (see [3] for details). In the 1980's, Sullivan proved that every Fatou component of a 1-dimensional rational maps $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is eventually periodic. The next section is devoted to this result.

Note that for higher dimensional dynamics, the classification of periodic Fatou components is still not complete. While working on this classification, Misha Lyubich and Han Peters discovered an approach to proving the existence of holomorphic endomorphisms $f : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ with a wandering Fatou component. Such an example is presented in Section 2.6.

2.4. No wandering Fatou component in dimension 1

This section is devoted to the proof of the following theorem proved by Sullivan in the more general context of rational maps $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. The approach we present is due to Adam Epstein. We restrict to the case of polynomials $f : \mathbb{C} \to \mathbb{C}$ to avoid some technical details.

THEOREM 2.6 (No Wandering Domain). — *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Every Fatou component is eventually periodic.*

Proof. — The proof goes by contradiction. We assume that U is a wandering Fatou component, i.e. $f^{\circ m}(U) \neq f^{\circ n}(U)$ for all integers $m > n \geqslant 0$. If the grand orbit of a critical point c intersects U, then there are integers $m \geqslant 0$ and $n \geqslant 0$ such that $f^{\circ m}(c) \in f^{\circ n}(U)$. Replacing U by $f^{\circ(n+1)}(U)$ if necessary, we may assume that c is in an iterated preimage of U so that the forward orbit of U does not contain c. Since there are finitely many critical points, we may assume that all the critical points whose grand orbits intersect U are in iterated preimages of U. In that case, the forward orbit of U contains no critical point. Since f is a polynomial, U is simply connected and $f^{\circ k} : U \to f^{\circ k}(U)$ is an isomorphism for all $k \geqslant 1$.

Let $\mathcal{O}_f$ be the grand orbit of critical points of f. Let $\mathcal{U}$ be the grand orbit of U. The grand orbit of any point $z \in \mathcal{U}$ intersects U in a single point. In particular, the map $F : \mathcal{U} \to U$ which sends a point $z \in \mathcal{U}$ to the unique point of the grand orbit of z in U restricts to a covering map $F : \mathcal{U} \smallsetminus \mathcal{O}_f \to U \smallsetminus \mathcal{O}_f$. Indeed, for each connected component V of $\mathcal{U}$, there are integers m, n such that $f^{\circ m}(V) = f^{\circ n}(U) =: W$. Then, $f^{\circ n} : U \smallsetminus \mathcal{O}_f \to W \smallsetminus \mathcal{O}_f$ is an isomorphism, $f^{\circ m} : V \smallsetminus \mathcal{O}_f \to W \smallsetminus \mathcal{O}_f$ is a covering map, and

$$F|_{V \smallsetminus \mathcal{O}_f} = (f^{\circ n}|_{U \smallsetminus \mathcal{O}_f})^{-1} \circ f^{\circ m}|_{V \smallsetminus \mathcal{O}_f}.$$

Let $(B_n)_{n\geqslant 0}$ be the sequence of finite sets defined recursively by

$$B_0 := U \cap \mathcal{O}_f \quad \text{and} \quad B_{n+1} := f^{-1}(B_n) \cup B_n = f^{-1}(B_n) \cup B_0.$$

The sequence (B_n) is increasing. We set $B := \bigcup_{n\geqslant 0} B_n$. According to Proposition 2.3, the Julia set $\mathcal{J}_f$ is contained in $\overline{B}$, so that $U \smallsetminus \mathcal{O}_f$ is a connected component of $\widehat{\mathbb{C}} \smallsetminus \overline{B}$.

Recall that a quadratic differential $\mathbf{q}$ belongs to $\mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B})$ if it is integrable on $\widehat{\mathbb{C}}$ and holomorphic outside $\overline{B}$, in particular on $\mathcal{U} \smallsetminus \mathcal{O}_f$. Thus, there is a well defined push forward operator

$$F_* : \mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B}) \to \mathcal{Q}^1_{\mathrm{hol}}(U \smallsetminus \mathcal{O}_f)$$

defined by first restricting to $\mathcal{U} \smallsetminus \mathcal{O}_f$, then pushing forward via the covering map $F : \mathcal{U} \smallsetminus \mathcal{O}_f \to U \smallsetminus \mathcal{O}_f$.

Lemma 2.7. — *The linear map $F_* : \mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B}) \to \mathcal{Q}^1_{\mathrm{hol}}(U \smallsetminus \mathcal{O}_f)$ is continuous and surjective.*

Proof. — Restricting $\mathbf{q} \in \mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B})$ to $\mathcal{U} \smallsetminus \mathcal{O}_f$ does not increase the norm, and pushing forward by the covering map $F : \mathcal{U} \smallsetminus \mathcal{O}_f \to U \smallsetminus \mathcal{O}_f$ does not increase the norm either. Thus, $\|F_*\| \leqslant 1$ and $F_* : \mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B}) \to \mathcal{Q}^1_{\mathrm{hol}}(U \smallsetminus \mathcal{O}_f)$ is continuous.

Since $U \smallsetminus \mathcal{O}_f$ is a connected component of $\widehat{\mathbb{C}} \smallsetminus \overline{B}$, any $\mathbf{q} \in \mathcal{Q}^1_{\mathrm{hol}}(U \smallsetminus \mathcal{O}_f)$ can be extended by 0 to a quadratic differential $\hat{\mathbf{q}} \in \mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B})$. Then, $F_* \hat{\mathbf{q}} = \mathbf{q}$. This proves that $F_* : \mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B}) \to \mathcal{Q}^1_{\mathrm{hol}}(U \smallsetminus \mathcal{O}_f)$ is surjective. $\square$

We will prove that $F_*\big(\mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B})\big)$ has finite dimension, which yields a contradiction since $\mathcal{Q}^1_{\mathrm{hol}}(U) \subseteq \mathcal{Q}^1_{\mathrm{hol}}(U \smallsetminus \mathcal{O}_f)$ is isomorphic to $\mathcal{Q}^1_{\mathrm{hol}}(\mathbb{D})$ which has infinite dimension.

