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The dual boundary complex of the SL_2 character variety of a punctured sphere

Carlos Simpson[(1)]

Dedicated to Vadim Schechtman on the occasion of his 60th birthday

Résumé. — Soient $C_1, \ldots, C_k$ des classes de conjugaison génériques dans $SL_2(C)$. On considère la variété de caractères des systèmes locaux sur $P^1 - \{y_1, \ldots, y_k\}$ dont les transformations de monodromie autour des y_i sont dans les classes de conjugaison C_i respectives. On montre que le complexe dual du bord de cette variété est équivalent par homotopie à un sphère de dimension $2(k-3)-1$.

Abstract. — Suppose $C_1, \ldots, C_k$ are generic conjugacy classes in $SL_2(C)$. Consider the character variety of local systems on $P^1 - \{y_1, \ldots, y_k\}$ whose monodromy transformations around the punctures y_i are in the respective conjugacy classes C_i. We show that the dual boundary complex of this character variety is homotopy equivalent to a sphere of dimension $2(k-3)-1$.

1. Introduction

Given a smooth quasiprojective variety X, choose a normal crossings compactification $\overline{X} = X \cup D$ and define a simplicial set called the *dual boundary complex* $D\partial X$, containing the combinatorial information about multiple intersections of divisor components of D. Danilov, Stepanov and Thuillier have shown that the homotopy type of $D\partial X$ is independent of the choice of compactification, and this structure has been the subject of much study.

We consider the case when $X = \mathrm{M}_B(S; C_1, \ldots, C_k)$ is the character variety, of local systems on a punctured sphere $S \sim P^1 - \{y_1, \ldots, y_k\}$ such that

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the conjugacy classes of the monodromies around the punctures are given by $C_1, \ldots, C_k$ respectively [38]. If these conjugacy classes satisfy a natural genericity condition then the character variety is a smooth affine variety. We prove that its dual boundary complex is a sphere of the appropriate dimension (see Conjecture 11.1), for local systems of rank 2.

Theorem 1.1. — *Suppose $C_1, \ldots, C_k$ are conjugacy classes in $SL_2(C)$ satisfying the genericity Condition 7.1. Then the dual boundary complex of the character variety is homotopy equivalent to a sphere:*

$$D\partial \mathrm{M}_B(S; C_1, \ldots, C_k) \sim S^{2(k-3)-1}.$$

This statement is a part of a general conjecture about the boundaries of moduli spaces of local systems [30]. The conjecture says that the dual boundary complex of the character variety or "Betti moduli space" should be a sphere, and that it should furthermore be naturally identified with the sphere at infinity in the "Dolbeault" or Hitchin moduli space of Higgs bundles. We will discuss this topic in further detail in Section 11 at the end of the paper.

The case $k = 4$ of our theorem is a consequence of the Fricke-Klein expression for the character variety, which was indeed the motivation for the conjecture. The case $k = 5$ of Theorem 1.1 has been proven by Komyo [33].

1.1. Strategy of the proof

Here is the strategy of our proof. We first notice that it is possible to make some reductions, based on the following observation (Lemma 2.3): if $Z \subset X$ is a smooth closed subvariety of a smooth quasiprojective variety, such that the boundary dual complex is contractible $D\partial Z \sim *$, then the natural map $D\partial X \to D\partial(X - Z)$ is a homotopy equivalence. This allows us to remove some subvarieties which will be "negligeable" for the dual boundary complex. The main criterion is that if $Z = A^1 \times Y$ then $D\partial Z \sim *$ (Corollary 2.5). Together, these two statements allow us successively to remove a whole sequence of subvarieties (Proposition 2.6).

The main technique is to express the moduli space $\mathrm{M}_B(S; C_1, \ldots, C_k)$ in terms of a decomposition of S into a sequence of "pairs of pants" S_i which are three-holed spheres. The decomposition is obtained by cutting S along $(k - 3)$ circles denoted ρ_i. In each S_i, there is one boundary circle corresponding to a loop ξ_i around the puncture y_i, and two other boundary circles ρ_{i-1} and ρ_i along which S was cut. At the start and the end of the sequence, two of the circles correspond to ξ_1, ξ_2 or ξ_{k-1}, ξ_k and only one to a cut. One may say that ρ_1 and ρ_{k-1} are confused with the original boundary circles ξ_1 and ξ_k respectively.

We would like to use this decomposition to express a local system V on S as being the result of "glueing" together local systems $V|_{S_i}$ on each of the pieces, glueing across the circles ρ_i. A basic intuition, which one learns from the elementary theory of classical hypergeometric functions, is that a local system of rank 2 on a three-holed sphere is determined by the conjugacy classes of its three monodromy transformations. This is true generically, but one needs to take some care in degenerate cases involving potentially reducible local systems, as will be discussed below.

The conjugacy classes of the monodromy transformations around ρ_i are determined, except in some special cases, by their traces. The special cases are when the traces are 2 or -2.

If we assume for the moment the uniqueness of $V|_{S_i}$ as a function of C_i and the traces t_{i-1} and t_i of the monodromies around ρ_{i-1} and ρ_i respectively, then the local system V is roughly speaking determined by specifying the values of these traces $t_2, \ldots, t_{k-2}$, plus the glueing parameters. The glueing parameters should respect the monodromy transformations, and are defined modulo central scalars, so each parameter is an element of G_m. In this rough picture then, the moduli space could be viewed as fibering over $(A^1)^{k-3}$ with fibers G_m^{k-3}.

The resulting coordinates are classically known as *Fenchel-Nielsen coordinates*. Originally introduced to parametrize $PGL_2(R)$ local systems corresponding to points in Teichmüller space, they have been extended to the complex character variety by Tan [50].

In the above discussion we have taken several shortcuts. We assumed that the traces t_i determined the monodromy representations, and in saying that the glueing parameters would be in G_m we implicitly assumed that these monodromy transformations were diagonal with distinct eigenvalues. These conditions correspond to saying $t_i \neq 2, -2$.

We also assumed that the local system $V|_{S_i}$ was determined by C_i, t_{i-1} and t_i. This is not in general true if it can be reducible, which is to say if there is a non-genericity relation between the conjugacy classes. The locus where that happens is somewhat difficult to specify explicitly since there are several possible choices of non-genericity relation (the different choices of ϵ_i in Condition 4.3). We would therefore like a good way of obtaining such a rigidity even over the non-generic cases.

Such a property is provided by the notion of *stability*. One may envision assigning parabolic weights to the two eigenvalues of C_i and assigning parabolic weights zero over ρ_j. The parabolic weights induce a notion of *stable* local system over S_i. But in fact we don't need to discuss parabolic weights themselves since the notion of stability can also be defined directly: a local system V_i on S_i is *unstable* if it admits a rank 1 subsystem L such that the monodromy matrix in C_i acts on L by c_i^{-1} (a previously chosen one of the two eigenvalues of C_i). It is *stable* otherwise. Now, it becomes true that a stable local system is uniquely determined by C_i, t_{i-1} and t_i. This will be the basis of our calculations in Section 10, see Corollary 10.3.

The first phase of our proof is to use the possibility for reductions given by Proposition 2.6 to reduce to the case of the open subset

$$M' \subset \mathrm{M}_B(S; C_1, \ldots, C_k)$$

consisting of local systems V such that $t_i \in A^1 - \{2, -2\}$ and such that $V|_{S_i}$ is stable. In order to make these reductions, we show in Sections 7 and 8 that the strata where some t_i is 2 or -2, or where some $V|_{S_i}$ is unstable, have a structure of product with A^1, hence by Lemma 2.5 these strata are negligeable in the sense that Lemma 2.3 applies.

For the open set M', there is still one more difficulty. The glueing parameters depend *a priori* on all of the traces, so we don't immediately get a decomposition of M' as a product. A calculation with matrices and a change of coordinates allow us to remedy this and we show in Theorem 10.6 that $M' \cong \mathbf{Q}^{k-3}$ where $\mathbf{Q}$ is a space of choices of a trace t together with a point $[p, q]$ in a copy of G_m.

It turns out that this family of multiplicative groups (it is a group scheme) over $A^1 - \{2, -2\}$ is twisted: the two endpoints of the fibers G_m get permuted as t goes around 2 and -2. This twisting property is what makes it so that

$$D\partial\mathbf{Q} \sim S^1,$$

and therefore by [45, Lemma 6.2], $D\partial(\mathbf{Q}^{k-3}) \sim S^{2(k-3)-1}$. This calculates $D\partial M'$ and hence also $D\partial\mathrm{M}_B(S; C_1, \ldots, C_k)$ to prove Theorem 1.1.

We should consider the open subset M' as the natural domain of definition of the Fenchel-Nielsen coordinate system, and the components in the expression $M' \cong \mathbf{Q}^{k-3}$ are the Fenchel-Nielsen coordinates.

1.2. Relation with other work

What we are doing here is closely related to a number of things. Firstly, as pointed out above, our calculation relies on the Fenchel-Nielsen coordinate system coming from a pair of pants decomposition, and this is a well-known technique. Our only contribution is to keep track of the things which must be removed from the domain of definition, and of the precise form of the coordinate system, so as to be able to conclude the structure up to homotopy of the dual boundary complex.

A few references about Fenchel-Nielsen coordinates include [10] [15] [44] [54], and for the complex case Tan's paper [50]. Nekrasov, Rosly and Shatashvili's work on bend parameters [41] involves similar coordinates and is related to the context of polygon spaces [14]. The work of Hollands and Neitzke [27] gives a comparison between Fenchel-Nielsen and Fock-Goncharov coordinates within the theory of spectral networks [13]. Jeffrey and Weitsman [28] consider what is the effect of a decomposition, in arbitrary genus, on the space of representations into a compact group. Recently, Kabaya uses these decompositions to give algebraic coordinate systems and furthermore goes on to study the mapping class group action [29]. These are only a few elements of a vast literature.

Conjecture 11.1 relating the dual boundary complex of the character variety and the sphere at infinity of the Hitchin moduli space, should be viewed as a geometric statement reflecting the first weight-graded piece of the $P = W$ conjecture of de Cataldo, Hausel and Migliorini [7] [18]. This will be discussed a little bit more in Section 11 but should also be the subject of further study.

Komyo gave the first proof of the theorem that the dual boundary complex was a sphere, for rank 2 systems on the projective line minus 5 points [33]. He did this by constructing an explicit compactification and writing down the dual complex. This provides more information than what we get in our proof of Theorem 1.1, because we use a large number of reduction steps iteratively replacing the character variety by smaller open subsets.

I first heard from Mark Gross in Miami in 2012 about a statement, which he attributed to Kontsevich, that if X is a log-Calabi-Yau variety (meaning that it has a compactification $\overline{X} = X \cup D$ such that $K_{\overline{X}} + D$ is trivial), then $D\partial X$ should be a sphere. Sam Payne points out that this idea may be traced back at least to [35, Remark 4] in the situation of a degeneration.

Gross also stated that this property should apply to character varieties, that is to say that some or all character varieties should be log-CY. That has apparently been known folklorically in many instances cf [17].

Recently, much progress has been made. Notably, Kollár and Xu have proven that the dual boundary of a log-CY variety is a sphere in dimension

4, and they go a long way towards the proof in general [32]. They note that the correct statement, for general log-CY varieties, seems to be that $D\partial X$ should be a quotient of a sphere by a finite group. In our situation of character varieties, part of the statement of Conjecture 11.1 posits that this finite quotienting doesn't happen. This is supported by our theorem, but it is hard to say what should be expected in general.

De Fernex, Kollár and Xu have introduced a refined dual boundary complex [8], which is expected to be a sphere in the category of PL manifolds. That is much stronger than just the statement about homotopy equivalence. See also Nicaise and Xu [42]. For character varieties, as well as for more general cluster varieties and quiver moduli spaces, the Kontsevich-Soibelman wallcrossing picture could be expected to be closely related to this PL sphere, more precisely the Kontsevich-Soibelman chambers in the base of the Hitchin fibration should to correspond to cells in the PL sphere. One may witness this phenomenon by explicit calculation for SL_2 character varieties of the projective line minus 4 points, under certain special choices of conjugacy classes where the character variety is the Cayley cubic.

