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Harish-Chandra's c-function; 50 years later

Simon Gindikin[(1)]

Résumé. — Nous discutons différents aspects de la fonction c de Harish-Chandra, en soulignant ses interactions avec la transformée horosphérique.

Abstract. — We discuss different aspects of the c-function of Harish-Chandra with focus on its connection with the horospherical transform.

I have known Vadim Schechtman for many years, but it was only a few years ago that I found out that we share two strong interests: surprises which it is possible yet to mine in works of great mathematicians and product-formulas for some special functions on Lie groups. Vadim has collected a very impressive exhibition of such formulas (see unpublished notes on his webpage). It starts with the product-formula for the c-function which Karpelevich and I found more than 50 years ago [19] in the beginning of my mathematical life. Vadim's anniversary for me is a pleasant reason to talk about this old subject. Starting at least with Gauss (who continuously produced new proofs of the Fundamental Theorem of Algebra), old mathematicians like to return to subjects of their first mathematical love. Perhaps it has a similar nature to criminals, returning to the scene of the crime? What about the c-function, for the past several years, there were quite a few popular generalizations of the product-formula on arbitrary fields, but in these notes we will stay inside old fashioned real or complex considerations.

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Origin of the c-function

The c-function appeared in the publication of Harish-Chandra of 1958 [22] about the asymptotic of zonal spherical functions on Riemannian symmetric spaces of non compact type and its applications to the Plancherel formula on these spaces. So it appeared as an important, but intermediate object. Let us start from a discussion if it has an independent conceptual nature. Often such kind of questions are artificial, but I believe that here it is a perfectly legitimate. Let us remind of the definitions.

Let G be a simple connected semisimple Lie group; K be its maximal compact subgroup; A, N be complimentary Cartanian and maximal unipotent subgroups; $G = KAN$ be the Iwasawa decomposition; $\bar{N}$ be the unipotent subgroup opposite to N and

$$X = G/K$$

be the Riemannian symmetric space. Let $a(g)$ be the projection on the factor A at the Iwasawa decomposition, $\mathfrak{a}$ be the Lie algebra Lie of A, $\mathfrak{a}^*$ be the dual space, Σ, Π be the system of positive and simple roots, ρ be the half-sum of positive roots;

$$a^{\xi} = \exp(\xi, \log a), \xi \in \mathfrak{a}, \mathfrak{a} \in \mathfrak{A}.$$

We can extend this definition on $\xi \in C\mathfrak{a}$. Let $\mu(d\bar{n})$ be the invariant measure on $\bar{N}$. Now we can define the c-function

$$c(\lambda) = \int_{\bar{N}} a(\bar{n})^{-\rho - i\lambda} \mu(d\bar{n}).$$

It is possible to prove that this integral is absolutely convergent if $Re(i\lambda)$ lies in the positive Weyl chamber W_{+}.

Let M be the centralizer of A at K and we call

$$\Xi = G/M\bar{N}$$

the *horospherical* space. Points of Ξ parameterize non degenerate orbits $E(\xi)$ of unipotent subgroups conjugated to N which are called the *horospheres*. On Ξ we have a "left" action of A commutating with the action G, since A normalizes $M\bar{N}$. Then the horospherical space Ξ fibers on A-fibers over the *flag* manifold

$$F = \Xi/A = G/AM\bar{N}.$$

So the isotropy subgroup of F is parabolic.

In E.Cartan's conception the irreducible finite dimensional spherical representations on X are characterized by unique K-invariant elements - zonal spherical functions. Another possibility is to connect with them N-invariant elements - highest weight vectors. This duality for spherical representations extends from finite dimensional representations on some infinite dimensional representations. It requires some analytical justifications which we will not discuss here.

At Borel-Weil modification of highest weight method the irreducible representations are realized at sections of line bundles on the flag spaces F. Gelfand-Graev's conception of integral geometry suggests realizing of irreducible representations at functions on the horospherical space Ξ. If we decompose the representation at functions on Ξ relative to the "left" action of A we will obtain Borel-Weil realizations of irreducible components on F, which are parameterized by characters of A. This looks as a small modification, but since points of Ξ admit a geometrical realization at X, it opens a possibility for a new construction of geometric analysis. Gelfand and Graev considered the integral geometry corresponding to infinite dimensional representations [6]. We will discuss later a similar approach to finite dimensional representations.