Lemma 2.8. — *The dimension of $F_*\big(\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; B_n)\big)$ is at most* $\mathrm{card}(B_0)$.

Proof. — The sequence of sets B_n is increasing. It follows that the sequences of spaces $\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; B_n)$ and $F_*\big(\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; B_n)\big)$ are increasing. Without loss of generality, we may choose n sufficiently large so that the set $\mathcal{V}_f$ of critical values of f is contained in B_n. Setting $A_n := f^{-1}(B_n)$, the map $f : \widehat{\mathbb{C}} \smallsetminus A_n \to \widehat{\mathbb{C}} \smallsetminus B_n$ is a covering map. It induces a linear map $f_* : \mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; A_n) \to \mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; B_n)$.

The linear map

$$\nabla_f := \mathrm{id} - f_* : \mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; A_n) \to \mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; A_n \cup B_n)$$

is injective. Indeed, let Δ be a disk centered at 0 with sufficiently large radius so that $f^{-1}(\Delta) \Subset \Delta$ and observe that for $\mathbf{q} \in \mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; A_n)$, we have

$$\int_\Delta |f_*\mathbf{q}| \leqslant \int_{f^{-1}(\Delta)} |q| < \int_\Delta |q|,$$

so that $f_*\mathbf{q} \neq \mathbf{q}$.

By construction of the sequence (B_n), we have $A_n \cup B_n = B_{n+1} = A_n \cup B_0$. We have that

$$\dim\bigl(\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; A_n)\bigr) = \mathrm{card}(A_n) - 3$$

and

$$\dim\bigl(\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; A_n \cup B_n)\bigr) = \mathrm{card}(B_{n+1}) - 3 \leqslant \mathrm{card}(A_n) - 3 + \mathrm{card}(B_0).$$

It follows that the codimension of $\nabla_f\bigl(\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; A_n)\bigr)$ in $\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; B_{n+1})$ is at most $\mathrm{card}(B_0)$. In addition, $F \circ f = F$, thus

$$F_*\bigl(\nabla_f(\mathbf{q})\bigr) = F_*(\mathbf{q}) - F_*(f_*\mathbf{q}) = 0.$$

and $\nabla_f\bigl(\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; A_n)\bigr)$ is contained in the kernel of F_*. As a consequence,

$$\dim F_*\bigl(\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; B_{n+1})\bigr) \leqslant \mathrm{card}(B_0). \qquad \square$$

To prove that $F_*\bigl(\mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B})\bigr)$ has finite dimension, we now proceed as follows. The sequence of spaces $F_*\bigl(\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; B_n)\bigr)$ is increasing. According to the previous lemma, they are all contained in a subspace $\mathcal{Q}$ of $\mathcal{Q}^1_{\mathrm{hol}}(\mathcal{U} \smallsetminus \mathcal{O}_f)$ of dimension at most $\mathrm{card}(B_0)$. According to Theorem 1.22,

$$\mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; B) := \bigcup_{n \geqslant 0} \mathcal{Q}^1_{\mathrm{mero}}(\widehat{\mathbb{C}}; B_n)$$

is dense in $\mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B})$ for the L^1-norm. Since $F_* : \mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B}) \to \mathcal{Q}^1_{\mathrm{hol}}(U \smallsetminus \mathcal{O}_f)$ is continuous, we deduce that $F_*\bigl(\mathcal{Q}^1(\widehat{\mathbb{C}}; \overline{B})\bigr) \subseteq \mathcal{Q}$, thus has dimension at most $\mathrm{card}(B_0)$. $\square$

2.5. Parabolic implosion in dimension 1

In this section, we recall the main ingredients in the proof that the Julia set $\mathcal{J}_f$ does not depend continuously on f (for the Hausdorff topology on the space of compact subsets of $\mathbb{C}$). More precisely, if (J_n) is a sequence of compact subsets of $\mathbb{C}$, then

$$\limsup J_n := \bigcap_m \overline{\bigcup_{n \geqslant m} J_n}.$$

We then have the following result (see [2]).

Theorem 2.9. — *Assume $f : \mathbb{C} \to \mathbb{C}$ is a polynomial fixing 0 with $f(z) = z + z^2 + \mathrm{O}(z^3)$. Then,*

$$\mathcal{J}_f \subsetneq \limsup_{\delta \to 0} \mathcal{J}_{f+\delta}.$$

The proof is based on the description of limits of iterates of $f+\delta$ in terms of Fatou coordinates. Those limits are called Lavaurs maps.

2.5.1. Parabolic basin

In the remainder of Section 2.5, we assume that $f : \mathbb{C} \to \mathbb{C}$ is a polynomial whose expansion near 0 is of the form

$$f(z) = z + z^2 + az^3 + \mathrm{O}(z^4) \quad \text{with} \quad a \in \mathbb{C}.$$

To understand the local dynamics of f near 0, it is convenient to consider the change of coordinates $Z := -1/z$. In the Z-coordinate, the expression of f becomes

$$F(Z) = Z + 1 + \frac{b}{Z} + \mathrm{O}\left(\frac{1}{Z^2}\right) \quad \text{with} \quad b := 1 - a.$$

In particular, if $R > 0$ is large enough, F maps the right half-plane $\{\mathrm{Re}(Z) > R\}$ into itself, so that if $r > 0$ is close enough to 0, f maps the disk $D(-r, r)$ into itself. In addition, the orbit under f of any point $z \in D(-r, r)$ converges to 0 tangentially to the real axis. Similarly, if $r > 0$ is sufficiently close to 0, there is a branch of f^{-1} which maps the disk $D(r, r)$ into itself and the orbit under that branch of f^{-1} of any point $z \in D(r, r)$ converges to 0 tangentially to the real axis.