Recently, Gross, Hacking, Keel and Kontsevich [17] building on work of Gross, Hacking and Keel [16], have given an explicit combinatorial description of a boundary divisor for log-Calabi-Yau cluster varieties. Their description depends on a choice of toroidal cluster coordinate patches, and the combinatorics involve toric geometry. It should in principle be possible to conclude from their construction that $D\partial \mathrm{M}_B(S; C_1, \ldots, C_k)$ is a sphere, as is mentioned in [17, Remark 9.12]. Their technique, based in essence on the Fock-Goncharov coordinate systems, should probably lead to a proof in much greater generality than our Theorem 1.1.

1.3. Varying the conjugacy classes

In the present paper, we have been considering the conjugacy classes $C_1, \ldots, C_k$ as fixed. As Deligne pointed out, it is certainly an interesting next question to ask what happens as they vary. Nakajima discussed it long ago [40]. This has many different aspects and it would go beyond our current scope to enter into a detailed discussion.

I would just like to point out that the natural domain on which everything is defined is the space of choices of $C_1, \ldots, C_k$ which satisfy the Kostov genericity Condition 4.3. This is an open subset of G_m^k, the complement of a toric arrangement [3], a divisor K whose components are defined by multiplicative monomial equalities. It therefore looks like a natural multiplicative analogue of the hyperplane arrangement complements which enter into the theory of higher dimensional hypergeometric functions [52]. The variation with parameters of the moduli spaces $\mathrm{M}_B(S; C_1, \ldots, C_k)$ leads, at the very least, to some variations of mixed Hodge structure over $G_m^k - K$ which undoubtedly have interesting properties.

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1.5. Dedication

It is a great honor to dedicate this work to Vadim Schechtman. Vadim's interests and work have illuminated many aspects of the intricate interplay between topology and geometry in the de Rham theory of algebraic varieties. His work on hypergeometric functions [52] motivates our consideration of moduli spaces of local systems on higher-dimensional varieties. His work with Hinich on dga's in several papers such as [24] was one of the first instances of homotopy methods for algebraic varieties. His many works on the chiral de Rham complex have motivated wide developments in the theory of $\mathcal{D}$-modules and local systems. The ideas generated by these threads have been suffused throughout my own research for a long time.

2. Dual boundary complexes

Suppose X is a smooth quasiprojective variety over C. By resolution of singularities we may choose a normal crossings compactification $X \subset \overline{X}$ whose complementary divisor $D := \overline{X} - X$ has simple normal crossings. In fact, we may assume that it satisfies a condition which might be called *very simple normal crossings*: if $D = \bigcup_{i=1}^{m} D_i$ is the decomposition into irreducible components, then we can ask that any multiple intersection $D_{i_1} \cap \cdots \cap D_{i_k}$ be either empty or connected. If the compactification satisfies this condition, then we obtain a simplicial complex denoted $D\partial X$, the dual complex $D(D)$ of the divisor D, defined as follows: there are m vertices $e_1, \ldots, e_m$ of $D\partial X$, in one-to-one correspondence with the irreducible components $D_1, \ldots, D_m$ of D; and a simplex spanned by $e_{i_1}, \ldots, e_{i_k}$ is contained in $D\partial X$ if and only if $D_{i_1} \cap \cdots \cap D_{i_k}$ is nonempty.

This defines a simplicial complex, which could be considered as a simplicial set, but which for the present purposes we shall identify with its topological realization which is the union of the span of those simplicies in R^m with e_i being the standard basis vectors.

The simplicial complex $D\partial X$ goes under several different terminologies and notations. We shall call it the *dual boundary complex* of X. It contains the purely combinatorial information about the divisor compactifying X. The main theorem about it is due to Danilov [4]:

Theorem 2.1 (Danilov). — *The homotopy type of $D\partial X$ is independent of the choice of compactification.*

The papers of Stepanov [48] [49], concerning the analogous question for singularities, started a lot of renewed activity. Following these, a very instructive proof, which I first learned about from A. Ducros, was given by Thuillier [51]. He interpreted the homotopy type of $D\partial X$ as being equivalent to the homotopy type of the *Berkovich boundary* of X, namely the set of points in the Berkovich analytic space [2] associated to X (over the trivially valued ground field), which are valuations centered at points outside of X itself.

Further refinements were given by Payne [45] and de Fernex, Kollár and Xu [8]. Payne showed that the simple homotopy type of $D\partial X$ was invariant, and proved several properties crucial to our arguments below. De Fernex, Kollár and Xu defined in some cases a special choice of compactification leading to a boundary complex $D\partial X$ whose PL homeomorphism type is invariant. Nicaise and Xu show in parallel, in the case of a degeneration at least, that the essential skeleton of the Berkovich space is a pseudomanifold [42]. Manon considers an embedding of "outer space" for character varieties, into the Berkovich boundary [39]. These refined versions provide very interesting objects of study but for the present paper we just use the homotopy type of $D\partial X$.

Our goal will be to calculate the homotopy type of the dual boundary complex of some character varieties. To this end, we describe here a few important reduction steps allowing us to modify a variety while keeping its dual boundary complex the same.

One should be fairly careful when manipulating these objects, as some seemingly insignificant changes in a variety could result in quite different boundary complexes. For example, in the case of an isotrivial fibration it isn't enough to know the homotopy types of the base and the fiber—essentially, the fibration should be locally trivial in the Zariski rather than étale topology in order for that kind of reasoning to work. The space $\mathbf{Q}$ to be considered at the end of the paper provides an example of this phenomenon.

In a similar vein, I don't know of a good notion of dual boundary complex for an Artin stack. It is possible that a theory of Berkovich stacks could remedy this problem, but that seems difficult. Payne has suggested, in answer to the problem of étale isotrivial fibrations, to look at an equivariant notion of isotrivial fibration which could give a natural group action on a dual boundary complex such that the quotient would be meaningful. This type of theory might give an alternate approach to some of our problems.

Let us get now to the basic properties of dual boundary complexes. The first step is to note that if $U \subset X$ is an open subset of a smooth quasiprojective variety, then we obtain a map $D\partial X \to D\partial U$.

Lemma 2.2 (Stepanov-Payne). — *If X is an irreducible smooth quasiprojective variety and $Z \to X$ is obtained by blowing up a smooth center, then it induces a homotopy equivalence on dual boundary complexes $D\partial Z \sim D\partial X$.*

Proof. — See [45, Theorem 5.3], where more generally boundary complexes of singular varieties are considered but we only need the smooth case. Then it is a corollary of Stepanov's lemma (cf [45, §5] once we choose a compactification that has simple normal crossings with the smooth subvariety that needs to be blown up. □

It follows from this lemma that if $U \subset X$ is an open subset of a smooth quasiprojective variety, it induces a natural map of boundary complexes $D\partial X \to D\partial U$. Indeed, for that we may assume by resolving singularities and applying the previous lemma, that U is the complement of a divisor $B \subset X$, and furthermore there is a very simple normal crossings compactification $\overline{X} = X \cup D$ such that $B \cup D$ also has very simple normal crossings. Then $D\partial U$ is the dual complex of the divisor $B \cup D$, which contains $D\partial X$, the dual complex of D, as a subcomplex.

Following up on this idea, here is our main reduction lemma:

Lemma 2.3. — *Suppose $U \subset X$ is an open subset of an irreducible smooth quasiprojective variety, obtained by removing a smooth irreducible closed subvariety of smaller dimension $Y = X - U \subset X$. Suppose that $D\partial Y \sim *$ is contractible. Then the map $D\partial X \to D\partial U$ is a homotopy equivalence.*

Proof. — Let $X^{\mathrm{Bl}Y}$ be obtained by blowing up Y. From the previous lemma, $D\partial X^{\mathrm{Bl}Y} \sim D\partial X$. Let $\mathrm{Bl}(Y) \subset X^{\mathrm{Bl}Y}$ be the inverse image of Y. It is an irreducible smooth divisor, and U is also the complement of this divisor in $X^{\mathrm{Bl}Y}$. By resolution of singularities we may choose a compactification $\overline{X^{\mathrm{Bl}Y}}$ such that the boundary divisor D, plus the closure $B := \overline{\mathrm{Bl}(Y)}$, form a very simple normal crossings divisor. This combined divisor is therefore a boundary divisor for U, so

$$D\partial U \sim D(D \cup B).$$

Now this bigger dual complex $D(D \cup B)$ has one more vertex than $D(D)$, corresponding to the irreducible component B. The star of this vertex is the cone over $D\partial\mathrm{Bl}(Y) = D(B \cap D)$. The cone is attached to $D(D)$ via its base $D(B \cap D)$, to give $D(B \cup D)$.

We would like to show that $D\partial\mathrm{Bl}(Y) \sim *$. The first step is to notice that $\mathrm{Bl}(Y) \to Y$ is the projective space bundle associated to the vector bundle $N_{Y/X}$ over Y.

We claim in general that if V is a vector bundle over a smooth quasiprojective variety Y, then $D\partial(P(V)) \sim D\partial(Y)$. The proof of this claim is that there exists a normal crossings compactification $\overline{Y}$ of Y such that the vector bundle V extends to a vector bundle on $\overline{Y}$. That may be seen by choosing a surjection from the dual of a direct sum of very ample line bundles to V, getting V as the pullback of a tautological bundle under a map from Y to a Grassmanian. The compactification may be chosen so that the map to the Grassmanian extends. We obtain a compactification of $P(V)$ wherein the boundary divisor is a projective space bundle over the boundary divisor of Y, and with these choices $D\partial P(V) = D\partial Y$. It follows from Danilov's theorem that for any other choice, there is a homotopy equivalence.

Back to our situation where $\mathrm{Bl}(Y) = P(N_{Y/X})$, and assuming that $D\partial Y \sim *$, we conclude that $D\partial\mathrm{Bl}(Y) \sim *$ too. Therefore the dual complex $D(B \cap D)$ is contractible.

Now $D\partial U = D(B \cup D)$ is obtained by attaching to $D(D)$ the cone over $D(B \cap D)$. As we have seen above $D(B \cap D)$ is contractible, so coning it off doesn't change the homotopy type. This shows that the map

$$D\partial X = D(D) \to D(B \cup D) = D\partial U$$

is a homotopy equivalence. □

In order to use this reduction, we need a criterion for the condition $D\partial Y \sim *$. Note first the following general property of compatibility with products.

LEMMA 2.4 (Payne). — *Suppose X and Y are smooth quasiprojective varieties. Then $D\partial(X\times Y)$ is the* join *of $D\partial(X)$ and $D\partial(Y)$, in other words we have a homotopy cocartesian diagram of spaces*

$$\begin{array}{ccc} D\partial(X)\times D\partial(Y) & \to & D\partial(Y) \\ \downarrow & & \downarrow \\ D\partial(X) & \to & D\partial(X\times Y)\,. \end{array}$$

Proof. — This is [45, Lemma 6.2]. □

COROLLARY 2.5. — *Suppose Y is a smooth quasiprojective variety. Then $D\partial(A^1\times Y)\sim *$.*

Proof. — Setting $X := A^1$ in the previous lemma, we have $D\partial(X)\sim *$, so in the homotopy cocartesian diagram the top arrow is an equivalence and the left vertical arrow is the projection to $*$; therefore the homotopy pushout is also $*$. □

PROPOSITION 2.6. — *Suppose $U\subset X$ is a nonempty open subset of a smooth irreducible quasiprojective variety, and suppose the complement $Z := X-U$ has a decomposition into finitely many locally closed subsets Z_j such that $Z_j\cong A^1\times Y_j$. Suppose that this decomposition can be ordered into a stratification, that is to say there is a total order on the indices such that $\bigcup_{j\leqslant a} Z_j$ is closed for any a. Then $D\partial(X)\sim D\partial(U)$.*

Proof. — We first prove the proposition under the additional hypothesis that the Y_j are smooth. Proceed by induction on the number of pieces in the decomposition. Let Z_0 be the lowest piece in the ordering. The ordering hypothesis says that Z_0 is closed in X. Let $X' := X-Z_0$. Now $U\subset X'$ is the complement of a subset $Z'=\bigcup_{j>0} Z_j$ decomposing in the same way, with a smaller number of pieces, so by induction we know that $D\partial(X')\sim D\partial(U)$.

By hypothesis $Z_0\cong A^1\times Y_0$. Lemma 2.5 tells us that $D\partial(Z_0)\sim *$ and now Lemma 2.3 tells us that $D\partial(X)\sim D\partial(X')$, so $D\partial(X)\sim D\partial(U)$. This completes the proof of the proposition under the hypothesis that Y_j are smooth.