So irreducible spherical representations can be realized either on X or on Ξ. The principal moment of Gelfand-Graev's approach was that since spectrums on X and Ξ coincide it must be an invariant intertwining operator between functions on X and Ξ. This is Radon's type operator - the integration along horospheres - the *horospherical transform*:

$$\mathcal{H}f(\xi) = \int_{E(\xi)} f(x)\mu(dn), \xi \in \Xi,$$

where $\mu(dn)$ is the invariant measure on $\bar{N}$ translated on the horospheres. The first area of applications was the Plancherel formula on complex semisimple Lie groups and some homogeneous spaces: since the decomposition on irreducible ones is reduced to commutative Fourier transform on A, we will have the decomposition as soon as we can invert the horospherical transform. Later we will return to this possibility.

Heuristically, the horospherical transform is scalar on irreducible constituents if we identify them on X and Ξ. Modulo all complications connected with the continuous spectrum and non L^2 eigenfunctions *the c-function represents the eigenvalues of the horospherical transform under a natural normalization.* I believe that this heuristic view is crucial for the understanding of the nature of the c-function. Its appearance at the asymptotic of the zonal spherical function is already a secondary event.

As test functions for the definition of eigenvalues let us consider the "highest weight vectors" on X and Ξ:

$$\chi(x|\lambda) = a(x)^{-\rho-i\lambda}, x \in X; \chi(\xi|\lambda) = a(\xi)^{-\rho-i\lambda}, \xi \in \Xi;$$

Here $a(\xi)$ is the projection on A in the decomposition $\bar{N}MAN$ at the open part of G and, as a consequence, of Ξ. Then $\chi(\xi|\lambda)$ can be interpreted as boundary values of $\chi(x|\lambda)$.

The direct computation shows that

$$\mathcal{H}\chi(x|\lambda) = c(-\lambda)\chi(\xi|\lambda)$$

if $Re(-i\lambda)$ is in the positive Weyl chamber W_+ and we take boundary values for real λ.

There is a natural intertwining operator acting from Ξ to X - the *dual horospherical transform*

$$\mathcal{P}F(x) = \int_{S(x)} F(\xi)\mu(d\xi).$$

Here $S(x)$ is the set of parameters of horospheres passing through $x \in X$. We have $S(x) = K/M$ and $\mu(d\xi)$ is the invariant measure. So in the dual horospherical transform we have the integration on the compact manifold K/M. Let us remark that $\chi(\xi|\lambda)$ is defined only on an open set of Ξ and the integral converges only for $Re(i\lambda) \in W_+$. On the highest weight vectors it is scalar:

$$\mathcal{P}\chi(\xi|\lambda) = d(\lambda)\chi(x|\lambda).$$

The direct computation shows that

$$d(\lambda) = c(\lambda), Re(i\lambda) \in W_+.$$

We need just to apply the Harish-Chandra's trick of the replacement of integrations along compact orbits by integrations along unipotent ones [23]. He applied it for the integral representation of zonal spherical functions on the unipotent subgroup. It corresponds to the appearance of the factor $c(\lambda)$ at the Poisson formula for reconstructing eigenfunctions of invariant differential operators on X through the boundary values on the boundary F ([23, 24] .

The integrals defining the direct and dual horospherical transforms $\mathcal{P}, \mathcal{H}$ have no joint convergence area, but we can in both cases consider the boundary values for real λ and then their composition will have the eigenvalues

$$c(\lambda)c(-\lambda) = |c(\lambda)|^2.$$

So if we add the operator $\mathcal{L}$ with the eigenvalues

$$p(\lambda) = 1/|c(\lambda)|^2$$

we have the identical operator $\mathcal{LPH}$ and the operator $\mathcal{LP}$ inverts $\mathcal{H}$. It illustrates the principal Harish-Chandra's observation that $p(\lambda)$ is the density in the Plancherel formula and is the reason why in the Plancherel density appears the square of modulus of the c-function.

It would be interesting to transform these heuristic discussions into rigorous statements. It has two sides. Firstly, to make exact the connections between the horospherical transform and the c-functions, as its eigenvalues, without an appeal to zonal spherical functions. It must not go too far from a standard technology with generalized eigenfunctions for the continuous spectrum. Another direction is to define a horospherical transform in such a way that it would make sense for eigen functions of invariant differential operators which grow and an integral definition does not work for them.

Such a way was suggested in [14, 15]. All eigen functions can be holomorphically extended to some complex neighborhood (the crown) $Crown(X)$ of X at its complexification $Z \subset X$. The crown was constructed in my paper with Akhiezer [1]. It is possible to define a version of the horospherical transform for holomorphic functions at the crown with values at $\bar{\partial}$-cohomology at $C\Xi\backslash\Xi$. If for holomorphic functions the integral in the definition of horospherical transform converges then the two definitions are compatible. On other side, the eigenfunctions have hyperfunctional boundary values on the boundary so that also $\bar{\partial}$-cohomology at $C\Xi\backslash\Xi$. To finish the picture we need to verify that these two cohomology classes are proportional and the coefficients are the c-function.