Definition 2.10. — *The basin $\mathcal{B}_f$ is the open set of points whose orbit under iteration of f intersects the disks $D(-r, r)$ for all $r > 0$.*

2.5.2. Fatou coordinates

In order to understand further the local dynamics of f near 0, it is customary to use local attracting and repelling Fatou coordinates. In the case of a polynomial, those Fatou coordinates have global properties.

Proposition 2.11. — *There exists a (unique)* attracting Fatou coordinate *$\Phi_f : \mathcal{B}_f \to \mathbb{C}$ which semi-conjugates $f : \mathcal{B}_f \to \mathcal{B}_f$ to the translation $T_1 : \mathbb{C} \ni Z \mapsto Z + 1 \in \mathbb{C}$:*

$$\Phi_f \circ f = T_1 \circ \Phi_f.$$

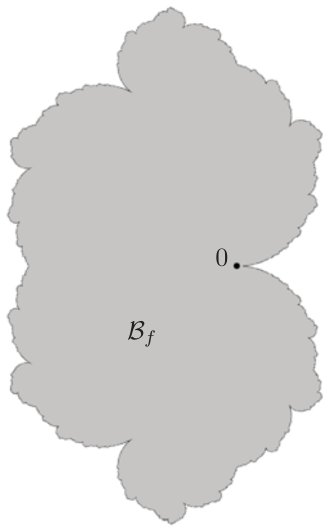

Figure 2.2. For the cubic polynomial $f(z) = z + z^2 + 0.95z^3$, the basin $\mathcal{B}_f$ (grey) is the interior of the filled-in Julia set $\mathcal{K}_f$.

and satisfies the normalization:

$$\Phi_f(z) = -\frac{1}{z} - b\log\left(-\frac{1}{z}\right) + \mathrm{o}(1) \quad \textit{as} \quad \mathrm{Re}\left(-\frac{1}{z}\right) \to +\infty.$$

Proof. — For $z \in \mathcal{B}_f$, set

$$Z := -\frac{1}{z}, \quad Z_n := -\frac{1}{f^{\circ n}(z)} \quad \text{and} \quad \varPhi_n(z) := Z_n - n - b\log Z_n.$$

We have that

$$\mathrm{Re}(Z_n) \to +\infty \quad \text{and} \quad Z_{n+1} = Z_n + 1 + \frac{b}{Z_n} + \mathrm{O}\left(\frac{1}{Z_n^2}\right),$$

so that

$$\frac{1}{Z_n} = \mathrm{O}\left(\frac{1}{Z+n}\right) \quad \text{as} \quad n \to +\infty.$$

As a consequence,

$$\varPhi_{n+1}(z) - \varPhi_n(z) = Z_{n+1} - Z_n - 1 - b\log\frac{Z_{n+1}}{Z_n} = \mathrm{O}\left(\frac{1}{Z_n^2}\right) = \mathrm{O}\left(\frac{1}{(Z+n)^2}\right).$$

So, the sequence $(\varPhi_n)$ converges to a limit $\Phi_f : \mathcal{B}_f \to \mathbb{C}$ which satisfies

$$\Phi_f(z) = \varPhi_0(z) + \mathrm{O}\left(\frac{1}{Z}\right) = -\frac{1}{z} - b\log\left(-\frac{1}{z}\right) + \mathrm{o}(1) \quad \text{as} \quad \mathrm{Re}\left(-\frac{1}{z}\right) \to +\infty.$$

Passing to the limit on the relation $\Phi_n \circ f = T_1 \circ \Phi_{n+1}$ yields the required result. $\square$

Figure 2.3 gives a rough idea of the behavior of the Fatou coordinate Φ_f for the cubic polynomial $f(z) := z + z^2 + 0.95z^3$. The basin $\mathcal{B}_f$ contains the two critical points of f. Those points and their iterated preimages form the critical points of Φ_f. Denote by c^+ the critical point with positive imaginary part and by c^- its complex conjugate. Set $v^\pm := \Phi_f(c^\pm)$. Points $z \in \mathcal{B}_f$ are colored according to the location $\Phi_f(z)$:

- dark grey when $\mathrm{Im}\big(\Phi_f(z)\big) < \mathrm{Im}(v^-)$,
- light grey when $\mathrm{Im}(v^-) < \mathrm{Im}\big(\Phi_f(z)\big) < \mathrm{Im}(v^+)$ and
- medium grey when $\mathrm{Im}(v^+) < \mathrm{Im}\big(\Phi_f(z)\big)$.