Now we prove the proposition in general. Proceed as in the first paragraph of the proof with the same notations: by induction we may assume that $D\partial(X') \sim D\partial(U)$ where $X' = X - Z_0$ such that Z_0 is closed and isomorphic to $A^1 \times Y_0$. Choose a totally ordered stratification of Y_0 by smooth locally closed subvarieties $Y_{0,i}$. Set $Z_{0,i} := A^1 \times Y_{0,i}$. This collection of subvarieties of X now satisfies the hypotheses of the proposition and the pieces are smooth. Their union is Z_0 and its complement in X is the open subset X'. Thus, the first case of the proposition treated above tells us that $D\partial(X) \sim D\partial(X')$. It follows that $D\partial(X) \sim D\partial(U)$, completing the proof. □

Caution. — A simple example shows that the condition of ordering, in the statement of the propostion, is necessary. Suppose X is a smooth projective surface containing two projective lines $D_1, D_2 \subset X$ such that their intersection $D_1 \cap D_2 = \{p_1, p_2\}$ consists of two distinct points. Then we could look at $Z_1 = D_1 - \{p_1\}$ and $Z_2 = D_2 - \{p_2\}$. Both Z_1 and Z_2 are affine lines. Setting $U := X - (D_1 \cup D_2) = X - (Z_1 \cup Z_2)$ we get an open set which is the complement of a subset $Z = Z_1 \sqcup Z_2$ decomposing into two affine lines; but $D\partial X = \emptyset$ whereas $D\partial U \sim S^1$.

3. Hybrid moduli stacks of local systems

The moduli space of local systems is different from the moduli stack, even at the points corresponding to irreducible local systems. Indeed, the open substack of the moduli stack parametrizing irreducible GL_r-local systems is a G_m-gerbe over the corresponding open subset of the moduli space. Even by considering SL_r-local systems we can only reduce this to being a μ_r-gerbe.

However, it is usual and convenient to consider the moduli space instead. In this section, we mention a construction allowing to define what we call a *hybrid moduli stack* in which the central action is divided out, making it so that for irreducible points it is the same as the moduli space. This is a special case of the *rigidification* procedure introduced by Abramovich, Corti and Vistoli [1].

Our initial discussion will use some simple 2-stacks, however the reader wishing to avoid these may refer to Proposition 3.1 which gives an equivalent definition in more concrete terms.

Consider a reductive group G with center Z. The fibration sequence of 1-stacks

$$BZ \to BG \to B(G/Z)$$

may be transformed into the cartesian diagram

$$\begin{array}{ccc} BG & \to & B(G/Z) \\ \downarrow & & \downarrow \\ * & \to & K(Z,2) \end{array} \tag{3.1}$$

of Artin 2-stacks on the site $\mathrm{Aff}^{ft,et}_C$ of affine schemes of finite type over C with the étale topology.

Suppose now that S is a space or higher stack. Then we may consider the relative mapping stack

$$M(S,G) := \underline{Hom}(S, B(G/Z)/K(Z,2)) \to K(Z,2).$$

It may be defined as the fiber product forming the middle arrow in the following diagram where both squares are cartesian:

$$\begin{array}{ccccc} \underline{Hom}(S,BG) & \to & M(S,G) & \to & \underline{Hom}(S,B(G/Z)) \\ \downarrow & & \downarrow & & \downarrow \\ * & \to & K(Z,2) & \to & \underline{Hom}(S,K(Z,2)) \end{array}.$$

Here the bottom right map is the "constant along S" construction induced by pullback along $S \to *$.

The bottom left arrow $* \to K(Z,2)$ is the universal Z-gerbe, so its pullback on the upper right is again a Z-gerbe. We have thus constructed a stack $M(S,G)$ over which $\underline{Hom}(S,BG)$ is a Z-gerbe. From the definition it is *a priori* a 2-stack, and indeed $M(\emptyset,G) = K(Z,2)$, but the following alternate characterization tells us that $M(S,G)$ is almost always a usual 1-stack.

Proposition 3.1. — *Suppose S is a nonempty connected CW-complex with basepoint x. Then the hybrid moduli stack may be expressed as the stack-theoretical quotient*

$$M(S,G) = \mathrm{Rep}(\pi_1(S,x),G)/\!\!/(G/Z).$$

In particular, it is an Artin 1*-stack.*

Proof. — The representation space may be viewed as a mapping stack

$$\mathrm{Rep}(\pi_1(S,x),G) = \underline{Hom}((S,x),(BG,o)).$$

The internal $\underline{Hom}$ can be taken in the Jardine model category of simplicial presheaves, where S denotes the constant simplicial presheaf whose values are the singular complex of the space S, and BG denotes the fibrant replacement of the Eilenberg-MacLane simplicial presheaf associated to the sheaf of groups G.

Consider the big diagram

$$
\begin{array}{ccccc}
\underline{Hom}((S,x),(BG,o)) & \rightarrow & \underline{Hom}((S,x),(B(G/Z),o)) & & \\
\downarrow & & \downarrow & & \\
* & \rightarrow & \underline{Hom}((S,x),(K(Z,2),o)) & \rightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
K(Z,2) & \rightarrow & \underline{Hom}(S,K(Z,2)) & \rightarrow & K(Z,2)
\end{array}
$$

where the bottom right map is evaluation at x. The pointed mapping 2-stack in the middle is defined by the condition that the bottom right square is homotopy cartesian. The composition along the bottom is the identity, so if we take the homotopy fiber product on the bottom left, the full bottom rectangle is a pullback too so that homotopy fiber product would be $*$ as is written in the diagram. In other words, the bottom left square is also homotopy cartesian. The middle horizontal map on the left sends the point to the map $S \rightarrow o \hookrightarrow K(Z,2)$, indeed it is constant along S because it comes from pullback of the bottom left map, and its value at x is o because of the right vertical map. Now, the upper left square is homotopy cartesian, just the result of applying the pointed mapping stack to the diagram (3.1). It follows that the whole left rectangle is homotopy cartesian.

Consider, on the other hand, the diagram

$$
\begin{array}{ccccc}
\underline{Hom}((S,x),(BG,o)) & \rightarrow & \underline{Hom}((S,x),(B(G/Z),o)) & \rightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
M(S,G) & \rightarrow & \underline{Hom}(S,B(G/Z)) & \rightarrow & B(G/Z) \\
\downarrow & & \downarrow & & \\
K(Z,2) & \rightarrow & \underline{Hom}(S,K(Z,2)) & & .
\end{array}
$$

The bottom square is homotopy-cartesian by the definition of $M(S,G)$. We proved in the previous paragraph that the full left rectangle is homotopy cartesian. In this 2-stack situation note that a commutative rectangle constitutes a piece of data rather than just a property. In this case, these data for the left squares are obtained by just considering the equivalence found in the previous paragraph, from $\underline{Hom}((S,x),(BG,o))$ to the homotopy pullback in the full left rectangle which is the same as the composition of the homotopy pullbacks in the two left squares. In particular, the upper left square is homotopy-cartesian. It now follows that the upper full rectangle is homotopy-cartesian. That exactly says that we have an action of G/Z on $\underline{Hom}((S,x),(BG,o)) = \mathrm{Rep}(\pi_1(S,x),G)$ and $M(S,G)$ is the quotient. □

The hybrid moduli stacks also satisfy the same glueing or factorization property as the usual ones.

LEMMA 3.2 *Suppose $S = S_1 \cup S_2$ with $S_{12} := S_1 \cap S_2$ excisive. Then*

$$M(S,G) \cong M(S_1,G) \times_{M(S_{12},G)} M(S_2,G).$$

Proof. — The mapping stacks entering into the definition of $M(S,G)$ as a homotopy pullback, satisfy this glueing property. Notice that this is true even for the constant functor which associates to any S the stack $K(Z,2)$. The homotopy pullback therefore also satisfies the glueing property since fiber products commute with other fiber products. □

Suppose $G = GL_r$ so $Z = G_m$ and $G/Z = PGL_r$, and suppose S is a connected CW-complex. Let

$$\underline{Hom}(S, BGL_r)^{\mathrm{irr}} \subset \underline{Hom}(S, BGL_r)$$

denote the open substack of irreducible local systems. It is classical that the stack $\underline{Hom}(S, BGL_r)$ has a *coarse moduli space* $\mathrm{M}_B(S, GL_r)$, and that the open substack $\underline{Hom}(S, BGL_r)^{\mathrm{irr}}$ is a G_m-gerbe over the corresponding open subset of the coarse moduli space $\mathrm{M}_B(S,G)^{\mathrm{irr}}$.

PROPOSITION 3.3. — *In the situation of the previous paragraph, we have a map*

$$M(S, GL_r) \to \mathrm{M}_B(S, GL_r)$$

which restricts to an isomorphism

$$M(S, GL_r)^{\mathrm{irr}} \cong \mathrm{M}_B(S, GL_r)^{\mathrm{irr}}$$

between the open subsets parametrizing irreducible local systems.

Indeed, comparing with the description of Proposition 3.1, the coarse moduli space is the universal categorical quotient of the space of representations by either G or G/Z, and for irreducible representations the action of G/Z is faithful.

The same holds for $G = SL_r$.

Remark 3.4. — The determinant map $GL_r \stackrel{\det}{\to} G_m$ induces a cartesian diagram

$$\begin{array}{ccc} M(S, SL_r) & \to & M(S, GL_r) \\ \downarrow & & \downarrow \\ * & \to & M(S, G_m) \end{array}$$

which essentially says that $M(S, SL_r)$ is the substack of $M(S, GL_r)$ parametrizing local systems of trivial determinant. Note that $M(S, G_m)$ is isomorphic to the quasiprojective variety $\mathrm{Hom}(H_1(S), G_m)$.

In what follows, we shall use these stacks $M(S, GL_r)$ which we call *hybrid moduli stacks* as good replacements intermediary between the moduli stacks of local systems and their coarse moduli spaces.

4. Boundary conditions

Let S denote a 2-sphere with k open disks removed. It has k boundary circles denoted $\xi_1, \ldots, \xi_k \subset S$ and

$$\partial S = \xi_1 \sqcup \cdots \sqcup \xi_k.$$

From now on we consider rank 2 local systems on this surface S.

Fix complex numbers $c_1, \ldots, c_k$ all different from 0, 1 or -1. Let

$$C_i := \left\{ P \begin{pmatrix} c_i & 0 \\ 0 & c_i^{-1} \end{pmatrix} P^{-1} \right\}$$

denote the conjugacy class of matrices with eigenvalues c_i, c_i^{-1}.

Consider the hybrid moduli stack $M(S, GL_2)$ constructed above, and let

$$M(S; C) \subset M(S, GL_2)$$

denote the closed substack consisting of local systems such that the monodromy transformation around ξ_i is in the conjugacy class C_i. See [38].

If we choose a basepoint $x \in S$ and paths γ_i going from x out by straight paths to the boundary circles, around once and then back to x, then $\pi_1(S, x)$ is generated by the γ_i subject to the relation that their product is the identity.

Therefore, the moduli stack of framed local systems is the affine variety

$$\underline{Hom}((S, x), (BGL_2, o)) = \mathrm{Rep}(\pi_1(S, x), GL_2)$$

$$= \{(A_1, \ldots, A_k) \in (GL_2)^k \text{ s.t. } A_1 \cdots A_k = 1\}.$$

The unframed moduli stack is the stack-theoretical quotient

$$\underline{Hom}(S, BGL_2) = \mathrm{Rep}(\pi_1(S, x), GL_2) /\!\!/ GL_2$$

by the action of simultaneous conjugation.

The center $G_m \subset GL_2$ acts trivially on $\mathrm{Rep}(\pi_1(S,x), GL_2)$ so the action of GL_2 there factors through an action of PGL_2.

Proposition 3.1 may be restated as

LEMMA 4.1. — *The hybrid moduli stack $M(S, GL_2)$ may be described as the stack-theoretical quotient*

$$M(S, GL_2) = \mathrm{Rep}(\pi_1(S,x), GL_2) /\!\!/ PGL_2.$$

Let $\mathrm{Rep}(\pi_1(S,x), GL_2; C) \subset \mathrm{Rep}(\pi_1(S,x), GL_2)$ denote the closed subscheme of representations which send γ_i to the conjugacy class C_i. These conditions are equivalent to the equations $\mathrm{Tr}(\rho(\gamma_i)) = c_i + c_i^{-1}$. We have

$$\mathrm{Rep}(\pi_1(S,x), GL_2; C) = \{(A_1, \ldots, A_k)\,\text{s.t.}\, A_i \in C_i \,\text{and}\, A_1 \cdots A_k = 1\}.$$

COROLLARY 4.2. — *The hybrid moduli stack with fixed conjugacy classes is given by*

$$M(S; C) = \mathrm{Rep}(\pi_1(S,x), GL_2; C) /\!\!/ PGL_2$$
$$= \{(A_1, \ldots, A_k) \; s.t. \; A_i \in C_i \; and \; A_1 \cdots A_k = 1\} /\!\!/ PGL_2.$$

It is also isomorphic to the stack one would have gotten by using the group SL_2 rather than GL_2.