Intermediate horospherical transforms and the wonderful boundary

The principal fact about the c-functions is their decomposition in a product of Euler's beta-functions corresponding to the decomposition of X on symmetric spaces of rank 1 [19, 21]. In [19] it is the result of a direct transformation of the integral representation of the c-function. In [21] it was developed in a more conceptual approach when the c-function is embedded in a more general family of special functions which are parameterized by elements of the Weyl group and relative to this parameter they satisfy a functional equation which gives as a consequence the decomposition. This approach is parallel to the intertwining operators of Knapp-Stein [25].

I believe that this conception can be more complete expressed in the language of horospherical transforms [17] when we can connect in one knot the triad {*wonderful boundary, horospherical transforms, c-functions*}. Let Π be a subsystem of prime roots, $\pi \subset \Pi$ and $\bar{\pi}$ be the supplement to π at Π. Let $A_\pi, A_{\bar{\pi}}$ be subgroups of A corresponding to π,$\bar{\pi}$. Let $\Sigma(\pi)$ be the result of removing from Σ all roots which are decomposed only on prime roots from π. Let $\bar{N}(\pi)$ be the unipotent subgroup generated by such roots α that $(-\alpha) \in \Sigma(\pi)$. Let M_π be the centralizer of $A_{\bar{\pi}}$ at H. Define

$$\Xi_\pi = G/M_\pi \bar{N}(\pi), F_\pi = \Xi_\pi / A_{\bar{\pi}} = G/P(\pi), P(\pi) = A_{\bar{\pi}} M_\pi \bar{N}(\pi)$$

and call them correspondingly *π-horospherical and π-flag spaces.* We have $\Xi_\emptyset = \Xi, \Xi_\Pi = F_\Pi = Z$. *Then F_π are components of wonderful boundary of* X [28, 5]. The dimensions of F_π are decreased if π decreases and $F_{\pi'}$ lies on the boundary of F_π if $\pi' \subset \pi$.

The π-horospherical spaces Ξ_π all have the same dimension as $X = \Xi_\Pi$. They were defined in the case when X is a group in [29]. It is an important technical moment to extend the horospherical transform on the whole system of the homogeneous spaces Ξ_π. The crucial moment is a geometrical duality between any pair of these spaces. Points $\xi \in \Xi = \Xi_\emptyset$ parameterize horospheres $E(\xi)$. Let us connect with points $\eta \in \Xi_\pi$ the non generate orbits $E_\pi(\eta), \eta \in \Xi_\pi$, of the subgroup $\bar{N}(\pi)$ and its conjugated ones. Let us call them *the intermediate horospheres or more specifically π-horospheres.* We will see that the integral geometry "lives" on Ξ_π rather than on the components F_π of the wonderful boundary (cf. the discussion above of the importance in the Gelfand-Graev construction of the jump from F on Ξ). Horospheres $E(\xi)$ can be considered as $E_\emptyset(\xi)$.

Dimensions of E_π increase when sizes of π decrease. Let $\pi' \subset \pi$; then each π'-horosphere $E_{\pi'}(\eta')$ is fibered on π-horospheres $E_\pi(\eta)$. The set $e_{\pi'|\pi}(\eta') \subset \Xi_\pi$ of their parameters we call *the small horospheres.* The small horospheres $e_{\pi'|\pi}$ on Ξ_π are orbits of minimal dimension of the subgroup $\bar{N}(\pi')$ and the conjugated subgroups. We can interpret E_π as $e_{\pi|\Pi}$ and small horospheres $e_{\emptyset|\pi}$ are orbits of the subgroup $\bar{N}$ and conjugated subgroups. So $e_{\emptyset|\Pi}$ correspond to the usual horospheres $E(\xi)$.

