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Exponential complexes, period morphisms, and characteristic classes

A. B. Goncharov[1]

To Vadim Schechtman on the occasion of his 60th birthday

Résumé. – Nous introduisons des complexes exponentiels de faisceaux sur une variété. Il s'agit de résolutions des faisceaux (Tate-twistés) constants de nombres rationnels généralisant la suite exacte courte exponentielle. Il existe des applications canoniques de ces complexes vers le complexe de de Rham. A l'aide de celles-ci, et en calculant la cohomologie de Deligne rationnelle, nous introduisons de nouveaux complexes que nous appelons complexes de Deligne exponentiels. L'avantage de ces derniers est qu'au moins au point générique d'une variété complexe on peut définir l'application de régulateur de Beilinson vers la cohomologie de Deligne rationnelle au niveau des complexes. En particulier, nous définissons des morphismes de périodes à l'aide desquels nous construisons des homomorphismes entre les complexes motiviques et les complexes de Deligne exponentiels en un point générique. En combinant cette construction avec celle des classes de Chern à coefficients dans des bicomplexes, nous obtenons une formule explicite, à l'aide de polylogarithmes, pour les classes de Chern à valeurs dans la cohomologie de Deligne rationnelle, en degré ≤ 4.

Abstract. – We introduce a *weight n exponential complex* of sheaves $\mathbb{Q}^\bullet_{\mathbf{E}}(n)$ on a manifold X:

$$\mathcal{O}(n-1) \longrightarrow \mathcal{O}^* \otimes \mathcal{O}(n-2) \longrightarrow ... \longrightarrow \otimes^{n-1}\mathcal{O}^* \otimes \mathcal{O} \longrightarrow \otimes^n \mathcal{O}^*. \quad (1)$$

It is a resolution of the constant sheaf $\mathbb{Q}(n)$, generalising the classical exponential sequence:

$$\mathbb{Z}(1) \longrightarrow \mathcal{O} \stackrel{\exp}{\longrightarrow} \mathcal{O}^*, \ \mathbb{Z}(1) := 2\pi i\mathbb{Z}.$$

There is a canonical map from the complex $\mathbb{Q}^\bullet_{\mathbf{E}}(n)$ to the de Rham complex $\Omega^\bullet$ of X. Using it, we define a *weight n exponential Deligne complex*, calculating rational Deligne cohomology:

$$\Gamma_{\mathcal{D}}(X; n) := \mathrm{Cone}\Big(\mathbb{Q}^\bullet_{\mathbf{E}}(n) \oplus F^{\geq n}\Omega^\bullet \longrightarrow \Omega^\bullet\Big)[-1].$$

Its main advantage is that, at least at the generic point $\mathcal{X}$ of a complex variety X, it allows to define Beilinson's regulator map to the rational Deligne cohomology on the level of complexes. (A regulator map to real Deligne complexes for any regular complex variety is known [18]).

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Namely, we define a weight n *period morphism*. We use it to define a map of complexes

$$\textit{a weight } n \textit{ motivic complex of } \mathcal{X} \longrightarrow \textit{the weight } n \textit{ exponential complex of } \mathcal{X}. \tag{2}$$

We show that it gives rise to a map of complexes

$$\textit{a weight } n \textit{ motivic complex of } \mathcal{X} \longrightarrow \textit{the weight } n \textit{ exponential Deligne complex of } \mathcal{X}. \tag{3}$$

It induces Beilinson's regulator map on the cohomology.

Combining the map (3) with the construction of Chern classes with coefficients in the bigrassmannian complexes [17], we get a local explicit formula for the n-th Chern class in the rational Deligne cohomology via polylogarithms, at least for $n \leq 4$. Equivalently, we get an explicit construction for the universal Chern class in the rational Deligne cohomology

$$c_n^{\mathcal{D}} \in H^{2n}(BGL_N(\mathbb{C}), \Gamma_{\mathcal{D}}(n)),\ n \leq 4.$$

In particular, this gives explicit formulas for Cech cocycles for the topological Chern classes.

Contents

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1. Introduction, main definitions, and examples

1.1. A motivation: local construction of Chern classes

Topological invariants can often be localised by introducing additional structures of local nature.

For example, the topological Chern classes of a vector bundle E on a manifold can be localised by introducing a connection ∇ on E: the differential form $(2\pi i)^{-n}\mathrm{Tr}(F_\nabla^n)$, where F_∇ is the curvature of ∇, is a de Rham representative of the Chern class $c_n(E)$.

In this paper we address the problem of a local construction of explicit Cech cocycles representing the Chern classes. A construction of Chern classes with values in the bigrassmannian complex was given in [17]. To get from there a local formula for topological Chern classes, or Chern classes in the rational Deligne cohomology, one needs a transcendental construction relying on polylogarithms. It should handle the complicated multivalued nature of polylogarithms.

We develop such a construction. We define a weight n exponential complex, which is a resolution of the constant sheaf $\mathbb{Q}(n)$ on a manifold. Using it, we define a new complex calculating the rational Deligne cohomology, and construct a period morphism, which gives rise to a regulator map on the level of complexes at the generic point of a complex algebraic variety. Yet, more work needs to be done to find a local construction of the Chern classes $c_n(E)$ when $n > 4$.

Let us now look at the problem in detail in the simplest possible case.

1. The first Chern class. Let E be a complex line bundle on a real manifold X. Here is a classical construction of a Cech cocycle representing the first Chern class

$$c_1(E) \in H^2(X, \mathbb{Z}(1)).$$

Take a cover of X by open subsets U_i such that all intersections $U_{i_0} \cap ... \cap U_{i_k}$ are empty or contractible. The restriction of E to U_i is trivial, so we may choose a nonvanishing section s_i. The ratio s_i/s_j is an invertible function on $U_i \cap U_j$. Choose a branch of $\log(s_i/s_j)$. Then there is a 2-cocycle in the Cech complex of the cover, whose cohomology class is $c_1(E)$:

$$U_i \cap U_j \cap U_k \longmapsto \log(s_i/s_j) - \log(s_j/s_k) + \log(s_k/s_i) \in 2\pi i\mathbb{Z}.$$

Equivalently, take the short exact exponential sequence of sheaves on X, where $\mathcal{O}$ is the structure sheaf of continuous complex valued functions:

$$\mathbb{Z}(1) \longrightarrow \mathcal{O} \stackrel{\exp}{\longrightarrow} \mathcal{O}^*.$$

Then the above construction just means the following:

1. We assign to a complex line bundle E on X a Cech cocycle representing its class

$$\mathrm{cl}(E) \in H^1(X, \mathcal{O}^*).$$

2. We calculate the coboundary map in the exponential complex:

$$\delta : H^1(X, \mathcal{O}^*) \longrightarrow H^2(X, \mathbb{Z}(1)).$$

Then

$$\delta(\mathrm{cl}(E)) = c_1(E) \in H^2(X, \mathbb{Z}(1)).$$

For an arbitrary vector bundle E, $c_1(E) := c_1(\det(E))$. The construction works the same way for complex manifolds.

The first step is algebraic: the class $\mathrm{cl}(E) \in H^1(X, \mathcal{O}^*)$ makes sense in Zariski topology.

The second step is transcendental. The very existence of the integral class $c_1(E)$ reflects the failure of the complex logarithm $\log(z)$ to satisfy the functional equation. And yet the functional equation $\log(xy) = \log x + \log y$ is satisfied on the real positive axis, and determines the logarithm uniquely.

Our starting point was the following problem:

Find similar in spirit "local formulas" for all Chern classes of a vector bundle on X.

2. The second Chern class. The next is a local formula for the second Chern class. It is much deeper. We discuss in Section 1.7 the case when the vector bundle is two-dimensional - the case of an arbitrary vector bundle requires additional ideas, and postponed till Section 4.

Here we see a similar phenomenon: the local formula for the second Chern class in $H^4(X, \mathbb{Z}(2))$ requires the dilogarithm function, and reflects all its beautiful properties at once:

- The monodromy of the dilogarithm.
- The differential equation of the dilogarithm.
- Abel's five term relation, or better the failure of the complex dilogarithm to satisfy it. Yet, the five term relation on the real positive locus is clean, and determines the dilogarithm.

The relevance of the real dilogarithm for the first Pontryagin class was discovered by Gabrielov, Gelfand and Losik [14]. Few years later, the relevance of the complex dilogarithm for the codimension two algebraic cycles and regulators was discovered by Spencer Bloch [8], [9].

Our formula for the second Chern class of a two-dimensional vector bundle is in the middle.

The construction of the universal second motivic Chern class from [17] had several applications in low dimensional geometry and mathematical physics, e.g. [12]. It provides a motivic point of view on the Chern-Simons theory. It is of cluster nature, and can be quantised using the quantum dilogarithm [13]. The present paper just clarifies its Hodge-theoretic aspect.

The local formula for the third motivic Chern class has the same level of precision. In particular it is of cluster nature. However, strangely enough, it did not have any application in geometry yet. Its quantisation is a tantalising open problem.

1.2. Exponential complexes

Definition 1.1. — *The weight n exponential complex $\mathbb{Q}^\bullet_{\mathbf{E}}(n)$ is the following complex of sheaves on a manifold X, concentrated in degrees $[0, n]$:*

$$\mathcal{O}(n-1) \longrightarrow \mathcal{O}^* \otimes \mathcal{O}(n-2) \longrightarrow ... \longrightarrow \otimes^{n-1}\mathcal{O}^* \otimes \mathcal{O} \longrightarrow \otimes^n \mathcal{O}^*. \quad (1.1)$$

The differential is

$$d : \underbrace{\mathcal{O}^* \otimes \ldots \otimes \mathcal{O}^*}_{k\text{-}1 \text{ times}} \otimes \mathcal{O} \otimes \underbrace{2\pi i \otimes \ldots \otimes 2\pi i}_{n\text{-}k \text{ times}} \longrightarrow \underbrace{\mathcal{O}^* \otimes \ldots \otimes \mathcal{O}^*}_{k \text{ times}} \otimes \mathcal{O} \otimes \underbrace{2\pi i \otimes \ldots \otimes 2\pi i}_{n\text{-}k\text{-}1 \text{ times}}, \quad (1.2)$$

$$a_1 \otimes ... \otimes a_{k-1} \otimes b \otimes \underbrace{2\pi i \otimes \ldots \otimes 2\pi i}_{n\text{-}k \text{ times}} \longmapsto$$
$$a_1 \otimes ... \otimes a_{k-1} \otimes \exp(b) \otimes 2\pi i \otimes \underbrace{2\pi i \otimes \ldots \otimes 2\pi i}_{n\text{-}k\text{-}1 \text{ times}}. \quad (1.3)$$

To check that we get a complex, observe that each map d^2 involves a factor $\exp(2\pi i) = 1$:

$$\ldots\otimes b\otimes 2\pi i\otimes 2\pi i\ldots \stackrel{d}{\longrightarrow} \ldots\otimes\exp(b)\otimes 2\pi i\otimes 2\pi i\ldots \stackrel{d}{\longrightarrow} \ldots\exp(b)\otimes\exp(2\pi i)\otimes 2\pi i\ldots.$$

For example, $\mathbb{Z}^\bullet_{\mathbf{E}}(1)$ is the classical exponential resolution of $\mathbb{Z}(1)$.

The complex $\mathbb{Q}^\bullet_{\mathbf{E}}(2)$ looks as follows:

$$\mathcal{O}(1) \longrightarrow \mathcal{O}^*\otimes\mathcal{O} \longrightarrow \mathcal{O}^*\otimes\mathcal{O}^*.$$

$$b\otimes 2\pi i \longmapsto \exp(b)\otimes 2\pi i,\ \ a\otimes b \longmapsto a\otimes\exp(b).$$

The map $\mathbb{Q}(n) \hookrightarrow \mathcal{O}(n-1)$ gives rise to a map of complexes $\mathbb{Q}(n) \longrightarrow \mathbb{Q}^\bullet_{\mathbf{E}}(n)$. The cone of this map is acyclic. So the exponential complex is a resolution of the constant sheaf $\mathbb{Q}(n)$.

The holomorphic de Rham complex on a complex manifold X is a resolution of the constant sheaf $\mathbb{C}$:

$$\Omega^\bullet := \ \Omega^0 \stackrel{d}{\longrightarrow} \Omega^1 \stackrel{d}{\longrightarrow} \Omega^2 \stackrel{d}{\longrightarrow} ... \tag{1.4}$$

Let X be a regular complex algebraic variety. Take a compactification $\overline{X}$ of X such that $\mathrm{D} := \overline{X} - X$ is a normal crossing divisor. The de Rham complex $\Omega^\bullet_{\log}$ of forms with logarithmic singularities at infinity is a complex of sheaves in the classical topology on X, given by the forms with logarithmic singularities at D.

The canonical embedding $\mathbb{Q}(n) \hookrightarrow \mathbb{C}$ gives rise to a canonical morphism of the resolutions

$$\Omega^{(\bullet)}_n : \mathbb{Q}^\bullet_{\mathbf{E}}(n) \ \longrightarrow\ \Omega^\bullet_{\log} \tag{1.5}$$

defined in the next Lemma.

Lemma 1.2. — *There is a canonical morphism of complexes of sheaves on X:*

$$\begin{array}{ccccccccc}
\mathcal{O}(n-1) & \longrightarrow & \mathcal{O}^*\otimes\mathcal{O}(n-2) & \longrightarrow \ldots \longrightarrow & \otimes^{n-1}\mathcal{O}^*\otimes\mathcal{O} & \longrightarrow & \otimes^n\mathcal{O}^* \\
\downarrow \Omega^{(0)}_n & & \downarrow \Omega^{(1)}_n & \ldots & \downarrow \Omega^{(n-1)}_n & & \downarrow \Omega^{(n)}_n \\
\Omega^0 & \stackrel{d}{\longrightarrow} & \Omega^1_{\log} & \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} & \Omega^{n-1}_{\log} & \longrightarrow & \Omega^n_{\log,cl}
\end{array}$$

Here

$$\Omega^{(m)}_n((2\pi i)^{n-m-1}\cdot f_1\otimes ...\otimes f_m\otimes g) :=$$
$$(2\pi i)^{n-m-1}(-1)^m g\cdot d\log f_1\wedge ...\wedge d\log f_m, \qquad m<n,$$
$$\Omega^{(n)}_n(f_1\otimes ...\otimes f_n) := (-1)^n d\log f_1\wedge ...\wedge d\log f_n.$$

1.3. Exponential Deligne complexes.

Let X be a complex manifold. Consider a subcomplex of the holomorphic de Rham complex:

$$F^n\Omega^\bullet := \Omega^n \to \Omega^{n+1} \to \ldots \subset \Omega^\bullet. \tag{1.6}$$

The weight n rational Deligne complex on X is defined as a complex of sheaves

$$\mathbb{Q}_{\mathcal{D}}(n) := \mathrm{Cone}\Big(\mathbb{Q}(n) \oplus F^n\Omega^\bullet \longrightarrow \Omega^\bullet\Big)[-1]. \tag{1.7}$$

Complex (1.7) is quasiisomorphic to

$$\mathbb{Q}(n) \hookrightarrow \Omega^0 \stackrel{d}{\longrightarrow} \Omega^1 \stackrel{d}{\longrightarrow} ... \stackrel{d}{\longrightarrow} \Omega^{n-1}. \tag{1.8}$$

Let X be a regular complex algebraic variety. The Beilinson-Deligne complex $\mathbb{Q}_{\mathcal{D}}(n)$ [2] is a complex of sheaves in the classical topology on X given by the total complex of the bicomplex

$$\mathbb{Q}_{\mathcal{D}}(n) := \begin{array}{ccccccccccc} \mathbb{Q}(n) & & & & & & \Omega^n_{\log} & \stackrel{d}{\longrightarrow} & \Omega^{n+1}_{\log} & \stackrel{d}{\longrightarrow} & \ldots \\ \downarrow & & & & & & \downarrow= & & \downarrow= & & \\ \Omega^0_{\log} & \stackrel{d}{\longrightarrow} & \Omega^1_{\log} & \stackrel{d}{\longrightarrow} & \ldots & \stackrel{d}{\longrightarrow} & \Omega^n_{\log} & \stackrel{d}{\longrightarrow} & \Omega^{n+1}_{\log} & \stackrel{d}{\longrightarrow} & \ldots \end{array}$$

Definition 1.3. — *The weight n exponential Deligne complex is a complex*

$$\Gamma_{\mathcal{D}}(n) := \mathrm{Cone}\Big(\mathbb{Q}^\bullet_{\mathbf{E}}(n) \oplus F^n\Omega^\bullet_X \longrightarrow \Omega^\bullet_X\Big)[-1] \tag{1.9}$$

obtained by replacing $\mathbb{Q}(n)$ in (1.7) by its exponential resolution $\mathbb{Q}^\bullet_{\mathbf{E}}(n)$, and using the map (1.5).

For example, when $n = 2$ we get the total complex of the following bicomplex:

$$\begin{array}{ccccccccccc} & \mathcal{O}(1) & \longrightarrow & \mathcal{O}^* \otimes \mathcal{O} & \longrightarrow & \mathcal{O}^* \otimes \mathcal{O}^* & \oplus & \Omega^2_{\log} & \stackrel{d}{\longrightarrow} & \Omega^3_{\log} & \stackrel{d}{\longrightarrow} & \ldots \\ \Gamma_{\mathcal{D}}(2) := & \downarrow \Omega_2^{(0)} & & \downarrow \Omega_2^{(1)} & & \Omega_2^{(2)} \searrow & & \swarrow= & & \downarrow= & & \\ & \Omega^0 & \stackrel{d}{\longrightarrow} & \Omega^1_{\log} & \stackrel{d}{\longrightarrow} & & \Omega^2_{\log} & & \stackrel{d}{\longrightarrow} & \Omega^3_{\log} & \stackrel{d}{\longrightarrow} & \ldots \end{array}$$

The quotient of complex (1.9) by the acyclic subcomplex $\mathrm{Cone}\Big(F^n\Omega^\bullet_X \longrightarrow F^n\Omega^\bullet_X\Big)[-1]$ is a quasiisomorphic complex

$$\begin{array}{ccccccccc} \mathcal{O}(n-1) & \longrightarrow & \mathcal{O}^* \otimes \mathcal{O}(n-2) & \longrightarrow \ldots \longrightarrow & \otimes^{n-1}\mathcal{O}^* \otimes \mathcal{O} & \longrightarrow & \otimes^n \mathcal{O}^* \\ \downarrow & & \downarrow & & \downarrow & & \\ \Omega^0 & \stackrel{d}{\longrightarrow} & \Omega^1_{\log} & \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} & \Omega^{n-1}_{\log} & & \end{array}$$

1.4. Period morphisms

Recall the basic fact, reviewed in Section 2.1, that the equivalence classes of variations of framed mixed $\mathbb{Q}$-Hodge-Tate structures on a complex manifold X give rise to a sheaf of graded commutative Hopf algebras over $\mathbb{Q}$:

$$\mathcal{H}_* = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

One has $\mathcal{H}_0 = \mathbb{Q}$, $\mathcal{H}_1 = \mathcal{O}^*_{\mathbb{Q}} := \mathcal{O}^* \otimes \mathbb{Q}$. The reduced coproduct $\Delta' : \mathcal{H}_{>0} \longrightarrow \otimes^2 \mathcal{H}_{>0}$ give rise to the reduced cobar complex, graded by the weight:

$$\mathcal{H}_{>0} \stackrel{\Delta'}{\longrightarrow} \mathcal{H}_{>0} \otimes \mathcal{H}_{>0} \stackrel{\Delta'}{\longrightarrow} ... \stackrel{\Delta'}{\longrightarrow} \otimes^n \mathcal{H}_{>0}.$$

In Section 3 we present our main construction, valid in the category of complex manifolds:

THEOREM 1.4. — *There is a canonical map of complexes of sheaves, called the <u>period morphism</u>:*

$$\textit{the weight } n \textit{ part of cobar complex of } \underline{\mathcal{H}}_* \longrightarrow$$
$$\textit{the weight } n \textit{ exponential complex } \mathbb{Q}^{\bullet}_{\mathbf{E}}(n). \tag{1.10}$$

In a more elaborate form, it looks as follows:

$$\begin{array}{ccccccccc}
\mathcal{H}_n & & \stackrel{\Delta'}{\longrightarrow} & & (\mathcal{H}_{>0} \otimes \mathcal{H}_{>0})_n & & \stackrel{\Delta'}{\longrightarrow} ... \stackrel{\Delta'}{\longrightarrow} & & \otimes^n \mathcal{H}_1 \\
\downarrow P^1_n & & & & \downarrow P^2_n & & & & =\downarrow P^n_n \\
\mathcal{O}(n-1) & \longrightarrow & \mathcal{O}^* \otimes \mathcal{O}(n-2) & \longrightarrow & \otimes^2 \mathcal{O}^* \otimes \mathcal{O}(n-3) & \longrightarrow ... \longrightarrow & & & \otimes^n \mathcal{O}^*_{\mathbb{Q}}
\end{array} \tag{1.11}$$

The map (1.11) has the following properties:

(1) *After the identification $\mathcal{H}_1 = \mathcal{O}^*_{\mathbb{Q}}$ the map P^n_n is the identity map.*
(2) *The map P^1_n is the big period map from* [21].
(3) *The composition $\Omega^k_n \circ P^k_n$ is zero unless $k = n$, i.e. everywhere except on the very right.*

Condition 3) just means that the following composition is zero:

$$\begin{array}{ccccccc}
\mathcal{H}_n & \stackrel{\Delta'}{\longrightarrow} & (\mathcal{H}_{>0} \otimes \mathcal{H}_{>0})_n & \stackrel{\Delta'}{\longrightarrow} ... \stackrel{\Delta'}{\longrightarrow} & (\otimes^{n-1} \mathcal{H}_{>0})_n \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}^* \otimes \mathcal{O}(n-2) & \longrightarrow & \mathcal{O}^* \otimes \mathcal{O}^* \otimes \mathcal{O}(n-3) & \longrightarrow ... \longrightarrow & \otimes^{n-1} \mathcal{O}^* \otimes \mathcal{O} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1 & \stackrel{d}{\longrightarrow} & \Omega^2 & \stackrel{d}{\longrightarrow} ... \stackrel{d}{\longrightarrow} & \Omega^{n-1}
\end{array}$$

Example: $n = 2$.— Then we have a map

$$\begin{array}{ccccc} & & \mathcal{H}_2 & \longrightarrow & \otimes^2\mathcal{H}_1 \\ & & \downarrow & & \downarrow = \\ \mathcal{O}(1) & \longrightarrow & \mathcal{O}^* \otimes \mathcal{O} & \longrightarrow & \otimes^2\mathcal{O}^*_{\mathbb{Q}} \end{array}$$

The period morphism to the exponential Deligne complex. Let us use period morphism (1.11) and its properties provided by Theorem 1.4 to define a map of complexes of sheaves

$$\begin{array}{l} \textit{the weight } n \textit{ part of the reduced cobar complex of } \mathcal{H}_\bullet \longrightarrow \\ \textit{the weight } n \textit{ exponential Deligne complex } \Gamma_{\mathcal{D}}(X; n). \end{array} \tag{1.12}$$

Let us recall that

$$\Gamma_{\mathcal{D}}(X; n) = \mathrm{Cone}\Big(\mathbb{Q}^\bullet_{\mathbf{E}}(X; n) \oplus F^n\Omega^\bullet_X \longrightarrow \Omega^\bullet_X\Big)[-1]. \tag{1.13}$$

Therefore a map to the complex (1.13) has three components:

(1) The exponential complex $\mathbb{Q}^\bullet_{\mathbf{E}}(X, n)$ component;
(2) The Hodge filtration $F^n\Omega^\bullet$ component;
(3) The de Rham complex $\Omega^\bullet_X$ component.

We define these components as follows.

(1) The $\mathbb{Q}^\bullet_{\mathbf{E}}(X, n)$-component is just the period morphism:

$$\begin{array}{ccccccccc} & & \mathcal{H}_n & \stackrel{\Delta'}{\longrightarrow} & (\mathcal{H}_{>0} \otimes \mathcal{H}_{>0})_n & \stackrel{\Delta'}{\longrightarrow} \ldots \stackrel{\Delta'}{\longrightarrow} & \otimes^n\mathcal{H}_1 \\ & & \downarrow P^1_n & & \downarrow P^2_n & & = \downarrow P^n_n \\ \mathcal{O}(n-1) & \longrightarrow & \mathcal{O}^* \otimes \mathcal{O}(n-2) & \longrightarrow & \otimes^2\mathcal{O}^* \otimes \mathcal{O}(n-3) & \longrightarrow \ldots \longrightarrow & \otimes^n\mathcal{O}^*_{\mathbb{Q}} \end{array}$$

(2) The $F^n\Omega^\bullet$-component is given by the map

$$-\otimes^n d\log : \otimes^n\mathcal{H}_1 \longrightarrow \Omega^n_X,\ (f_1, ..., f_n) \longmapsto -d\log f_1 \wedge \ldots \wedge d\log f_n. \tag{1.14}$$

(3) The de Rham complex component is zero.