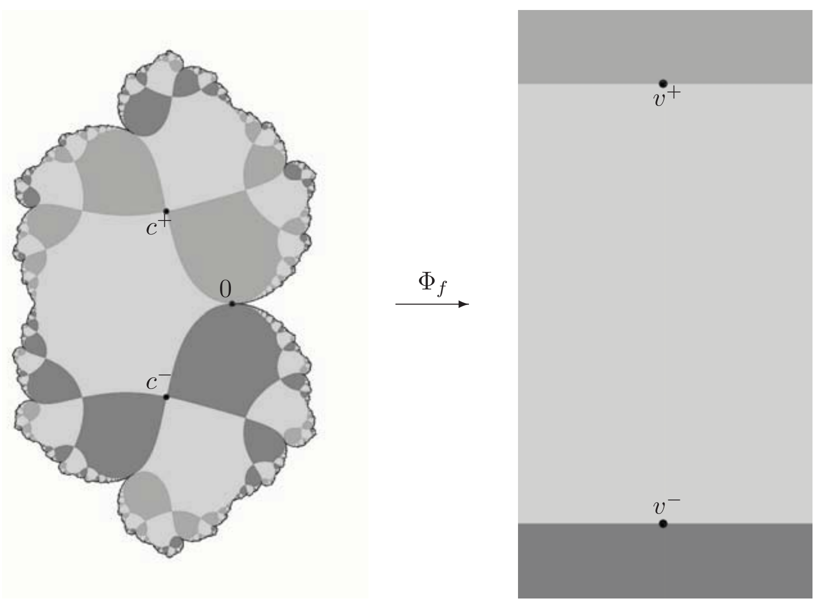

Figure 2.3. Behavior of Φ_f for $f(z) := z + z^2 + 0.95z^3$. Right: three regions delimited by the horizontal lines passing through the critical values of Φ_f. Left: the basin $\mathcal{B}_f$ is tiled by the preimages of those three regions by Φ_f.

Proposition 2.12. — *There exists a (unique)* repelling Fatou parametrization $\Psi_f : \mathbb{C} \to \mathbb{C}$ *which semi-conjugates* $T_1 : \mathbb{C} \to \mathbb{C}$ *to* $f : \mathbb{C} \to \mathbb{C}$*:*

$$\Psi_f \circ T_1 = f \circ \Psi_f.$$

and satisfies the normalization:

$$\Psi_f(Z) = -\frac{1}{Z + b\log(-Z) + \mathrm{o}(1)} \quad \textit{as} \quad \mathrm{Re}(Z) \to -\infty.$$

Proof. — Choose $r > 0$ sufficiently close to 0 so that there is a branch g of f^{-1} which maps the disk $D(r,r)$ into itself. For $z \in D(r,r)$, set

$$Z := -\frac{1}{z}, \quad Z_n := -\frac{1}{g^{\circ n}(z)} \quad \text{and} \quad \mathit{\Phi}_n(z) := Z_n + n - b\log(-Z_n).$$

Note that

$$Z_n = Z_{n+1} + 1 + \frac{b}{Z_{n+1}} + \mathrm{O}\left(\frac{1}{Z_{n+1}^2}\right).$$

So, as in the previous proof

$$\begin{aligned}\mathit{\Phi}_{n+1}(z) - \mathit{\Phi}_n(z) &= Z_{n+1} - Z_n + 1 + b\log\frac{Z_{n+1}}{Z_n}\\ &= \mathrm{O}\left(\frac{1}{Z_{n+1}^2}\right) = \mathrm{O}\left(\frac{1}{(Z-n)^2}\right).\end{aligned}$$

The sequence $\mathit{\Phi}_n$ converges in the left half-plane $\{\mathrm{Re}(Z) < -1/(2r)\}$ to a limit Φ_g which satisfies

$$\Phi_g(z) = \mathit{\Phi}_0(z) + \mathrm{O}\left(\frac{1}{Z}\right) = Z - b\log(-Z) + \mathrm{o}(1) \quad \text{as} \quad \mathrm{Re}\left(-\frac{1}{z}\right) \to -\infty.$$

Passing to the limit on the equation $\mathit{\Phi}_{n+1} \circ f = T_1 \circ \Phi_n$ shows that Φ_g conjugates f to T_1. The inverse Ψ_f of Φ_g conjugates T_1 to f and

$$\begin{aligned}Z = \Phi_g \circ \Psi_f(Z) &= -\frac{1}{\Psi_f(Z)} - b\log\left(-\frac{1}{\Psi_f(Z)}\right) + \mathrm{o}(1)\\ &= -\frac{1}{\Psi_f(Z)} - b\log(-Z) + \mathrm{o}(1)\end{aligned}$$

as $\mathrm{Re}(Z) \to -\infty$. □

2.5.3. Lavaurs maps

Definition 2.13. — *The Lavaurs map $\mathcal{L}_f : \mathcal{B}_f \to \mathbb{C}$ is the map*

$$\mathcal{L}_f := \Psi_f \circ \Phi_f : \mathcal{B}_f \to \mathbb{C}.$$

Note that the Lavaurs map $\mathcal{L}_f$ commutes with f. Indeed, $\Phi_f \circ f = T_1 \circ \Phi_f$ and $\Psi_f \circ T_1 = f \circ \Psi_f$, so that

$$\mathcal{L}_f \circ f = \Psi_f \circ \Phi_f \circ f = \Psi_f \circ T_1 \circ \Phi_f = f \circ \Psi_f \circ \Phi_f = f \circ \mathcal{L}_f.$$

Figure 2.4 gives a rough idea of the behavior of the Lavaurs map $\mathcal{L}_f$ for the cubic polynomial $f(z) := z + z^2 + 0.95z^3$. Points in the basin $\mathcal{B}_f$ are colored according to the location of their image by $\mathcal{L}_f$:

- grey when $\mathcal{L}_f(z) \in \mathcal{B}_f$,
- white otherwise.