Proof. — Our conjugacy classes have been defined as having determinant one. Since the γ_i generate the fundamental group, if the $\rho(\gamma_i)$ have determinant one then the representation ρ goes into SL_2. As $PGL_2 = PSL_2$, the hybrid moduli stack for GL_2 is the same as for SL_2. □

Recall the following Kostov-genericity condition [37] on the choice of the numbers c_i.

CONDITION 4.3. — *For any choice of $\epsilon_1, \ldots, \epsilon_k \in \{1, -1\}$ the product*

$$c_1^{\epsilon_1} \cdots c_k^{\epsilon_k}$$

is not equal to 1.

The following basic lemma has been observed by Kostov and others, see [37, Remark 5] for example.

LEMMA 4.4. — *If Condition 4.3 is satisfied then any representation in $\mathrm{Rep}(\pi_1(S,x), GL_2; C)$ is irreducible. In particular, the automorphism group of the corresponding GL_2 local system is the central G_m.*

The set of $(c_1, \ldots, c_k)$ satisfying this condition is a nonempty open subset of $(G_m - \{1, -1\})^k$. We also speak of the same condition for the sequence of conjugacy classes C.

Proposition 4.5. — *Suppose C satisfy Condition 4.3. The hybrid moduli stack $M(S; C)$ is an irreducible smooth affine variety. It is equal to the coarse, which is indeed fine, moduli space $\mathrm{M}_B(S; C_1, \ldots, C_k)$ of local systems with our given conjugacy classes.*

Proof. — The representation space $\mathrm{Rep}(\pi_1(S, x), GL_2; C)$ is an affine variety, call it $\mathrm{Spec}(A)$, on which the group PGL_2 acts. The moduli space is by definition

$$\mathrm{M}_B(S; C_1, \ldots, C_k) := \mathrm{Spec}(A^{PGL_2}).$$

By Lemma 4.4 and using the hypothesis 4.3 it follows that the stabilizers of the action are trivial. Luna's étale slice theorem (see [9]) implies that the quotient map

$$\mathrm{Spec}(A) \to \mathrm{Spec}(A^{PGL_2})$$

is an étale fiber bundle with fiber PGL_2. Therefore this quotient is also the stack-theoretical quotient:

$$\mathrm{Spec}(A^{PGL_2}) = \mathrm{Spec}(A) /\!\!/ PGL_2.$$

By Corollary 4.2 that stack-theoretical quotient is $M(S; C)$, completing the identification between the hybrid moduli stack and the moduli space required for the proposition.

Smoothness of the moduli space has been noted in many places, see for example [19] [38]. Irreducibility is proven in a general context in [20] [38] as a consequence of computations of E-polynomials, and a different proof is given in [47] using moduli stacks of parabolic bundles. In our case irreducibility could also be obtained by including dimension estimates for the subvarieties which will be removed in the course of our overall discussion. □

This proposition says that our hybrid moduli stack $M(S; C)$ is the same as the usual moduli space. A word of caution is necessary: we shall also be using $M(S', C)$ for subsets $S' \subset S$, and those are in general stacks rather than schemes, for example when Condition 4.3 doesn't hold over S'.

5. Interior conditions and factorization

We now define some conditions concerning what happens in the interior of the surface S. These conditions will serve to define a stratification of

$M(S;C)$. The biggest open stratum denoted M', treated in detail in Section 10, turns out to be the main piece, contributing the essential structure of the dual boundary complex. The smaller strata will be negligeable for the dual boundary complex, in view of Lemmas 2.3 and 2.5 as combined in Proposition 2.6.

Divide S into closed regions denoted $S_2, \ldots, S_{k-1}$ such that $S_i \cap S_{i+1} = \rho_i$ is a circle for $2 \leqslant i \leqslant k-2$, and the regions are otherwise disjoint. We assume that S_i encloses the boundary circle ξ_i, so it is a 3-holed sphere with boundary circles ρ_{i-1}, ξ_i and ρ_i. The orientation of ρ_{i-1} is reversed when it is viewed from inside S_i. The end piece S_2 has boundary circles ξ_1, ξ_2 and ρ_2 while the end piece S_{k-1} has boundary circles ρ_{k-2}, ξ_{k-1} and ξ_k. This is a "pair of pants" decomposition.

Factorization properties, related to chiral algebra cf [11] [12], are a kind of descent. As explained in Theorem 5.4 below, we will be applying the factorization property of Lemma 3.2 to the decomposition of our surface into pieces S_i. This classical technique in geometric topology was also used extensively in the study of the Verlinde formula. The factorization is often viewed as coming from a degeneration of the curve into a union of rational lines with three marked points.

For our argument it will be important to consider strata of the moduli space defined by fixing additional combinatorial data with respect to our decomposition. To this end, let us consider some nonempty subsets $\sigma_i \subset \{0,1\}$ for $i = 2, \ldots, k-1$, and conjugacy-invariant subsets $G_2, \ldots, G_{k-2} \subset SL_2$. We denote by $\alpha = (\sigma_1, \ldots, \sigma_{k-1}; G_2, \ldots, G_{k-2})$ this collection of data. The subsets G_i will impose conditions on the monodromy around the circles ρ_i, while the σ_i will correspond to the following *stability condition* on the restrictions of our local system to S_i. Recall that a local system $V \in M(S;C)$ is required to have monodromy around ξ_i with eigenvalues c_i and c_i^{-1}. We are making a choice of orientation of these boundary circles, and $c_i \neq c_i^{-1}$ by hypothesis, so the c_i^{-1} eigenspace corresponds to a well-defined rank 1 sub-local system of $V|_{\xi_i}$.

Definition 5.1. — *We say that a local system $V|_{S_i}$ on S_i, satisfying the conjugacy class condition, is* unstable *if there exists a rank* 1 *subsystem $L \subset V|_{S_i}$ such that the monodromy of L around ξ_i is c_i^{-1}. Say that $V|_{S_i}$ is* stable *otherwise.*

An irreducible local system $V|_{S_i}$ is automatically stable; one which decomposes as a direct sum is automatically unstable. If $V|_{S_i}$ is a nontrivial extension with a unique rank 1 subsystem L, then $V|_{S_i}$ is unstable if $L|_{\xi_i}$ is the c_i^{-1}-eigenspace of the monodromy, whereas it is stable if $L|_{\xi_i}$ is the c_i-

eigenspace. We will later express these conditions more concretely in terms of vanishing or nonvanishing of a certain matrix coefficient.

Definition 5.2. — *Given $\alpha = (\sigma_1, \ldots, \sigma_{k-1}; G_2, \ldots, G_{k-2})$, denote by $M^\alpha(S; C) \subset M(S; C)$ the locally closed substack of local systems V satisfying the following conditions:*

- *if $\sigma_i = \{0\}$ then $V|_{S_i}$ is required to be unstable; if $\sigma_i = \{1\}$ then it is required to be stable; and if $\sigma_i = \{0, 1\}$ then there is no condition; and*
- *the monodromy of V around ρ_i should lie in G_i.*

Consider a subset $S' \subset S$ made up of some or all of the S_i or the circles. Let $M^\alpha(S'; C)$ denote the moduli stack of local systems on S' satisfying the above conditions where they make sense (that is, for the restrictions to those subsets which are in S').

Notation. — In the case of the inner boundary circles we may just use the notation $M^\alpha(\rho_i)$ since the choices of conjugacy classes C_i, corresponding to circles ξ_i, don't intervene.

In the case of S_i, only the conjugacy class C_i matters so we may use the notation $M^\alpha(S_i; C_i)$.

Suppose $S' \subset S$ is connected and $x \in S'$. Let

$$\mathrm{Rep}^\alpha(\pi_1(S', x), GL_2; C) \subset \mathrm{Rep}(\pi_1(S', x), GL_2)$$

denote the locally closed subscheme of representations which satisfy conjugacy class conditions corresponding to C and the conditions corresponding to α, that is to say whose corresponding local systems are in $M^\alpha(S'; C)$. Proposition 3.1 says:

Lemma 5.3. — *The simultaneous conjugation action of GL_2 on the space of representations $\mathrm{Rep}^\alpha(\pi_1(S', x), GL_2; C)$ factors through an action of PGL_2 and*

$$M^\alpha(S'; C) = \mathrm{Rep}^\alpha(\pi_1(S', x), GL_2; C) /\!\!/ PGL_2$$

is the stack-theoretical quotient.

The hybrid moduli stacks allow us to state a glueing or *factorization property*, expressing the fact that a local system L on S may be viewed as being obtained by glueing together its pieces $L|_{S_i}$ along the circles ρ_i.

Theorem 5.4. — *We have the following expression using homotopy fiber products of stacks:*

$$M^{\alpha}(S;C) = M^{\alpha}(S_2;C_2) \times_{M^{\alpha}(\rho_i)} M^{\alpha}(S_3;C_3) \cdots \times_{M^{\alpha}(\rho_{k-2})} M^{\alpha}(S_{k-1};C_{k-1}).$$

Proof. — Apply Lemma 3.2. □

Proposition 4.5 has the following corollary:

Corollary 5.5. — *Suppose the requirements given for the boundary pieces of $\partial S'$ (which are circles either of the form ξ_i or ρ_i) satisfy Condition 4.3 for S'. Then the moduli stack $M^{\alpha}(S';C)$ is in fact a quasiprojective variety.*

Proof. — This follows from Proposition 4.5 applied to S'. □

6. Universal objects

Let us return for the moment to the general situation of Section 3, of a space S and a group G. If $x \in S$ is a basepoint, then we obtain a principal (G/Z)-bundle over $\underline{Hom}(S, B(G/Z))$, and this pulls back to a principal (G/Z)-bundle denoted $F(S,x) \to M(S,G)$. It may be viewed as the bundle of frames for the local systems, up to action of the center Z.

If $y \in S$ is another point, and γ is a path from x to y then it gives an isomorphism of principal bundles $F(S,x) \cong F(S,y)$ over $M(S,G)$. In particular, $\pi_1(S,x)$ acts on $F(S,x)$ in a tautological representation.

Suppose $S = S_a \cup S_b$ is a decomposition into pieces (in the application these will be some unions of pieces of the form S_i considered above), such that the intersection $S_{ab} = S_a \cap S_b$ is connected. Choose a basepoint $x \in S_{ab}$. This yields principal (G/Z)-bundles $F(S_a,x)$ and $F(S_b,x)$ over $M(S_a,G)$ and $M(S_b,G)$ respectively. The fundamental group $\pi_1(S_{ab},x)$ acts on both of these. We may restate the glueing property of Lemma 3.2 in the following way.

Proposition 6.1. — *We have an isomorphism of stacks lying over the product $M(S_a,G) \times M(S_b,G)$,*

$$M(S,G) \cong \mathrm{Iso}_{\pi_1(S_{ab},x)-G/Z}(p_1^* F(S_a,x), p_2^* F(S_b,x))$$

where on the right is the stack of isomorphisms, relative to $M(S_a,G) \times M(S_b,G)$, of principal G/Z-bundles provided with actions of $\pi_1(S_{ab},x)$.

Return now to the notation from the immediately preceding sections. There are several ways of dividing our surface S into two or more pieces, various of which shall be used in the next section.

Choose basepoints x_i in the interior of S_i, and s_i on the boundary circles ρ_i. Connect them by paths, nicely arranged with respect to the other paths γ_i, recalling that the γ_i are the paths gong out to the original boundary circles ξ_i, around and back to the basepoint.

Then, over any subset S' containing a basepoint x_i, we obtain a principal PGL_2-bundle $F(S', x_i) \to M(S', C)$, and the same for s_i. Our paths, when in S', give isomorphisms between these principal bundles.

It will be helpful to think of the description of glueing given by Proposition 6.1, using these basepoints and paths. The following local triviality property is useful.