Let us supply all intermediate and small horospheres by the invariant collection of measures $\mu_\pi(dn), \mu_{\pi'|\pi}(dn)$ and define the intermediate horospherical transforms

$$\mathcal{H}_\pi f(\eta) = \int_{E_\pi(\eta)} f(x)\mu_\pi(dn), \eta \in \Xi_\pi$$

or, more generally,

$$\mathcal{H}_{\pi'|\pi} f(\eta') = \int_{e_{\pi'|\pi}(\eta')} f(\eta)\mu_{\pi'|\pi}(dn), \eta' \in \Xi'_{\pi}.$$

The principal fact is *the composition law for the intermediate horospherical transforms. If $\pi' \subset \pi$ then*

$$\mathcal{H}_{\pi'} = \mathcal{H}_{\pi'|\pi} \circ \mathcal{H}_{\pi}.$$

It is just a corollary of the computation with invariant measures in [21] and reformulation of the functional equation from there. Using this relation we can decompose the horospherical transform $\mathcal{H}$ on the intermediate factors for cases when π differs from π' on just one simple root. It is the way to compute the c-function.

Similarly we define dual intermediate horospheres: cycles $S_\pi(x) \subset \Xi_\pi, x \in X$, are sets of parameters of π-horospheres passing through x. They are isomorphic to K/M_π. More generally, for $\pi' \subset \pi$ we define the small dual intermediate horospheres $s_{\pi'|\pi}(\eta) \subset \Xi_{\pi'}, \eta \in \Xi_\pi$, which are isomorphic to $M_\pi/M_{\pi'}$. We define compatible invariant measures $\mu_{\pi'|\pi}(dk)$ on $S_{\pi'|\pi}$ and dual intermediate horospherical transforms:

$$\mathcal{P}_\pi f(x) = \int_{S_\pi(\eta)} f(\eta)\mu_\pi(dk), x \in X$$

or, more generally,

$$\mathcal{P}_{\pi'|\pi} f(\eta) = \int_{s_{\pi'|\pi}(\eta)} f(\eta')\mu_{\pi'|\pi}(d\eta'), \eta \in \Xi_\pi.$$

So relative to the operators $\mathcal{H}$ the dual operators $\mathcal{P}$ act in the opposite direction. Correspondingly, the functional equation is

$$\mathcal{P}_\pi = \mathcal{P}_{\pi'|\pi} \circ \mathcal{P}_{\pi'}.$$

Now we want to define the functions $c_{\pi'|\pi}(\lambda)$, including $c_\pi(\lambda)$, such that $c_{\pi'|\pi}(-\lambda)$ will be the eigen functions of the intermediate horospherical transforms $\mathcal{H}_{\pi'|\pi}$ and $c_{\pi'|\pi}(\lambda)$ of dual intermediate operators $\mathcal{P}_{\pi'|\pi}$. For this aim we take on the open part the decomposition

$$G^0 = \bar{N}(\pi)M_\pi AN$$

and let $a_\pi(g)$ be the projection on A which we will consider on Ξ_π. We define the highest weight functions on Ξ_π:

$$\chi_\pi(\eta|\lambda) = a_\pi(\eta)^{-\rho-i\lambda}, \eta \in \Xi_\pi$$

and will use them as test eigen functions:

$$\mathcal{H}_{\pi'|\pi}\chi_\pi(\eta|\lambda) = c_{\pi'|\pi}(-\lambda)\chi_{\pi'}(\eta'|\lambda), \mathcal{P}_{\pi'|\pi}\chi_{\pi'}(\eta'|\lambda) = c_{\pi'|\pi}(\lambda)\chi_\pi(\eta|\lambda).$$

We have

$$c_{\pi'|\pi}(\lambda) = \int_{\bar{N}(\pi'|\pi)} a_\pi^{-\rho-i\lambda}(\bar{n})\mu(d\bar{n})$$

where the subgroup $\bar{N}(\pi'|\pi)$ corresponds to the roots $-\alpha$ with positive roots α which decompose on simple roots from π excluding roots which decompose only on roots from π'.

The composition law for $\mathcal{H}, \mathcal{P}$ gives the basic functional equations for intermediate c-functions:

$$c_\pi(\lambda) = c_{\pi'|\pi}(\lambda)c_{\pi'}(\lambda)$$

which gives the possibility to compute them, including $c(\lambda)$, by the reduction to the case when π', π differ on one root.

The inverse horospherical transform. Connections with Radon's transform

We discussed that the operator $\mathcal{PH}$ has the eigenvalues $p(\lambda) = 1/|c(\lambda)|^2$, $\lambda \in \mathfrak{a}^*$, which coincides with the Plancherel density. It means that the inverse horospherical transform is the composition of $\mathcal{P}$ and the operator $\mathcal{L}$ with the eigen values $p(\lambda)$:

$$\mathcal{H}^{-1}f(x_0) = \int_{S(x_0)} \int_{\mathfrak{a}} K(a)\mathcal{H}f(\exp a, k))\mu(dk)\nu(da), a \in \mathfrak{a}.$$

Using the exponential eigenfunctions χ we can interpret the kernel $K(a)$ as the inverse Fourier transform of the Plancherel density $p(\lambda)$ [20]. So the inverse horospherical transforms have the densities $p(\lambda)$ as symbols. If all roots multiplicities m_α are even, then functions $p(\lambda)$ are polynomials and we apply to $\mathcal{H}$ along $\mathfrak{a}$ the differential operators $p(D_a)$. The product formula for the c-function gives an explicit formula for these polynomial symbols [13], but they look quite complicated for big multiplicities m_α.