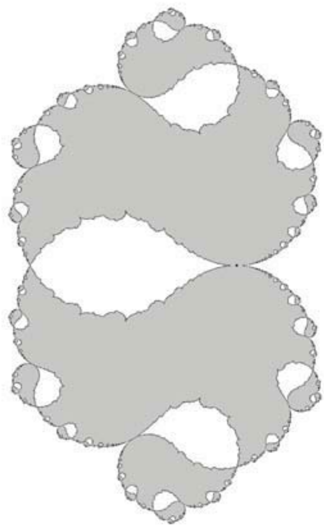

Figure 2.4. Behavior of $\mathcal{L}_f$ for $f(z) := z + z^2 + 0.95z^3$. The set of points $z \in \mathcal{B}_f$ whose image by $\mathcal{L}_f$ remains in $\mathcal{B}_f$ is colored in grey. The restriction of $\mathcal{L}_f$ to each bounded white domain is a covering of $\mathbb{C} \setminus \overline{\mathcal{B}_f}$

2.5.4. Discontinuity of the Julia set

Set

$$\mathcal{K}(\mathcal{L}_f) := \bigcap_{n \geqslant 0} \mathcal{L}_f^{-n}(\mathcal{K}_f) \quad \text{and} \quad \mathcal{J}(\mathcal{L}_f) := \partial \mathcal{K}(\mathcal{L}_f).$$

Figure 2.5 shows $\mathcal{K}(\mathcal{L}_f)$ for the cubic polynomial $f(z) = z + z^2 + 0.95z^3$.

The following result may be considered as the main reason why Lavaurs maps are studied in holomorphic dynamics.

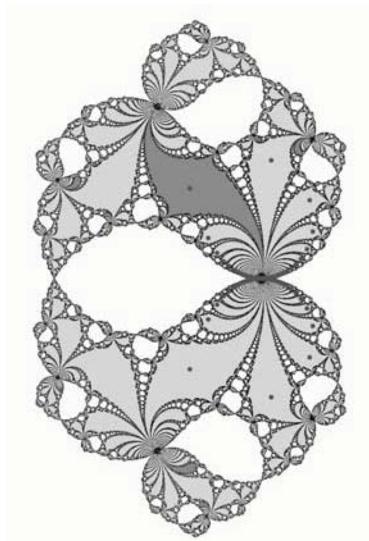

Figure 2.5. The set $\mathcal{K}(\mathcal{L}_f)$ for $f(z) = z + z^2 + 0.95z^3$. The Lavaurs map $\mathcal{L}_f$ has two complex conjugate sets of attracting fixed points. The fixed points of $\mathcal{L}_f$ are indicated and their basins of attraction are colored (dark grey for one of the fixed points, and light grey for the others). Those basins form the interior of $\mathcal{K}(\mathcal{L}_f)$. The black set $\mathcal{J}(\mathcal{L}_f)$ is the topological boundary of $\mathcal{K}(\mathcal{L}_f)$.

Proposition 2.14 (Lavaurs). — *Assume $f : \mathbb{C} \to \mathbb{C}$ is a polynomial whose expansion at 0 is $f(z) = z + z^2 + \mathrm{O}(z^3)$. Let (N_n) be a sequence of integers tending to $+\infty$ and (ε_n) be a sequence of complex numbers tending to 0, such that*

$$N_n - \frac{\pi}{\varepsilon_n} \to 0.$$

Then,

$$(f + \varepsilon_n^2)^{\circ N_n} \to \mathcal{L}_f \quad \text{locally uniformly on } \mathcal{B}_f.$$

In addition,

$$\mathcal{J}_f \subsetneq \mathcal{J}(\mathcal{L}_f) \subseteq \liminf \mathcal{J}_{f+\varepsilon_n^2} \quad \text{and} \quad \limsup \mathcal{K}_{f+\varepsilon_n^2} \subseteq \mathcal{K}(\mathcal{L}_f) \subsetneq \mathcal{K}_f.$$

Proof. — See [2] or [4]. □

We do not present the proof of this result which is rather technical. Instead, we will show that the result holds for the Möbius transformation

$$g(z) = \frac{z}{1-z} = z + z^2 + \mathrm{O}(z^3).$$

In that case, all maps involved are Möbius transformations and the computations are explicit. If we perform the change of coordinates $Z = -1/z$, the Möbius transformation g gets conjugated to T_1. Thus,

$$\Phi_g(z) = -\frac{1}{z} \quad \text{and} \quad \Psi_g(Z) = -\frac{1}{Z}, \quad \text{so that} \quad \mathcal{L}_g = \mathrm{id}\,.$$

Note that $g + \varepsilon^2$ is also a Möbius transformation. It has two fixed points

$$\alpha^{\pm} = \pm \mathrm{i}\varepsilon + \mathrm{O}(\varepsilon^2) \quad \text{with multipliers} \quad \lambda^{\pm} = \exp\bigl(\pm 2\mathrm{i}\varepsilon + \mathrm{O}(\varepsilon^3)\bigr).$$

So, if $N \to +\infty$ and

$$N - \frac{\pi}{\varepsilon} \to 0, \quad \text{so that} \quad \varepsilon = \frac{\pi}{N + \mathrm{o}(1)} = \frac{\pi}{N} + \mathrm{o}\left(\frac{1}{N^2}\right),$$

then $(g + \varepsilon^2)^{\circ N}$ is a Möbius transformation fixing $\alpha^{\pm}$ with multipliers

$$\mu^{\pm} := (\lambda^{\pm})^N = \exp\left(\pm 2\pi\mathrm{i} + \mathrm{o}(1/N)\right) = 1 + \mathrm{o}\left(\frac{1}{N}\right).$$

This Möbius transformation is

$$z \mapsto \alpha^+ + \frac{\mu^+ \cdot (z - \alpha^+)}{1 - \dfrac{\mu^+ - 1}{\alpha^+ - \alpha^-}(z - \alpha^+)}.$$

Since

$$\mu^+ - 1 = \mathrm{o}\left(\frac{1}{N}\right) = \mathrm{o}(\alpha^+ - \alpha^-),$$

we see that indeed,

$$(g + \varepsilon^2)^{\circ N} \underset{N \to +\infty}{\longrightarrow} \mathcal{L}_g.$$

2.5.5. Lavaurs maps with an attracting fixed point

Proposition 2.15. — *Consider the cubic polynomial $f : \mathbb{C} \to \mathbb{C}$ defined by*

$$f(z) := z + z^2 + az^3 \quad \textit{with} \quad a \in \mathbb{C}.$$

If $r > 0$ is sufficiently close to 0 and $a \in D(1 - r, r)$, then the Lavaurs map $\mathcal{L}_f : \mathcal{B}_f \to \mathbb{C}$ admits an attracting fixed point.