Lemma 6.2. — *Suppose S' has at most one boundary circle of the form ρ_i, and suppose that the conjugacy classes determining the moduli problem on $M^{\alpha}(S', C)$ satisfy Condition 4.3, and suppose that $x \in S'$ is one of our basepoints. Then the principal PGL_2-bundle $F(S', x) \to M^{\alpha}(S', C)$ is locally trivial in the Zariski topology of the moduli space $M^{\alpha}(S', C)$, and Zariski locally $F(S', x)$ may be viewed as the projective frame bundle of a rank 2 vector bundle.*

Proof. — Consider a choice of three loops $(\gamma_{j_1}, \gamma_{j_2}, \gamma_{j_3})$ and a choice of one of the two eigenvalues of the conjugacy class C_{j_1}, C_{j_2}, or C_{j_3} for each of them respectively. This gives three rank 1 eigenspaces in V_x for any local system V. Over the Zariski open subset of the moduli space where these three subspaces are distinct, they provide the required projective frame. Notice that the eigenspaces of the γ_j cannot all be aligned since these loops generate the fundamental group of S', by the hypothesis that there is at most one other boundary circle ρ_i. Therefore, as our choices of triple of loops and triple of eigenvalues range over the possible ones, these Zariski open subsets cover the moduli space. We get the required frames. A framed PGL_2-bundle comes from a vector bundle so $F(S', x)$ locally comes from a GL_2-bundle. $\square$

7. Splitting along the circle ρ_i

In this section we consider one of the circles ρ_i which divides S into two pieces. Let

$$S_{<i} := \bigcup_{j<i} S_j \ , \quad S_{>i} := \bigcup_{j>i} S_j,$$

and similarly define $S_{\leqslant i}$ and $S_{\geqslant i}$. We have the decomposition

$$S = S_{\leqslant i} \cup S_{>i}$$

into two pieces intersecting along the circle ρ_i. Thus,

$$M^\alpha(S; C) = M^\alpha(S_{\leqslant i}; C) \times_{M^\alpha(\rho_i)} M^\alpha(S_{>i}; C).$$

This factorization will allow us to analyze strata where G_i is a unipotent or trivial conjugacy class. The following condition will be in effect:

Condition 7.1. — *We assume that the sequence of conjugacy classes $C_1, \ldots, C_k$ is* very generic*, meaning that for any i the partial sequences $C_1, \ldots, C_i$ and $C_i, \ldots, C_k$ satisfy Condition 4.3, and they also satisfy that condition if we add the scalar matrix -1. That is to say, no product of eigenvalues or their inverses should be equal to either* 1 *or* -1.

Suppose that $G_i = \{1\}$. Then $M^\alpha(\rho_i) = B(PGL_2)$. On the other hand, Condition 7.1 means that the sequences of conjugacy classes defining the moduli problems on $S_{\leqslant i}$ and $S_{>i}$ themselves satisfy Condition 4.3. Therefore Proposition 4.5 applies saying that the moduli stacks $M^\alpha(S_{\leqslant i}; C)$ and $M^\alpha(S_{>i}; C)$ exist as quasiprojective varieties.

The projective frame bundles over a basepoint of ρ_i are principal PGL_2-bundles denoted

$$F_{\leqslant i} \to M^\alpha(S_{\leqslant i}; C)$$

and

$$F_{>i} \to M^\alpha(S_{>i}; C).$$

These principal bundles may be viewed as given by pullbacks of the universal principal bundle on $B(PGL_2)$, along the maps

$$M^\alpha(S_{\leqslant i}; C) \to M^\alpha(\rho_i) = B(PGL_2) \leftarrow M^\alpha(S_{>i}; C).$$

These principal bundles are locally trivial in the Zariski topology by Lemma 6.2.

The principal bundle description of the moduli space in Proposition 6.1 now says

$$M^\alpha(S; C) = \mathrm{Iso}(p_1^*(F_{\leqslant i}), p_1^*(F_{\leqslant i})) \text{ over } M^\alpha(S_{\leqslant i}; C) \times M^\alpha(S_{>i}; C).$$

The bundle of isomorphisms between our two principal bundles, is a fiber bundle with fiber PGL_2, locally trivial in the Zariski topology because the two principal bundles are Zariski-locally trivial. We may sum up this conclusion with the following lemma, noting that the argument also works the same way if $G_i = \{-1\}$.

Lemma 7.2. — *Under the assumption that $G_i = \{1\}$, the moduli space $M^\alpha(S;C)$ is a fiber bundle over $M(S_{\leqslant i};C) \times M(S_{>i};C)$, locally trivial in the Zariski topology, with fiber PGL_2. The same holds true if $G_i = \{-1\}$.*

Consider the next case: suppose that G_i is the conjugacy class of matrices conjugate to a nontrivial unipotent matrix

$$\mathcal{U} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In that case, $M^\alpha(\rho_i) = BG_a$. The situation is the same as before: the moduli spaces $M^\alpha(S_{\leqslant i};C)$ and $M^\alpha(S_{>i};C)$ are quasiprojective varieties, and we have principal bundles $F_{\leqslant i}$ and $F_{>i}$. This time, these principal bundles have unipotent automorphisms denoted R' and R respectively, in the conjugacy class of $\mathcal{U}$. We have

$$M^\alpha(S;C) = \mathrm{Iso}_{M^\alpha(S_{\leqslant i};C)\times M^\alpha(S_{>i};C)}(p_1^*(F_{\leqslant i}, R'), p_2^*(F_{>i}, R)).$$

This means the relative isomorphism bundle of the principal bundles together with their automorphisms.

We claim that these principal bundles together with their automorphisms may be trivialized locally in the Zariski topology. For the principal bundles themselves this is Lemma 6.2. The unipotent endomorphisms then correspond, with respect to these local trivializations, to maps into PGL_2/G_a. One can write down explicit sections of the projection $PGL_2 \to PGL_2/G_a$ locally in the Zariski topology of the base, and these give the claimed local trivializations. One might alternatively notice here that a G_a-torsor for the étale topology is automatically locally trivial in the Zariski topology by "Hibert's theorem 90".

From the result of the previous paragraph, $M^\alpha(S;C)$ is a fiber bundle over $M^\alpha(S_{\leqslant i};C) \times M^\alpha(S_{>i};C)$, locally trivial in the Zariski topology, with fiber the centralizer $Z(R) \subset PGL_2$ of a unipotent element $R \in PGL_2$. This centralizer is $G_a \cong A^1$. We obtain the following statement.

Lemma 7.3. — *Under the assumption that G_i is the unipotent conjugacy class, the moduli space $M^\alpha(S;C)$ is a fiber bundle over $M^\alpha(S_{\leqslant i};C) \times M^\alpha(S_{>i};C)$, locally trivial in the Zariski topology, with fiber A^1. The same holds true if G_i is the conjugacy class of matrices conjugate to $-\mathcal{U}$.*

We may sum up the conclusion of this section as follows.

Proposition 7.4. — *With the hypothesis of Condition 7.1 in effect, suppose that the datum α is chosen such that for some i, G_i is one of* *the following four conjugacy classes*

$$\{1\},\ \{-1\},\ \{P\mathcal{U}P^{-1}\},\ \textit{or}\ \{-P\mathcal{U}P^{-1}\},$$

that is to say the conjugacy classes whose traces are 2 *or* -2. *Then the dual boundary complex of the* α*-stratum is contractible:*

$$D\partial M^{\alpha}(S,C) \sim *.$$

Proof. — In all four cases, covered by Lemmas 7.2 and 7.3 above, the space $M^{\alpha}(S,C)$ admits a further decomposition into locally closed pieces all of which have the form $A^1 \times Y$. Therefore, Corollary 2.5 and Proposition 2.6 apply to show that the dual boundary complex is contractible. □

8. Decomposition at S_i in the unstable case

Define the function $t_i : M(S,C) \to A^1$ sending a local system to the trace of its monodromy around the circle ρ_i. In the previous section, we have treated any strata which might be defined in such a way that at least one of the G_i is a conjugacy class with t_i equal to 2 or -2. Therefore, we may now assume that all of our subsets G_i consist entirely of matrices with trace different from $2, -2$. In particular, these matrices are semisimple with distinct eigenvalues.

If G_i consists of a single conjugacy class, it is possible to choose one of the two eigenvalues. But in general, this is not possible. However, in the situation considered in the present section, where one of the σ_i indicates an unstable local system, then the destabilizing subsystem serves to pick out a choice of eigenvalue.

In the case where one of the σ_i is $\{0\}$ stating that $V|_{S_i}$ should be unstable, we will again obtain a structure of decomposition into a product with A^1 locally over a stratification, essentially by considering the extension class of the unstable local system. Some arguments are needed in order to show that this leads to direct product decompositions.

8.1. Some cases with G_{i-1} and G_i fixed

We suppose in this subsection that G_{i-1} and G_i are single conjugacy classes, with traces different from $2, -2$, and furthermore chosen so that the moduli problem for $M^{\alpha}(S_{>i};C)$ on one side is Kostov-generic. Hence, that moduli stack is a quasiprojective variety. Furthermore we assume that $\sigma_i =$

$\{0\}$. Therefore, $M^\alpha(S_i;C_i)$ is the moduli stack of unstable local systems on S_i. The elements here are local systems V fitting into an exact sequence

$$0 \to L \to V \to L' \to 0$$

such that the monodromy of L on ξ_i has eigenvalue c_i^{-1}. We assume that $M^\alpha(S_i;C_i)$ is nonempty.

LEMMA 8.1. — *If we are given the conjugacy classes G_{i-1} and G_i such that there exists an unstable local system V on S_i, then the eigenvalues b_{i-1} of L on ρ_{i-1}, and b_i of L on ρ_i, are uniquely determined.*

Proof. — The conjugacy classes G_{i-1}, G_i determine the pairs (b_{i-1}, b_{i-1}^{-1}) and (b_i, b_i^{-1}) respectively. The instability condition says that L has eigenvalue c_i^{-1} along ξ_i. Suppose that $b_{i-1}c_i^{-1}b_i = 1$ so there exists a local system L with eigenvalues b_{i-1} and b_i. We show that the other products with either b_{i-1}^{-1} or b_i^{-1} or both, are different from 1. For example, $b_{i-1}c_i^{-1}b_i^{-1} = b_i^{-2}$, but $b_i^2 \neq 1$ since we are assuming that G_i is a conjugacy class with distinct eigenvalues. Thus $b_{i-1}c_i^{-1}b_i^{-1} \neq 1$. Similarly, $b_{i-1}^{-1}c_i^{-1}b_i \neq 1$. Also, $b_{i-1}^{-1}c_i^{-1}b_i^{-1} = c_i^{-2} \neq 1$. This shows that if there is one possible combination of eigenvalues for a sub-local system, then it is unique. □

From the assumption that $M^\alpha(S_i;C_i)$ is nonempty and the previous remark, we may denote by b_{i-1} and b_i the eigenvalues of L on ρ_{i-1} and ρ_i respectively.

We are assuming a genericity condition implying that $M^\alpha(S_{>i};C)$ is a quasiprojective variety. It has a universal principal bundle $F_{>i}$ over it, and this has an automorphism R corresponding to the monodromy transformation around ρ_i. The eigenvalues of R are b_i and b_i^{-1}.

Restrict to a finer stratification of $M^\alpha(S_{>i};C)$ into some strata denoted $M^\alpha(S_{>i})^a$ on which $(F_{>i}, R)$ is trivial. Let $M^\alpha(S;C)^a$ be the inverse image of $M^\alpha(S_{>i};C)^a$ under the map $M^\alpha(S;C) \to M^\alpha(S_{>i};C)$.

PROPOSITION 8.2. — *We have*

$$M^\alpha(S;C)^a = M^\alpha(S_{>i};C)^a \times M^\alpha(S_{\leqslant i};C)^{\mathrm{fr},R}$$

where $M^\alpha(S_{\leqslant i};C)^{\mathrm{fr},R}$ is the moduli space of framed *local systems, that is to say local systems with a projective framing along ρ_i compatible with the monodromy and having the specified eigenvalues (b_i, b_i^{-1}).*

Proof. — Use Proposition 6.1. □

Without the conditions $\alpha = (\sigma., G.)$, the framed moduli space is just the space of sequences of group elements $A_1, \ldots, A_i$, in conjugacy classes $C_1, \ldots, C_i$ respectively, such that $A_1 \cdots A_i R = 1$. Denote this space by

$$\mathrm{Rep}(C_1, \ldots, C_i; R).$$

The moduli space $M^\alpha(S_{\leqslant i}, C)^{\mathrm{fr},R}$ is the subspace of $\mathrm{Rep}(C_1, \ldots, C_i; R)$ given by the conditions $\sigma.$ and $G.$.