Here is how the map (1.12) looks in the weight two. The top row is the weight 2 reduced cobar complex. The second and third rows provide us a bicomplex whose total is the weight two exponential Deligne complex. The

map is given by the first row of vertical arrows:

$$\begin{array}{ccccccccccc}
 & & & & \mathcal{H}_2 & \stackrel{\Delta'}{\longrightarrow} & \mathcal{H}_1 \otimes \mathcal{H}_1 & & & & \\
 & & & & \downarrow P_2^1 & \swarrow = & \searrow -\otimes^2 d\log & & & & \\
\mathcal{O}(1) & \longrightarrow & \mathcal{O}^* \otimes \mathcal{O} & \longrightarrow & \mathcal{O}^* \otimes \mathcal{O}^* & \oplus & \Omega^2_{\log} & \stackrel{d}{\longrightarrow} & \Omega^3_{\log} & \stackrel{d}{\longrightarrow} & \dots \\
\downarrow = & & \downarrow \Omega_2^{(1)} & & \Omega_2^{(2)} \searrow & & \swarrow = & & \downarrow = & & \\
\Omega^0 & \stackrel{d}{\longrightarrow} & \Omega^1_{\log} & \stackrel{d}{\longrightarrow} & & \Omega^2_{\log} & & \stackrel{d}{\longrightarrow} & \Omega^3_{\log} & \stackrel{d}{\longrightarrow} & \dots
\end{array}$$

THEOREM 1.5. — *The map defined by the components 1)-3) is a homomorphism of complexes.*

Proof. — By Theorem 1.4, the component 1) is a homomorphism of complexes. The component 2) is also a homomorphism of complexes. Indeed, the forms in the image of the map (1.14) are evidently closed. So the statement reduces to the claim that the following composition is zero:

$$\Big(\otimes^{n-1}\mathcal{H}_{>0}\Big)_n \stackrel{\Delta'}{\longrightarrow} \otimes^n \mathcal{H}_1 \longrightarrow \Omega^n.$$

This follows from the $n = 2$ case, telling that (see Theorem 2.11) thanks to the Griffith transversality, the following composition is zero:

$$\mathcal{H}_2 \stackrel{\Delta'}{\longrightarrow} \mathcal{H}_1 \otimes \mathcal{H}_1 \longrightarrow \Omega^1.$$

After that the Theorem reduces to properties 1) and 3) of the period map in Theorem 1.4. □

Alternatively, using the reduced model (1.11) for the exponential Deligne complex, the homomorphism (1.12) is given <u>just</u> by the period morphism:

$$\begin{array}{ccccccccc}
\mathcal{H}_n & \stackrel{\Delta'}{\longrightarrow} & \Big(\mathcal{H}_{>0} \otimes \mathcal{H}_{>0}\Big)_n & \stackrel{\Delta'}{\longrightarrow} \dots \stackrel{\Delta'}{\longrightarrow} & \Big(\otimes^{n-1}\mathcal{H}_{>0}\Big)_n & \stackrel{\Delta'}{\longrightarrow} & \otimes^n \mathcal{H}_1 \\
\downarrow P_n^{(1)} & & \downarrow P_n^{(2)} & & \downarrow P_n^{(n-1)} & & \downarrow P_n^{(n)} \\
\mathcal{O}(n-1) & \to & \mathcal{O}^* \otimes \mathcal{O}(n-2) & \to \dots \to & \otimes^{n-1}\mathcal{O}^* \otimes \mathcal{O} & \to & \otimes^n \mathcal{O}^*_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega^0 & \stackrel{d}{\longrightarrow} & \Omega^1_{\log} & \stackrel{d}{\longrightarrow} \dots \stackrel{d}{\longrightarrow} & \Omega^{n-1}_{\log} & &
\end{array}$$

1.5. A map: Bloch complex → weight two exponential Deligne complex

The Bloch complex as a "resolution" of Milnor's K_2. Given a field F, the Milnor group $K_2(F)$ is the quotient of the group $F^* \otimes F^*$ by a subgroup generated by *Steinberg relations* $(1-x)\otimes x$ where $x \in F^* - \{1\}$ [30]. Since $x \otimes y + y \otimes x$ is a sum of Steinberg relations,

$$K_2(F) = \frac{\wedge^2 F^*}{\text{subgroup generated by Steinberg relations}}. \tag{1.15}$$

In other words, the group $K_2(F)$ is the cokernel of the map

$$\delta : \mathbb{Z}[F^* - \{1\}] \longrightarrow \wedge^2 F^*, \qquad \{x\} \longmapsto (1-x) \wedge x.$$

where $\{x\}$ is the generator of $\mathbb{Z}[F^* - \{1\}]$ corresponding to an $x \in F^* - \{1\}$.

Recall the cross-ration of four points on the projective line:

$$r(s_1, s_2, s_3, s_4) := \frac{(s_1 - s_4)(s_2 - s_3)}{(s_1 - s_3)(s_2 - s_4)}. \tag{1.16}$$

Let $R_2(F)$ be the subgroup of $\mathbb{Z}[F^* - \{1\}]$ generated by the "five term relations"

$$\sum_{i=1}^{5} (-1)^i \{r(s_1, ..., \widehat{s_i}, ..., s_5)\}, \qquad s_i \in \mathrm{P}^1(F), \quad s_i \neq s_j. \tag{1.17}$$

It is well known that $\delta(R_2(F)) = 0$ (see Lemma 1.8). Let us set

$$B_2(F) := \frac{\mathbb{Z}[F^* - \{1\}]}{R_2(F)}.$$

Then the map δ gives rise to a homomorphism

$$\delta : B_2(F) \longrightarrow \wedge^2 F^*. \tag{1.18}$$

Let $\{x\}_2 \in B_2(F)$ be the image of $\{x\}$. We add $\{0\}_2 = \{1\}_2 = \{\infty\}_2 = 0$, annihilated by δ. We view (1.18) as a complex, called the *Bloch complex* [9], [31], [11], placed in degrees $[1, 2]$.

Consider a twin of the weight two exponential complex, which we call the weight two Lie-exponential complex,[1] which is a complex of sheaves on X in degrees $[0, 2]$:

$$\mathbb{Q}^\bullet_{\mathcal{E}}(2) := \ \mathcal{O}(1) \longrightarrow \Lambda^2 \mathcal{O} \stackrel{\wedge^2 \exp}{\longrightarrow} \Lambda^2 \mathcal{O}^*. \tag{1.19}$$

[1]The prefix Lie refers to the fact that the period map in this case is a map from the standard Chevalley-Eilenberg complex of the Lie coalgebra $\mathcal{L}_*$ associated with the Hopf algebra $\mathcal{H}_*$. See Section 2.4 for the definition of Lie-exponential complexes and discussion of the Lie-period maps for them.

The differentials are given as follows:

$$2\pi i \otimes a \longmapsto 2\pi i \wedge a, \ a \wedge b \longmapsto \exp(a) \wedge \exp(b). \tag{1.20}$$

There is a canonical map of complexes:

$$\mathbb{Q}(2) \longrightarrow \mathbb{Q}^{\bullet}_{\mathcal{E}}(2), \ (2\pi i)^2 \longmapsto 2\pi i \otimes 2\pi i.$$

Therefore one can easily see that $\mathbb{Q}^{\bullet}_{\mathcal{E}}(2)$ is a resolution of $\mathbb{Q}(2)$.

Let us sheafify the Bloch complex to a complex of sheaves on X:

$$\mathrm{B}^{\bullet}(2) := \ B_2(\mathcal{O}) \longrightarrow \Lambda^2\mathcal{O}^*. \tag{1.21}$$

Let us define a map of complexes

$$\begin{array}{ccccc} & & B_2(\mathcal{O}) & \longrightarrow & \Lambda^2\mathcal{O}^* \\ & & \downarrow p_2 & & \downarrow = \\ \mathcal{O}(1) & \longrightarrow & \Lambda^2\mathcal{O} & \stackrel{\wedge^2\exp}{\longrightarrow} & \Lambda^2\mathcal{O}^* \end{array} \tag{1.22}$$

To define the homomorphism p_2, we set

$$\mathrm{Li}_2(x) := \int_0^x \frac{dt}{1-t} \circ \frac{dt}{t}, \ -\log(1-x) = \int_0^x \frac{dt}{1-t}, \ \log x := \int_0^x \frac{dt}{t}. \tag{1.23}$$

Here all integrals are along the same path from 0 to x. The last one is regularised using the tangential base point at 0 dual to dt. When $|x| < 1$, we have standard power series expansions

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \ \mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Then we set, modifying slightly the original construction of Spencer Bloch [9],

$$\mathrm{L}_2(x) := \mathrm{Li}_2(x) + \frac{1}{2} \cdot \log(1-x) \log x + \frac{(2\pi i)^2}{24},$$

$$p_2(\{x\}_2) := \frac{1}{2} \cdot \log(1-x) \wedge \log x + 2\pi i \wedge \frac{1}{2\pi i} \mathrm{L}_2(x).$$

Notice that $2\pi i \wedge \frac{2\pi i}{24} = 0$ in $\Lambda^2\mathbb{C}$. Indeed, for any integer N we have $2\pi i \wedge \frac{2\pi i}{N} = -N \cdot \frac{2\pi i}{N} \wedge \frac{2\pi i}{N} = 0$. Yet it is handy to keep the summand $\frac{(2\pi i)^2}{24}$ in $\mathrm{L}_2(x)$, although it does not change $2\pi i \wedge \frac{1}{2\pi i} \mathrm{L}_2(x)$.

Lemma 1.6. — *i) The map p_2 is well defined on $\mathbb{Z}[\mathbb{C}^* - \{1\}]$, i.e. does not depend on the monodromy of the logarithms and the dilogarithm along the path γ in (1.23).*

ii) The map p_2 sends the five term relations to zero.

Proof.— The part i) is easy to check using well known monodromy properties of the dilogarithm.

Let us prove the five term relation. Recall the map

$$\delta_2 : \mathbb{Z}[\mathbb{C}(t)^* - \{1\}] \longrightarrow \mathbb{C}(t)^* \wedge \mathbb{C}(t)^*, \ \{x\} \longmapsto (1-x) \wedge x.$$

Then we have a commutative diagram:

$$\begin{array}{ccccc} \text{Ker } \delta_2 & \longrightarrow & \mathbb{Z}[\mathbb{C}(t)^* - \{1\}]] & \stackrel{\delta_2}{\longrightarrow} & \mathbb{C}(t)^* \wedge \mathbb{C}(t)^* \\ \downarrow & & \downarrow p_2 & & \downarrow = \\ \mathbb{C}(t)(1) & \longrightarrow & \mathbb{C}(t) \wedge \mathbb{C}(t) & \stackrel{\exp}{\longrightarrow} & \mathbb{C}(t)^* \wedge \mathbb{C}(t)^*. \end{array}$$

It implies that $p_2(\text{Ker } \delta_2) \subset 2\pi i \wedge \mathbb{C}(t)$. Next, let us consider a map

$$\omega : \Lambda^2 \mathcal{O} \longrightarrow \Omega^1, \ f \wedge g \longmapsto \frac{1}{2}(fdg - gdf). \tag{1.24}$$

The differential equation for the dilogarithm function is

$$d\mathrm{L}_2(x) = \frac{1}{2} \cdot \Big(-\log(1-x) \ d\log x + \log x \ d\log(1-x) \Big). \tag{1.25}$$

It just means that the following composition is zero:

$$\mathbb{Z}[\mathbb{C}(t)^* - \{1\}]] \stackrel{p_2}{\longrightarrow} \Lambda^2 \mathbb{C}(t) \stackrel{\omega}{\longrightarrow} \Omega^1_t. \tag{1.26}$$

The kernel of the map $\omega : 2\pi i \wedge \mathbb{C}(t) \longrightarrow \Omega^1$ is $2\pi i \wedge \mathbb{C}$. This implies that

$$p_2(\text{Ker } \delta_2) \in 2\pi i \wedge \mathbb{C}.$$

Given a configuration of five distinct points $(x_1, ..., x_5)$ on $\mathbb{CP}^1$, denote by $R_2(x_1, ..., x_5) \in \mathbb{Z}[\mathbb{C}]$ the corresponding five-term relation element (1.17). Since it lies in the kernel of the map δ_2, applying the map p_2 to it we get a constant:

$$c(x_1, ..., x_5) := p_2\Big(R_2(x_1, ..., x_5)\Big) \in 2\pi i \wedge \mathbb{C}.$$

Let us calculate this constant. Similar argument shows that we have constants

$$b(x) := p_2(\{x\}_2 + \{1-x\}_2) \in 2\pi i \wedge \mathbb{C}, \ c(x) := p_2(\{x\}_2 + \{x^{-1}\}_2) \in 2\pi i \wedge \mathbb{C}.$$

One has $b(x) = c(x)$. Indeed, they tautologically coincide if x solves the equation $1 - x = x^{-1}$. Thus they must coincide for any $x \in \mathbb{C}^* - 1$. On the other hand, switching the last two points in the cross-ratio we get $r(x_1, x_2, x_3, x_4) = r(x_1, x_2, x_4, x_3)^{-1}$. Therefore

$$c(x_1, x_2, x_3, x_4, x_5) + c(x_1, x_2, x_3, x_5, x_4) = c(x).$$

Finally, $b(x) = 0$ for $x \in (0, 1)$. Indeed, $\log(1-x) \wedge \log x + \log x \wedge \log(1-x) = 0$, and each term of $\mathrm{L}_2(x)$ is well defined if $x \in (0, 1)$. So it is sufficient to

show that $\mathrm{L}_2(x) + \mathrm{L}_2(1-x) = 0$. One has $d(\mathrm{L}_2(x) + \mathrm{L}_2(1-x)) = 0$. The limit of $\mathrm{L}_2(x) + \mathrm{L}_2(1-x)$ as $x \to 1$ is 0 due to $\mathrm{Li}_2(1) = \pi^2/6$. □

Recall that the weight two rational Deligne complex $\mathbb{Q}^\bullet_{\mathcal{D}}(2)$ is a complex of sheaves on X in degrees $[0, 2]$:

$$\mathbb{Q}^\bullet_{\mathcal{D}}(2) := \begin{array}{ccc} \mathbb{Q}(2) & & \\ \downarrow & & \\ \mathcal{O} & \stackrel{d}{\longrightarrow} & \Omega^1 \end{array}$$

Consider the following version of the exponential Deligne complex, which we call the weight two Lie-exponential Deligne complex, and abusing notation denote also by $\Gamma_{\mathcal{D}}(2)$, obtained by replacing the constant sheaf $\mathbb{Q}(2)$ by its Lie-exponential resolution $\mathbb{Q}^\bullet_{\mathcal{E}}(2)$. It is a complex of sheaves in the classical topology on X associated with the following bicomplex:

$$\Gamma_{\mathcal{D}}(2) := \begin{array}{ccccc} \mathcal{O}(1) & \longrightarrow & \mathcal{O} \wedge \mathcal{O} & \longrightarrow & \mathcal{O}^* \wedge \mathcal{O}^* \\ \downarrow = & & \downarrow \omega & & \\ \mathcal{O} & \stackrel{d}{\longrightarrow} & \Omega^1 & & \end{array}$$

Proposition 1.7. — *There is a canonical morphism of complexes of sheaves*

$$r_{\mathcal{D}} : \mathrm{B}^\bullet(2) \longrightarrow \Gamma_{\mathcal{D}}(2). \tag{1.27}$$

Proof.— Let us define the map (1.27) as a morphism of complexes:

$$\begin{array}{ccccc} & & B_2(\mathcal{O}) & \longrightarrow & \Lambda^2\mathcal{O}^* \\ & & \downarrow p_2 & & \downarrow \mathrm{Id} \\ \mathcal{O}(1) & \longrightarrow & \Lambda^2\mathcal{O} & \longrightarrow & \Lambda^2\mathcal{O}^* \\ \downarrow = & & \downarrow \omega & & \\ \mathcal{O} & \longrightarrow & \Omega^1 & & \end{array}$$

Here the top raw is the sheafified Bloch complex, and the bottom two raws describe the weight two Lie-exponential Deligne complex. The morphism of the first raw to the second is given by the maps (p_2, Id). The other components of the morphism are zero.

To show that this is a map of complexes we use two facts:

1. The top right square is commutative by the definition of the map p_2.

2. The composition $B_2(\mathcal{O}) \stackrel{p_2}{\longrightarrow} \Lambda^2\mathcal{O} \stackrel{\omega}{\longrightarrow} \Omega^1$ is zero by the differential equation for the dilogarithm.

□

Applications to regulators. Let us look at the dilogarithm regulator map for Spec($\mathbb{C}$):

$$\begin{array}{ccccc} & & B_2(\mathbb{C}) & \longrightarrow & \Lambda^2\mathbb{C}^* \\ & & \downarrow p_2 & & \downarrow = \\ \mathbb{C}^*(1) & \longrightarrow & \Lambda^2\mathbb{C} & \stackrel{\wedge^2\exp}{\longrightarrow} & \Lambda^2\mathbb{C}^* \end{array} \tag{1.28}$$

It implies that there is a canonical map

$$\mathrm{Ker}\Big(B_2(\mathbb{C}) \longrightarrow \Lambda^2\mathbb{C}^*\Big) \longrightarrow \mathbb{C}^*(1) = \mathrm{Ker}\Big(\Lambda^2\mathbb{C} \longrightarrow \Lambda^2\mathbb{C}^*\Big).$$

According to a theorem of Suslin [32], one has

$$\mathrm{Ker}\Big(B_2(\mathbb{C}) \longrightarrow \Lambda^2\mathbb{C}^*\Big) \otimes \mathbb{Q} = K_3^{\mathrm{ind}}(\mathbb{C}) \otimes \mathbb{Q}.$$

So we get an explicit construction of Beilinson's regulator map

$$K_3^{\mathrm{ind}}(\mathbb{C}) \longrightarrow \mathbb{C}^*(1).$$

1.6. Regulator maps: motivic complexes → exponential Deligne complexes

Motivic complexes and regulators. According to Beilinson [3], for any scheme X over $\mathbb{Q}$, and for each integer $n \geq 0$, one should have a complex of sheaves $\mathbb{Z}_{\mathcal{M}}(X; n)$ in the Zariski topology on X, called the *weight n motivic complex of sheaves on* X, well defined in the derived category. For example, $\mathbb{Z}_{\mathcal{M}}(X; 0) = \mathbb{Z}$, and $\mathbb{Z}_{\mathcal{M}}(X; 1) = \mathcal{O}_X^*[-1]$. Beilinson's formula relates its cohomology to the weight n pieces for the Adams filtration on Quillen's K-groups of X:

$$H^i(\mathbb{Z}_{\mathcal{M}}(X; n) \otimes \mathbb{Q}) \stackrel{?}{=} \mathrm{gr}_\gamma^n K_{2n-i}(X) \otimes \mathbb{Q}. \tag{1.29}$$

Beilinson defined higher regulator maps, with the source understood by (1.29):

$$H^i_{\mathrm{Zar}}(\mathbb{Z}_{\mathcal{M}}(X; n) \otimes \mathbb{Q}) \longrightarrow H^i(X, \mathbb{Q}_{\mathcal{D}}(n)).$$

Let X be a regular complex algebraic variety. We want to have higher regulator maps on the level of complexes. Motivic complexes are complexes of sheaves in the Zarisky topology on X, while the Beilinson-Deligne complexes are complexes of sheaves in the classical topology on X. To relate them, let us consider a map of sites

$$\pi : \text{Classical site} \longrightarrow \text{Zariski site}.$$

Then the problem is interpreted as a problem of of construction of a map of complexes

$$\mathbb{Z}_{\mathcal{M}}(X; n) \longrightarrow \mathrm{R}\pi_*\mathbb{Q}_{\mathcal{D}}(X; n). \tag{1.30}$$

We address this problem at the generic point $\mathcal{X}$ of X – this is sufficient for local explicit formulas for the Chern classes. Notice that the $\mathrm{R}\pi_*$ is highly non-trivial since the constant sheaf $\mathbb{Q}_{\mathcal{X}}$ has complicated cohomology at the generic point.

It is unlikely that one can construct a map just to the Beilinson-Deligne complex on $\mathcal{X}$.

Our point is that replacing the constant sheaf $\mathbb{Q}_{\mathcal{X}}$ by its exponential resolution and considering the exponential Deligne complex $\Gamma_{\mathcal{D}}(\mathcal{X}; n)$, one should be able to define a map of complexes

$$\mathbb{Z}_{\mathcal{M}}(\mathcal{X}; n) \longrightarrow \pi_*\Gamma_{\mathcal{D}}(\mathcal{X}; n). \tag{1.31}$$

Combining it with the map $\pi_*\Gamma_{\mathcal{D}}(\mathcal{X}; n) \to R\pi_*\Gamma_{\mathcal{D}}(\mathcal{X}; n)$ we get a regulator map (1.30) for $\mathcal{X}$.

Here is our strategy to define a map (1.31). We make the following assumption:

The motivic complex $\mathbb{Q}_{\mathcal{M}}(\mathcal{X}; n)$ can be constructed as the weight n part of the cobar complex of a graded commutative Hopf algebra $\mathcal{A}_*(\mathcal{X})$, the motivic Tate Hopf algebra, graded by $\mathbb{Z}_{\geq 0}$.

Then the Hodge realisation provides a map of Hopf algebras

$$\mathcal{A}_*(\mathcal{X}) \longrightarrow \mathcal{H}_*(\mathcal{X}).$$

It induces a map of their cobar complexes:

$$\begin{aligned} &\textit{the weight } n \textit{ part of the cobar complex of } \mathcal{A}_*(\mathcal{X}) \longrightarrow \\ &\textit{the weight } n \textit{ part of the cobar complex of } \mathcal{H}_*(\mathcal{X}). \end{aligned} \tag{1.32}$$

Composing (1.10) and (1.12) we arrive at a map of complexes

$$\mathcal{P} : \mathbb{Q}_{\mathcal{M}}(\mathcal{X}; n) \longrightarrow \textit{the weight } n \textit{ exponential Deligne complex } \Gamma_{\mathcal{D}}(\mathcal{X}; n). \tag{1.33}$$

The induced map on the cohomology provides higher regulators.

Using the polylogarithmic complexes, we can avoid assumptions about the existence of the motivic Hopf algebra when $n \leq 3$. So in this case the construction goes through unconditionally. In general there is the Bloch-Kriz construction of the motivic Tate Hopf algebra [10]. However their Hodge realisation map deserves a more explicit construction.

1.7. A local formula for the second Chern class of a two-dimensional vector bundle

We consider complex vector bundles on real manifolds, and produce a local formula for a Cech cocycle representing the topological second Chern class, as well as the second Chern class in the integral Deligne cohomology. All constructions can be applied to vector bundles over complex manifolds. The algebraic part of the construction makes sense in Zariski topology.

Given a two-dimensional vector bundle E on a manifold X, pick a cover $\{U_i\}$ of X by open sets such that all intersections $U_{i_0 \dots i_k} := U_{i_0} \cap \dots \cap U_{i_k}$ are empty or contractible. Choose a non-zero regular section s_i on each open set U_i. Then,

- For a three open sets U_1, U_2, U_3 there are three sections s_1, s_2, s_3 over U_{123}. They provide a section

$$l_2(s_1, s_2, s_3) \in \mathcal{O}^*_{U_{123}} \otimes_{\mathbb{Z}} \mathcal{O}^*_{U_{123}}.$$

  Namely, pick a volume form $\omega \in \det(E^\vee_{U_{123}})$ on the restriction of E to U_{123}. Set

$$\Delta(s_i, s_j) := \langle \omega, s_i \wedge s_j \rangle,$$

$$l_2(s_1, s_2, s_3) := \Delta(s_1, s_2) \wedge \Delta(s_2, s_3) + \Delta(s_2, s_3) \wedge \Delta(s_1, s_3) + \Delta(s_1, s_3) \wedge \Delta(s_1, s_2). \tag{1.34}$$

  This expression does not depend on the choice of the volume form ω.
- For any four open sets U_1, U_2, U_3, U_4 take the cross-ratio of the restriction of the four sections to U_{1234}:

$$r(s_1, s_2, s_3, s_4) := \frac{\Delta(s_1, s_4)\Delta(s_2, s_3)}{\Delta(s_1, s_3)\Delta(s_2, s_4)} \in \mathcal{O}^*_{U_{1234}}. \tag{1.35}$$

  The Plücker identity implies that it satisfies the crucial relation

$$(1 - r(s_1, s_2, s_3, s_4)) \wedge r(s_1, s_2, s_3, s_4) = \tag{1.36}$$

$$l_2(s_2, s_3, s_4) - l_2(s_1, s_3, s_4) + l_2(s_1, s_2, s_4) - l_2(s_1, s_2, s_3).$$

Recall the map $\delta : \mathbb{Z}[F^* - \{1\}] \longrightarrow \wedge^2 F^*$, given by $\{x\} \longmapsto (1 - x) \wedge x$.