Numerical experiments suggest that the value $a = 0.95$ works (see Figure 2.5).

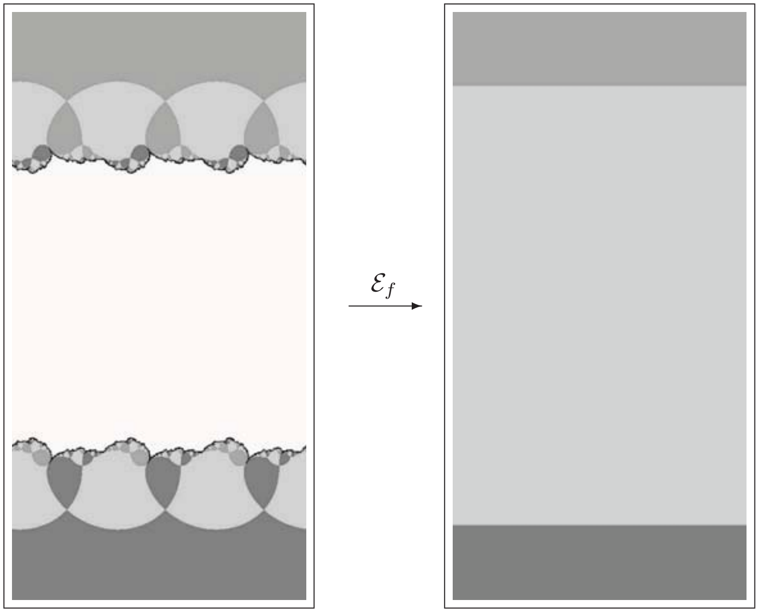

Figure 2.6. Behavior of the map $\mathcal{E}_f$ for $f(z) = z + z^2 + 0.95z^3$. The domain $\mathcal{U}_f$ has two connected components, one containing an upper half-plane and the other containing a lower half-plane. The domain is tiled according to the behavior of $\mathcal{E}_f$. The restriction of Ef to each tile is univalent. The image of medium grey tiles is the medium grey upper half-plane on the right. The image of dark grey tiles is the dark grey lower half-plane on the right. The image of light grey tiles is the horizontal light grey strip on the right.

Proof. — We outline the proof; see [1] for details. Set

$$\mathcal{U}_f := \Psi_f^{-1}(\mathcal{B}_f) \quad \text{and} \quad \mathcal{E}_f := \Phi_f \circ \Psi_f : \mathcal{U}_f \to \mathbb{C}.$$

The open set $\mathcal{U}_f$ contains an upper half-plane and a lower half-plane. Figure 2.6 shows the domain $\mathcal{U}_f$, for $f(z) = z+z^2+0.95z^3$, tiled by the preimages by $\mathcal{E}_f$ of three regions defined as in Figure 2.3.

The open set $\mathcal{U}_f$ is invariant by T_1 and the map $\mathcal{E}_f$ commutes with T_1. So, $\mathcal{E}_f - \mathrm{id}$ is periodic of period 1 and has a Fourier expansion which holds in a upper half-plane:

$$\mathcal{E}_f(Z) = Z + \sum_{k\geqslant 0} c_k \mathrm{e}^{2\pi \mathrm{i} kZ}.$$

An elementary computation using the expansion of Φ_f and Ψ_f near infinity yields that $c_0 = -\pi\mathrm{i}(1-a)$:

$$\begin{aligned}\mathcal{E}_f(Z) &= \Phi_f \circ \Psi_f(Z)\\ &= Z + (1-a)\log(-Z) + \mathrm{o}(1)\\ &\qquad - (1-a)\log\big(Z + (1-a)\log(-Z) + \mathrm{o}(1)\big) + \mathrm{o}(1)\\ &= Z + (1-a)\log(Z) - \pi\mathrm{i}(1-a) - (1-a)\log(Z) + \mathrm{o}(1)\\ &= Z - \pi\mathrm{i}(1-a) + \mathrm{o}(1).\end{aligned}$$

A more elaborate argument, based on the notion of finite type analytic map introduced by Adam Epstein, shows that:

$$\mathcal{E}_f(Z) = Z - \pi\mathrm{i}(1-a) + c_1\mathrm{e}^{2\pi\mathrm{i}Z} + \mathrm{o}(\mathrm{e}^{2\pi\mathrm{i}Z}) \quad \text{with} \quad c_1 \neq 0.$$

It follows that when $a \neq 1$ is close to 1, $\mathcal{E}_f$ has a fixed point Z_f with multiplier ρ_f satisfying

$$c_1\mathrm{e}^{2\pi\mathrm{i}Z} \sim \pi\mathrm{i}(1-a) \quad \text{and} \quad \rho_f - 1 \sim 2\pi\mathrm{i}c_1\mathrm{e}^{2\pi\mathrm{i}Z_f} \sim -2\pi^2(1-a) \quad \text{as} \quad a \to 1.$$

It follows easily that when $r > 0$ is sufficiently close to 0 and $a \in D(1-r, 1)$, the multiplier ρ_f belongs to the unit disk and Z_f is an attracting fixed point of $\mathcal{E}_f$.