Notice here that, since we don't know a genericity condition for $(C_1, \ldots, C_i, G_i)$ the moduli space might not be smooth. Even though we are considering framed representations, at a reducible representation the space will in general have a singularity. Furthermore, the conditions G_j might, in principle, introduce other singularities.

Theorem 8.3. — *With the above notations, let R' be an element in the conjugacy class G_{i-1}. We have*

$$M^\alpha(S_{\leqslant i}; C)^{\mathrm{fr},R} \cong A^1 \times M^\alpha(S_{\leqslant i-1}; C)^{\mathrm{fr},R'}.$$

Proof. — It isn't too hard to see that the moduli space is an A^1-bundle over the second term on the right hand side, where the A^1-coordinate is the extension class. The statement that we would like to show, saying that there is a natural decomposition as a direct product, is a sort of commutativity property.

Let $\mathrm{Rep}(C_1, \ldots, C_i; R)^u$ denote the subspace of $\mathrm{Rep}(C_1, \ldots, C_i; R)$ consisting of representations which are unstable on S_i. This is equivalent to saying that A_i fixes, and acts by c_i^{-1} on the eigenvector of R of eigenvalue b_i. We will show an isomorphism

$$\mathrm{Rep}(C_1, \ldots, C_i; R)^u \cong A^1 \times \mathrm{Rep}(C_1, \ldots, C_{i-1}; R'),$$

and this isomorphism will preserve the conditions $(\sigma., G.)$ over S_{i-1} so it restricts to an isomorphism between the moduli spaces as claimed in the theorem.

Write

$$R = \begin{pmatrix} b_i^{-1} & 0 \\ 0 & b_i \end{pmatrix}.$$

Then $\mathrm{Rep}(C_1, \ldots, C_i; R)^u$ is the space of sequences $(A_1, \ldots, A_i)$ such that

$$A_1 \cdots A_i R = 1$$

and

$$A_i = \begin{pmatrix} c_i & 0 \\ y & c_i^{-1} \end{pmatrix} \tag{8.1}$$

for some $y \in A^1$.

Similarly, write

$$R' = \begin{pmatrix} b_{i-1}^{-1} & 0 \\ 0 & b_{i-1} \end{pmatrix},$$

and $\mathrm{Rep}(C_1, \ldots, C_{i-1}; R')$ is the space of sequences $(A'_1, \ldots, A'_{i-1})$ such that

$$A'_1 \cdots A'_{i-1} R' = 1.$$

Suppose $(A_1, \ldots, A_i)$ is a point in $\mathrm{Rep}(C_1, \ldots, C_i; R)^u$ and let $y \in A^1$ be the lower left coefficient of A_i from (8.1). Note that $c_i^{-1} b_i = b_{i-1}$ so

$$A_i R = \begin{pmatrix} b_i^{-1} c_i & 0 \\ b_i^{-1} y & c_i^{-1} b_i \end{pmatrix} = \begin{pmatrix} b_{i-1}^{-1} & 0 \\ b_i^{-1} y & b_{i-1} \end{pmatrix}.$$

Let

$$U := \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$$

be chosen so that $UA_iRU^{-1} = R'$, which happens if and only if

$$b_{i-1}^{-1} u + b_i^{-1} y - b_{i-1} u = 0,$$

in other words

$$u := \frac{-b_i^{-1} y}{b_{i-1}^{-1} - b_{i-1}}.$$

The denominator is nonzero because we are assuming the trace of G_{i-1} is different from 2 or -2, which is equivalent to asking $b_{i-1} \neq b_{i-1}^{-1}$.

Then put $A'_j := UA_jU^{-1}$. From the equation $UA_iRU^{-1} = R'$ we get

$$A'_1 \cdots A'_{i-1} R' = U(A_1 \cdots A_{i-1})U^{-1}(UA_iRU^{-1}) = 1.$$

Hence, $(y, (A'_1, \ldots, A'_{i-1}))$ is a point in $A^1 \times \mathrm{Rep}(C_1, \ldots, C_{i-1}; R')$. This defines the map

$$\mathrm{Rep}(C_1, \ldots, C_i; R)^u \to A^1 \times \mathrm{Rep}(C_1, \ldots, C_{i-1}; R'),$$

Its inverse is obtained by mapping $(y, (A'_1, \ldots, A'_{i-1}))$ to $(A_1, \ldots, A_i)$ where for $1 \leqslant j \leqslant i-1$ we put $A_j = U^{-1} A'_j U$ with U defined as above using y, and A_i is the upper triangular matrix (8.1). We obtain the claimed isomorphism. $\square$

By symmetry the same holds in case of Kostov-genericity on the other side, giving a statement written as

$$M^{\alpha}(S_{\geqslant i-1};C)^{\mathrm{fr},R'} \cong A^1 \times M^{\alpha}(S_{\geqslant i};C)^{\mathrm{fr},R}.$$

8.2. Open G_{i-1} and G_i

If $\sigma_i = \{0\}$ and the moduli space is nonempty, then we cannot have both sides being Kostov-nongeneric at once. Also, if either one of G_{i-1} or G_i are single conjugacy classes, then it constrains the other one to be in one of finitely many conjugacy classes, and we return to the situation of the previous subsection.

Therefore, the remaining case is when G_{i-1} and G_i are open sets which are unions of all but finitely many conjugacy classes (that is to say, allowing all traces but a finite number), such that the moduli problems on both $S_{<i}$ and $S_{>i}$ are Kostov-generic. In this situation, which we now assume, the moduli spaces $M^{\alpha}(S_{<i};C)$ and $M^{\alpha}(S_{>i};C)$ exist and have principal bundles $F_{<i}$ and $F_{>i}$ respectively.

We have a map

$$M^{\alpha}(S_{<i};C) \times M^{\alpha}(S_{>i};C) \to G_{i-1} \times G_i.$$

Consider the étale covering space $\widetilde{G_i}$ which parametrizes matrices with a choice of one of the two eigenspaces. This was considered extensively by Kabaya [29]. Let

$$\widetilde{M}^{\alpha}(S_{>i};C) := M^{\alpha}(S_{>i};C) \times_{G_i} \widetilde{G_i}$$

and similarly for $\widetilde{M}^{\alpha}(S_{<i};C)$.

Our hypothesis that $\sigma_i = \{0\}$, in other words that for any local system V in $M^{\alpha}(S;C)$ the restriction is unstable, provides a factorization of the projection map through

$$M^{\alpha}(S;C) \to \widetilde{M}^{\alpha}(S_{<i};C) \times \widetilde{M}^{\alpha}(S_{>i};C).$$

Indeed the destabilizing rank one subsystem is uniquely determined by the condition that the monodromy around ξ_i have eigenvalue c_i^{-1}, and this rank one subsystem serves to pick out the eigenvalues of the matrices for ρ_{i-1} and ρ_i.

Now the same argument as before goes through. We may choose a stratification such that on each stratum the principal bundles have framings such that the automorphisms R' and R are diagonal (note, however, that the eigenvalues are now variable).

We reduce to the following situation: Z is a quasiprojective variety with invertible functions b_{i-1} and b_i such that $b_{i_1}^{-1}c_i^{-1}b_i = 1$, and we look at the moduli space of quadruples (z, V_i, β', β) such that $z \in Z$, V_i is an unstable local system on S_i, and

$$\beta' : V|_{\rho_{i-1}} \cong (V, R'(z)),$$

$$\beta : V|_{\rho_i} \cong (V, R(z))$$

where

$$R'(z) = \begin{pmatrix} b_{i-1}(z)^{-1} & 0 \\ 0 & b_{i-1}(z) \end{pmatrix} \quad R(z) = \begin{pmatrix} b_i(z)^{-1} & 0 \\ 0 & b_i(z) \end{pmatrix}.$$

The map $Y = \beta'\beta^{-1}$ is an automorphism of V (defined up to scalars, so it is a group element in PGL_2) and it preserves the marked subspace, so it is a lower-triangular matrix. It uniquely determines the data (V_i, β, β') up to isomorphism. Indeed we may consider $V_i \cong V$ using for example β', then our local system is (R', A_i, YRY^{-1}) where A_i is specified by the condition $(R')^{-1}A_iYRY^{-1} = 1$. As the group of lower triangular matrices in PGL_2 is isomorphic to $G_m \times G_a$ we obtain an isomorphism between our stratum and $Z \times G_m \times G_a$.

Alternatively, one could just do a parametrized version of the proof of Theorem 8.3.

8.3. Synthesis

We may gather together the various cases that have been treated in this section so far. Let G^u denote the set of matrices with $\mathrm{Tr} = \pm 2$ and let G^v denote its complement, the open subset of matrices with trace $\neq 2, -2$.

Theorem 8.4. — *Suppose α is any datum such that for some i we have $\sigma_i = \{0\}$. If $M^\alpha(S; C)$ is nonempty, then $D\partial M^\alpha(S; C) \sim *$.*

Proof. — In the previous section we have treated the cases where any G_i is one of the four conjugacy classes of trace 2 or -2, that is to say we have treated the conjugacy classes in G^u. Therefore we may assume that G_{i-1} and G_i are contained in G^v.

Suppose that G_{i-1} and G_i are conjugacy classes chosen so that the sequences $(C_1, \ldots, C_{i-1}, G_{i-1})$ and $(G_i, C_{i+1}, \ldots, C_k)$ are both Kostov-nongeneric. Under the hypothesis $\sigma_i = \{0\}$ and supposing $M^\alpha(S; C)$ nonempty, containing say a local system V, then an eigenvalue of G_{i-1} is the product of an eigenvalue of C_i and an eigenvalue of G_i, since there exists a rank one subsystem of $V|_{S_i}$. The same holds for the other eigenvalue of G_{i-1}. Combining with the nongenericity relations among eigenvalues of $(C_1, \ldots, C_{i-1}, G_{i-1})$ and $(G_i, C_{i+1}, \ldots, C_k)$, we obtain a nongenericity relation for $(C_1, \ldots, C_k)$. This contradicts the hypothesis of Condition 4.3 for C. Therefore, we conclude that if $M^\alpha(S; C)$ is nonempty, then for any specific choice of conjugacy classes G_{i-1} and G_i, at least one of the moduli problems over $S_{<i}$ or $S_{>i}$ has to satisfy Condition 4.3. These cases are then covered by Theorem 8.3 above.

There are finitely many choices of single conjugacy classes G_{i-1} (resp. G_i) such that $(C_1, \ldots, C_{i-1}, G_{i-1})$ (resp. $(G_i, C_{i+1}, \ldots, C_k)$) is Kostov nongeneric. We may therefore isolate these choices and treat them by Theorem 8.3 according to the previous paragraph. Let now G_{i-1} and G_i be the complement in G^v of these nongeneric conjugacy classes. These are open subsets such that for any conjugacy classes therein, the moduli problems on $S_{<i}$ and $S_{>i}$ satisfy Condition 4.3. The discussion of subsection 8.2 now applies to give the conclusion that this part of $M^\alpha(S, C)$ has contractible dual boundary complex. □

9. Reduction to M'

In this section, we put together the results of the previous sections to obtain a reduction to the main biggest open stratum. Recall from Condition 7.1 that we are assuming that C is very generic.

Recall also that $G = G^v \sqcup G^u$ where G^v be the open subset of matrices with trace $\neq 2, -2$, with its complement G^u of matrices with $\mathrm{Tr} = \pm 2$.

Let the datum α' consist of the following choices: for all i, $\sigma'_i = \{1\}$ and $G_i = G^v$. Then we put

$$M' := M^{\alpha'}(S, C).$$

It is an open subset of $M(S, C)$ since stability, and the conditions on the traces, are open conditions.