Recall the subgroup $R_2(F) \subset \mathbb{Z}[F^* - \{1\}]$ generated by the "five term relations" (1.17).

LEMMA 1.8. — *One has* $\delta(R_2(F)) = 0$.

Proof.— Denote by $C_n(k)$ the free abelian group generated by the configurations of n vectors in generic position in a k-dimensional vector space over

a field F. It follows from (1.36) that there is a map of complexes

$$\begin{array}{ccccc} C_5(2) & \stackrel{d}{\longrightarrow} & C_4(2) & \stackrel{d}{\longrightarrow} & C_3(2) \\ \downarrow l_0 & & \downarrow l_1 & & \downarrow l_2 \\ R_2(F) & \hookrightarrow & \mathbb{Z}[F^* - \{1\}] & \stackrel{\delta}{\longrightarrow} & \Lambda^2 F^* \end{array} \tag{1.37}$$

Here the map l_2 is given by (1.34), the map l_1 is given by $(s_1, ..., s_4) \longmapsto \{r(s_1, ..., s_4)\}$, and the map l_0 assigns to a configuration of five generic vectors $(s_1, ..., s_5)$ the configuration of the corresponding five points on P^1. $\square$

Setting $B_2(F) := \mathbb{Z}[F^* - \{1\}]/R_2(F)$ we get the Bloch complex $\delta : B_2(F) \longrightarrow \Lambda^2 F^*$.

Our construction delivers a Cech cochain $C_\bullet$ for the covering $\{U_i\}$ of total degree four with values in the sheafified Bloch complex

$$\mathrm{B}^\bullet(2) = \ B_2(\mathcal{O}) \longrightarrow \Lambda^2 \mathcal{O}^*. \tag{1.38}$$

It has two components given by (1.34) and (1.35):

$$C_3(U_i, U_j, U_k) \in \Lambda^2 \mathcal{O}^*_{U_{ijk}} \text{ and } C_4(U_i, U_j, U_k, U_l) \in B_2(\mathcal{O}_{U_{ijkl}}).$$

Condition (1.36) plus the five term relations (1.17) just mean that it is a cocycle. It represents the *second motivic Chern class* of the vector bundle E:

$$c_2^M(E) \in H^4(X, \mathrm{B}^\bullet(2)). \tag{1.39}$$

Remark.— The name refers to a construction of the second universal motivic Chern class of Milnor's simplicial model $BGL_{2\bullet}$ of the classifying space BGL_2:

$$c_2^{\mathcal{M}} \in H^4(BGL_{2\bullet}, \mathbb{Z}_{\mathcal{M}}(2)).$$

Here $\mathbb{Z}_{\mathcal{M}}(2)$ is the weight two motivic complex, which is a complex of sheaves in Zariski topology on the simplicial scheme $BGL_{2\bullet}$. It is defined by applying the Gersten resolution to the Bloch complex at the generic point. A complex two dimensional vector bundle on a manifold X equipped with a Cech cover can be described as the pull back of the universal bundle over $BGL_{2\bullet}$. Then the class $c_2^{\mathcal{M}}$ pulls back to the class (1.39).

We use the classical topology, aiming at a local formula for the topological Chern class

$$c_2(E) \in H^4(X, \mathbb{Z}(2)).$$

To get it from the motivic one (1.39) is a non-trivial problem. Although it is asking for the dilogarithm, we have to deal with its complicated multivalued nature. We employ the weight two Lie-exponential complex to handle this

problem, and construct a cocycle representing the second Chern class in the weight two Lie-exponential Deligne complex

$$c_2^{\mathcal{D}}(E) \in H^4(X, \Gamma_{\mathcal{D}}(2)). \tag{1.40}$$

Recall the weight two Lie-exponential complex of sheaves:

$$\mathbb{Q}^{\bullet}_{\mathcal{E}}(2) := \ \mathcal{O}(1) \longrightarrow \Lambda^2\mathcal{O} \stackrel{\wedge^2\exp}{\longrightarrow} \Lambda^2\mathcal{O}^*. \tag{1.41}$$

$$2\pi i \otimes a \longmapsto 2\pi i \wedge a, \ a \wedge b \longmapsto \exp(a) \wedge \exp(b). \tag{1.42}$$

It is a resolution of $\mathbb{Q}(2)$. Recall a map of complexes (1.22):

$$\begin{array}{ccccc} & & B_2(\mathcal{O}) & \longrightarrow & \Lambda^2\mathcal{O}^* \\ & & \downarrow p_2 & & \downarrow = \\ \mathcal{O}(1) & \longrightarrow & \Lambda^2\mathcal{O} & \stackrel{\wedge^2\exp}{\longrightarrow} & \Lambda^2\mathcal{O}^* \end{array} \tag{1.43}$$

Here

$$\mathrm{L}_2(x) := \mathrm{Li}_2(x) + \frac{1}{2} \cdot \log(1-x)\log x + \frac{(2\pi i)^2}{24},$$

$$p_2(\{x\}_2) := \frac{1}{2} \cdot \log(1-x) \wedge \log x + 2\pi i \wedge \frac{1}{2\pi i}\mathrm{L}_2(x).$$

Recall the weight two Lie-exponential Deligne complex $\Gamma_{\mathcal{D}}(2)$:

$$\Gamma_{\mathcal{D}}(2) := \begin{array}{ccccc} \mathcal{O}(1) & \longrightarrow & \mathcal{O} \wedge \mathcal{O} & \longrightarrow & \mathcal{O}^* \wedge \mathcal{O}^* \\ \downarrow = & & \downarrow \omega & & \downarrow \\ \mathcal{O} & \stackrel{d}{\longrightarrow} & \Omega^1 & \longrightarrow & 0 \end{array}$$

By Proposition 1.7, there is a canonical morphism of complexes of sheaves

$$r_{\mathcal{D}} : \mathrm{B}^{\bullet}(2) \longrightarrow \Gamma_{\mathcal{D}}(2). \tag{1.44}$$

The Cech cocycle (C_3, C_4) representing a class in $H^4(X, \mathrm{B}^{\bullet}(2))$, combined with a morphism of complexes (1.44), delivers a Cech cocycle representing the second Chern class $c_2^{\mathcal{D}}(E)$ in (1.40).

Namely, we start with the Cech cocycle (C_3, C_4) with values in the Bloch complex:

$$C_4 \in B_2(\mathcal{O}) \stackrel{\delta}{\longrightarrow} C_3 \in \Lambda^2\mathcal{O}^*.$$

Let us define a Cech cochain $(C_3, \widetilde{C}_3, \widetilde{C}_4, \widetilde{C}_5)$ with values in the weight two Lie-exponential complex, organised as follows:

$$\widetilde{C}_5 \in \mathbb{Z}(2) \ \stackrel{d}{\longrightarrow} \ \widetilde{C}_4 \in \mathcal{O}(1) \ \stackrel{d}{\longrightarrow} \ \widetilde{C}_3 \in \Lambda^2\mathcal{O} \ \stackrel{\wedge^2\exp}{\longrightarrow} \ C_3 \in \Lambda^2\mathcal{O}^* \tag{1.45}$$

- For any three open sets U_1, U_2, U_3, let us define

$$\widetilde{C}_3(U_1, U_2, U_3) \in \Lambda^2 \mathcal{O}_{U_{123}}.$$

Namely, we choose a branch of each $\log \Delta(s_i, s_j)$ on U_{123}, and set

$$\widetilde{C}_3(U_1, U_2, U_3) := \log \Delta(s_1, s_2) \wedge \log \Delta(s_2, s_3) \tag{1.46}$$

$$+ \log \Delta(s_2, s_3) \wedge \log \Delta(s_1, s_3) + \log \Delta(s_1, s_3) \wedge \log \Delta(s_1, s_2). \tag{1.47}$$

- We assign to any four open sets U_1, U_2, U_3, U_4 an element

$$\widetilde{C}_4(U_1, U_2, U_3, U_4) \in \mathcal{O}_{U_{1234}}(1).$$

To define it, we use an isomorphism, see (1.19) - (1.20):

$$\mathcal{O}_U(1)/\mathbb{Z}(2) \ \widetilde{=} \ \mathbb{Z}(1) \wedge \mathcal{O}_U(1) \ = \ \mathrm{Ker}\Big(\Lambda^2 \mathcal{O}_U \stackrel{\wedge^2 \exp}{\longrightarrow} \Lambda^2 \mathcal{O}_U^*\Big).$$

So we exhibit an element in $\Lambda^2 \mathcal{O}_{U_{1234}}$ which is in the kernel of the $\wedge^2 \exp$ map:

$$\widetilde{C}_4(U_1, U_2, U_3, U_4) :=$$

$$(\delta_{\mathrm{Cech}} \circ \widetilde{C}_3)(U_1, U_2, U_3, U_4) - 2\pi i \wedge \frac{1}{2\pi i} \mathrm{L}_2(r(s_1, s_2, s_3, s_4))$$

$$+ \log(1 - r(s_1, s_2, s_3, s_4)) \wedge \log r(s_1, s_2, s_3, s_4). \tag{1.48}$$

To find $\widetilde{C}_4(U_1, U_2, U_3, U_4)$ explicitly we start with an equality in $\wedge^2 \mathcal{O}^*_{U_{1234}}$:

$$(\delta_{\mathrm{Cech}} \circ C_3)(U_1, U_2, U_3, U_4) + (\delta_{\mathrm{Bloch}} \circ C_4)(U_1, U_2, U_3, U_4) = 0, \tag{1.49}$$

which is just equivalent to (1.36). It follows that

$$(\delta_{\mathrm{Cech}} \circ \widetilde{C}_3)(U_1, U_2, U_3, U_4) + (p_2 \circ C_4)(U_1, U_2, U_3, U_4)$$

$$= 2\pi i \wedge \frac{1}{2\pi i} \mathrm{L}_2(r(s_1, s_2, s_3, s_4)) + 2\pi i \wedge \log \ F.$$

So we set, "dropping" $2\pi i \wedge$ in the last formula:

$$\widetilde{C}_4(U_1, U_2, U_3, U_4) := \frac{1}{2\pi i} \mathrm{L}_2(r(s_1, s_2, s_3, s_4)) + \log \ F.$$

The "correction term" $2\pi i \wedge \log \ F$ shows up as follows. Since $\log(fg) - \log(f) - \log(g)$ is a locally constant function with values in $2\pi i \mathbb{Z}$, an equality $\sum_i f_i \wedge g_i = 0$, which in our case is just the equality (1.49), implies only that, after we choose branches of $\log(f_i)$ and $\log(g_i)$ on a contractible set, $\sum_i \log(f_i) \wedge \log(g_i) = 2\pi i \wedge \log \ F$. Notice that in our case the choices of the branches of log consist of the choices made in (1.47) and (1.48)

- Finally, to any five open sets U_1, U_2, U_3, U_4, U_5 we assign an element

$$\widetilde{C}_5(U_1, U_2, U_3, U_4, U_5) := \sum_{i=1}^{5} (-1)^i \widetilde{C}_4(U_1, \dots, \widehat{U}_i, \dots, U_5) \in (2\pi i)^2 \mathbb{Q}.$$

A priory this sum lives in $\mathcal{O}_{U_{12345}}$. We claim that it is annihilated by the differential in the exponential complex. Indeed, the Cech coboundary of the first line (1.48) is zero due to the five term relation for the $\Lambda^2\mathbb{C}$-valued dilogarithm, that is since the map p_2 sends the five term relation to zero. For the second line this is just $\delta^2_{\text{Cech}} = 0$. Therefore $C_5^{\mathcal{D}} \in (2\pi i)^2\mathbb{Q}$.

We get a cocycle in the Cech complex with coefficients in the Lie-exponential Deligne complex. It represents the second Chern class $c_2^{\mathcal{D}}(E)$, and hence the usual Chern class.

1.8. Explicit formulas for the universal Chern classes.

Let us formulate our approach to local formulas for the Chern classes. Denote by $BGL^*_{N\bullet}$ the classifying space for GL_N. The $*$ stands for an open "generic" part $BGL^*_\bullet$ of Milnor's $BGL_\bullet$, which is a model of the classifying space. One should construct universal Chern classes of $BGL^*_{N\bullet}$ with values in the exponential Deligne complex:

$$c_n^{\mathcal{D}} \in H^{2n}(BGL^*_{N\bullet}, \Gamma_{\mathcal{D}}(n)). \tag{1.50}$$

They induce explicit cocycles for the Chern classes in a given Cech cover.

We define the universal Chern classes in three steps.

(1) An explicit formula for the Chern classes with values in the *bigrassmannian complexes* $\mathrm{BC}(n)$ [17].

(2) A map from the bigrassmannian complex to a motivic complex.

There are several flavors of the problem, depending on our choice of the motivic complex.

When $n = 1, 2, 3$ there is a map of the bigrassmannian complex to the polylogarithmic motivic complexes $\mathrm{B}^\bullet(n)$. The latter reflects the motivic nature of the classical polylogarithms. For example, $\mathrm{B}^\bullet(2)$ is the Bloch complex. For $n = 4$ there is also an explicit map to the motivic complex. So for $n \leq 4$ there is a satisfactory construction.

So we should get the universal motivic Chern class[2]

$$c_n^{\mathcal{M}} \in H^{2n}_{\text{Zar}}(BGL_{N\bullet}, \mathbb{Z}_{\mathcal{M}}(n)). \tag{1.51}$$

[2]Here we do not need to restrict to BGL^*_N.

(3) A map from the motivic complex to the weight n exponential complex $\mathbb{Q}^\bullet_{\mathcal{E}}(n)$, which allows to promote the class $c_n^{\mathcal{M}}$ (1.51) to the universal Chern class (1.50).

One could probably combine Steps 2 and 3, to define an explicit map from the bigrassmannian complex to the weight n exponential complex. Its most non-trivial part follows from the motivic construction of the Grassmannian n-logarithm in [25]. However the problem is open for $n > 4$.

In contrast with this, the problem of explicit construction of Chern classes with values in the real Deligne cohomology is solved for all weights n: one combines the Step 1 with the construction of a map from bigrassmannian complex to the real Deligne complex given in [18].

An approach to construction of Grassmannian polylogarithms was developed by Hanamura and MacPherson [26].

Organisation of the paper. In Section 2 we recall the definition of the fundamental Hodge-Tate Hopf algebra $\mathcal{H}_*$, and then construct the period morphism.

In Section 3 we calculate the period morphism from the polylogarithmic motivic complexes of weights ≤ 4 to the Lie-exponential complexes.

Section 4 mostly borrowed from [17]. We recall the construction of characteristic classes using the bigrassmannian complex, articulating the role of the hypersimplices, and then recall the map from the bigrassmannian complex to the motivic complexes of weights ≤ 4. Combining with the construction of the period morphisms from Section 2 we get an explicit construction of the universal Chern classes of weights ≤ 4.

Section 5 is a continuation of Section 2: we show that the $\mathbb{C}/\mathbb{R}(n)$-part of the canonical map

$$\omega_n^\bullet : \text{ the Lie-exponential complex} \longrightarrow \text{the de Rham complex } \Omega^\bullet$$

is homotopic to zero, and construct the homotopy, getting a regulator map to the real Deligne complex.

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2. Period morphisms

2.1. The $\mathbb{Q}$-Hodge-Tate Hopf algebra, and the period morphisms

The algebra background. Consider a graded commutative Hopf algebra over $\mathbb{Q}$ with a unit:

$$A_* = \oplus_{k=0}^{\infty} A_k. \tag{2.1}$$

Let $\Delta : A_* \longrightarrow A_* \otimes A_*$ be the coproduct. The quotient

$$\mathrm{CoLie}(A_*) := \frac{A_*}{A_{>0} \cdot A_{>0}}$$

is a graded Lie coalgebra with the cobracket δ induced by the coproduct Δ on A_*. Let $\mathrm{Lie}(A_*)$ be its graded dual. Then the universal enveloping algebra of the Lie algebra $\mathrm{Lie}(A_*)$ is the graded dual to the Hopf algebra A_*, assuming that all graded components are finite dimensional.

Consider the reduced coproduct

$$\Delta' := \Delta - (\mathrm{Id} \otimes 1 + 1 \otimes \mathrm{Id}) : A_* \longrightarrow A_{>0} \otimes A_{>0}$$

The reduced cobar complex of the Hopf algebra A_* is the following complex starting in degree 1:

$$A_* \stackrel{\Delta'}{\longrightarrow} A_* \otimes A_* \stackrel{\Delta'}{\longrightarrow} ... \stackrel{\Delta'}{\longrightarrow} \otimes^n A_* \stackrel{\Delta'}{\longrightarrow} \dots.$$

$$\Delta'(a_1 \otimes \ldots \otimes a_n) := \sum_{k=1}^{n} (-1)^k a_1 \otimes \ldots \otimes \Delta'(a_k) \otimes \ldots \otimes a_n.$$

The standard cochain complex of the Lie coalgebra $\mathrm{CoLie}(A_*)$ is given by

$$\mathrm{CoLie}(A_*) \stackrel{\delta}{\longrightarrow} \Lambda^2 \mathrm{CoLie}(A_*) \stackrel{\delta}{\longrightarrow} \Lambda^3 \mathrm{CoLie}(A_*) \stackrel{\delta}{\longrightarrow} \dots.$$

These two complexes are canonically quasiisomorphic. The degree $n > 0$ part of either of them calculates $\mathrm{RHom}_{A_*}(\mathbb{Q}(0), \mathbb{Q}(n))$ in the category of graded A_*-comodules, or, what is the same, graded $\mathrm{CoLie}(A_*)$-comodules, where $\mathbb{Q}(n)$ is the trivial one dimensional comodule in degree $-n$.

The fundamental Hopf algebra of the category of mixed Hodge-Tate structures. For the convenience of the reader I recall some definitions from [BGSV]. See details in [21, Section 4].

A a mixed $\mathbb{Q}$-Hodge structure H is *Hodge-Tate* if its weight factors are isomorphic to $\oplus \mathbb{Q}(k)$. A *n-framing* on H is a choice of a nonzero maps $v_0 : \mathbb{Q}(0) \to gr_0^W H$ and $f^n : gr_{-2n}^W H \to \mathbb{Q}(n)$. Consider the equivalence relation $\sim$ on the set of all n-framed Hodge-Tate structures induced by the

following: if there is a map $H_1 \to H_2$ compatible with frames, then $H_1 \sim H_2$. In particular, any n-framed Hodge-Tate structure is equivalent to a one H with $W_{-2n-2}H = 0$, $W_0H = H$. Let $\mathcal{H}_n$ be the set of equivalence classes. We define on $\mathcal{H}_n$ an abelian group structure as follows:

$$(f^n, H, v_0) + (\widetilde{f}^n, \widetilde{H}, \widetilde{v}_0) := (f^n + \widetilde{f}^n, H \oplus \widetilde{H}, v_0 + \widetilde{v}_0);$$
$$-(f^n, H, v_0) := (f^n, H, -v_0).$$

The tensor product of mixed Hodge structures induces the commutative multiplication

$$\mu : \mathcal{H}_k \otimes \mathcal{H}_\ell \to \mathcal{H}_{k+\ell}.$$

Let us define a coproduct

$$\Delta = \bigoplus_{k+\ell=n} \Delta_{k\ell} : \mathcal{H}_n \to \bigoplus_{k+\ell=n} \mathcal{H}_k \otimes \mathcal{H}_\ell. \tag{2.2}$$

Let $(f^n, H, v_0) \in \mathcal{H}_n$. Choose a basis $\{v_k^{(i)}\}$ in $\mathrm{Hom}(\mathbb{Q}(k), gr^W_{-2k}H)$ and the dual basis $\{f^k_{(i)}\}$ in $\mathrm{Hom}(gr^W_{-2k}H, \mathbb{Q}(k))$. Then

$$\Delta_{k,n-k}(f^n, H, v_0) := \sum_i (f^n, H, v_k^{(i)}) \otimes (f^k_{(i)}, H, v_0).$$

The graded $\mathbb{Q}$-vector space

$$\mathcal{H}_* := \oplus_{n=0}^{\infty} \mathcal{H}_n,$$

has a natural structure of a graded Hopf algebra over $\mathbb{Q}$ with the commutative multiplication μ and the comultiplication Δ.

THEOREM 2.1. — *The category of mixed $\mathbb{Q}$-Hodge-Tate structures is canonically equivalent to the category of finite-dimensional graded $\mathcal{H}_*$-comodules.*

Let Δ'_n be the restriction of the restricted coproduct Δ' to $\mathcal{H}_n$. Then for $n > 0$ we have

$$\mathrm{Ker}(\Delta'_n) = \quad \frac{\mathbb{C}}{(2\pi i)^n \mathbb{Q}} = \mathrm{Ext}^1_{MHS/\mathbb{Q}}(\mathbb{Q}(0), \mathbb{Q}(n)).$$

In [21] we constructed a canonical homomorphism, called the *big period map*

$$P_n : \mathcal{H}_n \quad \longrightarrow \quad \mathbb{C}^* \otimes_{\mathbb{Q}} \mathbb{C}(n-2). \tag{2.3}$$

The restriction of P_n to the subgroup $\mathrm{Ker}(\Delta'_n)$ provides an isomorphism

$$\frac{\mathbb{C}}{(2\pi i)^n \mathbb{Q}} = \mathrm{Ker}(\Delta'_n) \quad \longrightarrow \quad \mathbb{C}^* \otimes (2\pi i)^{n-1}.$$

Period morphisms. The same construction as above for the category of variations of framed mixed Hodge-Tate structures over a manifold X delivers a sheaf $\mathcal{H}_*$ of graded Hopf algebras in the *classical* topology on X. Consider a complex of sheaves $\mathcal{H}^\bullet(n)$ given by the weight n part of the reduced cobar complex of $\mathcal{H}_*$, placed in degrees $[1, n]$:

$$\mathcal{H}_n \stackrel{\Delta'}{\longrightarrow} (\mathcal{H} \otimes \mathcal{H})_n \stackrel{\Delta'}{\longrightarrow} ... \stackrel{\Delta'}{\longrightarrow} \otimes^n \mathcal{H}_1.$$

For $n > 0$ one has a quasiisomorphism of complexes of sheaves in the classical topology on X:

$$\mathrm{RHom}_{\mathrm{MHS}_X}(\mathbb{Q}(0)_X, \mathbb{Q}(n)_X) = \mathcal{H}_n \stackrel{\Delta'}{\longrightarrow} (\mathcal{H} \otimes \mathcal{H})_n \stackrel{\Delta'}{\longrightarrow} ... \stackrel{\Delta'}{\longrightarrow} \otimes^n \mathcal{H}_1.$$

We can state now precisely Theorem 1.4.

THEOREM 2.2. — *There exists a canonical morphism of complexes of sheaves*

$$P_n^\bullet : \mathcal{H}^\bullet(n) \longrightarrow \mathbb{Q}_{\mathbf{E}}^\bullet(n),$$

called the period morphism, which satisfies the properties 1)-3) in Theorem 1.4.

A proof of Theorem 2.2 is given in Section 2.3.

2.2. The period homomorphism of algebras $\mathrm{P}' : \mathcal{H}_* \longrightarrow \mathbb{C} \otimes \mathbb{C}$

This Section is an elaborate exposition of Section 4 of [21].

1. The period operator and the period matrix. Let H be a mixed Hodge-Tate structure over $\mathbb{Q}$. Then there is an isomorphism

$$H_{\mathbb{C}} = \oplus_p F^p H_{\mathbb{C}} \cap W_{2p} H_{\mathbb{C}}. \tag{2.4}$$

Furthermore, the following canonical map is an isomorphism:

$$F^p H_{\mathbb{C}} \cap W_{2p} H_{\mathbb{C}} \stackrel{\sim}{\longrightarrow} gr_{2p}^W H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}. \tag{2.5}$$

Using isomorphisms (2.5) and (2.4) we get a canonical morphism

$$S_{HT} : \oplus_p gr_{2p}^W H_{\mathbb{Q}} \longrightarrow H_{\mathbb{C}}.$$

On the other hand a splitting of the weight filtration on $H_{\mathbb{Q}}$ also provides us a morphism

$$S_W : \oplus_p gr_{2p}^W H_{\mathbb{Q}} \longrightarrow H_{\mathbb{C}}.$$

Both maps became isomorphisms when extended to $\oplus_p gr_{2p}^W H_{\mathbb{C}}$. Therefore a splitting of the weight filtration on $H_{\mathbb{Q}}$ provides a map, called the *period operator*:

$$S_{HT}^{-1} \circ S_W : \oplus_p gr_{2p}^W H_{\mathbb{C}} \longrightarrow \oplus_p gr_{2p}^W H_{\mathbb{C}}.$$

Let (f_n, H, v_0) be a Hodge-Tate structure over $\mathbb{Q}$, framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(n)$. Choose a splitting s over $\mathbb{Q}$ of the weight filtration on $H_{\mathbb{Q}}$. We define the period of the splitted framed Hodge-Tate structure $(f_n, H, v_0; s)$ as the matrix coefficient of the period operator:

$$p(f_n, H, v_0; s) := < v_0 | S_{HT}^{-1} \circ S_W | f^n \rangle.$$

Choose a basis in each $\mathbb{Q}$-vector space $gr_{2p}^W H_{\mathbb{Q}}$, providing a basis in their direct sum. The *period matrix* is the matrix of the period operator in this basis. One can define a mixed Hodge-Tate structure by exhibiting its period matrix. See an example below.