In [7, 16] there was suggested a direct way to invert the horospherical transform in the case of even multiplicities by a simple reduction to the inversion of the Radon-John transform (the integration over k-planes at R^n). The inversion formulas under this approach include a much simpler differential operator, namely the operator for the similar problem in the flat model at the tangent space to X, which is just the Radon-John transform. The operator is

$$\prod_{\alpha\in\Sigma} D_\alpha^{m_\alpha},$$

where we take the composition of differentiations along positive roots α with multiplicities m_α. In this approach we apply this differential operator to a modified version of the horospherical transform

$$\check{f}(a,k) = a^\rho \mathcal{H}f(\exp a, k)).$$

Correspondingly, we can rewrite the Plancherel density in a more simple form as

$$p(\lambda) = \prod_{\alpha\in\Sigma}(i\alpha+\rho,\lambda).$$

This phenomenon demonstrates a very important conceptual moment that from the point of view of the horospherical approach, harmonic analysis on these symmetric spaces is equivalent in a sense to the flat model.

In the general case we can present the density as a product of a polynomial part $p_l(\lambda)$ which looks similar to the above case of even multiplicities and which produces in a similar way the differential operator $p(D_\alpha)$ in the inversion formula and a non polynomial factor $p_n(\lambda)$ which is responsible for the non local part in the inversion formula and which is (under a small restriction) the product of

$$\tanh(\pi\frac{\alpha}{(\alpha,\alpha)})$$

over all positive roots $\alpha\in\Sigma$ with odd multiplicities [20]. The problem is that we need to compute the Fourier transform of the product of m factors which is bigger than the rank $l = \dim A$ (the dimension of the integration). In [20] a trick was suggested, using the addition formula for tanh, which consequently decreases the number of factors. As the result in the computations for a broad class of classical symmetric spaces it is possible to reach a sum of products of not more than l factors. Then it is possible to write explicitly the inversion of the horospherical transform. Then in [2], it was proved that for all symmetric spaces it is possible to continue this process, such that not more than l factors it will be left. In such a way the explicit inversion is always possible but the choice of reduced root systems is not unique: in [2] there is a sufficient list of such reduced root systems. The problem was solved and the consideration of reduced root systems is a nice combinatoric problem, but the possible inversion formulas were different for different roots systems. A universal transparent formula which works in the general case was not found.

A construction of such a formula was discussed in [16].The problem of a unification of local and non local inversion formulas of integral geometry starts from the Radon transform. There the inversion is an averaging of a differential operator for odd dimensions (local formula) and of a pseudodifferential operator for even dimensions (non local formula). Using the distribution $(x-i0)^{-k}$ it is possible to unify these two formulas at a unique formula, independent of the dimension. It turns out that a similar possibility exists in the case of symmetric spaces. For simplicity let us put a (non essential) restriction that if a root $\alpha/2$ exists, its multiplicity is even. Then we can rewrite the Plancherel density as

$$p(r) = cp_l(r)\{\prod_{\alpha\in\Pi}(1+\tanh(\pi\frac{s\alpha}{(\alpha,\alpha)}r)\}_{s\in W}$$

where $p_l(r)$ is the described above the polynomial and we take the alternation over the Weyl group W. After the inverse Fourier transform we obtain the inversion formula

$$f(x_0) = c\int_{S(x_0)}\mu(dk)\int_{\mathfrak{a}}\prod_{\alpha\in\Pi}[\sinh((\alpha,s))-i0]^{-1}p_l(D)(\exp(\rho,s)\mathcal{H}f(k,\exp s)ds.$$

We keep only one term in the alternated sum since we apply it to the W-symmetric function. We receive quite a transparent formula operated only with habitual systems of roots: positive and simple ones. It looks like the way to compute the Plancherel density through the c-functions, where we decompose the density through eigenvalues of 2 operators, does not give an optimal representation. Since the image of the spherical Fourier transform is symmetric relative to the Weyl group the expression for the Plancherel measure is not unique, but the possibility to seriously simplify the Harish-Chandra's representation was a surprise for me. In the case of the polynomial factor we found a direct way which gives the formula in a more transparent form coinciding with the formula in the flat model. Since we removed now magic roots systems in non local part as well it looks realistic to find a direct way also for the non local part (again as in the flat model).