Theorem 9.1. — *There exist collections of data denoted α^j such that*

$$M(S, C) = M' \sqcup \coprod_j M^{\alpha^j}(S, C)$$

is a stratification, i.e. a decomposition into locally closed subsets admitting a total order satisfying the closedness condition of 2.6. Furthermore, this *admits a further refinement into a stratification with M' together with pieces denoted $Z^{j,a} \subset M^{\alpha^j}(S,C)$, such that all of the pieces $Z^{j,a}$ have the form*

$$Z^{j,a} = Y^{j,a} \times A^1.$$

Proof. — Let α^j run over the 2^{2k-3} choices of $(\sigma_2, \ldots, \sigma_{k-1}; G_2, \ldots, G_{k-2})$ with σ_i either $\{0\}$ or $\{1\}$, and G_i either G^u or G^v. The locally closed pieces $M^{\alpha^j}(S,C)$ are disjoint and their union is $M(S,C)$. Furthermore, the set of indices is partially ordered with the product order induced by saying that $\{0\} < \{1\}$ and $G^u < G^v$ and $j_1 \leqslant j_2$ if each component of α^{j_1} is $\leqslant$ the corresponding component of α^{j_2}. If J is a downward cone in this partial ordering then $\bigcup_{j\in J} M^{\alpha^j}(S,C)$ is closed, because specialization decreases the indices (stable specializes to unstable and G^v specializes to G^u). Choosing a compatible total ordering we obtain the required closedness property.

The highest element in the partial ordering is the datum α' considered above, so M' is the open stratum of the stratification.

The discussion of the previous two sections allows us to further decompose all of the other strata $M^{\alpha^j}(S,C)$, in a way which again preserves the ordered closedness condition, into pieces of the form $Z^{j,a} = Y^{j,a} \times A^1$. □

Corollary 9.2. — *The natural map $D\partial M(S,C) \to D\partial M'$ is a homotopy equivalence.*

Proof. — Apply Proposition 2.6 to the stratification given by the theorem. Note that M' is nonempty and the full moduli space is irreducible so the other strata are subvarieties of strictly smaller dimension. □

10. Fenchel-Nielsen coordinates

We are now reduced to the main case $M' = M^{\alpha}(S;C)$ for α' such that all $\sigma_i = \{1\}$ and all $G_i = G^v$. We would like to get an expression for M' allowing us to understand its dual boundary complex by inspection. We will show $M' \cong \mathbf{Q}^{k-3}$ where $\mathbf{Q}$ is defined near the end of this section, such that $D\partial(\mathbf{Q}) \sim S^1$. The conclusion $D\partial M' \sim S^{2(k-3)-1}$ then follows from Lemma 2.4.

This product decomposition is a system of Fenchel-Nielsen coordinates for the open subset M' of the moduli space.

10.1. Local systems on the three-punctured sphere

One of the main things we learn from the basic theory of the classical hypergeometric function is that a rank two local system on $P^1 - \{0,1,\infty\}$ is heuristically determined by the three conjugacy classes of the monodromy transformations at the punctures. This general principle is not actually true, in cases where there might be a reducible local system. But, imposing the condition of stability provides a context in which this rigidity holds precisely. This is the statement of Corollary 10.3 below.

For convenience, we keep our standard notations for any one of our pieces S_i which is a sphere with three punctures. For this subsection, let us view S_i as the complement of three open discs in the sphere; the three boundary circles are denoted ρ_{i-1}, ρ_i and ξ_i.

Let t_{i-1} and t_i be points in $A^1 - \{2, -2\}$. We will write down a stable local system $V_i(t_{i-1}, t_i)$ on S_i, whose monodromy traces around ρ_{i-1} and ρ_i are t_{i-1} and t_i respectively, and whose monodromy around ξ_i is in the conjugacy class C_i. Furthermore, any stable local system with these traces is isomorphic to $V_i(t_{i-1}, t_i)$ in a unique way up to scalars.

Construct $V_i(t_{i-1}, t_i)$ together with a basis at the basepoint x_i, by exhibiting monodromy matrices R'_{i-1}, R_i and A_i in SL_2. Set

$$A_i := \begin{pmatrix} c_i & 0 \\ 0 & c_i^{-1} \end{pmatrix} \text{ and } R_i := \begin{pmatrix} u_i & 1 \\ w_i & (t_i - u_i) \end{pmatrix}$$

with u_i given by the formula (10.1) to be determined below, and $w_i := u_i(t_i - u_i) - 1$ because of the determinant one condition.

We could just write down the formula for u_i but in order to motivate it let us first calculate

$$R'_{i-1} = A_i R_i = \begin{pmatrix} c_i u_i & c_i \\ c_i^{-1} w_i & c_i^{-1}(t_i - u_i) \end{pmatrix}.$$

We need to choose u_i such that

$$t_{i-1} = \operatorname{Tr}(R'_{i-1}) = c_i u_i + c_i^{-1}(t_i - u_i).$$

This gives the formula

$$u_i = \frac{t_{i-1} - c_i^{-1} t_i}{c_i - c_i^{-1}}. \tag{10.1}$$

The denominator is nonzero since by hypothesis $c_i \neq c_i^{-1}$.

Lemma 10.1. — *Suppose V_i is an SL_2 local system with traces t_{i-1} and t_i. Suppose V_i is given a frame at the base point x_i, such that the monodromy matrix around the loop γ_i is diagonal with c_i in the upper left, and such that the monodromy matrix around ρ_i (via the path going from x_i to $s_i \in \rho_i$) has* *a 1 in the upper right corner. Then the three monodromy matrices of V_i are the matrices R'_{i-1}, R_i and A_i defined above.*

Proof. — The matrix A_i is as given, by hypothesis. The matrix R_i has trace t_i and upper right entry 1 by hypothesis, so it too has to look as given. Now the calculation of the trace t_{i-1} as a function of u_i has a unique inversion: the value of u_i must be given by (10.1) as a function of t_{i-1}, t_i and c_i. This determines the matrices. $\square$

LEMMA 10.2. — *Suppose V_i is an SL_2 local system with traces t_{i-1} and t_i different from 2 or -2, and suppose V_i is stable. Then, up to a scalar multiple, there is a unique frame for V_i over the basepoint x_i satisfying the conditions of the previous lemma.*

Proof. — Let e_1 and e_2 be eigenvectors for the monodromy around γ_i, with eigenvalues c_i and c_i^{-1} respectively. They are uniquely determined up to a separate scalar for each one. We claim that the upper right entry of the monodromy around ρ_i is nonzero. If it were zero, then the subspace generated by e_2 would be fixed, with the monodromy around ξ_i being c_i^{-1}; that would contradict the assumption of stability.

Now since the upper right entry of the monodromy around ρ_i is nonzero, we may adjust the vectors e_1 and e_2 by scalars such that this entry is equal to 1. Once that condition is imposed, the only further allowable change of basis vectors is by multiplying e_1 and e_2 by the same scalar. $\square$

COROLLARY 10.3. — *Suppose V_i is a local system on S_i, with conjugacy class C_i around ξ_i, stable, and whose traces around ρ_{i-1} and ρ_i are t_{i-1} and t_i respectively. Then there is up to a scalar a unique isomorphism $V_i \cong V_i(t_{i-1}, t_i)$ with the system constructed above.*

10.2. Preliminary equations

We now put together the discussions of the previous subsection for the pieces S_i, to obtain a first explicit description of the moduli space.

Suppose V is a point in M', and let t_i denote the traces of the monodromies of V around the loops ρ_i. Then by the definition of the datum α', $t_i \neq 2, -2$ and the restriction to each S_i is stable, so by the corollary there is up to scalars a unique isomorphism $h_i : V|_{S_i} \cong V_i(t_{i-1}, t_i)$.

Recall that x_i is a basepoint in S_i, and that we have chosen a path in S_i from x_i to a basepoint s_i in ρ_i, and then a path in S_{i+1} from s_i to x_{i+1}. Let ψ_i denote composed the path from x_i to x_{i+1}, and use the same symbol to denote the transport along this path which is an isomorphism $\psi_i : V_{x_i} \cong V_{x_{i+1}}$. The stalk of the local system $V_i(t_{i-1}, t_i)$ at x_i is by construction C^2, and the same at x_{i+1}, so the map

$$P_i := h_{i+1}\psi_i h_i^{-1} : V_i(t_{i-1}, t_i)_{x_i} \to V_{i+1}(t_i, t_{i+1})_{x_{i+1}}$$

is a matrix $P_i : C^2 \to C^2$ well-defined up to scalars, that is $P_i \in PGL_2$.

By the factorization property of M', the local system V is determined by these glueing isomorphisms P_i, subject to the constraint that they should intertwine the monodromies around the circle ρ_i for V_i and V_{i+1}. We have used the notation R'_i for the monodromy of the local system V_{i+1} around the circle ρ_i, whereas R_i denotes the monodromy of V_i around here. We will have made sure to use the same paths from x_i or x_{i+1} to the basepoint $s_i \in \rho_i$ in order to define these monodromy matrices as were combined together to make the path ψ_i. Therefore, the compatibility condition for P_i says

$$R'_i \circ P_i = P_i \circ R_i. \tag{10.2}$$

The frames for V_{x_i} are only well-defined up to scalars, so the matrices P_i are only well-defined up to scalars and conversely if we change them by scalars then it doesn't change the isomorphism class of the local system. Putting together all of these discussions, we obtain the following preliminary description of M'.

Lemma 10.4. — *The moduli space M' is isomorphic to the space of $(t_2, \ldots, t_{k-2}) \in (A^1 - \{2, -2\})^{k-3}$ and $(P_2, \ldots, P_{k-2}) \in (PGL_2)^{k-3}$ subject to the equations (10.2), where R'_i and R_i are given by the previous formulas in terms of the t_j.*

For the end pieces, one should formally set $t_1 := c_1 + c_1^{-1}$ and $t_{k-1} := c_k + c_k^{-1}$.

At this point, we have not yet obtained a good "Fenchel-Nielsen" style coordinate system, because the equation (10.2) for P_i contains R'_i which depends on t_{i+1} as well as t_i, and R_i which depends on t_{i-1} as well as t_i.

10.3. Decoupling

In order to remedy this point, let us proceed to decouple the equations. The strategy is to introduce the matrices

$$T_i := \begin{pmatrix} 0 & 1 \\ -1 & t_i \end{pmatrix}$$

which serve as a canonical normal form for matrices with given traces t_i, not requiring the marking of one of the two eigenvalues. Notice that if we set

$$U_i := \begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix}$$

then

$$U_i^{-1} T_i U_i = \begin{pmatrix} 1 & 0 \\ -u_i & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & t_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix} = \begin{pmatrix} u_i & 1 \\ w_i & t_i - u_i \end{pmatrix}$$

with w_i as before. Therefore, using the formula (10.1) for u_i and including the dependence of R_i on t_{i-1} and t_i in the notation, we may write

$$R_i(t_{i-1}, t_i) = U_i^{-1} T_i U_i.$$

Now

$$R'_{i-1} = A_i R_i = A_i U_i^{-1} T_i U_i = U_i^{-1}(U_i A_i U_i^{-1} T_i) U_i.$$

Furthermore, $U_i A_i U_i^{-1}$ is lower triangular with c_i and c_i^{-1} along the diagonal, and when we multiply with T_i it gives a matrix of the form

$$U_i A_i U_i^{-1} T_i = \begin{pmatrix} c_i & 0 \\ * & c_i^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & t_i \end{pmatrix} = \begin{pmatrix} 0 & c_i \\ -c_i^{-1} & * \end{pmatrix}.$$

However, we know that u_i was chosen so that this matrix has trace t_{i-1} (it is conjugate to R'_{i-1}), therefore in fact

$$U_i A_i U_i^{-1} T_i = \begin{pmatrix} 0 & c_i \\ -c_i^{-1} & t_{i-1} \end{pmatrix}$$

as could alternately be seen by direct computation. By inspection this matrix is conjugate to T_{i-1} as it should be from its trace. Interestingly enough, the conjugation is by the matrix

$$A_i^{\frac{1}{2}} := \begin{pmatrix} c_i^{\frac{1}{2}} & 0 \\ 0 & c_i^{-\frac{1}{2}} \end{pmatrix},$$

with

$$U_i A_i U_i^{-1} T_i = A_i^{\frac{1}{2}} T_{i-1} A_i^{-\frac{1}{2}}.$$

This half-power seems also to occur somewhere in the classical treatments of the Fenchel-Nielsen coordinates,

We obtain

$$R'_{i-1} = U_i^{-1}(U_i A_i U_i^{-1} T_i) U_i = U_i^{-1} A_i^{\frac{1}{2}} T_{i-1} A_i^{-\frac{1}{2}} U_i.$$

Recall that the equation (10.2) for P_{i-1} reads

$$R'_{i-1} \circ P_{i-1} = P_{i-1} \circ R_{i-1},$$

and using the above formula for R'_{i-1} as well as $R_{i-1} = U_{i-1}^{-1} T_{i-1} U_{i-1}$, this equation reads

$$U_i^{-1} A_i^{\frac{1}{2}} T_{i-1} A_i^{-\frac{1}{2}} U_i \circ P_{i-1} = P_{i-1} \circ U_{i-1}^{-1} T_{i-1} U_{i-1}. \tag{10.3}$$

Set

$$Q_{i-1} := A_i^{-\frac{1}{2}} U_i P_{i-1} U_{i-1}^{-1}.$$

This is a simple change of variables of the matrix P_{i-1}, with the matrices entering into the change of variables depending however on t_{i-2}, t_{i-1} and t_i. Notice that the coefficients of Q_{i-1} are linear functions of the coefficients of P_{i-1}, in particular the action of scalars is the same on both.