We define an equivalence relation on the set of all splitted framed $\mathbb{Q}$-Hodge-Tate structures as the finest equivalence relation for which any morphism of mixed $\mathbb{Q}$-Hodge structure $H \to H'$ respecting the splittings and the frames is an equivalence.

Let $\widetilde{\mathcal{H}}_n$ be the set of equivalence classes of splitted n-framed Hodge-Tate structures. Then $\widetilde{\mathcal{H}}_* := \oplus_n \widetilde{\mathcal{H}}_n$ is equipped in the usual way with a structure of a graded Hopf algebra. For instance $\widetilde{\mathcal{H}}_1 = \mathbb{C} \otimes \mathbb{Q}$. In particular there is a coproduct map $\Delta : \widetilde{\mathcal{H}}_* \to \widetilde{\mathcal{H}}_* \otimes \widetilde{\mathcal{H}}_*$.

Let $H \to H'$ be a morphism of Hodge-Tate structures respecting the frames and splittings. Then the periods of H and H' are the same, so we get the period homomorphism

$$\widetilde{p}_n : \widetilde{\mathcal{H}}_n \to \mathbb{C}.$$

2. The big period map. Let A and B be operators in a $\mathbb{Q}$-vector space V. Let $\{v_k\}$ be a $\mathbb{Q}$-basis in V, and $\{f^k\}$ be the dual basis. Define

$$\langle f^n | B \otimes_{\mathbb{Q}} A | v_0 \rangle := \sum_{v_k} \langle f^n | B | v_k \rangle \otimes_{\mathbb{Q}} \langle f^k | A | v_0 \rangle \ \in \ \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C},$$

where the sum is over all basis vectors v_k. It is well defined.

Definition 2.3. — *Let $(f^n, H, v_0; s)$ be a splitted framed $\mathbb{Q}$-Hodge-Tate structure, and $\mathcal{M}$ the period operator on $\oplus_k gr_{-2k}^W H_{\mathbb{Q}}$. Then we set*

$$\mathrm{P}'_n(f^n, H, v_0; s) := \langle f^n | \mathcal{M} \otimes_{\mathbb{Q}} \mathcal{M}^{-1} | v_0 \rangle \in \mathbb{C} \otimes \mathbb{C}. \tag{2.6}$$

Lemma 2.4. — *The element (2.6) does not depend on the choice of splitting.*

Proof. — The normalised period matrix corresponding to a different splitting is given by $\mathcal{M}\mathrm{N}$, where N is a rational unipotent upper triangular

matrix. One has

$$\langle f^n|\mathcal{M}\mathrm{N}\otimes_{\mathbb{Q}}(\mathcal{M}\mathrm{N})^{-1}|v_0\rangle \quad = \quad \langle f^n|\mathcal{M}\otimes_{\mathbb{Q}}\mathcal{M}^{-1}|v_0\rangle.$$

□

Notice that $\mathbb{C}\otimes_{\mathbb{Z}}\mathbb{C}$ is an algebra: $(a\otimes b)\cdot(a'\otimes b')=aa'\otimes bb'$.

Lemma 2.5 tells that the big period map P'_n is multiplicative: it takes the tensor product of the splitted framed Hodge -Tate structures into the product in $\mathbb{C}\otimes_{\mathbb{Z}}\mathbb{C}$.

Lemma 2.5. — *Let* M *and* M′ *be splitted framed Hodge-Tate structures of weights* m *and* m'*. Then*

$$\mathrm{P}'_{m+m'}(f^m\otimes f'^{m'},\mathrm{M}\otimes\mathrm{M}',v_0\otimes v'_0;s\otimes s')=\mathrm{P}'_m(f^m,\mathrm{M},v_0;s)\cdot\mathrm{P}'_{m'}(f'^{m'},\mathrm{M}',v'_0;s').$$

Proof.— Let $\mathcal{M}$ (respectively $\mathcal{M}'$) be the normalised period matrix for the splitted framed Hodge-Tate structure M (respectively M′). Then the normalised period matrix describing $\mathrm{M}\otimes\mathrm{M}'$ is just the tensor product $\mathcal{M}\otimes\mathcal{M}'$ of the normalised period matrices $\mathcal{M}$ and $\mathcal{M}'$. Evidently,

$$\langle f^p\otimes f'^q\mid\mathcal{M}\otimes\mathcal{M}'\mid e_0\otimes e'_0\rangle=\langle f^p|\ \mathcal{M}\mid e_0\rangle\langle f'^q\mid\mathcal{M}'\mid e'_0\rangle.$$

The claim follows immediately from this remark. □

Definition 2.6. — *The big period map* P_n *is the composition of the map* P'_n *with the map*

$$\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C}\longrightarrow\mathbb{C}^*\otimes_{\mathbb{Q}}\mathbb{C}(n-2),\qquad a\otimes b\longmapsto\exp(2\pi i\cdot a)\otimes 2\pi i\cdot b\otimes(2\pi i)^{n-2}. \quad (2.7)$$

Let U be a complex domain. There is a map

$$\omega:\mathcal{O}_U\otimes_{\mathbb{Q}}\mathcal{O}_U\longrightarrow\Omega^1_U,\quad f\otimes g\longmapsto(df)g.$$

Theorem 2.7. — *a) The map* P'_n *is a homomorphism of abelian groups* $\mathcal{H}_n\longrightarrow\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C}$.

Given $\mathrm{H}_m\in\mathcal{H}_m$ *and* $\mathrm{H}_n\in\mathcal{H}_n$ *one has*

$$\mathrm{P}'_{n+m}(\mathrm{H}_m\otimes\mathrm{H}_n)=\mathrm{P}'_m(\mathrm{H}_m)\cdot\mathrm{P}'_n(\mathrm{H}_n).$$

So the collection of the maps $\{\mathrm{P}'_n\}$ *gives rise to an algebra homomorphism*

$$\mathrm{P}':\mathcal{H}_*\longrightarrow\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C}.$$

b) The restriction of the map P_n *to* $\mathrm{Ker}(\Delta'_n)$ *coincides with the natural isomorphism*

$$\frac{\mathbb{C}}{(2\pi i)^n\mathbb{Q}} \quad = \quad \mathbb{C}^*_{\mathbb{Q}}\otimes(2\pi i)^{n-1}. \quad (2.8)$$

c) Let H_U be a variation of framed mixed Hodge-Tate structures over a domain U. Then there is a section $\mathrm{P}'(H_U) \in \mathcal{O}_U \otimes_{\mathbb{Q}} \mathcal{O}_U$, and the following composition is zero:

$$H_U \stackrel{\mathrm{P}'}{\longrightarrow} \mathcal{O}_U \otimes \mathcal{O}_U \stackrel{\omega}{\longrightarrow} \Omega^1_U, \ \omega \circ \mathrm{P}' = 0.$$

Proof.— To prove the part a) of Theorem 2.7 we rewrite the map P'_n in terms of the Hopf algebra $\mathcal{H}_*$. This is done in the Appendix. The second statement follows then from Lemma 2.5.

b) Clear from the definitions.

c) We will prove it in paragraph 5 below. □

3. Explicit formulas. Given a variation of splitted framed $\mathbb{Q}$-Hodge-Tate structure H, choose a basis $\{v_i\}$ over $\mathbb{Q}$ in each fiber of a variation. We assume that basis vectors v_i are of pure weight $\mathrm{wt}(v_i)$. Denote by $\{f^i\}$ the dual basis. We use notation $\langle f|\mathcal{M}|v\rangle$ for $p(f, H, v; s)$.

We usually assume that the framing is given by basis vectors (v_0, f^n).

Set $\mathcal{M} = 1 + \mathcal{M}_0$. Since $\mathcal{M}_0$ is nilpotent, expanding $(1 + \mathcal{M}_0)^{-1} = \sum_{k\geq 0}(-1)^k \mathcal{M}_0^k$ we get

$$\langle f^n|\mathcal{M} \otimes_{\mathbb{Q}} \mathcal{M}^{-1}|v_0\rangle \ = \ \sum_{k\geq 0}(-1)^k \langle f^n|\mathcal{M} \otimes_{\mathbb{Q}} \mathcal{M}_0^k|v_0\rangle. \tag{2.9}$$

By (2.9), the big period of a splitted framed Hodge-Tate structure $(H; f^n, v_0; s)$ is

$$\mathrm{P}'(f^n, H, v_0; s) \in \mathbb{C} \otimes \mathbb{C}.$$

$$\mathrm{P}'(f^n, H, v_0; s) = \langle f^n|\mathcal{M}|v_0\rangle \otimes 1 + \tag{2.10}$$

$$\sum_{k\geq 2} \sum_{0<i_1<\ldots<i_{k-1}<n} (-1)^{k-1} \langle f^n|\mathcal{M}|v_{i_{k-1}}\rangle \otimes \langle f^{i_{k-1}}|\mathcal{M}|v_{i_{k-2}}\rangle \cdot \ldots \cdot \langle f^{i_1}|\mathcal{M}|v_0\rangle +$$

$$\sum_{k\geq 1} \sum_{0<i_1<\ldots<i_{k-1}<n} (-1)^{k} \cdot 1 \otimes \langle f^n|\mathcal{M}|v_{i_{k-1}}\rangle \langle f^{i_{k-1}}|\mathcal{M}|v_{i_{k-2}}\rangle \cdot \ldots \cdot \langle f^{i_1}|\mathcal{M}|v_0\rangle. \tag{2.11}$$

The sum is over all nonempty chains of basis vectors $v_i \in gr^W_{-2i} H_{\mathbb{Q}}$, $0 < i < n$.

Since the term (2.11) disappears after the projection $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathbb{C}^* \otimes_{\mathbb{Q}} \mathbb{C}(n-2)$, we have

$$(2\pi i)^{-n+2}\mathrm{P}(f^n, H, v_0; s) \in \mathbb{C}^* \otimes \mathbb{C}.$$

$$(2\pi i)^{-n+2}\mathrm{P}(f^n, H, v_0; s) = \exp(2\pi i \cdot \langle f^n|\mathcal{M}|v_0\rangle) \otimes 2\pi i + \tag{2.12}$$

$$\sum_{k\geq 2}(-1)^{k-1}\sum_{0<i_1<\ldots<i_{k-1}<n}\exp(2\pi i\cdot\langle f^{i_n}|\mathcal{M}|v_{i_{k-1}}\rangle)\otimes 2\pi i\cdot\langle f^{i_{k-1}}|\mathcal{M}|v_{i_{k-2}}\rangle$$
$$\cdot\ldots\cdot\langle f^{i_1}|\mathcal{M}|v_0\rangle.$$

4. Examples. 1. Let us define a Hodge-Tate structure M using a normalised period matrix:

$$\mathcal{M}:=\begin{pmatrix}1&&&\\x_1&1&&\\x_2&y_1&1&\\x_3&y_2&z_1&1\end{pmatrix},\ x_i,y_j,z_1\in\mathbb{C}.$$

Let $\mathcal{I}$ be the matrix of the operator acting by $(2\pi i)^{-k}$ on $gr^W_{2k}H_{\mathbb{Q}}$. Then the period matrix is

$$\widetilde{\mathcal{M}}:=\mathcal{M}\mathcal{I}=\begin{pmatrix}1&&&\\2\pi i\cdot x_1&2\pi i&&\\(2\pi i)^2\cdot x_2&(2\pi i)^2\cdot y_1&(2\pi i)^2&\\(2\pi i)^3\cdot x_3&(2\pi i)^3\cdot y_2&(2\pi i)^3\cdot z_1&(2\pi i)^3\end{pmatrix}.$$

Remark.— The matrix $\widetilde{\mathcal{M}}$ is the period matrix which appear naturally in algebraic geometry. The normalized period matrix $\mathcal{M}$ is more convenient when we work with the big period.

Precisely, if M is the Hodge realization of a mixed Tate motive, the entries of the canonical period matrix are periods of rational algebraic differential forms over relative cycles. The $\widetilde{\mathcal{M}}$ is the matrix of the comparison isomorphism $M_{DR}\otimes\mathbb{C}\longrightarrow M_{Betti}\otimes\mathbb{C}$ in the natural $\mathbb{Q}$-bases in M_{DR} and M_{Betti}.

Let C_i be the i-th column of the matrix $\widetilde{\mathcal{M}}$. Let e_{-j} be the column whose only non zero entry is 1 on j-th place. We define the weight filtration $W_\bullet$ and the Hodge filtration $F^\bullet$ by

$$W_{-6}M=\langle C_3\rangle_{\mathbb{Q}},\ W_{-4}M=\langle C_2,C_3\rangle_{\mathbb{Q}},\ W_{-2}M=\langle C_1,C_2,C_3\rangle_{\mathbb{Q}},$$
$$W_0M=\langle C_0,C_1,C_2,C_3\rangle_{\mathbb{Q}}.$$
$$F^0M=\langle e_0\rangle,\ F^{-1}M=\langle e_0,e_{-1}\rangle,\ F^{-2}M=\langle e_0,e_{-1},e_{-2}\rangle,$$
$$F^{-3}M=\langle e_0,e_{-1},e_{-2},e_{-3}\rangle.$$

The splitted Hodge-Tate structure M has a framing given by e_0 and $(2\pi i)^{-3}f^3$. Its period is x_3. The big period is

$$\mathrm{P}'_3(M)=x_3\otimes 1+y_2\otimes(-x_1)\tag{2.13}$$
$$+z_1\otimes(-x_2+x_1y_1)+1\otimes(-x_3+x_1y_2+x_2z_1-x_1y_1z_1)\in\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C}.$$

The period P_3 is given by

$$\mathrm{P}'_3(M) = \exp(2\pi i \cdot x_3) \otimes 2\pi i + \exp(2\pi i \cdot y_2) \otimes \exp(2\pi i \cdot (-x_1))$$

$$+\exp(2\pi i \cdot z_1) \otimes \exp(2\pi i \cdot (-x_2 + x_1 y_1)).$$

2. Here is a classical example of the period matrix for the variation of Hodge-Tate structures related to the dilogarithm (due to Deligne):

$$\widetilde{\mathcal{L}}_2 := \quad \begin{pmatrix} 1 & 0 & \\ -\mathrm{Li}_1(z) & 2\pi i & \\ -\mathrm{Li}_2(z) & 2\pi i \cdot \log z & (2\pi i)^2 \end{pmatrix}.$$

Then

$$\mathrm{P}'_2(\mathcal{L}_2) = -\frac{\mathrm{Li}_2(z)}{(2\pi i)^2} \otimes 1 + \frac{\log z}{2\pi i} \otimes \frac{\log(1-z)}{2\pi i} + 1 \otimes \frac{\mathrm{Li}_2(z) - \log z \log(1-z)}{(2\pi i)^2} \in \mathbb{C} \otimes \mathbb{C}.$$

$$\mathrm{P}_2(\mathcal{L}_2) = \exp(\frac{-\mathrm{Li}_2(z)}{2\pi i}) \otimes 2\pi i + z \otimes \log(1-z) \in \mathbb{C}^* \otimes \mathbb{C}.$$

The invariant $\mathrm{P}_2(\mathcal{L}_2)$ was first written by S. Bloch [Bl3]. Generalizing it, $Sym^{n-1}_{\mathbb{Q}}\mathbb{C} \otimes \mathbb{C}^*$-valued invariants of the Hodge-Tate structures related to classical n-logarithms where constructed in [BD] and [Bl4]. However the approach of these papers is different from ours; it uses the specific structure of the Hodge-Tate structures related to classical polylogarithms, which can not be generalized to other mixed Tate motives.

5. Differential equations on periods and the Griffiths transversality condition. A variation of mixed Hodge structures satisfies the Griffiths transversality condition. We say that a partial period $\langle f^l | \mathcal{M} | v_k \rangle$, where $v_k \in gr^W_{-2k} H$ and $f^l \in (gr^W_{-2l} H)^*$, has *amplitude* $l - k$.

THEOREM 2.8. — *Let $\mathcal{M}$ be a normalised period matrix of a variation of splitted framed Hodge-Tate structures $(H_U; v_0, f^n; s)$. Then*

i) The connection ∇ on the variation is given by

$$\nabla(v_k) = -\sum_{\{v_{k+1}\}} \langle f^{k+1} | d\mathcal{M} | v_k \rangle \cdot v_{k+1}. \tag{2.14}$$

The sum is over basis vectors $\{v_{k+1}\}$ of weight $-2(k+1)$.

ii) The Griffiths transversality condition is equivalent to the following differential equations on the entries of the normalised period matrix $\mathcal{M}$:

$$\langle f^{k+s} | \mathcal{M}^{-1} d\mathcal{M} \Big| v_k \rangle \;=\; 0 \;\forall s > 1. \tag{2.15}$$

iii) The period $\langle v_0|\mathcal{M}|f^n\rangle$ *satisfies a differential equation*

$$d\langle f^n|\mathcal{M}|v_0\rangle = \sum_{\{v_{n-1}\}} \langle f^n|\mathcal{M}|v_{n-1}\rangle d\langle f^{n-1}|\mathcal{M}|v_0\rangle. \tag{2.16}$$

iv) The Griffiths transversality is equivalent to differential equations (2.16) for all partial periods of amplitudes ≥ 2.

Proof.— The vectors $\sum_j \langle f^j|\mathcal{M}|v_i\rangle v_j$ are flat sections of the connection ∇ on then variation:

$$0 = \nabla\Big(\sum_j \langle f^j|\mathcal{M}|v_i\rangle v_j\Big) = \sum_j \langle f^j|\mathcal{M}|v_i\rangle \cdot \nabla(v_j) + \sum_j d\langle f^j|\mathcal{M}|v_i\rangle \cdot v_j.$$

Therefore

$$\nabla(v_i) = -\sum_j \langle f^j|\mathcal{M}^{-1}d\mathcal{M}|v_i\rangle \cdot v_j.$$

i) To check (2.14) notice that the only way $\mathcal{M}^{-1}d\mathcal{M}$ can have a non-zero matrix coefficient of amplitude 1 is that it is the matrix coefficient of amplitude 1 of $d\mathcal{M}$.

ii) The Griffiths transversality just means that all matrix coefficients of $\mathcal{M}^{-1}d\mathcal{M}$ of amplitude bigger then 1 are zero, which is just what (2.15) says.

iii) - iv). Let us write, using (2.15) and assuming $n > 1$,

$$0 = \langle f^n|\mathcal{M}^{-1}d\mathcal{M}|v_0\rangle =$$

$$\langle f^n|d\mathcal{M}|v_0\rangle - \sum_{\{v_{n-1}\}} d\langle f^n|\mathcal{M}|v_{n-1}\rangle\langle f^{n-1}|\mathcal{M}|v_0\rangle + \sum_{k\leq n-2} \langle f^n|\mathcal{M}^{-1}d\mathcal{M}|v_k\rangle\langle f^k|\mathcal{M}|v_0\rangle.$$

The last summand is zero by (2.15). So we get differential equation (2.16), and the Claim iv). □

Remark.— Define a homomorphism $\Omega_n : \widetilde{\mathcal{H}}_n \longrightarrow \Omega^1_U$ as the composition

$$\widetilde{\mathcal{H}}_n \stackrel{\Delta_{n-1,1}}{\longrightarrow} \widetilde{\mathcal{H}}_{n-1} \otimes \widetilde{\mathcal{H}}_1 \stackrel{pdp}{\longrightarrow} \Omega^1_U.$$

The differential equation (2.16) for the period $\langle f^n|\mathcal{M}|v_0\rangle$ can be rewritten as

$$d\langle f^n|\mathcal{M}|v_0\rangle = \Omega_n(H_U; f^n, v_0; s). \tag{2.17}$$

6. The big period map via the Hopf algebra $\mathcal{H}_*$. We use notation as in Section 2.1. Projecting to $\otimes^p A_{>0}$ the coproduct and the iterated coproduct, we get their reduced versions:

$$\Delta' : A_* \longrightarrow A_{>0} \otimes A_{>0}, \quad \Delta'^{(p)} : A \longrightarrow \otimes^p A_{>0}.$$

For $n \geq 2$, let us consider an algebra map

$$m_n : A \longrightarrow A_{>0} \otimes A_{>0}$$

given by the n-iterated reduced coproduct followed by the product of the first $n-1$ factors:

$$A_* \stackrel{\Delta'^{(n)}}{\longrightarrow} \otimes^{n-1} A_{>0} \otimes A_{>0} \stackrel{\mu^{(n-1)} \otimes id}{\longrightarrow} A_{>0} \otimes A_{>0}.$$

Let $m_1 : A_* \longrightarrow A_* \otimes A_*$, $a \longmapsto 1 \otimes a$. Now set:

$$m : A_* \longrightarrow A_* \otimes A_*, \quad m := \sum_{n \geq 1} (-1)^{n-1} m_n.$$

Let us define a map $\widetilde{m} : A_* \longrightarrow A_* \otimes 1 \hookrightarrow A_* \otimes A_*$ by setting

$$\widetilde{m} := \sum_{n \geq 1}^{\infty} (-1)^n \mu^{(n)} \circ \Delta^{(n)} : A_* \longrightarrow A_* = A_* \otimes 1 \hookrightarrow A_* \otimes A_*,$$

$$\Delta^{(1)} = \mu^{(1)} = \mathrm{id}.$$

We apply this to the Hopf algebra $\mathcal{H}_*$. We get a map

$$m + \widetilde{m} : \mathcal{H}_* \longrightarrow \mathcal{H}_* \otimes \mathcal{H}_*.$$

The explicit formula for the map P'_n just means the following.

Lemma 2.9. — *The big period map P'_n is equal to a composition*

$$\mathcal{H}_* \quad \stackrel{m+\widetilde{m}}{\longrightarrow} \quad \mathcal{H}_* \otimes \mathcal{H}_* \quad \stackrel{2\pi i \cdot p \otimes 2\pi i \cdot p}{\longrightarrow} \quad \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Then P_n is the composition

$$\mathcal{H}_* \quad \stackrel{m}{\longrightarrow} \quad \mathcal{H}_* \otimes \mathcal{H}_* \quad \stackrel{p \otimes p}{\longrightarrow} \quad \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}^*(n-2). \tag{2.18}$$

The term (2.11) corresponding to $\widetilde{m}$ disappears after the projection $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}^*(n-2)$.

2.3. Construction of period morphisms and proof of Theorem 2.2

Step 1. The map $\mathrm{P}^\bullet_n$. Let us define a homomorphism of abelian groups

$$P^k_n : \otimes^k \mathcal{H}_* \longrightarrow \underbrace{\mathcal{O}^* \otimes \ldots \otimes \mathcal{O}^*}_{\text{k times}} \otimes \mathcal{O} \otimes \underbrace{2\pi i \otimes \ldots \otimes 2\pi i}_{\text{n-k-1 times}}. \tag{2.19}$$

First, there is an associative algebra structure on $\otimes^{\bullet-1}\mathcal{O}$ given by

$$(\otimes^{k+1}\mathcal{O}) * (\otimes^{l+1}\mathcal{O}) \longrightarrow \otimes^{k+l+1}\mathcal{O},$$

$$(a_0 \otimes ... \otimes a_k) * (b_0 \otimes ... \otimes b_l) \longmapsto a_0 \otimes ... \otimes a_k \cdot b_0 \otimes ... \otimes b_l.$$

Let $H_i \in \mathcal{H}_*$. We set

$$\mathrm{P'}^k_n(H_1 \otimes ... \otimes H_k) := P'_n(H_1) * ... * P'_n(H_k) \in \otimes^{k+1}\mathcal{O}.$$

Next, consider a map

$$\mathrm{Exp}^{(k)} \otimes 2\pi i \cdot \mathrm{Id} : \ \otimes^{k+1}\mathcal{O} \longrightarrow \underbrace{\mathcal{O}^* \otimes \ldots \otimes \mathcal{O}^*}_{\text{k times}} \otimes \mathcal{O}. \tag{2.20}$$

$$\mathrm{Exp}^{(k)} := \underbrace{\exp(2\pi i \cdot *) \otimes ... \otimes \exp(2\pi i \cdot *)}_{\text{k times}}.$$

We define the map (2.19) by setting

$$P^k_n := \underbrace{2\pi i \otimes \ldots \otimes 2\pi i}_{\text{n-k-1 times}} \otimes \Big(\mathrm{Exp}^{(k)} \otimes (2\pi i \cdot \mathrm{Id})\Big) \circ \mathrm{P'}^k_n. \tag{2.21}$$

Step 2. The maps $\{(-1)^k P^k_n\}$ *provide a morphism of complexes.* Equivalently, we have to show that the following diagram is a bicomplex, where Δ' is the restricted coproduct:

$$\begin{array}{ccccccc}
\mathcal{H}_n & \xrightarrow{\Delta'} & (\mathcal{H}\otimes\mathcal{H})_n & \xrightarrow{\Delta'} \ldots \xrightarrow{\Delta'} & \otimes^n\mathcal{H}_1 \\
\downarrow \mathrm{P}^1_n & & \downarrow \mathrm{P}^2_n & & \downarrow \mathrm{P}^n_n \\
\mathcal{O}^*\otimes\mathcal{O}(n-2) & \xrightarrow{d} & \mathcal{O}^*\otimes\mathcal{O}^*\otimes\mathcal{O}(n-3) & \xrightarrow{d} \ldots \xrightarrow{d} & \otimes^n\mathcal{O}^*
\end{array}$$

Let us show that the left square is anticommutative.