Note that $\Psi_f : \mathcal{U}_f \to \mathcal{B}_f$ semi-conjugates $\mathcal{E}_f$ to $\mathcal{L}_f$. So, when Z_f is an attracting fixed point of $\mathcal{E}_f$, the point $\Psi_f(Z_f)$ is an attracting fixed point of $\mathcal{L}_f$. $\square$

2.6. Wandering Fatou components in dimension 2

We will now complete these notes by a sketch of a joint work with M. Astorg, R. Dujardin, H. Peters and J. Raissy. In [1], we prove that there exist polynomial maps $F : \mathbb{C}^2 \to \mathbb{C}^2$ having a wandering Fatou component. The approach was initially suggested by Misha Lyubich and Han Peters.

2.6.1. Main result

Theorem 2.16. — *Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be polynomials of the form*

$$f(z) = z + z^2 + \mathrm{O}(z^3) \quad \textit{and} \quad g(w) = w - w^2 + \mathrm{O}(w^3). \tag{2.1}$$

If the Lavaurs map $\mathcal{L}_f : \mathcal{B}_f \to \mathbb{C}$ has an attracting fixed point, then the skew-product $F : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$F(z,w) := \left(f(z) + \frac{\pi^2}{4}w, g(w)\right) \tag{2.2}$$

admits a wandering Fatou component.

Note that if f and g have the same degree, F extends to an endomorphism of $\mathbb{P}^2(\mathbb{C})$. As a corollary of this theorem and Proposition 2.15, we derive the existence of polynomial skew-products with a wandering Fatou component.

Corollary 2.17. — *If $r > 0$ is sufficiently small and $a \in D(1-r, r)$, then the polynomial skew-product*

$$F(z,w) := \left(z + z^2 + az^3 + \frac{\pi^2}{4}w, w - w^2 + w^3\right)$$

admits a wandering Fatou component.

2.6.2. An analog of the Lavaurs estimates in the case of skew products

Let $\mathcal{B}_f$ and $\mathcal{B}_g$ be the parabolic basins of 0 under iteration of respectively f and g. We want to choose $(z_0, w_0) \in \mathcal{B}_f \times \mathcal{B}_g$ so that the first coordinate of $F^{\circ m}(z_0, w_0)$ returns close to the attracting fixed point of $\mathcal{L}_f$ infinitely many times. The proof is designed so that the return times are the integers n^2 for $n \geqslant n_0$. So, we have to analyze the orbit segment between n^2 and $(n+1)^2$, which is of length $2n+1$. The main step for proving Theorem 2.16 is the following proposition which is an analog of Proposition 2.14 in the setting of skew-products.

Proposition 2.18. — *As $n \to +\infty$, the sequence of maps*

$$\mathbb{C}^2 \ni (z,w) \mapsto F^{\circ 2n+1}\big(z, g^{\circ n^2}(w)\big) \in \mathbb{C}^2$$

converges locally uniformly in $\mathcal{B}_f \times \mathcal{B}_g$ to the map

$$\mathcal{B}_f \times \mathcal{B}_g \ni (z,w) \mapsto \big(\mathcal{L}_f(z), 0\big) \in \mathbb{C} \times \{0\}.$$

Proof. — We outline the proof; see [1] for details. If $w \in \mathcal{B}_g$, then for large m,

$$g^{\circ m}(w) \simeq \frac{1}{m}.$$

We want to analyze the behavior of F starting at $\big(z, g^{\circ n^2}(w)\big)$ during $2n+1$ iterates. For large n, the first coordinate of F along this orbit segment is approximately

$$f(z) + \varepsilon^2 \quad \text{with} \quad \varepsilon^2 \simeq \frac{\pi^2}{4n^2}, \quad \text{so that} \quad \frac{\pi}{\varepsilon} \simeq 2n.$$

Proposition 2.14 asserts that if $\pi/\varepsilon = 2n$, then for large n, the $(2n)^{\text{th}}$ iterate of $f + \varepsilon^2$ is approximately equal to $\mathcal{L}_f$ on $\mathcal{B}_f$.

Our setting is slightly different since ε keeps decreasing along the orbit. Indeed on the first coordinate we are taking the composition of $2n+1$ transformations

$$(f+\varepsilon_k^2)_{1\leqslant k\leqslant 2n+1} \quad \text{with} \quad \varepsilon_k^2 \simeq \frac{\pi^2}{4(n^2+k)}, \quad \text{so that} \quad \frac{\pi}{\varepsilon_k} \simeq 2n+\frac{k}{n}.$$

The main step of the proof of the proposition consists in a detailed analysis of this perturbed situation, proving that the decay of ε_k is counterbalanced by taking *exactly* one additional iterate of F. $\square$