It is possible to modify the horospherical transform including in it the factors $\prod_{\alpha\in\Pi}[\sinh((\alpha,s))-i0]^{-1}$. We replace δ-function along horospheres on a version of Cauchy kernel (cf. our considerations below). Then the inversion formula will be local.

Horospherical Cauchy transform and c-functions for finite-dimensional representations

Weyl's formula for dimensions of irreducible representations is a first example of a product-formula for root systems. Since there is a natural connection of the dimension and the Plancherel density it was natural to connect these two product-formulas: to connect the dimensions with the c-functions and in such a way to reproduce such Weyl's formula. It was done [23] through analytic extension and a regularization of the c-function (see subsequent discussions in [27]). Let us follow the above stated view that each time when a c-function appears, we need to seek an appropriate version of horospherical transform. In this case we can do it, considering complex symmetric spaces using tools of complex analysis [11, 15].

Complex symmetric spaces are homogeneous spaces $Z = G/H$ where G is a simply connected semisimple complex Lie group and H is its involutive subgroup relative to a holomorphic involution. Let A, N be, transversal to H, Cartanian and maximal unipotent subgroups and we take the Iwasawa decomposition on a Zariski open part of G:

$$G^0 = HAN.$$

Let M be the centralizer of A at H. Then we call

$$\Xi = G/MN, F = G/AMN$$

correspondingly the horospherical and flag spaces; Ξ is fibering on A-fibers over F; $\dim Z = \dim \Xi$. There is a natural duality between Z, Ξ corresponding to the double fibering of the homogeneous space G/M over them. Correspondingly, the points $\zeta \in \Xi$ parameterize the horospheres $E(\zeta)$ at Z - orbits of subgroups conjugated to N and points $z \in Z$ parameterize pseudospheres $S(z)$ - orbits of subgroups, conjugated to H; $S(z)$ are isomorphic to H/M and they are Stein submanifolds.

The principal fact of the finite-dimensional harmonic analysis is that the representations on Z and Ξ have the same simple spectrum corresponding to the spherical constituents. Under the algebraic approach we consider the representations at regular functions on Z, Ξ, but from the analytical point of view it is natural to consider the representations at spaces of holomorphic functions $\mathcal{O}(Z), \mathcal{O}(\Xi)$. The coincidence of spectrums makes it suspicious that these spaces are G-isomorphic (let us remark that Z is a Stein manifold, but Ξ is not). Some version of horospherical transforms -*the horospherical Cauchy transform* - realizes this isomorphism.

Since we want to define a horospherical transform on holomorphic functions we, of course, can not integrate them along horospheres. Instead we will apply a Cauchy kind singular operator with singularities on horospheres [11, 15]. We need to modify some way $a^\lambda(x)$ for the holomorphic and finite-dimensional situation. So we take characters of A and extend them using the Iwasawa decomposition on the open part Z^0. Let us consider the semigroup of spherical highest weight characters and its generators $\Delta_j(z), j \leqslant l$ (l is the rank of Z). We call them *Sylvester's functions* since in the case when Z is the manifold of non degenerated symmetric matrices they are principal minors and they participate in the Sylvester condition. It is important that they can be holomorphically extended from Z^0 to the whole Z. So the highest weights correspond to

$$\Delta^m(z) = \Delta_1^{m_1}(z) \cdots \Delta_l^{m_l}(z),$$

where m_j are non negative integers. These Sylvester's functions are associated with the subgroup N. Let us connect them with other unipotent subgroups which are parameterized by points $\zeta \in \Xi$. If ζ_0 correspond to N and $\zeta = g(\zeta_0)$ then we put

$$\Delta_j(\zeta|z) = \Delta(g^{-1} \cdot z).$$

The system of equations for a fixed ζ:

$$\Delta_j(\zeta|z) = 1, 1 \leqslant j \leqslant l$$

defines the horosphere $E(\zeta)$. Let us fix the holomorphic invariant n-form $\omega(dz)$ (it is unique up to a constant factor).