For the restricted coproduct Δ', we have the following element in $\mathcal{O} \otimes \mathcal{O} \otimes \mathcal{O}$:

$$\mathrm{P'}^2_n(\Delta'(\mathcal{M})) = \sum \langle f^n|\mathcal{M}|v_l\rangle \otimes \langle f^l|\mathcal{M}^{-1}|v_m\rangle \cdot \langle f^m|\mathcal{M}|v_k\rangle \otimes \langle f^k|\mathcal{M}^{-1}|v_0\rangle. \tag{2.22}$$

The sum is over all basis vectors v_i satisfying the following two conditions

$$wt(v_n) \leq wt(v_l) \leq wt(v_m) \leq wt(v_k) \leq 0. \tag{2.23}$$

$$wt(v_n) < wt(v_m) < 0. \tag{2.24}$$

Condition (2.24) results from taking the restricted coproduct Δ' rather then the coproduct Δ.

Let us compute the image of element (2.22) under the map

$$\mathrm{Exp}^{(2)} \otimes 2\pi i : \mathcal{O} \otimes \mathcal{O} \otimes \mathcal{O} \longrightarrow \mathcal{O}^* \otimes \mathcal{O}^* \otimes \mathcal{O}, \tag{2.25}$$

$$a \otimes b \otimes c \longmapsto \exp(2\pi i a) \otimes \exp(2\pi i b) \otimes 2\pi i c.$$

Observe that since $\mathcal{M}\mathcal{M}^{-1} = I$, we have

$$\sum_{wt(v_l) \leq wt(v_m) \leq wt(v_k)} \langle f_l | \mathcal{M}^{-1} | v_m \rangle \cdot \langle f^m | \mathcal{M} | v_k \rangle = \delta_{0,l}. \tag{2.26}$$

Now there are three cases of the summation.

i) If $wt(v_k) < 0$, and $wt(v_n) < wt(v_l)$, then thanks to conditions (2.23) - (2.24) and formula (2.26), the corresponding sum in (2.22) collapses to

$$\sum \langle f^n | \mathcal{M} | v_k \rangle \otimes 1 \otimes \langle f^k | \mathcal{M}^{-1} | v_0 \rangle. \tag{2.27}$$

ii) If $wt(v_k) = 0$, then, thanks to (2.24) the corresponding sum in (2.22) is

$$\sum_{v_l} \sum_{v_m \neq v_0} \langle f^n | \mathcal{M} | v_l \rangle \otimes \langle f^l | \mathcal{M}^{-1} | v_m \rangle \cdot \langle f^m | \mathcal{M} | v_0 \rangle \otimes 1 \stackrel{(2.26)}{=}$$

$$- \sum_{v_l} \langle f^n | \mathcal{M} | v_l \rangle \otimes \langle f^l | \mathcal{M}^{-1} | v_0 \rangle \otimes 1. \tag{2.28}$$

Indeed, since $\langle f_0 | \mathcal{M} | v_0 \rangle = 1$, formula (2.26) implies

$$\sum_{v_m \neq v_0} \langle f^l | \mathcal{M}^{-1} | v_m \rangle \cdot \langle f^m | \mathcal{M} | v_0 \rangle = - \langle f^l | \mathcal{M}^{-1} | v_0 \rangle.$$

iii) If $wt(v_l) = wt(v_n)$, we similarly get

$$\sum_{v_m \neq v_n} 1 \otimes \langle f^n | \mathcal{M}^{-1} | v_m \rangle \cdot \langle f^m | \mathcal{M} | v_k \rangle \otimes \langle f^k | \mathcal{M}^{-1} | v_0 \rangle.$$

$$- \sum_{v_k} 1 \otimes \langle f^n | \mathcal{M} | v_k \rangle \otimes \langle f^k | \mathcal{M}^{-1} | v_0 \rangle. \tag{2.29}$$

Since $\exp(2\pi i) = 1$ is the neutral element in $\mathcal{O}^*$, (2.27) and (2.29) contributes zero after applying map (2.25). So applying the map (2.25) to the expression (2.28) we get

$$-\sum_{v_l} \exp(2\pi i \cdot \langle f^n|\mathcal{M}|v_l\rangle) \otimes \exp(2\pi i \cdot \langle f^l|\mathcal{M}^{-1}|v_0\rangle) \otimes 2\pi i.$$

On the other hand, by the definition of P'_n in (2.6),

$$\mathrm{P}'_n(\mathcal{M}) = \sum_{v_l} \langle f^n|\mathcal{M}|v_l\rangle \otimes \langle f^l|\mathcal{M}^{-1}|v_0\rangle.$$

Therefore thanks to the definition of $\mathrm{P}_n = \mathrm{P}^1_n$ in (2.7), and the definition of d in (1.2),

$$d \circ \mathrm{P}^1_n(\mathcal{M}) = \sum_{v_l} \exp(2\pi i \cdot \langle f^n|\mathcal{M}|v_l\rangle) \otimes \exp(2\pi i \cdot \langle f^l|\mathcal{M}^{-1}|v_0\rangle) \otimes 2\pi i.$$

We conclude that

$$\mathrm{P}^2_n(\Delta'(\mathcal{M})) + d \circ \mathrm{P}^{(1)}_n(\mathcal{M}) = 0.$$

In general we have to check that the composition of the restricted coproduct

$$\mathcal{M}_1 \otimes ... \otimes \mathcal{M}_k \longrightarrow \sum (-1)^{i-1} \mathcal{M}_1 \otimes ... \otimes \Delta'(\mathcal{M}_i) \otimes ... \otimes \mathcal{M}_k$$

with the map P^{k+1}_n is equal to $-d \circ \mathrm{P}^k_n(\mathcal{M}_1 \otimes ... \otimes \mathcal{M}_k)$. Notice that

$$\mathrm{P}^{k+1}_n(\mathcal{M}_1 \otimes ... \otimes \Delta'(\mathcal{M}_i) \otimes ... \otimes \mathcal{M}_k) = 0 \qquad if \quad i > 1.$$

Indeed, $\mathcal{M}_1 \otimes ... \otimes \Delta'(\mathcal{M}_i) \otimes ... \otimes \mathcal{M}_k$ has three terms, just like (2.27), (2.28), and (2.29). Each of them has the j-th factor 1, where $j = i+2, i+1, i$. So each of them vanishes when we apply the $\mathrm{Exp}^{(k)} \otimes (2\pi i \cdot \mathrm{Id})$ map (2.20). In the case $i = k$ only the very right factor survives, contributing $-d \circ \mathrm{P}^k_n(\mathcal{M}_1 \otimes ... \otimes \mathcal{M}_k)$.

Step 3. The composition $\Omega^k_n \circ \mathrm{P}^k_n = 0$ *for* $k < n$. It is enough to check an equivalent claim for $d\mathrm{P}'^{\,k}_n$. For $k = 1$, Theorem 2.8ii) implies, since $n > 1$,

$$d\mathrm{P}'^{\,1}_n(\mathcal{M}) = \sum_k \langle f^n|d\mathcal{M}|v_k\rangle \otimes \langle f^k|\mathcal{M}^{-1}|v_0\rangle = 0.$$

For $k = 2$ we have

$$d\mathrm{P}'^{\,2}_n(\mathcal{M} \otimes \mathcal{N}) =$$

$$\sum \langle f^n|d\mathcal{N}|v_l\rangle \otimes \langle f^l|\mathcal{N}^{-1}|v_m\rangle \cdot \langle f^m|d\mathcal{M}|v_k\rangle \otimes \langle f^k|\mathcal{M}^{-1}|v_0\rangle +$$

$$\sum \langle f^n|d\mathcal{N}|v_l\rangle \otimes \langle f^l|d\mathcal{N}^{-1}|v_m\rangle \cdot \langle f^m|\mathcal{M}|v_k\rangle \otimes \langle f^k|\mathcal{M}^{-1}|v_0\rangle.$$

Since $n > k = 2$, either $m > 1$ or $n - m > 1$. Then in the first line is zero since the factor of amplitude > 1 is zero by Theorem 2.8ii). The second line is always zero.

For general k we proceed just as in the case $k = 2$. The expression $d\mathrm{P}'^{\,k}_n$ consists of sums of k factors. If one of them has two differentials, it is zero. Otherwise each has just one differential, and one is of amplitude > 1, and so vanishes by Theorem 2.8ii). Theorem 2.2 is proved.

2.4. A variant: Lie-exponential complexes and Lie-period morphisms

Let X be a manifold, either a real or a complex analytic one.

Definition 2.10. — *The weight n Lie-exponential complex $\mathbb{Q}^\bullet_{\mathcal{E}}(n)$ is a complex of sheaves on X, concentrated in degrees $[0, n]$:*

$$\mathcal{O}(n-1) \longrightarrow \Lambda^2\mathcal{O}(n-2) \longrightarrow ... \longrightarrow \Lambda^n\mathcal{O} \stackrel{\wedge^n\exp}{\longrightarrow} \Lambda^n\mathcal{O}^*. \tag{2.30}$$

The differentials are given by

$$(2\pi i)^{n-k} \otimes a_1 \wedge ... \wedge a_k \longmapsto (2\pi i)^{n-k-1} \otimes 2\pi i \wedge a_1 \wedge ... \wedge a_k, \;\; k < n,$$

$$a_1 \wedge ... \wedge a_n \longmapsto \exp(a_1) \wedge ... \wedge \exp(a_n).$$

For example, the complex $\mathbb{Q}^\bullet_{\mathcal{E}}(2)$ is

$$\mathcal{O}(1) \stackrel{\delta}{\longrightarrow} \Lambda^2\mathcal{O} \stackrel{\wedge^2\exp}{\longrightarrow} \Lambda^2\mathcal{O}^*.$$

Take the n-th symmetric power of the complex $\mathbb{Q}(1) \hookrightarrow \mathcal{O}$ in degrees $[0, 1]$. It is augmented by the exponential map to $\Lambda^n\mathcal{O}^*[-n]$. There is an isomorphism of complexes

$$\mathbb{Q}(n) \longrightarrow \mathbb{Q}^\bullet_{\mathcal{E}}(n) \;=\; \mathrm{Cone}\Big(\mathrm{Sym}^n\Big(\mathbb{Q}(1) \hookrightarrow \mathcal{O}\Big) \stackrel{\wedge^n\exp}{\longrightarrow} \Lambda^n\mathcal{O}^*[-n]\Big). \tag{2.31}$$

Therefore the complex (2.30) is a resolution of the constant sheaf $\mathbb{Q}(n)$.

Mapping Lie-exponential complexes to differential forms. Recall the holomorphic de Rham complex $\Omega^\bullet$ on a complex manifold X. There is a natural map from the weight n Lie-exponential complex to the holomorphic de Rham complex:

$$\omega^\bullet_n : \mathbb{Q}^\bullet_{\mathcal{E}}(n) \;\longrightarrow\; \Omega^\bullet.$$

Precisely, we have the following Lemma, proved by a simple check, which is left to a reader.

LEMMA 2.11. — *There is a canonical morphism of complexes of sheaves on X:*

$$\begin{array}{ccccccccc}
\mathcal{O}(n-1) & \longrightarrow & \Lambda^2\mathcal{O}(n-2) & \longrightarrow \ldots \longrightarrow & \Lambda^n\mathcal{O} & \longrightarrow & \Lambda^n\mathcal{O}^* \\
\downarrow \omega_n^{(0)} & ... & \downarrow \omega_n^{(1)} & & \downarrow \omega_n^{(n-1)} & & \downarrow \omega_n^{(n)} \\
\Omega^0 & \stackrel{d}{\longrightarrow} & \Omega^1 & \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} & \Omega^{n-1} & \longrightarrow & \Omega^n_{cl}
\end{array}$$

Here

$$\omega_n^{(m)}\Big((2\pi i)^{n-m-1} \otimes (f_0 \wedge f_1 \wedge ... \wedge f_m)\Big) :=$$

$$(2\pi i)^{n-m-1} m! \sum_{j=0}^{m} (-1)^j f_j \; df_0 \wedge \ldots \wedge \widehat{df_j} \wedge \ldots \wedge df_m, \; 0 \leq m < n,$$

$$\omega_n^{(n)}(f_1 \wedge ... \wedge f_n) := n! \; d\log f_1 \wedge ... \wedge d\log f_n.$$

Lie-*period morphisms of complexes.* The graded commutative Hopf algebra $\mathcal{H}_*$ gives rise to a graded Lie coalgebra $(\mathcal{L}_*, \delta)$:

$$\mathcal{L}_* := \frac{\mathcal{H}_{>0}}{\mathcal{H}_{>0} \cdot \mathcal{H}_{>0}}.$$

Let $\mathcal{L}^\bullet(n)$ be the weight n part of the standard cochain complex of the graded Lie coalgebra $\mathcal{L}_*$:

$$\mathcal{L}^\bullet(n) := \; \mathcal{L}_n \stackrel{\delta}{\longrightarrow} (\Lambda^2\mathcal{L})_n \stackrel{\delta}{\longrightarrow} (\Lambda^3\mathcal{L})_n \stackrel{\delta}{\longrightarrow} \ldots$$

CONJECTURE 2.12. — *There exists a canonical morphism of complexes of sheaves on X*

$$p_n^\bullet : \mathcal{L}^\bullet(n) \longrightarrow \mathbb{Q}_{\mathcal{E}}^\bullet(n),$$

called the Lie-*period morphism:*

$$\begin{array}{ccccccccccc}
 & & \mathcal{L}_n & \stackrel{\delta}{\to} & (\mathcal{L}\wedge\mathcal{L})_n & \stackrel{\delta}{\to} ... \stackrel{\delta}{\to} & (\Lambda^{n-1}\mathcal{L})_n & \stackrel{\delta}{\to} & \Lambda^n\mathcal{L}_1 \\
 & & \downarrow p_n^1 & & \downarrow p_n^2 & & \downarrow p_n^{n-1} & & \downarrow p_n^n \\
\mathcal{O}(n-1) & \to & \Lambda^2\mathcal{O}(n-2) & \to & \Lambda^3\mathcal{O}(n-3) & \to ... \to & \Lambda^n\mathcal{O} & \stackrel{\exp}{\to} & \Lambda^n\mathcal{O}^*
\end{array} \tag{2.32}$$

such that

(1) *After the identification $\mathcal{L}_1 = \mathcal{O}^*$ the map p_n^n is the identity map.*
(2) *The composition $\omega_n^\bullet \circ p_n^\bullet$ is zero everywhere except on the very right.*

The condition 2) just means that the following composition is zero:

$$\begin{array}{ccccccc} \mathcal{L}_n & \stackrel{\delta}{\longrightarrow} & (\mathcal{L}\wedge\mathcal{L})_n & \stackrel{\delta}{\longrightarrow} ... \stackrel{\delta}{\longrightarrow} & (\Lambda^{n-1}\mathcal{L})_n \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda^2\mathcal{O}(n-2) & \longrightarrow & \Lambda^3\mathcal{O}(n-3) & \longrightarrow ... \longrightarrow & \Lambda^n\mathcal{O} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^1_U & \stackrel{d}{\longrightarrow} & \Omega^2_U & \stackrel{d}{\longrightarrow} ... \stackrel{d}{\longrightarrow} & \Omega^{n-1}_U \end{array}$$

Let us explain the meaning of "canonical" in Conjecture 2.12. A canonical map of complexes

$$\mathcal{L}^\bullet(n) \longrightarrow \mathbb{R}_{\mathcal{D}}(n) \tag{2.33}$$

was defined in [24]. There we defined, on the level of appropriate complexes, a product $\mathbb{R}_{\mathcal{D}}(a)\otimes\mathbb{R}_{\mathcal{D}}(b) \longrightarrow \mathbb{R}_{\mathcal{D}}(a+b)$ making $\oplus_{a=0}^{\infty}\mathbb{R}_{\mathcal{D}}(a)$ into a DG commutative algebra. The key property of the map (2.33) is that its components describe a map DG commutative algebras

$$S^\bullet(\mathcal{L}^\bullet[-1]) \longrightarrow \oplus_{a=0}^{\infty}\mathbb{R}_{\mathcal{D}}(a).$$

So this map is completely determined its restriction to the Lie coalgebra $\mathcal{L}^\bullet$. The map $p^\bullet_n$, combined with a map $s^\bullet_n$ from Section 6 or its modification, should deliver canonical map (2.33).

2.5. The Lie-period map

Recall the graded commutative Hopf algebra over $\mathbb{Q}$ with a unit $A_\bullet = \oplus_{k=0}^{\infty}A_k$, see (2.1), coming with a product $\mu : A_\bullet \otimes A_\bullet \longrightarrow A_\bullet$ and a coproduct $\Delta : A_\bullet \longrightarrow A_\bullet \otimes A_\bullet$.

Let $\mu^{(p)} : A_\bullet^{\otimes p} \longrightarrow A_\bullet$ be the product map: $a_1 \otimes ... \otimes a_p \longmapsto a_1 \cdot ... \cdot a_p$.

Let us consider the iterated coproduct maps

$$\Delta^{(p)} : A_\bullet \longrightarrow A_\bullet^{\otimes p}.$$

They are defined inductively:

$$\Delta^{(p)} := (\Delta \otimes \mathrm{Id}^{(p-2)}) \circ \Delta^{(p-1)},$$

$$(\Delta \otimes \mathrm{Id}^{(p-2)})(a_1 \otimes \ldots \otimes a_{p-1}) := \Delta(a_1) \otimes a_2 \otimes \ldots \otimes a_{p-1}.$$

Equivalently, they are dual to the product maps $\mu^{(n)}$ for the dual Hopf algebra.

Let us consider the following map:

$$l : A_\bullet \longrightarrow A_\bullet,\ l(M) := \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \mu^{(p)} \circ \Delta^{(p)}(M).$$

Elaborating this:

$$l : M \longmapsto M - \frac{1}{2}\mu^{(2)} \circ \Delta^{(2)}(M) + \frac{1}{3}\mu^{(3)} \circ \Delta^{(3)}(M) - \dots .$$

The map l has the following geometric interpretation. Denote by G the pro-nilpotent group with the Lie algebra $\mathrm{Lie}(A_\bullet)$. Then $\mathcal{O}(G) = A_\bullet$ as algebras. Let Log be the inverse of the exponential map. Then the map l reads as follows:

$$l : \mathcal{O}(G) \longrightarrow \mathcal{O}(G),\ l(F)(g) := \langle dF, \mathrm{Log}(g)\rangle.$$

So evidently the map l is zero on $A_{>0} \cdot A_{>0}$. Therefore we get a canonical map of graded spaces

$$\mathrm{CoLie}(A_\bullet) \longrightarrow A_\bullet.$$

Let us define a map, which we call the Lie-*period map*:

$$\mathcal{P}_n : \mathrm{CoLie}_n(\mathcal{H}_\bullet) \longrightarrow \Lambda^2\mathbb{C}.$$

Consider the composition of the map l with the big period map P'_n:

$$\mathcal{P}_n = \mathrm{P}'_n \circ l : \mathcal{H}_n \xrightarrow{l} \mathcal{H}_n \xrightarrow{\mathrm{P}'_n} \mathbb{C} \otimes \mathbb{C}.$$

Proposition 2.13. — *The map $\mathcal{P}_n$ provides a map*

$$\mathcal{P}_n : \mathrm{CoLie}_n(\mathcal{H}_\bullet) \longrightarrow \Lambda^2\mathbb{C}.$$

Proof. — The map $\mathcal{P}_n$ is a map $\mathrm{CoLie}_n(\mathcal{H}_\bullet) \to \mathbb{C} \otimes \mathbb{C}$. We need to check that its image lies in $\Lambda^2\mathbb{C}$. □

Functions $\mathrm{L}_n(z)$ *obtained from classical polylogarithms via the* Lie-*period map.* Set

$$\sum_{k\geq 0} \beta_k t^k = \frac{t}{e^t - 1},\ \beta_k := \frac{B_k}{k!}.$$

So $\beta_{2m+1} = 0$ for $m \geq 1$, and $\beta_0 = 1, \beta_1 = -\frac{1}{2}, \beta_2 = \frac{1}{12}, \beta_4 = -\frac{1}{720}, \dots$ Let us consider a function

$$\mathrm{L}_n(z) := \sum_{k=0}^{n-1} \beta_k \mathrm{Li}_{n-k}(z) \log^k z.$$

The right hand side is defined as follows. Take a path γ from $a \in (0,1)$ to a point $z \in \mathbb{C}$ and continue analytically along this path the functions $\mathrm{Li}_1(z), \dots, \mathrm{Li}_n(z)$ using the inductive formula $\mathrm{Li}_m(z) := \int_\gamma \mathrm{Li}_{m-1}(t) d\log t$. Then make the sum on the right hand using these brunches. So

$$\mathrm{L}_2(z) = \mathrm{Li}_2(z) - \frac{1}{2}\mathrm{Li}_1(z)\log z.$$

$$\mathrm{L}_3(z) = \mathrm{Li}_3(z) - \frac{1}{2}\mathrm{Li}_2(z)\log z + \frac{1}{12}\mathrm{Li}_1(z)\log^2 z.$$
$$\mathrm{L}_4(z) = \mathrm{Li}_4(z) - \frac{1}{2}\mathrm{Li}_3(z)\log z + \frac{1}{12}\mathrm{Li}_2(z)\log^2 z.$$

The real version $\pi_n\Big(\sum_{k=0}^{n-1}\beta_k \mathrm{Li}_{n-k}(z)\log^k|z|\Big)$ of the function $\mathrm{L}_n(z)$, where $\pi_n(a+ib) = a$ for odd n and ib for even n, was considered by Zagier in [34], who showed that it is single valued. Its Hodge-theoretic interpretation was given by Beilinson and Deligne in [4].

Denote by $\langle \mathrm{Li}_n(z)\rangle = (f^n, \mathrm{Li}_n(z), v_0)$ the n-framed Hodge-Tate structure assigned to the classical n-logarithm, whose normalised period matrix is given as follows:

$$\begin{pmatrix}
1 & & & & & \\
-\frac{\mathrm{Li}_1(z)}{2\pi i} & 1 & & & & \\
-\frac{\mathrm{Li}_2(z)}{(2\pi i)^2} & \frac{\log z}{2\pi i} & 1 & & & \\
-\frac{\mathrm{Li}_3(z)}{(2\pi i)^3} & \frac{\log^2 z}{2\cdot(2\pi i)^2} & \frac{\log z}{2\pi i} & 1 & & \\
-\frac{\mathrm{Li}_4(z)}{(2\pi i)^4} & \frac{\log^3 z}{3!\cdot(2\pi i)^3} & \frac{\log^2 z}{2\cdot(2\pi i)^2} & \frac{\log z}{2\pi i} & 1 & \\
-\frac{\mathrm{Li}_5(z)}{(2\pi i)^5} & \frac{\log^4 z}{4!\cdot(2\pi i)^4} & \frac{\log^3 z}{3!\cdot(2\pi i)^3} & \frac{\log^2 z}{2\cdot 2\pi i} & \frac{\log z}{2\pi i} & 1
\end{pmatrix}$$

Notice that in the normalised period matrix all entries are of weight zero.

PROPOSITION 2.14. — *The maximal period of n-framed Hodge-Tate structure $l\langle \mathrm{Li}_n(z)\rangle$ is:*

$$\langle f^n \mid l\langle \mathrm{Li}_n(z)\rangle \mid v_0\rangle = -\mathrm{L}_n(z).$$

Proof. — Let us do an example first, the 5-logarithm. The calculation gives $(2\pi i)^{-5}$ times

$$\mathrm{Li}_5 - \frac{1}{2}\cdot\Big(\mathrm{Li}_4(z)\log z + \mathrm{Li}_3(z)\frac{\log^2 z}{2} + \mathrm{Li}_2(z)\frac{\log^3 z}{3!} + \mathrm{Li}_1(z)\frac{\log^4 z}{4!}\Big)$$
$$+\frac{1}{3}\cdot\Big(\mathrm{Li}_3(z)\log^2 z + (\frac{1}{2}\cdot 1 + 1\cdot\frac{1}{2})\cdot\mathrm{Li}_2(z)\log^3 z + (\frac{1}{3!}\cdot 1 + 1\cdot\frac{1}{3!} + \frac{1}{2}\cdot\frac{1}{2})\cdot\mathrm{Li}_1(z)\log^3 z\Big)$$
$$-\frac{1}{4}\cdot\Big(\mathrm{Li}_2(z)\log^3 z + (\frac{1}{2}\cdot 1\cdot 1 + 1\cdot\frac{1}{2}\cdot 1 + 1\cdot 1\cdot\frac{1}{2})\cdot\mathrm{Li}_1(z)\log^4 z\Big) + \frac{1}{5}\cdot\mathrm{Li}_1(z)\log^4 z.$$

In general we need to sum the following series in x (where $x = \log z$ in our application):

$$S(x) := 1 - \frac{1}{2}\cdot(e^x - 1) + \frac{1}{3}\cdot(e^x - 1)^2 - \frac{1}{4}\cdot(e^x - 1)^3 + \frac{1}{5}\cdot(e^x - 1)^5 - ...$$

One has $S(x)(e^x - 1) = \log(1 + e^x - 1) = x$. Therefore $S(x) = \frac{x}{e^x - 1}$. □

Calculating the monodromy. Let γ_0 (resp. γ_1) be a small counterclockwise loop around 0 (resp. 1). Let T_{γ_0} (resp. T_{γ_1}) be the monodromy operator around loop γ_0 (resp. γ_1).