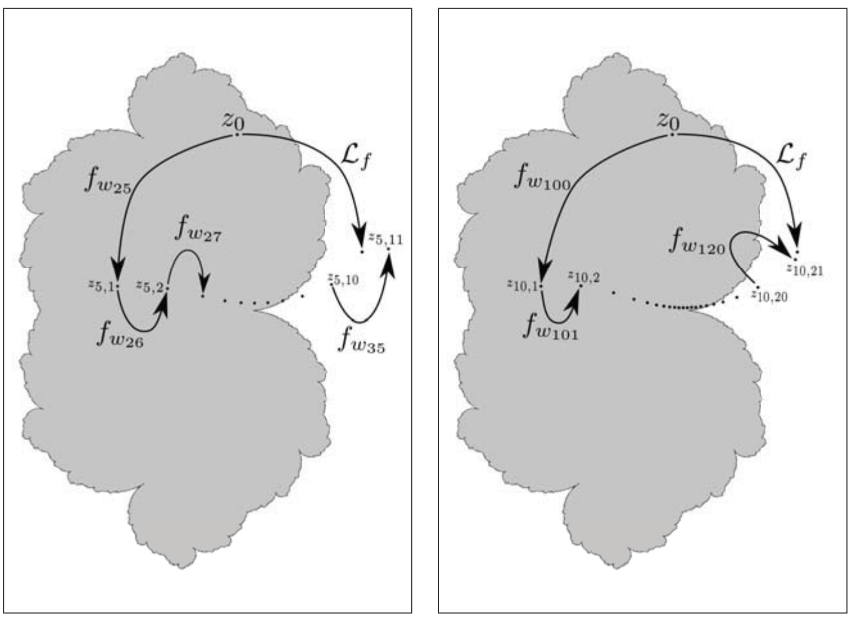

Figure 2.7. Illustration of Proposition 2.18 for $f(z)=z+z^2+0.95z^3$ and $g(w)=w-w^2$. The parabolic basin $\mathcal{B}_f$ is colored in grey. It is invariant under f, but not under $f_w := f+\frac{\pi^2}{4}w$ for $w\neq 0$. The Lavaurs map $\mathcal{L}_f$ is defined on $\mathcal{B}_f$. The point $z_0=-0.05+0.9\mathrm{i}$ and its image $\mathcal{L}_f(z_0)$ are indicated. The other points are the points $z_{n,k}$ which are defined by $F^{\circ k}(z_0,w_{n^2})=(z_{n,k},w_{n^2+k})$ for $1\leqslant k\leqslant 2n+1$ and $w_m=g^{\circ m}(1/2)$. If n is large enough, the point $z_{n,2n+1}$ is close to $\mathcal{L}_f(z_0)$. Left: $n=5$, there are 11 points of the orbit. Right: $n=10$, there are 21 points of the orbit.

2.6.3. Proof of Theorem 2.16

Let $\xi \in \mathcal{B}_f$ be an attracting fixed point of the Lavaurs map $\mathcal{L}_f$. Let V be a disk centered at ξ, chosen that $\mathcal{L}_f(V)$ is compactly contained in V. It follows that $\mathcal{L}_f^{\circ k}(V)$ converges to ξ as $k \to +\infty$. Let also $W \Subset \mathcal{B}_g$ be an arbitrary disk.

Denote by $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ the first coordinate projection, that is $\pi_1(z,w) := z$. According to Proposition 2.18, there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$,

$$\pi_1 \circ F^{\circ(2n+1)}(V \times g^{\circ n^2}(W)) \Subset V.$$

Let U be a connected component of the open set $F^{-n_0^2}\big(V \times g^{\circ n_0^2}(W)\big)$.

Lemma 2.19. — *The sequence $(F^{\circ n^2})_{n \geqslant 0}$ converges locally uniformly to $(\xi, 0)$ on U.*

Proof. — An easy induction shows that for every integer $n \geqslant n_0$,

$$F^{\circ n^2}(U) \subseteq V \times g^{\circ n^2}(W). \tag{2.3}$$

Indeed this holds by assumption for $n = n_0$. Now if the inclusion is true for some $n \geqslant n_0$, then

$$\begin{aligned}\pi_1 \circ F^{\circ(n+1)^2}(U) = \pi_1 \circ F^{\circ(2n+1)} &\Big(F^{\circ n^2}(U)\Big)\\ &\subset \pi_1 \circ F^{\circ(2n+1)}\Big(V \times g^{\circ n^2}(W)\Big) \subset V,\end{aligned}$$

from which (2.3) follows.

From this we get that the sequence $(F^{\circ n^2})_{n \geqslant 0}$ is uniformly bounded, hence normal, on U. Also, any cluster value of this sequence of maps is constant and of the form $(z, 0)$ for some $z \in V$. In addition, $(z, 0)$ is a limit value (associated to a subsequence (n_k)) if and only if $\big(\mathcal{L}_f(z), 0\big)$ is a limit value (associated to the subsequence $(1 + n_k)$). We infer that the set of cluster limits is totally invariant under $\mathcal{L}_f : V \to V$, therefore it must be reduced to the attracting fixed point ξ of $\mathcal{L}_f$, and we are done. $\square$

Corollary 2.20. — *The domain U is contained in the Fatou set of F.*

Proof. — It is well-known in our context that the sequence $(F^{\circ m})_{m \geqslant 0}$ is locally bounded on U if and only if there exists a subsequence (m_k) such that $(F^{\circ m_k}|_U)_{k \geqslant 0}$ has the same property. Indeed since $\overline{W}$ is compact, there exists $R > 0$ such that if $|z| > R$, then for every $w \in W$, (z, w) escapes locally uniformly to infinity under iteration. The result follows. $\square$

Proof of Theorem 2.16. — Let Ω be the component of the Fatou set $\mathcal{F}_F$ containing U. According to Lemma 2.19, for any integer $j \geqslant 0$, the sequence $(F^{\circ(n^2+j)})_{n\geqslant 0}$ converges locally uniformly to $F^{\circ j}(\xi, 0) = \big(f^{\circ k}(\xi), 0\big)$ on U, hence on Ω. Therefore, the sequence $(F^{\circ n^2})_{n\geqslant 0}$ converges locally uniformly to $\big(f^{\circ j}(\xi), 0\big)$ on $F^{\circ j}(\Omega)$.

As a consequence, if j, k are nonnegative integers such that $F^{\circ j}(\Omega) = F^{\circ k}(\Omega)$, then $f^{\circ j}(\xi) = f^{\circ k}(\xi)$, from which we deduce that $j = k$. Indeed, ξ belongs to the parabolic basin $\mathcal{B}_f$, and so, it is not (pre)periodic under iteration of f. This shows that Ω is not (pre)periodic under iteration of F: it is a wandering Fatou component for F. $\square$

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