We can now define the horospherical Cauchy kernel as

$$K(\zeta|z) = \prod_{1 \leqslant j \leqslant l} \frac{1}{1 - \Delta_j(\zeta|z)}$$

and then the horospherical Cauchy transform as

$$\mathcal{H}f(\zeta) = \int_\Gamma K(\zeta|z) f(z) \omega(dz), \zeta \in \Xi, f \in \mathcal{O}(Z)$$

where Γ is a cycle of (real) dimension n which avoids the singularities of the kernel. The manifold Z can be contracted on its compact form $X = U/K$ where U, K are compact form of G, H for the same involution. Any such a form we can take as a cycle Γ if it does not have singularities for this ζ. It is possible to prove that there is a domain of ζ for which K has no singularities on Γ [11]. Deforming this cycle we can define the transform for all $\zeta \in \Xi$. Apparently it will be an intertwining operator

$$\mathcal{H} : \mathcal{O}(Z) \to \mathcal{O}(\Xi).$$

We see that the horosphere $E(\zeta)$ is the edge of singularities of the kernel, but the complete singular set is bigger. It is impossible to take the residue of the integrand on the horosphere (it has a trivial topology).

The horospherical Cauchy transform $\mathcal{H}$ intertwines irreducible constituents at Z and Ξ. Using the decomposition on an open part $G^0 = \hat{N}MAN$ we can define highest weight vectors $\delta^m(\zeta) = \Delta^m(z^0|\zeta)$. Then

$$\mathcal{H}\Delta^m(z) = c^*(m)\delta^m(\zeta)$$

and this coefficient $c^*(m)$ is the eigen value of $\mathcal{H}$ on the corresponding irreducible constituent. We can see that the heuristic interpretation of c-functions as eigenvalues in the infinite dimensional case in the finite dimensional case is exact. We have

$$c^*(2m) = \int_{U/K} \Delta^m(z)\omega(dz).$$

The function $c^*(m)$ can be expressed through the c-function of Harish-Chandra as in [27]. It would be interesting to produce in this case constructions similar to the ones discussed above: to define intermediate Cauchy transforms with the similar composition laws and to directly find product-formulas.

Let us define the dual horospherical Cauchy transform. It would be possible to define it as an integral along some cycles similar to the integral for $\mathcal{H}$ using the same kernel, but for fixed $z \in Z$ with the edge of singularities on the pseudospheres $S(z)$. However the geometry of Ξ is dramatically different from the geometry of Z. This time we can take the residue on $S(Z)$ and consider nonsingular integrals. Pseudospheres $S(z)$ are Stein manifolds and they are isomorphic to U/K. Define on them invariant system of holomorphic forms $\mu_z(d\zeta)$ of maximal dimension n-l. Then we define the dual horospherical transform as

$$\mathcal{P}F(z) = \int_{\gamma \subset S(z)} F(\zeta)\mu_z(d\zeta), z \in Z, F(\zeta) \in \mathcal{O}(\Xi).$$

Here the cycle γ is any generating cycle of the Stein submanifold $S(z)$; it can be any flag manifold which is a real form of $S(z)$. Again we can define the dual c- functions as the eigenvalues

$$\mathcal{P}\delta^m(z) = c(m)\delta^m(z),$$

where

$$c(m) = \int_{K/M} \delta^m(\zeta)(\mu_{z_0}(dz).$$

This coefficient can be express through the c-function of Harish-Chandra but it would be interesting to define intermediate dual transforms and to find product-formulas on this way. The operator $\mathcal{PH}$ has the eigen values $c^*(m)c(m)$. Then

$$d(m) = \frac{1}{c^*(m)c(m)}$$

is a polynomial and coincides with the dimension of the corresponding finite dimensional representation. It can be presented using Weyl's formula. If we were to develop direct ways to produce the product-formulas for $c(m), c^*(m)$ we would be have an alternative method to produce Weyl's formulas for the dimensions and their specifications for spherical representations. On the group A we define the differential operator $W(D_a)$ with constant coefficients (at logarithmic coordinates) with the symbol $d(m)$. Let us translate this operator on fibers of the fibering $\Xi \to F$. Then

$$\mathcal{H}^{-1} = \mathcal{P}W(D_a).$$

This inversion formula for the horospherical Cauchy transform is a variant of the multidimensional integral Cauchy formula. Let us clarify this analogy. Let $\Pi(w) = \{(z, \zeta), \zeta \in S(w)\}$. It is equivalent to $w \in E(\zeta)$. On $\Pi(w)$ we define the Cauchy kernel-form

$$C(w|z, \zeta) = W(D_a)(\frac{1}{\prod_{1 \le j \le l}(\Delta_j(z|\zeta) - 1)})\omega(z; dz) \wedge \mu_w(\zeta; d\zeta).$$

This form is closed and

$$\int_{\delta \subset \Pi(w)} C(w|z, \zeta)f(z) = cf(w), f \in \mathcal{O}(Z).$$

Here the constant depends on the homology class of the cycle δ. In the case of the inversion of the horospherical case, we take a special cycle $\delta = \delta(\gamma)$ which corresponds to a cycle $\gamma \subset S(w)$ and for each $\zeta \in \gamma$ we take a cycle $\Gamma(\zeta)$ at Z which avoids the singularities of the kernel. However since the kernel C is the closed form the integration over arbitrary cycle $\delta \subset \Pi(w)$ of the dimension $2n - l$, avoiding singularities, reconstructs the holomorphic functions.