LEMMA 2.15. — *One has*

$$\frac{1}{2\pi i}(T_{\gamma_1} - \mathrm{Id}): \quad \mathrm{L}_n(x) \longmapsto -(-1)^{n-1}\beta_{n-1}\log^{n-1}(x). \tag{2.34}$$

Proof.— One has

$$\frac{1}{2\pi i}(T_{\gamma_1} - \mathrm{Id}) : -\mathrm{Li}_n(z) \longmapsto \frac{\log^{n-1} z}{(n-1)!}.$$

From the definition of the Bernoulli polynomials, $\frac{xe^{tx}}{e^x-1} = \sum_{k=0}^{\infty} \frac{B_n(t)}{n!}x^n$. So

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_k t^{n-k}, \quad \text{and} \quad B_n(1) = (-1)^{n-1}B_n \quad \text{for } n \geq 1.$$

Therefore $\sum_{k=0}^{n-1} \frac{B_k}{k!(n-k-1)!} = \frac{B_{n-1}}{(n-1)!}$. Using this identity we get the formula (2.34). □

Examples.—

$$\frac{1}{2\pi i}(T_{\gamma_0} - \mathrm{Id}): \quad \log x \longmapsto 1, \quad \mathrm{L}_1(x) \longmapsto 0, \quad \mathrm{L}_2(x) \longmapsto -\frac{1}{2}\cdot \mathrm{L}_1(x),$$
$$\mathrm{L}_3(x) \longmapsto -\frac{1}{2}\cdot \mathrm{L}_2(x) - \frac{1}{12}\cdot \mathrm{L}_1(x)\cdot \log x + \frac{1}{12}\cdot \mathrm{L}_1(x),$$
$$\mathrm{L}_4(x) \longmapsto -\frac{1}{2}\cdot \mathrm{L}_3(x) - \frac{1}{12}\cdot \mathrm{L}_2(x)\cdot \log x + \frac{1}{12}\cdot \mathrm{L}_2(x).$$

$$\frac{1}{2\pi i}(T_{\gamma_1} - \mathrm{Id}): \quad \log x \longmapsto 0, \quad \mathrm{L}_1(x) \longmapsto -1, \quad \mathrm{L}_2(x) \longmapsto -\frac{1}{2}\cdot \log x,$$
$$\mathrm{L}_3(x) \longmapsto -\frac{1}{12}\cdot \log^2 x, \quad \mathrm{L}_4(x) \longmapsto 0.$$

3. **Period morphisms on polylogarithmic motivic complexes of weights ≤ 4**

Given a field F, let us recall the inductive definition of the groups $\mathcal{B}_n(F)$ [16]. One can set $\mathcal{B}_2(F) = B_2(F)$. There is a map

$$\mathbb{Z}[F^* - \{1\}] \xrightarrow{\delta_n} \mathcal{B}_n(F) \otimes F^*, \ \{x\} \longmapsto \{x\}_{n-1} \otimes x, \ n > 2.$$

Let us define a subgroup $\mathcal{A}_n(F) \subset \mathrm{Ker}\ \delta_n$. Given an element $\sum_i n_i\{f_i(t)\}$ in the kernel of δ_n for the field $F(t)$, the element $\sum_i n_i(\{f_i(t_0)\} - \{f_i(t_1)\})$,

where $t_0, t_1 \in F^* - \{1\}$, lies in $\mathcal{A}_n(F)$, and the subgroup $\mathcal{A}_n(F)$ is generated by such elements.

Our goal. We are going to construct, for $n \leq 4$, a morphism of complexes

$$\begin{array}{ccccccc} \mathcal{B}_n(\mathbb{C}) & \longrightarrow & \mathcal{B}_{n-1}(\mathbb{C}) \otimes \mathbb{C}^* & \longrightarrow ... \longrightarrow & \mathcal{B}_2(\mathbb{C}) \otimes \Lambda^{n-2}\mathbb{C}^* & \longrightarrow & \Lambda^n\mathbb{C}^* \\ \downarrow l^1_n & & \downarrow l^2_n & & \downarrow l^{n-1}_n & & \downarrow l^n_n \\ \Lambda^2\mathbb{C}(n-2) & \longrightarrow & \Lambda^3\mathbb{C}(n-3) & \longrightarrow ... \longrightarrow & \Lambda^n\mathbb{C} & \stackrel{\wedge^n \exp}{\longrightarrow} & \Lambda^n\mathbb{C}^* \end{array} \tag{3.1}$$

such that its composition with $\omega^\bullet_n$ is zero, and the map l^n_n is the identity.

Remark. — If $n = 4$ it will not be the canonical map from Conjecture 2.12.

We start with a few general observations which help to construct the map $l^\bullet_n$.

Proposition 3.1. — *Let $\mathcal{O} := \mathbb{C}(t)$. Let us suppose that we have maps l^1_n and l^2_n such that:*

i) The following diagram commutative:

$$\begin{array}{ccc} \mathbb{Z}[\mathcal{O}^* - \{1\}] & \stackrel{\delta_n}{\longrightarrow} & \mathcal{B}_{n-1}(\mathcal{O}) \otimes \mathcal{O}^* \\ \downarrow l^1_n & & \downarrow l^2_n \\ \Lambda^2\mathcal{O}(n-2) & \longrightarrow & \Lambda^3\mathcal{O}(n-3) \end{array}$$

ii) The following composition is zero:

$$\omega \circ l^1_n : \mathbb{Z}[\mathcal{O}^* - \{1\}] \longrightarrow \Lambda^2\mathcal{O}(n-2) \longrightarrow \Omega^1, \ \omega(f \wedge g) = fdg - gdf. \tag{3.2}$$

Then

$$l^1_n(\mathcal{A}_n(\mathcal{O})) = 0. \tag{3.3}$$

Proof. — Consider the following diagram:

$$\begin{array}{ccccccc} \mathcal{A}_n(\mathcal{O}) & \longrightarrow & \mathrm{Ker}\ \delta_n & \longrightarrow & \mathbb{Z}[\mathcal{O}] & \stackrel{\delta_n}{\longrightarrow} & \mathcal{B}_{n-1}(\mathcal{O}) \otimes \mathcal{O}^* \\ & & \downarrow & & \downarrow l^1_n & & \downarrow l^2_n \\ & & \mathcal{O}(n-2) & \longrightarrow & \Lambda^2\mathcal{O}(n-2) & \longrightarrow & \Lambda^3\mathcal{O}(n-3) \end{array}$$

Then

$$l^1_n(\mathrm{Ker}\ \delta_n) \subset \mathrm{Ker}\Big(\Lambda^2\mathcal{O}(n-2) \longrightarrow \Lambda^3\mathcal{O}(n-3)\Big) = \mathcal{O}(n-2).$$

The kernel of the restriction of the map $\omega : \Lambda^2\mathcal{O} \longrightarrow \Omega^1,\ f\wedge g \longmapsto fdg = gdf$ to the subgroup $2\pi i\wedge\mathcal{O} \subset \Lambda^2\mathcal{O}$ is $2\pi i\wedge\mathbb{C}$. Therefore, since the composition (3.2) is zero, we have $l^1_n(\mathrm{Ker}\ \delta_n) \subset \mathbb{C}$. This implies (3.3). □

Lemma 3.2. — *Let $U \subset \mathbb{C}^* - \{1\}$ be an open subset. Suppose that we have a commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}[U] & \stackrel{\delta_n}{\longrightarrow} & \mathcal{B}_{n-1}(\mathbb{C}) \otimes \mathbb{C}^* \\ \downarrow l^1_n & & \downarrow l^2_n \\ \Lambda^2\mathbb{C}(n-2) & \longrightarrow & \Lambda^3\mathbb{C}(n-3) \end{array}$$

where the maps l^1_n, l^2_n are given by products of $\log z, \mathrm{Li}_k(z)$, and the following composition is zero:

$$\omega \circ l^1_n : \mathbb{Z}[U] \longrightarrow \Lambda^2\mathbb{C}(n-2) \longrightarrow \Omega^1_{\mathbb{C}/\mathbb{Q}},\ \omega(f\wedge g) = fdg - gdf. \qquad (3.4)$$

Then the map l^1_n extends to a well defined map on $\mathbb{Z}[\mathbb{C}]$.

Proof. — Since the map l^1_n is given by polylogarithms, it can be analytically continued to a multivalued function on $\mathbb{C}^* - \{1\}$ with values in $\Lambda^2\mathbb{C}(n-2)$. we need to prove that this function is single-valued. Take the monodromy around some loop minus the identity map. We get a multivalued function on $\mathbb{C}^* - \{1\}$ with values in $\Lambda^2\mathbb{C}(n-2)$ which is annihilated by the map $\Lambda^2\mathbb{C}(n-2) \longrightarrow \Lambda^3\mathbb{C}(n-3)$, and thus takes values in $\mathbb{C}(n-1)$. Since it is also killed by ω, it is a constant, and then one can easily see that it must be zero. □

Non-associative $*$-product. Any ring A provides a $*$ - product

$$\Lambda^{k+1}A * \Lambda^{l+1}A \longrightarrow \Lambda^{k+l+1}A,$$

$$(a_0\wedge...\wedge a_k)*(b_0\wedge...\wedge b_l) := \sum(-1)^{k-j+i}a_0\wedge...\wedge\widehat{a}_i\wedge...\wedge a_k\wedge a_i\cdot b_j\wedge b_0\wedge...\wedge\widehat{b}_j\wedge...\wedge b_l.$$

For instance

$$(a_0\wedge a_1)*(b_0\wedge b_1) = a_0\wedge a_1b_0\wedge b_1 - a_1\wedge a_0b_0\wedge b_1 + a_0\wedge a_1b_1\wedge b_0 - a_0\wedge a_1b_1\wedge b_0.$$

If A is a commutative ring, then $(\Lambda^{\bullet-1}A, *)$ is a supercommutative non-associative algebra:

$$(a_0\wedge ...\wedge a_m) * (b_0\wedge ...\wedge b_n) = (-1)^{mn}(b_0\wedge ...\wedge b_n) * (a_0\wedge ...\wedge a_m).$$

We define the $*$-product on $\Lambda^{\bullet-1}\mathcal{O}$ using the following algebra structure on $\mathcal{O}$:

$$a * b := \frac{1}{2\pi i} ab.$$

It is useful to note the following formula:

$$(2\pi i \wedge a_1 \wedge ... \wedge a_m) * (2\pi i \wedge b_1 \wedge ... \wedge b_n) = (m+n+1) \cdot 2\pi i \wedge a_1 \wedge ... \wedge a_m \wedge b_1 \wedge ... \wedge b_n.$$

Example 1. — Let us define a homomorphism of complexes

$$\begin{array}{ccc} \mathcal{B}_2(\mathbb{C}) & \longrightarrow & \Lambda^2\mathbb{C}^* \\ \downarrow \mathbb{L}_2 & & \downarrow = \\ \Lambda^2\mathbb{C} & \longrightarrow & \Lambda^2\mathbb{C}^* \end{array}$$

$$\mathbb{L}_2 : \frac{1}{2} \cdot \{x\}_2 \longmapsto 2\pi i \wedge \frac{1}{2\pi i}\, \mathrm{L}_2(x) - \frac{1}{2} \mathrm{Li}_1(x) \wedge \log x.$$

Example 2. — Set

$$l_n^{n-1} : \{x\}_2 \otimes y_1 \wedge ... \wedge y_{n-2} \longmapsto \mathbb{L}_2(x) * (2\pi i \wedge \log y_1 \wedge ... \wedge \log y_{n-2}). \quad (3.5)$$

Lemma 3.3. — *The map (3.5) gives rise to a group homomorphism*

$$l_n^{n-1} : \mathcal{B}_2(\mathbb{C}) \otimes \Lambda^{n-2}\mathbb{C}^* \longrightarrow \Lambda^n\mathbb{C}.$$

It makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{B}_2(\mathbb{C}) \otimes \Lambda^{n-2}\mathbb{C}^* & \longrightarrow & \Lambda^n\mathbb{C}^* \\ \downarrow l_n^{n-1} & & \downarrow = \\ \Lambda^n\mathbb{C} & \stackrel{\exp}{\longrightarrow} & \Lambda^n\mathbb{C}^* \end{array} \quad (3.6)$$

The following composition is zero:

$$\omega_n^{n-1} \circ l_n^{n-1} : \mathcal{B}_2(\mathbb{C}) \otimes \Lambda^{n-2}\mathbb{C}^* \longrightarrow \Lambda^n\mathbb{C} \longrightarrow \Omega^{n-1}_{\mathbb{C}/\mathbb{Q}}. \quad (3.7)$$

Proof. — The maps $y_1 \wedge \ldots \wedge y_m \longmapsto 2\pi i \wedge \log y_1 \wedge ... \wedge \log y_m$ and $\mathbb{L}_2 : \mathcal{B}_2(\mathbb{C}) \longrightarrow \Lambda^2\mathbb{C}$ are well defined group homomorphisms. Therefore the map (3.5) is a well defined group homomorphism. The commutativity is evident.

Let us check that the composition (3.7) is zero. We write $d((a_1 \wedge a_2) * (b_1 \wedge ... \wedge b_m))$ as a sum with certain coefficients λ, μ, skewsymmetrising with respect to $\{a_1, a_2\}$ as well as $\{b_1, ..., b_m\}$:

$$\begin{aligned} &\lambda \cdot \mathrm{Alt}_{(a_1,a_2),(b_1,...,b_m)} \Big((b_1 a_1) \wedge da_2 - d(b_1 a_1) \wedge a_2 \Big) \wedge db_2 \wedge ... \wedge db_m \\ &+ \mu \cdot \mathrm{Alt}_{(a_1,a_2),(b_1,...,b_m)} \Big(d(b_1 a_1) \wedge da_2 \Big) \wedge b_1 \wedge db_2 \wedge ... \wedge db_m. \end{aligned} \quad (3.8)$$

In our case

$$a_1 da_2 - a_2 da_1 = 0, \text{ and } db_1 \wedge db_2 \wedge ... \wedge db_m = 0.$$

The first condition implies that the second line is zero. The first and second condition imply that the first line is zero. □

Example 3. — Let us define a homomorphism of complexes

$$\begin{array}{ccccc} \mathcal{B}_3(\mathbb{C}) & \longrightarrow & \mathcal{B}_2(\mathbb{C}) \otimes \mathbb{C}^* & \longrightarrow & \Lambda^3\mathbb{C}^* \\ \downarrow l_3^1 & & \downarrow l_3^2 & & \downarrow = \\ \Lambda^2\mathbb{C}(1) & \longrightarrow & \Lambda^3\mathbb{C} & \longrightarrow & \Lambda^3\mathbb{C}^* \end{array}$$

Set

$$\begin{aligned} & l_3^2 : \{x\}_2 \otimes y \longmapsto \frac{1}{2} \cdot \mathbb{L}_2(\{x\}_2) * (2\pi i \wedge \log y) = \\ & \Big(2\pi i \wedge \frac{1}{2\pi i} \mathrm{L}_2(x) - \frac{1}{2} \cdot \mathrm{L}_1(x) \wedge \log x\Big) * (2\pi i \wedge \log y) = \\ & 3 \ \cdot \ 2\pi i \wedge \frac{1}{2\pi i} \mathrm{L}_2(x) \wedge \log y - \mathrm{L}_1(x) \wedge \log x \wedge \log y \\ & + \frac{1}{2} \ \cdot \ 2\pi i \wedge \Big(\frac{1}{2\pi i} \log y \ \mathrm{L}_1(x) \wedge \log x + \mathrm{L}_1(x) \wedge \frac{1}{2\pi i} \log y \log x\Big). \end{aligned} \tag{3.9}$$

By Lemma 3.3, the map l_3^2 is well defined, makes the second square commute, and $\omega_3^2 \circ l_3^2 = 0$.

Set

$$\mathbb{L}_3 : -\frac{1}{6} \cdot \{x\}_3 \longmapsto \quad 2\pi i \wedge \mathrm{L}_3(x) - \frac{1}{2} \cdot \frac{1}{2\pi i} \mathrm{L}_2(x) \wedge \log x - \frac{1}{12} \cdot \Big(\mathrm{L}_1(x) \wedge \log x\Big) * \log x.$$

One checks that $\omega_3^1 \circ l_3^1 = 0$ thanks to the differential equations for the polylogarithms.

This map makes the first square commutative. Indeed, we have

$$l_3^2 : \{x\}_2 \otimes x \longmapsto 2\pi i \wedge \Big(3 \cdot \mathrm{L}_2(x) \wedge \log x + \frac{1}{2} \cdot \Big(\mathrm{L}_1(x) \wedge \log x\Big) * \log x\Big).$$

Thanks to Lemma 3.2 the map $\mathbb{L}_3$ provides a single-valued map

$$\mathbb{L}_3 : \mathbb{C}^* - \{1\} \longrightarrow \Lambda^2\mathbb{C}(1).$$

Proposition 3.1 implies that it gives rise to a homomorphism

$$\mathbb{L}_3 : \mathcal{B}_3(\mathbb{C}) \longrightarrow \Lambda^2\mathbb{C}(1).$$

Therefore we get a well defined morphism of complexes.

Example 4. — We define a homomorphism of complexes

$$\begin{array}{ccccccc} \mathcal{B}_4(\mathbb{C}) & \longrightarrow & \mathcal{B}_3(\mathbb{C})\otimes\mathbb{C}^* & \longrightarrow & \mathcal{B}_2(\mathbb{C})\otimes\Lambda^2\mathbb{C}^* & \longrightarrow & \Lambda^4\mathbb{C}^* \\ \downarrow l_4^1 & & \downarrow l_4^2 & & \downarrow l_4^3 & & \downarrow l_4^4 \\ \Lambda^2\mathbb{C}(2) & \longrightarrow & \Lambda^3\mathbb{C}(1) & \longrightarrow & \Lambda^4\mathbb{C} & \longrightarrow & \Lambda^4\mathbb{C}^* \end{array}$$

We set

$$l_4^3(\{x\}_2\otimes y_1\wedge y_2) := \mathbb{L}_2(x) * (2\pi i\wedge\log y_1\wedge\log y_2) =$$

$$\Big(2\pi i\wedge\frac{1}{2\pi i}\mathrm{L}_2(x)-\frac{1}{2}\mathrm{L}_1(x)\wedge\log x\Big) * (2\pi i\wedge\log y_1\wedge\log y_2).$$

By Lemma 3.3 the map l_4^3 is well defined, makes the last square commute, and $\omega_4^3\circ l_4^3=0$.

Next, set

$$l_4^2(\{x\}_3\otimes y) :=$$

$$\begin{aligned} &2\pi i\wedge\Big(-12\cdot\frac{1}{(2\pi i)^2}\mathrm{L}_3(x)\wedge\log y-2\cdot(\frac{1}{2\pi i}\mathrm{L}_2(x)\wedge\log x)*\log y\\ &-\frac{1}{2}\cdot\frac{1}{2\pi i}\Big(\mathrm{L}_1(x)\log x\Big)\wedge\log x\log y-\frac{1}{2}\cdot\frac{1}{2\pi i}\Big(\mathrm{L}_1(x)\log y\Big)\wedge(\log x)^2\Big)\\ &+4\cdot\frac{1}{2\pi i}\mathrm{L}_2(x)\wedge\log x\wedge\log y+\frac{1}{2}\cdot\Big(\mathrm{L}_1(x)\wedge\log x\Big)*\Big(\log x\wedge\log y\Big). \end{aligned} \tag{3.10}$$

Direct check shows that the middle square has all the desired properties.

Finally, we set

$$\mathbb{L}_4:\frac{1}{24}\cdot\{x\}_4\longmapsto$$

$$\begin{aligned} &2\pi i\wedge\frac{1}{(2\pi i)^3}\mathrm{L}_4(x)-\frac{1}{2}\cdot\frac{1}{(2\pi i)^2}\mathrm{L}_3(x)\wedge\log x\\ &-\frac{1}{12}\cdot\Big(\frac{1}{2\pi i}\mathrm{L}_2(x)\wedge\log x\Big)*\log x-\frac{1}{24}\cdot\frac{1}{2\pi i}\Big(\mathrm{L}_1(x)\log x\Big)\wedge\log^2 x. \end{aligned} \tag{3.11}$$

One checks that the left square is formally commutative. Thanks to Lemma 3.2 and Proposition 3.1 the map $l_4^1:=\mathbb{L}_4$ is a well defined homomorphism of abelian groups. Finally, we check that $\omega_4^1\circ l_4^1=0$ by using the differential equations for the classical polylogarithms.

Example: the regulator map on the weight three motivic complex. Let X be a regular complex projective curve. Then the motivic complex $\mathbb{Z}_{\mathcal{M}}(X;3)$ is the total of the following complex, where $\mathcal{O}_{\mathcal{X}} := \mathbb{C}(X)$ and Res stands for the tame symbol on the right and the map $\{f\}_2 \otimes g \longmapsto \sum_{x\in X} \mathrm{val}_x(g)\{f(x)\}_2$ in the middle:

$$\begin{array}{ccccc} \mathcal{B}_3(\mathcal{O}_{\mathcal{X}}) & \stackrel{\delta}{\longrightarrow} & \mathcal{B}_2(\mathcal{O}_{\mathcal{X}}) \otimes \mathcal{O}^*_{\mathcal{X}} & \stackrel{\delta}{\longrightarrow} & \Lambda^3\mathcal{O}^*_{\mathcal{X}} \\ & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} \\ & & \coprod_{x\in X} \mathcal{B}_2(\mathbb{C}) & \longrightarrow & \coprod_{x\in X} \Lambda^2\mathbb{C}^* \end{array} \tag{3.12}$$

The top line is mapped to the weight three Lie-exponential complex at the generic point $\mathcal{X}$:

$$\begin{array}{ccccccc} & & \mathcal{B}_3(\mathcal{O}_{\mathcal{X}}) & \stackrel{\delta}{\longrightarrow} & \mathcal{B}_2(\mathcal{O}_{\mathcal{X}}) \otimes \mathcal{O}^*_{\mathcal{X}} & \stackrel{\delta}{\longrightarrow} & \Lambda^3\mathcal{O}^*_{\mathcal{X}} \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{X}}(2) & \longrightarrow & \Lambda^2\mathcal{O}_{\mathcal{X}}(1) & \longrightarrow & \Lambda^3\mathcal{O}_{\mathcal{X}} & \longrightarrow & \Lambda^3\mathcal{O}^*_{\mathcal{X}} \end{array}$$

An important property of the Lie-period is that the element $\mathbb{L}_3(f(x)) \in \Lambda^2\mathcal{O}_{\mathcal{X}}(1)$ is non-singular:

$$\mathbb{L}_3(f(x)) \in \Lambda^2\mathcal{O}(1).$$

So there is a map

$$\mathcal{B}_3(\mathcal{O}_{\mathcal{X}}) \longrightarrow \Lambda^2\mathcal{O}(1).$$

The element $l^2_3(\sum\{f_i(x)\}_2 \otimes g_i(x)) \in \Lambda^3\mathcal{O}_{\mathcal{X}}$ can have singularities at the divisors of the functions g_i. To guarantee that the singularity at $y \in X$ is absent it is sufficient to require that the residue of that element at y is zero. So there is a map on the kernel of the residue map:

$$\mathrm{Ker}\Big(\mathcal{B}_2(\mathcal{O}_{\mathcal{X}}) \otimes \mathcal{O}^*_{\mathcal{X}} \stackrel{\mathrm{Res}}{\longrightarrow} \coprod_{x\in X} \mathcal{B}_2(\mathbb{C})\Big) \longrightarrow \Lambda^3\mathcal{O}.$$

So we get a map

$$\begin{aligned} Z^2(\mathrm{B}^\bullet(X;3)) &:= \mathrm{Ker}\Big(\mathcal{B}_2(\mathcal{O}_{\mathcal{X}}) \otimes \mathcal{O}^*_{\mathcal{X}} \longrightarrow \coprod_{x\in X} \mathcal{B}_2(\mathbb{C}) \oplus \Lambda^3\mathcal{O}^*_{\mathcal{X}}\Big) \\ &\longrightarrow \mathrm{Ker}\Big(\Lambda^3\mathcal{O} \stackrel{\wedge^3\exp}{\longrightarrow} \Lambda^3\mathcal{O}^*\Big). \end{aligned} \tag{3.13}$$

It gives rise to an explicit map

$$H^2(\mathrm{B}^\bullet(X;3)) \longrightarrow H^2(X, \Gamma_{\mathcal{D}}(3)).$$

It can be described explicitly as follows. Take a cycle $A \in Z^2(\mathrm{B}^\bullet(X;3))$. Then we have

$$l^2_3(A) \in \mathrm{Ker}\Big(\Lambda^3\mathcal{O} \stackrel{\wedge^3\exp}{\longrightarrow} \Lambda^3\mathcal{O}^*\Big) = \Lambda^2\mathcal{O}(1).$$

Pick an open cover $\{U_i\}$ of X by small discs. On each cover we get can find an element

$$C(U_i) \in \Lambda^2\mathcal{O}_{U_i}(1): \ dC(U_i) = A_{|U_i} \in \Lambda^3\mathcal{O}_{U_i}.$$

Then $d(C(U_i) - C(U_j)) = 0$ on U_{ij}. So we can find a $C(U_i, U_j) \in \mathcal{O}_{U_{ij}}(2)$ such that $dC(U_i, U_j) = C(U_i) - C(U_j)$. Similarly we find $C(U_i, U_j, U_k) \in \mathbb{Q}(3)$. Now taking the image of the cocycle $(C(U_i), C(U_i, U_j), C(U_i, U_j, U_k))$ in the Lie-exponential Deligne complex $\Gamma_{\mathcal{D}}(X; 3)$ we get a cycle representing the regulator of A.

Unlike the de Rham complex, the exponential complex is exact in a trivial way: finding a primitive does not require integration. So our construction is effective.

One can generalize the above construction to the case when X is an arbitrary regular complex variety. In this case the motivic complex we use is the Gersten resolution of the weight three polylogarithmic complex. It is obtained by adding to (3.12) the contributions of the codimension two and three cycles. The construction remains the same.

4. A local combinatorial construction of characteristic classes

4.1. A map: decorated flags complex → Bigrassmannian complex

Configuration complexes. Let X be a set. Let G be a group acting on X. *Configurations* of m elements in X are orbits of the group G acting on X^m. The complex of configurations $C'_*(X)$ is the complex of the G-coinvariants of the chain complex of the simplex with the vertices parametrized by X:

$$\stackrel{d}{\longrightarrow} C'_m(X) \stackrel{d}{\longrightarrow} C'_{m-1}(X) \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} C'_1(X).$$

So $C'_m(X)$ is the free abelian group generated by configurations. Denote by $(x_1, ..., x_m)$ the generator provided by the configuration corresponding to the G-orbit of an m-tuple $\{x_1, ..., x_m\}$. The differential is

$$d: C'_{m+1}(X) \longrightarrow C'_m(X), \ (x_0, ..., x_m) \longmapsto \sum_{i=0}^{i}(-1)^i(x_0, ..., \widehat{x}_i, ..., x_m).$$

Let us assume now that X is an algebraic variety over $\mathbb{Z}$, and G an algebraic group over $\mathbb{Z}$ acting on X. Then for any field F there is a $\mathrm{G}(F)$-set $X(F)$. So we get complexes of configurations of $X(F)$. Abusing notation, we skip the field F from the notation.

Suppose that we have a notion of generic configurations of points in X, stable under the operation of forgetting a point. We assume that generic configurations of m points in X are parametrised by a variety $\mathrm{Conf}^*_m(X)$. So forgetting the i-th point provides a map

$$f_i : \mathrm{Conf}^*_m(X) \longrightarrow \mathrm{Conf}^*_{m-1}(X).$$

Consider the free abelian group generated by the F-points of $\mathrm{Conf}^*_m(X)$.

$$C_m(X) := \mathbb{Z}[\mathrm{Conf}^*_m(X)(F)].$$

We get a subcomplex of the complex $C'_\bullet(X)$, called the *complex of generic configurations*:

$$C_\bullet(X) : \stackrel{d}{\longrightarrow} C_m(X) \stackrel{d}{\longrightarrow} C_{m-1}(X) \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} C_1(X).$$

An example: Grassmannian complexes [32]. Let $\mathrm{Conf}^*_m(q)$ be the variety of generic configurations of m vectors in a vector space of dimension q. A configuration is generic if any $k \leq q$ of the vectors are linearly independent. Observe that the configuration spaces assigned to isomorphic vector spaces are *canonically* isomorphic.

The variety $\mathrm{Conf}^*_m(q)$ is defined over $\mathrm{Spec}(\mathbb{Z})$: a collection of generic vectors is given by a $q \times m$ matrix with non-zero principal minors. So we get abelian groups

$$C_m(q) := \mathbb{Z}[\mathrm{Conf}^*_m(q)(F)].$$

They form the *weight q Grassmannian complex*:

$$\stackrel{d}{\longrightarrow} C_m(q) \stackrel{d}{\longrightarrow} C_{m-1}(q) \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} C_1(q).$$

The Bigrassmannian [17]. Given a configuration of $(m+1)$ vectors $(l_0, ..., l_m)$ in a q-dimensional vector space V_q, there are two ways to get a configuration of m vectors:

(1) Forgetting the i-th vector l_i, we get a map

$$f_i : \mathrm{Conf}^*_{m+1}(q) \longrightarrow \mathrm{Conf}^*_m(q),\ (l_0, ..., l_m) \longmapsto (l_0, ..., \widehat{l_i}, ..., l_m).$$

(2) Projecting the vectors $(l_0, \ldots, \widehat{l_j}, \ldots, l_m)$ to the quotient $V_q/(l_j)$ by the subspace spanned by l_j, we get a map

$$p_j : \mathrm{Conf}^*_{m+1}(q) \longrightarrow \mathrm{Conf}^*_m(q-1),\ (l_0, ..., l_m) \longmapsto (l_j \mid l_0, ..., \widehat{l_i}, ...l_m).$$

Denote by $\mathbf{G}_m(q)$ the Grassmannian of q-dimensional subspaces in a vector space of dimension m with a given basis $(e_1, ..., e_m)$, in generic position to the coordinate hyperplanes.

There is a canonical isomorphism

$$\mathbf{G}_m(q) = \mathrm{Conf}^*_m(q).$$

It assigns to a generic q-plane π a configuration of vectors in the dual space π^* given by the restrictions of the linear coordinate functionals x_i dual to the basis.

Using this, we organise the spaces $\mathrm{Conf}^*_m(q)$ into a single object, the *Bigrassmannian*:

$$\begin{array}{ccccccccc}
 & & & & & & \dots & \stackrel{\longrightarrow}{\cdots} & \mathbf{G}_5(4) \\
 & & & & & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow \\
 & & & & \dots & \stackrel{\longrightarrow}{\cdots} & \mathbf{G}_5(3) & \stackrel{\longrightarrow}{\cdots} & \mathbf{G}_4(3) \\
 & & & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow \\
 & & \stackrel{\longrightarrow}{\cdots} & & \mathbf{G}_5(2) & \stackrel{\longrightarrow}{\cdots} & \mathbf{G}_4(2) & \stackrel{\longrightarrow}{\cdots} & \mathbf{G}_3(2) \\
\dots & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow \\
\mathbf{G}_5(1) & \stackrel{\longrightarrow}{\cdots} & \mathbf{G}_4(1) & \stackrel{\longrightarrow}{\cdots} & \mathbf{G}_3(1) & \stackrel{\longrightarrow}{\cdots} & \mathbf{G}_2(1) & &
\end{array} \tag{4.1}$$

Applying the functor $X \to \mathbb{Z}[X(F)]$ to the Bigrassmannian we get the *Grassmannian bicomplex*:

$$\begin{array}{ccccccccc}
 & & & & & & \dots & \stackrel{p}{\longrightarrow} & C_5(4) \\
 & & & & & & \downarrow p & & \downarrow p \\
 & & & & \dots & \stackrel{f}{\longrightarrow} & C_5(3) & \stackrel{f}{\longrightarrow} & C_4(3) \\
 & & & & \downarrow p & & \downarrow p & & \downarrow p \\
 & & \stackrel{f}{\longrightarrow} & & C_5(2) & \stackrel{f}{\longrightarrow} & C_4(2) & \stackrel{f}{\longrightarrow} & C_3(2) \\
\dots & & \downarrow p & & \downarrow p & & \downarrow p & & \downarrow p \\
C_5(1) & \stackrel{f}{\longrightarrow} & C_4(1) & \stackrel{f}{\longrightarrow} & C_3(1) & \stackrel{f}{\longrightarrow} & C_2(1) & &
\end{array}$$

Here the maps f and p are the alternating sums of the maps f_j, and p_i:

$$f = \sum_{s=0}^{m} (-1)^s f_s, \ p = \sum_{s=0}^{m} (-1)^s p_s.$$

Denote by BC_* the sum of the groups on the diagonals:

$$BC_m := \bigoplus_{q=1}^{m-1} C_m(q).$$

Changing the signs of the differentials in the bicomplex, we get the *Bigrassmannian complex*

$$\dots \longrightarrow BC_5 \longrightarrow BC_4 \longrightarrow BC_3 \longrightarrow BC_2.$$

Decorated flags.

DEFINITION 4.1. — *A decorated flag $F_\bullet$ in an N-dimensional vector space is a collection of subspaces*

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_N\,,\ \dim F_i = i, \tag{4.2}$$

together with a choice of a non-zero vector $f_i \in F_i/F_{i-1}$ for each $i = 1, ..., N$.

A collection of $m+1$ decorated flags $(F_{0,\bullet}, F_{1,\bullet}, ..., F_{m,\bullet})$ in an N-dimensional vector space V_N is *generic*, if for any integers $a_0, ..., a_m$ which sum to N one has an isomorphism

$$F_{0,a_0} \oplus ... \oplus F_{m,a_m} = V_N.$$

Denote by $\mathcal{A}_N$ the variety of all decorated flags in V_N, and by $\mathrm{Conf}^*_m(\mathcal{A}_N)$ the variety of generic configurations of m decorated flags. It is defined over $\mathrm{Spec}(\mathbb{Z})$. So for any field F there is the complex of generic configurations of decorated flags

$$\ldots \longrightarrow C_m(\mathcal{A}_N) \longrightarrow \ldots \longrightarrow C_2(\mathcal{A}_N) \longrightarrow C_1(\mathcal{A}_N).$$

From configurations of decorated flags to configurations of vectors. We start with a collection of $m+1$ generic decorated flags in an N-dimensional vector space V_N:

$$(F_{0,\bullet}, ..., F_{m,\bullet}). \tag{4.3}$$

Given a partition

$$\mathbf{a} = \{a_0, \ldots, a_m\}, \quad a_0 + \ldots + a_m = N - (q+1), \quad a_i \geq 0, \tag{4.4}$$

consider a codimension $q+1$ linear subspace of V_N given by the sum of the flag subspaces F_{i,a_i}:

$$F_{0,a_0} \oplus F_{1,a_1} \oplus \ldots \oplus F_{m,a_m} \subset V_N. \tag{4.5}$$

Take the quotient by this subspace

$$Q_{\mathbf{a}} = \frac{V_N}{F_{0,a_0} \oplus F_{1,a_1} \oplus \ldots \oplus F_{m,a_m}}. \tag{4.6}$$

We use the decorations to produce a configuration of $(m+1)$ vectors in the quotient $Q_{\mathbf{a}}$. Namely, the "next" decoration vector $f_{a_i+1} \in F_{i,a_i+1}/F_{i,a_i}$ in the decorated flag F_i provides a vector in the quotient, denoted by l_i. The vectors $\{l_0, ..., l_m\}$ in the space $Q_{\mathbf{a}}$ provide a configuration $(l_0, \ldots, l_m)$. So we put

$$\pi_{\mathbf{a}}(F_{0,\bullet} \ldots, F_{m,\bullet}) := (l_0, \ldots, l_m) \in \mathrm{Conf}^*_{m+1}(q+1).$$

So a partition $\mathbf{a}$ gives rise to a projection

$$\pi_{\mathbf{a}} : \mathrm{Conf}^*_{m+1}(\mathcal{A}_N) \longrightarrow \mathrm{Conf}^*_{m+1}(q+1).$$

The main construction [17, Section 2].

- *Given a configuration $(F_{0,\bullet}, F_{1,\bullet}, ..., F_{m,\bullet})$ of decorated flags in V_N, we assign to every partition $\mathbf{a}$ as in (4.4) the configuration of vectors $\pi_{\mathbf{a}}(F_{0,\bullet}, F_{1,\bullet}, ..., F_{m,\bullet})$ in a $(q+1)$-dimensional vector space, and take the* **sum over all q and all partitions $\mathbf{a}$**:

$$c_m : (F_{0,\bullet}, F_{1,\bullet}, ..., F_{m,\bullet}) \longrightarrow \sum_{\mathbf{a}} \pi_{\mathbf{a}}(F_{0,\bullet}, F_{1,\bullet}, ..., F_{m,\bullet}) \in BC_{m+1}.$$

We extend the map to a homomorphism of abelian groups

$$c_m : C_{m+1}(\mathcal{A}_N) \longrightarrow BC_{m+1}.$$

The following crucial result was proved in Lemma 2.1 from [17].

THEOREM 4.2. — *The collection of maps c_n gives rise to a homomorphism of complexes*

$$\begin{array}{ccccccccc} \longrightarrow & C_5(\mathcal{A}_N) & \longrightarrow & C_4(\mathcal{A}_N) & \longrightarrow & C_3(\mathcal{A}_N) & \longrightarrow & C_2(\mathcal{A}_N) & \\ & \downarrow c_4 & & \downarrow c_3 & & \downarrow c_2 & & \downarrow c_1 & (4.7) \\ \longrightarrow & BC_5 & \longrightarrow & BC_4 & \longrightarrow & BC_3 & \longrightarrow & BC_2 & \end{array}$$

Our next goal is to give an interpretation of this map via hypersimiplicial decompositions.

4.2. Hypersimplicial decompositions of simplices and a proof of Theorem 4.2

Hypersimplices [14]. Let $p, q \geq 0$ be a pair of non-negative integers. Set $p+q = m-1$. A hypersimplex $\Delta^{p,q}$ is a hyperplane section of the $(m+1)$-dimensional unit cube:

$$\Delta^{p,q} := \{(x_0, ..., x_m) \in [0,1]^{m+1} \mid \sum_{i=0}^{m} x_i = q+1\}, \ p+q = m-1.$$

The hypersimplex $\Delta^{p,q}$ is a convex polyhedron isomorphic to the convex hull of the centers of q-dimensional faces of an m-dimensional simplex.

The hypersimplices $\Delta^{p,0}$ and $\Delta^{0,q}$ are just simplices. The hypersimplex $\Delta^{1,1}$ is the octahedron. It is the convex hull of the centers of the edges of a tetrahedron.

The boundary of a hypersimplex $\Delta^{p,q}$ is a union of $m+1$ hypersimplices $\Delta^{p-1,q}$ and $m+1$ hypersimplices $\Delta^{p,q-1}$. They are given by the intersections with the hyperplanes $x_i = 1$ and $x_i = 0$ of the unit cube. For example, the boundary of the octahedron $\Delta^{1,1}$ consists of four $\Delta^{1,0}$-triangles and four $\Delta^{0,1}$-triangles.

Hypersimplicial N-decomposition of a simplex [12, Section 10.4]. It is a canonical decomposition of an m-dimensional simplex into hypersimplices which depend on an additional natural number N.

Consider the standard coordinate space $\mathbb{R}^{m+1}$. It contains the integral lattice $\mathbb{Z}^{m+1}$. The integral hyperplanes $x_i = s$, $s \in \mathbb{Z}$, cut the space into unit cubes with vertices at integral points. Take an m-dimensional simplex given by the intersection of the hyperplane $\sum x_i = N$ with the positive octant:

$$\Delta^m_{(N)} = \{(x_0, ..., x_m) \mid x_i \geq 0,\ \sum_{i=0}^{m} x_i = N\}.$$

The integral hyperplanes $x_i = s$ cut this simplex into a union of hypersimplices. Indeed, the hyperplane $\sum x_i = N$ intersects each of the standard unit lattice cubes either by an empty set, or by a hypersimplex. We call it *a hypersimplicial N-decomposition of an m-dimensional simplex*. A hypersimplicial N-decomposition of a simplex induces a hypersimplicial N-decomposition of each face of the simplex.

LEMMA 4.3. — *The $\Delta^{p,q}$-hypersimplices of the hypersimplicial N-decomposition of an m-simplex match partitions $\mathbf{a} = (a_0, ..., a_m)$, where*

$$a_0 + \ldots + a_m = N - (q+1),\ a_i \in \mathbb{Z}_{\geq 0}. \tag{4.8}$$

Proof. — The standard hypersimplex $\Delta^{p,q}$ consists of the points of the unit cube with coordinates $(x_0, ..., x_m)$ satisfying $x_0 + ... + x_m = q + 1$. So any partition $\mathbf{a}$ provides a hypersimplex

$$(a_0, ..., a_m) + (x_0, ..., x_m) \subset \Delta^m_{(N)}.$$

So we parametrise hypersimplices in $\Delta^m_{(N)}$ by the coordinates $(a_0, .., ., a_m)$ of their "lowest" vertices. □

Let $\Delta^{p,q}_{\mathbf{a}}$ be the hypersimplex of a hypersimplicial N-decomposition assigned to a partition $\mathbf{a}$.

Examples. — 1. The N-decomposition of a segment Δ^1 is a decomposition into N little $\Delta^{0,0}$-segments. They match partitions $a_0 + a_1 = N - 1$.

2. The N-decomposition of a triangle Δ^2 is a decomposition into triangles of two types, $\Delta^{1,0}$ and $\Delta^{0,1}$. The $\Delta^{1,0}$-triangles match partitions $a_0 + a_1 + a_2 = N - 1$. The $\Delta^{0,1}$-triangles match partitions $a_0 + a_1 + a_2 = N - 2$.

3. The N-decomposition of a tetrahedron Δ^3 has tetrahedrons of two types and octahedron. The $\Delta^{2,0}$-tetrahedrons match partitions $a_0+a_1+a_2+a_3 = N-1$. The $\Delta^{1,1}$-octahedrons match partitions $a_0+a_1+a_2+a_3 = N-2$. The $\Delta^{0,2}$-tetrahedrons match partitions $a_0 + a_1 + a_2 + a_3 = N - 3$.

Recall that a hypersimplex $\Delta^{p,q}$ has $2(m + 1)$ codimension one faces: $(m + 1)$ of them are hypersimplices of type $\Delta^{p-1,q}$, and the other $(m + 1)$ are hypersimplices of type $\Delta^{p,q-1}$.

Each hypersimplex $\Delta^{p,q}_{\mathbf{a}}$ is surrounded by $m+1$ hypersimplices $\Delta^{p+1,q-1}_{\mathbf{b}}$, sharing with it a codimension one face of type $\Delta^{p,q-1}$. The $\mathbf{b}$'s are obtained from $\mathbf{a}$ by adding 1 to one of the coordinates $(a_0, ..., a_m)$. So the collection of $\mathbf{b}$'s is

$$(a_0 + 1, a_1, a_2, ..., a_m),\ (a_0, a_1 + 1, a_2, ..., a_m),\ \dots\ , (a_0, a_1, a_2, ..., a_m + 1).$$

Each hypersimplex $\Delta^{p,q}_{\mathbf{a}}$ is also surrounded by $m + 1$ hypersimplices $\Delta^{p-1,q+1}_{\mathbf{c}}$, sharing with it a codimension one face of type $\Delta^{p-1,q}$. The $\mathbf{c}$'s are obtained from $\mathbf{a}$ by subtracting 1 from one of the coordinates $(a_0, ..., a_m)$. So the collection of $\mathbf{c}$'s is

$$(a_0 - 1, a_1, a_2, ..., a_m),\ (a_0, a_1 - 1, a_2, ..., a_m),\ \dots\ , (a_0, a_1, a_2, ..., a_m - 1).$$

The combinatorics of hypersimplices is related [15] to the geometry of the Grassmannians.

The Grassmannian $\mathbf{G}_{p+q+2}(q + 1)$ matches the hypersimplex $\Delta^{p,q}$.

Precisely, consider the action of the coordinate torus $\mathrm{T}_{p+q+2} = \mathbb{G}_m^{p+q+2}$ on the Grassmannian $\mathbf{G}_{p+q+2}(q + 1)$. Then the closure of each of the generic T_{p+q+2}-orbits is a $(p+q+1)$-dimensional toric variety, and combinatorics of its boundary strata coincides with the structure of the hypersimplex $\Delta^{p,q}$. Alternatively, it follows from the general Convexity Theorem of Atiyah [1] that the image of $\mathbf{G}_{p+q+2}(q + 1)$ under the moment map assigned to the torus action is the hypersimplex $\Delta^{p,q}$.

A proof of Theorem 4.2. Our key construction provides a map

$$\textit{Complex of decorated flags} \longrightarrow \textit{Bigrassmannian complex.} \tag{4.9}$$

To see that it commutes with differentials, we rephrase it as a correspondence from the variety $\mathrm{Conf}^*_{m+1}(\mathcal{A}_N)$ to the Bigrassmannian:

- *Given a generic configuration of* $(m+1)$ *decorated flags* $(F_{0,\bullet}, F_{1,\bullet}, ..., F_{m,\bullet})$ *in* V_N*, we define a collection of points in the Grassmannians* $\mathbf{G}_{m+1}(*)$*. These points are parametrised by the hypersimplices of the hypersimplicial* N*-decomposition of an* m*-dimensional simplex:*

  Each hypersimplex $\Delta^{p,q}_{\mathbf{a}} \subset \Delta^m_{(N)}$ *gives rise to a point of the Grassmannian* $\mathbf{G}_{m+1}(q+1)$*:*

$$\pi_{\mathbf{a}}(F_{0,\bullet}, F_{1,\bullet}, ..., F_{m,\bullet}) \in \mathbf{G}_{m+1}(q+1). \tag{4.10}$$

Furthermore, the $2(m+1)$ elements provided by the boundary of the element (4.10) match the ones assigned to the boundaries of the hypersimplex $\Delta^{p,q}_{\mathbf{a}} \subset \Delta^m_{(N)}$. The sum of the boundaries of all these hypersimplices is, of course, the boundary of the simplex $\Delta^m_{(N)}$ presented as a sum of its own hypersimplices. This just means that we get a homomorphism of complexes.

We defined homomorphisms of complexes (4.7):

$$\textit{Complexes of generic decorated flags in } V_N \longrightarrow \textit{the Bigrassmannian complex.} \tag{4.11}$$

We will review in Section 4.3 homomorphisms of complexes,

$$\textit{the Bigrassmannian complex} \longrightarrow \textit{weight } n \textit{ motivic complex}, \ n \leq 4. \tag{4.12}$$

Finally, we defined in Section 2.5 for $n \leq 4$ maps

$$\begin{aligned} &\textit{Weight } n \textit{ polylogarithmic complex} \longrightarrow \\ &\textit{weight } n \text{ Lie-}\textit{exponential complex}, \ n \leq 4. \end{aligned} \tag{4.13}$$

Combining these three maps, we get explicit cocycles for the Chern classes with values in the Deligne cohomology for $n \leq 3$. The $n = 4$ case needs a more general map (4.13), since the weight four motivic complex in (4.12) is no longer the polylogarithmic complex, it is rather, see [22]:

$$G_4(F) \longrightarrow \mathcal{B}_3(F) \otimes F^* \longrightarrow \mathcal{B}_2(F) \otimes \Lambda^2 F^* \longrightarrow \Lambda^4 F^*. \tag{4.14}$$

However, using the big period map on the $\mathcal{H}_4$, one can extend (4.13) to this case.

4.3. Maps Bigrassmannian complex $\to$ motivic complexes

1. Bigrassmannian complex $\longrightarrow$ Bloch complex. We construct a map of complexes

$$\begin{array}{ccccccc} BC_5 & \longrightarrow & BC_4 & \longrightarrow & BC_3 & \longrightarrow & BC_2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B}_2(F) & \longrightarrow & \Lambda^2 F^* & \longrightarrow & 0 \end{array} \tag{4.15}$$

It is defined at the Grassmannian bicomplex, raw by raw. The bottom raw goes to zero. The map on the second raw amounts to the following map of complexes, defined in (1.37), Section 1:

$$\begin{array}{ccccc} C_5(2) & \longrightarrow & C_4(2) & \longrightarrow & C_3(2) \\ \downarrow & & \downarrow l_1 & & \downarrow l_2 \\ 0 & \longrightarrow & \mathcal{B}_2(F) & \longrightarrow & \Lambda^2 F^* \end{array} \tag{4.16}$$

Combining the homomorphism (4.7)= (4.11) with the homomorphism (4.15), we arrive at a homomorphism from the complex of decorated flags in V_N to the Bloch complex:

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_5(\mathcal{A}_N) & \longrightarrow & C_4(\mathcal{A}_N) & \longrightarrow & C_3(\mathcal{A}_N) \\ & & \downarrow 0 & & \downarrow & & \downarrow \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{B}_2(F) & \longrightarrow & \Lambda^2 F^* \end{array} \tag{4.17}$$

It is the main ingredient of the cocycle for the second motivic Chern class in [17]:

$$C_2^{\mathcal{M}} \in H^4(BGL_N, \mathbb{Z}_{\mathcal{M}}(2)). \tag{4.18}$$

2. Bigrassmannian complex $\to$ weight three motivic complex. Let us construct a map of complexes

$$\begin{array}{ccccccccc} BC_7 & \longrightarrow & BC_6 & \longrightarrow & BC_5 & \longrightarrow & BC_4 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B}_3(F) & \longrightarrow & \mathcal{B}_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^* & \longrightarrow & 0 \end{array} \tag{4.19}$$

We define it by looking at the Grassmannian bicomplex, and defining the map raw by raw.