In such an approach the Cauchy formula is a result of some considerations in harmonic analysis on complex symmetric spaces but it possible to move in the opposite direction: to receive directly the Cauchy formula as a version of the Cauchy-Fantappiè integral formulas and take it as an initial point of the complex analysis on complex symmetric spaces which includes the horospherical Cauchy transform and as a consequence the harmonic analysis [12, 15]. Let us recall that H.Weyl when he made first steps in his analytic (transcendental) approach to the harmonic analysis on classical groups remarked that spherical functions on them are holomorphic on complex classical groups and can be treated by tools of complex analysis, but he suggested another way: to restrict representations on compact groups and then to apply the real analysis ("unitary trick"). I believe one of the reasons could be that appropriate methods of multidimensional complex analysis did not exist yet (of course, the other reason was the already existing theory of Peter-Weyl on compact groups). May be the suggested way to work with the Cauchy formula on symmetric spaces is a realization of the possibility to apply complex analysis, and this opens new interesting possibilities.

Other problems and possibilities

The central problem of harmonic analysis on symmetric spaces is the construction of such analysis on pseudo Riemannian semisimple symmetric spaces. Such spaces are real forms of complex symmetric spaces. The analogue of Plancherel's formula is known for such spaces, but the realization of a horospherical approach met substantial obstructions. The problem is that we always can consider the real horospherical transform, but it in the non Riemannian case, as rule, has a kernel which corresponds to discrete or partly discrete series of representations. So real horospherical transform corresponds only to the most continuous series.

I believe that the problem of integral geometry is for each series of representations to find on the complex horospherical space Ξ appropriate geometrical and analytic objects in which models of series (at first turn discrete ones) are realized. On each model an appropriate real Cartanian group acts commutating with the action G. It gives the decomposition of the model on irreducible representations. Then for each series we need to construct a horospherical transform on the model and its inversion. It explains the geometrical structure of the Plancherel formula.

This was realized so far in quite a few cases. We already talked about the case of Riemannian non compact symmetric spaces. For the compact symmetric space X we have a domain $\hat{X} \subset Z$ and the space of holomorphic functions $\mathcal{O}(\hat{X})$ is isomorphic to the space of hyperfunctions on $\hat{X}$ and there is a version of the horospherical Cauchy transform which intertwines these functional spaces [11].

The simplest case where discrete series have appeared is the group $SL(2;R)$. Correspondingly, in this case, the real horospherical transform has the kernel corresponding to the holomorphic and antiholomorphic discrete series. I suggested [9] to consider not only real horospheres, but also complex horospheres without real points. We connect with them a horospherical Cauchy transform which have singularities on these horospheres. The images will be spaces of holomorphic functions at 2 domains at Ξ - models of holomorphic and anti holomorphic series. This construction works only for a class of symmetric spaces. It turns out that in the general case we need to generalize the conception of horospherical Cauchy transform replacing it by a horospherical transform with values in higher $\bar{\partial}$-cohomology. In all cases some c-functions have appeared as eigen values of horospherical transforms.

There is a case where it is natural to have the appearance of the c-function, but it has not appeared yet. I mean spherical spaces - homogeneous spaces on which the Borelian subgroup has an open orbit. This class contains symmetric spaces and many facts of harmonic analysis on symmetric spaces can be generalized on spherical ones, sometimes under some restrictions. For spherical spaces there is a horospherical transform, but the concept of restricted roots can not be transfered on the spherical case in complete form. Nevertheless, in [18], under some restrictions, a variant of restricted roots sufficient for a product-formula for dimensions of spherical representations was developed. It gives a strong hope to obtain a similar product-formula for the c-functions, beginning with the Riemannian case and spaces with complex groups.

There are several cases where there exist analogues of the c-functions together with product-formulas: for buildings [26], for arbitrary fields [3, 4], Heckman-Opdam's hypergeometric functions for virtual root multiplicities [27]. It would be interesting to connect them with analogues of the horospherical transform.

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