We send the bottom two raws to zero. The map on the third raw amounts to a construction of the following map of complexes:

$$\begin{array}{ccccccc} C_7(3) & \longrightarrow & C_6(3) & \longrightarrow & C_5(3) & \longrightarrow & C_4(3) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B}_3(F) & \longrightarrow & \mathcal{B}_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^* \end{array} \tag{4.20}$$

This has been done in Section 3.2 in [19]), see some additions in Section 5 in [20].

Combining homomorphism (4.7) with the homomorphism (4.19) from the Bigrassmannian complex to the weight three motivic complex, we arrive at a homomorphism of complexes

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_6(\mathcal{A}_N) & \longrightarrow & C_5(\mathcal{A}_N) & \longrightarrow & C_4(\mathcal{A}_N) \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \mathcal{B}_3(F) & \longrightarrow & \mathcal{B}_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^* \end{array} \tag{4.21}$$

It is the main ingredient of the cocycle for the third motivic Chern class in [17]:

$$C_3^{\mathcal{M}} \in H^6(BGL_N, \mathbb{Z}_{\mathcal{M}}(3)). \tag{4.22}$$

Bigrassmannian complex $\longrightarrow$ weight four motivic complex. We will treat it in a different place, since it requires an elaborate exposition.

Remark. — Motivic Chern classes $C_n^{\mathcal{M}} \in H^{2n}(BGL_N, \mathbb{Z}_{\mathcal{M}}(n))$ are defined for $n \leq 4$ on Milnor's simplicial model of the classifying space BGL_N, and take values in the motivic complexes there. We construct cocycles representing these classes at the generic point of BGL_N. It is a key property of the construction that these cocycles extend to cocycles on the whole space BGL_N with the values in the motivic complex defined using the Gersten resolution, see details in [17] for the weights 2 and 3, and even more details in Section 4 of [19] for the weight 3.

Contrary to this, our construction of cocycles representing the Deligne cohomology classes

$$C_n^{\mathcal{D}} \in H^{2n}(BGL_N^*, \mathbb{Z}_{\mathcal{D}}(n))$$

works at the generic point only. This is sufficient for the goal, since BGL_N^* is a model of the classifying space for the GL_N. And this is sufficient to get explicit formulas for the Chern classes of vector bundles. Yet it is desired to extend the construction to BGL_N.

5. **Appendix: a map to the real Deligne complex**

An outline. Let $(S^\bullet, d)$ be the de Rham complex of *smooth* real valued forms on a manifold X. Recall that we constructed a map of complexes

$$\omega_n^{(\bullet)} : \mathbb{Q}_{\mathcal{E}}^\bullet(n) \longrightarrow \Omega^\bullet.$$

Consider the canonical projection:

$$\pi_n : \mathbb{C} \longrightarrow \mathbb{C}/\mathbb{R}(n) = \mathbb{R}(n-1);\ \pi_n(a+ib) := \begin{cases} a & n \text{ odd}, \\ ib & n \text{ even}. \end{cases}$$

The map π_n induces a projection of the de Rham complex of complex valued smooth forms to the de Rham complex of $\mathbb{R}(n-1)$-valued forms:

$$\pi_n : \Omega^\bullet \longrightarrow S^\bullet(n-1).$$

So we get a canonical map from the exponential complex:

$$\varphi_n^{(\bullet)} := \pi_n \circ \omega_n^{(\bullet)} : \mathbb{Q}_{\mathcal{E}}^\bullet(n) \longrightarrow S^\bullet(n-1). \tag{5.1}$$

We will show that the map $\varphi_n^{(\bullet)}$ is canonically homotopic to zero by constructing a homotopy

$$s_n^{(\bullet)} : \mathbb{Q}_{\mathcal{E}}^\bullet(n) \longrightarrow S^\bullet(n-1)[-1],\ d \circ s_n^{(\bullet)} + s_n^{(\bullet)} \circ d = \varphi_n^{(\bullet)}. \tag{5.2}$$

Let us assume that we have a map, conjectured in Conjecture 2.12, from the weight n part $\mathcal{L}^\bullet(n)$ of cochain complex of the $\mathbb{Q}$-Hodge-Tate Lie coalgebra $\mathcal{L}$ to the Lie-exponential complex:

$$p_n^{(\bullet)} : \mathcal{L}^\bullet(n) \longrightarrow \mathbb{Q}_{\mathcal{E}}^\bullet(n).$$

Recall an important feature of the map (5.1):

the composition $\varphi_n^{(\bullet)} \circ p_n^{(\bullet)} : \mathcal{L}^\bullet(n) \longrightarrow \mathbb{Q}_{\mathcal{E}}^\bullet(n) \longrightarrow S^\bullet(n-1)$ is zero.

Therefore the composition $s_n^{(\bullet)} \circ p_n^{(\bullet)}$ is a map of complexes:

$$s_n^{(\bullet)} \circ p_n^{(\bullet)} : \mathcal{L}^\bullet(n) \longrightarrow \mathbb{Q}_{\mathcal{E}}^\bullet(n) \longrightarrow S^\bullet(n-1)[-1]. \tag{5.3}$$

Recall that the weight n real Deligne complex is given by the cone

$$\mathbb{R}_{\mathcal{D}}(n) = \operatorname{Cone}\Big(\pi_n : F^n\Omega^\bullet \longrightarrow S^\bullet(n-1)[-1]\Big).$$

Lemma 5.1. — *The map (5.3) gives rise to a morphism to the weight n real Deligne complex:*

$$(s_n^{(\bullet)} \circ p_n^{(\bullet)}, \omega_n^{(n)} \circ p_n^{(n)}) : \mathcal{L}^\bullet(n) \longrightarrow \mathbb{R}_{\mathcal{D}}(n). \tag{5.4}$$

Proof. — The map (5.3) gives the component of the map (5.4) in $(S^0 \to ... \to S^{n-1})(n-1)[-1]$. The only other non-trivial component is the standard map $\omega_n^{(n)} \circ p_n^{(n)} : \Lambda^n\mathcal{L}_1 \longrightarrow \Omega^n$. □

In particular, combining this with a regulator map from the motivic complex to $\mathcal{L}^\bullet(n)$ we would get a homomorphism from the motivic complex to the weight n real Deligne complex.

The morphism $\varphi_n^{(\bullet)}$. It is a morphism of complexes which looks as follows:

$$\begin{array}{ccccccccc}
\mathcal{O}(n-1) & \to & \Lambda^2\mathcal{O}(n-2) & \stackrel{\delta}{\to} & ... & \stackrel{\delta}{\to} & \Lambda^n\mathcal{O} & \stackrel{\exp}{\to} & \Lambda^n\mathcal{O}^* \\
\downarrow \varphi_n^{(0)} & & \downarrow \varphi_n^{(1)} & & ... & ... & \downarrow \varphi_n^{(n-1)} & & \downarrow \varphi_n^{(n)} \\
S^0(n-1) & \to & S^1(n-1) & \stackrel{d}{\to} & ... & \stackrel{d}{\longrightarrow} & S^{n-1}(n-1) & \stackrel{d}{\to} & S^n(n-1)
\end{array} \tag{5.5}$$

Namely,

$$\varphi_n^{(n)} : \Lambda^n\mathcal{O}^* \longmapsto S^n(n-1),\ F_1 \wedge ... \wedge F_n \longmapsto \pi_n(d\log F_1 \wedge ... \wedge d\log F_n),$$

and for $k = 1, ..., n$,

$$\begin{array}{c}
\varphi_n^{(k-1)} : \Lambda^k\mathcal{O}(n-k) \longmapsto S^{k-1}(n-1), \\
(2\pi i)^{n-k} \cdot f_1 \wedge ... \wedge f_k \longmapsto \pi_n \circ d^{-1}\Big((2\pi i)^{n-k} \cdot df_1 \wedge ... \wedge df_k\Big) := \\
(k-1)!\ \pi_n\Big((2\pi i)^{n-k} \cdot \sum_{i=1}^{k} (-1)^i f_i\ df_1 \wedge ... \wedge \widehat{df_i} \wedge ... \wedge df_k\Big).
\end{array} \tag{5.6}$$

A homotopy $s_n^{(\bullet)}$. For example for $n = 2$ we are going to get a diagram of maps

$$\begin{array}{ccccccc}
 & & \mathcal{O}(1) & \stackrel{\delta}{\longrightarrow} & \Lambda^2\mathcal{O} & \stackrel{\exp}{\longrightarrow} & \Lambda^2\mathcal{O}^* \\
 & \swarrow s_2^{(0)} & \downarrow \varphi_2^{(0)} & \swarrow s_2^{(1)} & \downarrow \varphi_2^{(1)} & \swarrow s_2^{(2)} & \downarrow \varphi_2^{(2)} \\
0 & \longrightarrow & S^0(1) & \stackrel{d}{\longrightarrow} & S^1(1) & \stackrel{d}{\longrightarrow} & S^2(1)
\end{array} \tag{5.7}$$

Let $\mathrm{Alt}_n F(x_1, ..., x_n) := \sum_{\sigma \in S_n} (-1)^{|\sigma|} F(x_{\sigma(1)}, ..., x_{\sigma(n)})$, and $\mathrm{Im}(x + iy) := iy$. Let us set

$$\widetilde{r}_{n-1} : \Lambda^n\mathcal{O} \longrightarrow S^{n-1}(n-1), \quad f_1 \wedge ... \wedge f_n \longmapsto d^{-1} \circ \pi_n(df_1 \wedge ... \wedge df_n) :=$$

$$\mathrm{Alt}_n \sum_{j \geq 0} c_{j,n} \mathrm{Re} f_1\ d\mathrm{Re} f_2 \wedge ... \wedge d\mathrm{Re} f_{2j+1} \wedge d\mathrm{Im} f_{2j+2} \wedge ... \wedge d\mathrm{Im} f_n.$$

Here $c_{j,n} := \frac{1}{(2j+1)!(n-2j-1)!}$. For example,

$$\widetilde{r}_1 : \Lambda^2\mathcal{O} \longrightarrow S^1(1), \quad f_1 \wedge f_2 \longmapsto d^{-1} \circ \pi_2(df_1 \wedge df_2) := \mathrm{Re} f_1\ d\mathrm{Im} f_2 - \mathrm{Re} f_2\ d\mathrm{Im} f_1.$$

A primitive $d^{-1} \circ \pi_n(df_1 \wedge ... \wedge df_n)$ is not uniquely defined. Our choise has a property that

$$r_{k-1}(2\pi i \wedge f_2 \wedge ... \wedge f_k) = 0. \tag{5.8}$$

Set $s_n^{(n)} := \frac{1}{n!} r_{n-1}$. Let us define maps

$$s_n^{(k)} : \Lambda^{k+1}\mathcal{O}(n-k-1) \longrightarrow S^{k-1}(n-1) \qquad 1 \leq k \leq n-1$$

by setting

$$s_n^{(k)} : (2\pi i)^{n-k-1} \otimes f_0 \wedge ... \wedge f_k \longmapsto \frac{(2\pi i)^{n-k-1}}{n!} \mathrm{Alt}_{k+1}\Big(\mathrm{Im} f_0 \cdot \widetilde{r}_{k-1}(f_1 \wedge ... \wedge f_k)\Big).$$

THEOREM 5.2. — *The map $s_n^{(\bullet)}$ is a homotopy between the map $\varphi_n^{(\bullet)}$ and zero:*

$$s_n^{(k+1)} \circ \delta + d \circ s_n^{(k)} = \varphi_n^{(k)} \quad \textit{for} \quad 1 \leq k \leq n-1$$

Proof. — Let us prove the statement for the diagram

$$\begin{array}{ccc} \Lambda^k\mathcal{O}(n-k) & \stackrel{\delta}{\longrightarrow} & \Lambda^{k+1}\mathcal{O}(n-k-1) \\ \swarrow s_n^{(k-1)} & \downarrow \varphi_n^{(k-1)} & \swarrow s_n^{(k)} \\ S^{k-2}(n-1) & \stackrel{d}{\longrightarrow} & S^{k-1}(n-1) \end{array}$$

Thank to (5.8), for $k \leq n-1$ one has

$$\begin{aligned} & s_n^{(k)} \circ \delta\Big((2\pi i)^{n-k} \otimes f_1 \wedge ... \wedge f_k\Big) = \\ & s_n^{(k)}\Big((2\pi i)^{n-k-1} \otimes 2\pi i \wedge f_1 \wedge ... \wedge f_k\Big) = \\ & k! \ \frac{(2\pi i)^{n-k}}{n!} \widetilde{r}_{k-1}(f_1 \wedge ... \wedge f_k). \end{aligned} \tag{5.9}$$

It is easy to see that the same result is valid also for $k = n$. On the other hand

$$\begin{aligned} & d \circ s_n^{(k)}\Big((2\pi i)^{n-k} \otimes f_1 \wedge ... \wedge f_k\Big) = \\ & d\frac{(2\pi i)^{n-k}}{n!} \mathrm{Alt}_k\Big(\mathrm{Im} f_1 \cdot \widetilde{r}_{k-2}(f_2 \wedge ... \wedge f_k)\Big) = \\ & (k-1)! \ \frac{(2\pi i)^{n-k}}{n!} \cdot d\Big(\sum_{i=1}^{k} (-1)^{i-1} \mathrm{Im} f_i \cdot \widetilde{r}_{k-2}(f_1 \wedge ... \wedge \widehat{f_i} \wedge ... \wedge f_k)\Big). \end{aligned} \tag{5.10}$$

Putting together (5.6), (5.9) and (5.10), and dividing by $(k-1)!$, the statement reduces to the following basic identity:

$$d\Big(\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Im}f_i\cdot\widetilde{r}_{k-2}(f_1\wedge...\wedge\widehat{f_i}\wedge...\wedge f_k)\Big)+k\cdot\widetilde{r}_{k-1}(f_1\wedge...\wedge f_k)=$$
$$\pi_k\Big(\sum_{i=1}^{k}(-1)^{i-1}f_i\cdot df_1\wedge...\wedge\widehat{d}f_i\wedge...\wedge df_k\Big). \tag{5.11}$$

We can rewrite it in its natural form:

$$k\cdot\Big(\pi_k\circ d^{-1}-d^{-1}\circ\pi_k\Big)(f_1\wedge...\wedge f_k)=$$
$$d\Big(\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Im}f_i\cdot\widetilde{r}_{k-2}(f_1\wedge...\wedge\widehat{f_i}\wedge...\wedge f_k\Big). \tag{5.12}$$

Proof of the basic identity. — We need the following simple observation:

$\widetilde{r}_{k-1}(f_1\wedge...\wedge f_k)=\widetilde{r}_{k-2}(f_1\wedge...\wedge f_{k-1})\wedge d\mathrm{Im}f_k\ +$ terms without $d\mathrm{Im}f_k$.

We prove the basic identity by induction. Let $k=2$. Then it boils down to

$$d\Big(\mathrm{Im}f_1\mathrm{Re}f_2-\mathrm{Im}f_2\mathrm{Re}f_1\Big)+2\Big(\mathrm{Re}f_1d\mathrm{Im}f_2-\mathrm{Re}f_2d\mathrm{Im}f_1\Big)$$
$$\mathrm{Re}f_1d\mathrm{Im}f_2-\mathrm{Re}f_2d\mathrm{Im}f_1+\mathrm{Im}f_1d\mathrm{Re}f_2-\mathrm{Im}f_2d\mathrm{Re}f_1,$$

which is easy to check.

Let us assume that the identity was already proved for $k-1$. We compute first the parts of each of the sides containing the term $d\mathrm{Im}f_k$. The contribution of the right hand side is

$$\pi_{k-1}\Big(\sum_{i=1}^{k-1}(-1)^{i-1}f_i\cdot df_1\wedge...\wedge\widehat{d}f_i\wedge...\wedge df_k\Big)\wedge d\mathrm{Im}f_k.$$

By the induction assumption this is equal to

$$\Big(d\sum_{i=1}^{k-1}(-1)^{i-1}\mathrm{Im}f_i\cdot\widetilde{r}_{k-3}(f_1\wedge...\wedge\widehat{f_i}\wedge...\wedge f_{k-1})+(k-1)\widetilde{r}_{k-2}(f_1\wedge...\wedge f_{k-1})\Big)\wedge d\mathrm{Im}f_k.$$

We have to show that this expression is equal to the $d\mathrm{Im}f_k$-content of the left hand side of the basic equality, i.e. to

$$-\widetilde{r}_{k-2}(f_1\wedge...\wedge f_{k-1})\wedge d\mathrm{Im}f_k+k\widetilde{r}_{k-2}(f_1\wedge...\wedge f_{k-1})\wedge d\mathrm{Im}f_k+$$
$$\Big(\sum_{i=1}^{k-1}(-1)^{i-1}d\widetilde{r}_{k-2}(f_1\wedge...\wedge\widehat{f_i}\wedge...\wedge f_{k-1})\mathrm{Im}f_i\Big)\wedge d\mathrm{Im}f_k.$$

This is obvious. It remains to check that the $d\mathrm{Im}f_k$-free parts of the basic equality also coincide. The right hand side gives us

$$\sum_{i=1}^{k}(-1)^{i-1}\pi_k(f_i)(d\mathrm{Re}f_1 \wedge ... \wedge \widehat{d\mathrm{Re}f_i} \wedge ... \wedge d\mathrm{Re}f_k). \tag{5.13}$$

Let us assume first that k is odd. Then the left hand side is

$$\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Re}f_i(d\mathrm{Re}f_1 \wedge ... \wedge \widehat{d\mathrm{Re}f_i} \wedge ... \wedge d\mathrm{Re}f_k),$$

which coincides with (5.13) since $\pi_k(f_i) = \mathrm{Re}f_i$ if k is odd. If k is even the first term contributes

$$\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Im}f_i\pi_{k-1}(f_1 \wedge ... \wedge \widehat{f_i} \wedge ... \wedge f_k),$$

which coincides with (5.13) since $\mathrm{Im}f_i = \pi_k(f_i)$ in this case. □

Bibliography

[1] Atiyah (M.). — Convexity and commuting hamiltonians. Michale Atiyah Collected papers, Vol 5 (1982).

[2] Beilinson (A.A.). — Higher regulators and values of L-functions. VINITI 24, (1984). 181-238. J. Soviet Math. 30 p. 2036-2070 (1985).

[3] Beilinson (A.A.). — Height pairings between algebraic cycles Lect. Notes in Math 1289, p. 1-26 (1987).

[4] Beilinson (A.A.), Deligne (P.). — Interpretation motivique de la conjecture de Zagier reliant polylogarithms et regulateurs. Proc. Proc. of Symposia in Pure Math., Volume 55, Part 2. p. 97-122.

[5] Beilinson (A.A.), MacPherson (R.), Schechtman (V.V.). — Notes on motivic cohomology, Duke Math. J. 54, p. 679-710 (1987).

[6] Beilinson (A.A.), Goncharov (A.B.), Schechtman (V.V.), Varchenko (A.N.). Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of pairs of triangles on the plane. The Grothendieck Festschrift, Vol. I, p. 135-172, Progr. Math., 86, Birkhäuser Boston, Boston, MA, (1990).

[7] Bloch (S.). — K_2 and algebraic cycles. Annals of Math. 99, p. 349-379 (1974).

[8] Bloch (S.). — Applications of the dilogarithm function in algebraic K-theory and algebraic geometry. Proc. Int. Symposium on Algebraic Geometry, Kyoto, 1977. Kinokuniyo Book Store Ltd., Tokyo, Japan.

[9] Bloch (S.). — Higher regulators, Algebraic K-theory, and Zeta functions of elliptic curves. Irvine lecture notes. CRM Monograph series, AMS 2000. Original preprint of 1978.

[10] Bloch (S.), Kriz (I.). — Mixed Tate motives. Ann of Math. 140, p. 557-605 (1994).

[11] Dupont (J.), Sah (S.-H.). — Scissor congruences II. J. Pure Appl. Algebra 25, p. 159-195 (1982).

[12] Fock (V.V.), Goncharov (A.B.). — Moduli spaces of local systems and Higher Teichmuller theory. Publ. Math. IHES, n. 103 (2006) 1-212. ArXiv math.AG/0311149.

[13] Fock (V.V.), Goncharov (A.B.). — The quantum dilogarithm and representations of quantum cluster varieties. 175, Inventiones Math., p. 223-286 (2009). arXiv:math/0702397.

[14] Gabrielov (A.), Gelfand (I.M.), Losik (M.). — Combinatorial computation of characteristic classes I, II, Funct. Anal. i ego pril. 9, p. 12-28 (1975), ibid 9, p. 5-26 (1975). MR0410758 (53 14504a).

[15] Gelfand (I.M.), MacPherson (R.). — Geometry in Grassmannians and a generalisation of the dilogarithm, Adv. in Math., 44, p. 279-312 (1982). MR0658730 (84b:57014).

[16] Goncharov (A.B.). — Geometry of configurations, polylogarithms, and motivic cohomology. Adv. Math. 114, no. 2, p. 197-318 (1995).

[17] Goncharov (A.B.). — Explicit construction of characteristic classes. Advances in Soviet Mathematics, 16, v. 1, Special volume dedicated to I.M.Gelfand's 80th birthday, p. 169-210 (1993).

[18] Goncharov (A.B.). — Chow polylogarithms and regulators. Math. Res. Letters, 2, p. 99-114 (1995).

[19] Goncharov (A.B.). — Polylogarithms and motivic Galois groups. Proc. of Symposia in Pure Math., Volume 55, Part 2. p. 43-96.

[20] Goncharov (A.B.). — Deninger's conjecture on L-function of elliptic curve at $s = 3$. ArXiv:alg-geom/9512016.

[21] Goncharov (A.B.). — Volumes of hyperbolic manifolds and mixed Tate motives JAMS vol. 12, N2, p. 569-618 (1999).

[22] Goncharov (A.B.). — Geometry of the trilogarithm and the motivic Lie algebra of a field. Regulators in analysis, geometry and number theory, p. 127-165, Progr. Math., 171, Birkhäuser Boston, Boston, MA (2000).

[23] Goncharov (A.B.). — Polylogarithms, regulators and Arakelov motivic complexes. JAMS J. Amer. Math. Soc. 18, no. 1, p. 1-60 (2005).

[24] Goncharov (A.B.). — Hodge correlators ArXive:math/0803.0297.

[25] Goncharov (A.B.). — A simple construction of Grassmannian polylogarithms. Adv. in Math 2014. arXive

[26] Hain (R.), MacPherson (R.). — Higher Logarithms, Ill. J. of Math,, vol. 34, N2, p. 392-475 (1990).

[27] Hanamura (M.), MacPherson (R.). — Geometric construction of polylogarithms, Duke Math. J. 70, p. 481-516 (1993).

[28] Hanamura (M.), MacPherson (R.). — Geometric construction of polylogarithms, II, Progress in Math. vol. 132, p. 215-282 (1996).

[29] MacPherson (R.). — Gabrielov, Gelfand, Losic combinatorial formula for the first Pontryagin class Seminar Bourbaki 1976. MR0521763 (81a:57022).

[30] Milnor (J.). — Introduction to algebraic K-theory. Annals of mathematical studies 72, Princeton University Press. (1971-.

[31] Suslin (A.A.). — K-theory of a field and the Bloch group, Proc. Steklov Inst. Math. 4, p. 217 239 (1991).

[32] Suslin (A.A.). — Homology of GL_n, characteristic classes and Milnor's K-theory. Trudy Mat. Inst. Steklov, 165, p. 188-204 (1984).

[33] Yusin (B. V.). — Sur les formes $S^{p,q}$ apparaissant dans le calcul combinatoire de la deuxiéme classe de Pontriaguine par la méthode de Gabrielov, Gelfand et Losik. (French) C. R. Acad. Sci. Paris Sér. I Math. 292, no. 13, p. 641-644 (1981).

[34] Zagier (D.). — Polylogarithms, zeta-functions, and algebraic K-theory of fields. Progress Math Vol 89. Birkhauser, Boston, MA, p. 392-430 (1991).