Our equation which was previously (10.2) (but for $i-1$ instead of i), has become (10.3) which, after multiplying on the left by U_i then by $A_i^{-\frac{1}{2}}$ and on the right by U_{i-1}^{-1} and substituting Q_{i-1}, becomes:

$$T_{i-1} \circ Q_{i-1} = Q_{i-1} T_{i-1}. \tag{10.4}$$

A sequence of matrices Q_i satisfying these equations leads back to a sequence of matrices P_i satisfying (10.2) and vice-versa. Recall that the glueing for the local system depended on these matrices modulo scalars, that is to say in PGL_2. We may sum up with the following proposition:

Proposition 10.5. — *The moduli space M' is isomorphic to the space of choices of*

$$(t_2, \ldots, t_{k-2}) \in (A^1 - \{2, -2\})^{k-3} \text{ and } (Q_2, \ldots, Q_{k-2}) \in (PSL_2)^{k-3}$$

subject to the equations $T_i Q_i = Q_i T_i$, with T_i, depending on t_i, defined as at the start of this subsection.

10.4. The product description

The expression for the moduli space of the previous proposition is now decoupled, and furthermore the equations are in a nice and simple form. We can therefore write M' as a product.

Theorem 10.6. — *Let $\mathbf{Q}$ be the space of pairs $(t, [p : q]) \in A^1 \times P^1$ such that $t \neq 2, -2$ and*

$$p^2 + tpq + q^2 \neq 0. \tag{10.5}$$

Then we have

$$M' \cong \mathbf{Q}^{k-3}.$$

Proof. — This will follow from the previous proposition, once we calculate that the space of matrices Q_i in PGL_2 commuting with T_i, is equal to the space of points $[p, q] \in P^1$ such that $p^2 + t_i pq + q^2 \neq 0$. Write

$$Q_i = \begin{pmatrix} p & q \\ p' & q' \end{pmatrix}$$

then

$$Q_i T_i = \begin{pmatrix} p & q \\ p' & q' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & t_i \end{pmatrix} = \begin{pmatrix} -q & p + t_i q \\ -q' & p' + t_i q' \end{pmatrix}$$

whereas

$$T_i Q_i = \begin{pmatrix} 0 & 1 \\ -1 & t_i \end{pmatrix} \begin{pmatrix} p & q \\ p' & q' \end{pmatrix} = \begin{pmatrix} p' & q' \\ t_i p' - p & t_i q' - q \end{pmatrix}.$$

The equation $Q_i T_i = T_i Q_i$ thus gives from the top row

$$p' = -q, \quad q' = p + t_i q$$

and then, those actually make the other two equations hold automatically. Therefore a solution Q_i may be written

$$Q_i = \begin{pmatrix} p & q \\ -q & p + t_i q \end{pmatrix}.$$

The statement $Q_i \in PGL_2$ means that Q_i is taken up to multiplication by scalars, in other words $[p : q]$ is a point in P^1 (clearly those coordinates are not both zero); and

$$\det(Q_i) = p^2 + t_i pq + q^2 \neq 0.$$

We conclude that the space of $(t_i, Q_i) \in (A^1 - \{2, -2\}) \times PGL_2$ such that $T_i Q_i = Q_i T_i$ is isomorphic to $\mathbf{Q}$. Therefore Proposition 10.5 now says $M' \cong \mathbf{Q}^{k-3}$. □

The variety $\mathbf{Q}$ may be seen as a group scheme over $A^1 - \{2, -2\}$ in a few different ways, but those aren't needed for our current considerations. There are several different possible choices of identity section, including the families of points $(t, [0 : 1])$, $(t, [-\frac{t}{2}, 1])$, and the same with p and q interchanged. We leave it to the reader to write down the multiplication operations in these cases. The fibers are projective lines with two points removed, hence non-canonically isomorphic to G_m with a twist by the automorphism $z \mapsto z^{-1}$ when going around $t = \pm 2$.

Lemma 10.7. — *The dual boundary complex of* $\mathbf{Q}$ *is*

$$D\partial\mathbf{Q} \sim S^1.$$

Therefore

$$D\partial\mathbf{Q}^{k-3} \sim S^{2(k-3)-1}.$$

Proof. — Let $\Phi \subset P^1 \times A^1$ be the open subset defined by the same inequation (10.5). Then $\mathbf{Q} \subset \Phi$ is an open subset, whose complement is the disjoint union of two affine lines. Furthermore, $\overline{\Phi} := P^1 \times P^1$ is a (non simple) normal crossings compactification of Φ. The divisor at infinity is the union of two copies of P^1, namely the fiber over $t = \infty$ and the conic defined by $p^2 + tpq + q^2 = 0$. These intersect transversally in two points. Therefore, the incidence complex of $\Phi \subset \overline{\Phi}$ at infinity is a graph with two vertices and two edges joining them.

It follows that the incidence complex at infinity for $\mathbf{Q}$ is a circle. That may also be seen directly by blowing up two times over each ramification point of the conic lying over $t = \pm 2$.

Now applying Lemma 2.4 successively, and noting that the successive join of $k - 3$ times the circle is $S^{2(k-3)-1}$, we obtain the second statement. □

Corollary 10.8. — *Let C be a collection of conjugacy classes satisfying Condition 7.1. Then the moduli space $\mathrm{M}_B(S; C_1, \ldots, C_k)$ of rank 2 local systems with those prescribed conjugacy classes, has dual boundary complex homotopy equivalent to a sphere*

$$D\partial\mathrm{M}_B(S; C_1, \ldots, C_k) \sim S^{2(k-3)-1}.$$

Proof. — We have been working with the hybrid moduli stack $M(S; C)$ above, but Proposition 4.5 says that this is the same as the moduli space $\mathrm{M}_B(S; C_1, \ldots, C_k)$. By Corollary 9.2, $D\partial M(S; C) \sim D\partial M'$. By Theorem 10.6, $M' \cong \mathbf{Q}^{k-3}$, and by Lemma 10.7 $D\partial\mathbf{Q}^{k-3} \sim S^{2(k-3)-1}$. Putting these all together we obtain the desired conclusion. □

This completes the proof of Theorem 1.1.

Remark 10.9. — The space Φ^{k-3} itself has a modular interpretation: it is $M^\alpha(S, C)$ for α given by setting all σ_i to $\{1\}$ (requiring stability of each $V|_{S_i}$), but having $G_i = GL_2$ for all i, that is no longer constraining the traces.

11. A geometric $P = W$ conjecture

In this section we discuss briefly the relationship between the theorem proven above, and the Hitchin fibration. For this discussion, let us suppose that the eigenvalues c_i are n_i-th roots of unity, so the conjugacy classes C_i have finite order n_i. Fix points $y_1, \ldots, y_k \in P^1$ and let

$$X := P^1[\frac{1}{n_1}y_1, \ldots, \frac{1}{n_k}y_k]$$

be the *root stack* with denominators n_i at the points y_i respectively. It is a smooth proper Deligne-Mumford stack. The fundamental group of its topological realization [43] is generated by the paths $\gamma_1, \ldots, \gamma_k$ subject to the relations that $\gamma_1 \cdots \gamma_k = 1$ and $\gamma_i^{n_i} = 1$. We may also let S be a punctured sphere such as considered above, the complement of a collection of small discs in P^1 centered at the points y_i. Therefore, a local system on X^{top} is the same thing as a local system on S such that the monodromies around the boundary loops ξ_i have order n_i respectively. We have

$$\mathrm{M}_B(X^{\mathrm{top}}, GL_r) = \coprod_{(C_1,\ldots,C_k)} \mathrm{M}_B(S, C)$$

where the disjoint union runs over the sequences of conjugacy classes such that C_i has order n_i. Recall that if we assume that C satisfies the Kostov-genericity condition then the character variety with fixed conjugacy classes $\mathrm{M}_B(S, C)$ is the same as the hybrid moduli stack $M(S, C)$. It may be seen as a connected component of the character variety $\mathrm{M}_B(X^{\mathrm{top}}, GL_r)$.

Now we recall that there is a homeomorphism between the character variety $\mathrm{M}_B(X^{\mathrm{top}}, GL_r)$ and the Hitchin-Nitsure moduli space $\mathrm{M}_{Dol}(X^{\mathrm{top}}, GL_r)$ of Higgs bundles. One may consult for example [46], [34], [40] for the general theory in the open or orbifold setting. We denote by $\mathrm{M}_{Dol}(S, C)$ the connected component of $\mathrm{M}_{Dol}(X^{\mathrm{top}}, GL_r)$ corresponding to the choice of conjugacy classes, which it may be recalled corresponds to fixing appropriate parabolic weights for the parabolic Higgs bundles. Hitchin's equations give a homeomorphism, the "nonabelian Hodge correspondence"

$$\mathrm{M}_{Dol}(S, C)^{\mathrm{top}} \cong \mathrm{M}_B(S, C)^{\mathrm{top}}. \tag{11.6}$$

Recall that the resulting two complex structures on the same underlying moduli space, form a part of a hyperkähler triple [26].

In the smooth proper orbifold setting we have the same theory of the Hitchin map

$$\mathrm{M}_{Dol}(S, C) \to A^n$$

which is a Lagrangian fibration to the space of integrals of Hitchin's Hamiltonian system [25]. In particular, n is one-half of the complex dimension of the moduli space, that dimension being even because of the hyperkähler structure.

Fix a neighborhood of infinity in the Hitchin base $B^* \subset A^n$, and let N^*_{Dol} denote its preimage in $\mathrm{M}_{Dol}(S, C)$. Similarly, let N^*_B denote a neighborhood of infinity in $\mathrm{M}_B(S, C)$. The homeomorphism 11.6 gives a natural homotopy equivalence $N^*_{Dol} \sim N^*_B$.

The neighborhood at infinity $B^* \subset A^n$ has the homotopy type of the sphere S^{2n-1}, and indeed we may view S^{2n-1} as the quotient of B^* by radial scaling, so the Hitchin map provides a natural map

$$N^*_{Dol} \to S^{2n-1}.$$

On the other hand, there is a natural projection $N^*_B \to D\partial\mathrm{M}_B(S, C)$. This is a general phenomenon, indeed if we have chosen a very simple normal crossings compactification with divisor components $D_1, \ldots, D_m$ then we may choose an open covering of N^*_B by open subsets $U_1, \ldots, U_m$ punctured neighborhoods of the D_i, such that $U_{i_1} \cap \cdots \cap U_{i_r}$ is nonempty if and only if $D_{i_1} \cap \cdots \cap D_{i_r}$ is nonempty. Then, any partition of unity for this covering provides a map $N^*_B \to R^m$ which just goes into the subspace $D\partial\mathrm{M}_B(S, C)$.

Recall the following conjecture [30], which was motivated by consideration of the case $P^1 - \{y_1, y_2, y_3, y_4\}$.

Conjecture 11.1. — *There is a homotopy-commutative square*

$$\begin{array}{ccc} N^*_{Dol} & \overset{\sim}{\to} & N^*_B \\ \downarrow & & \downarrow \\ S^{2n-1} & \overset{\sim}{\to} & D\partial\mathrm{M}_B(S, C) \end{array}$$

where the top and side maps are those described above, such that the bottom map is a homotopy equivalence.

Our main theorem provides a homotopy equivalence such as the one which is conjectured to exist on the bottom of the square, for the group GL_2 on $P^1 - \{y_1, \ldots, y_k\}$. This was our motivation, and it was also the motivation for Komyo's proof in the case $k = 5$ [33].

We haven't shown anything about commutativity of the diagram. This is one of the motivations for looking at the geometric theory of harmonic maps to buildings developed in [30] [31]. A result in this direction is shown by Daskalopoulos, Dostoglou and Wentworth [5]. The Kontsevich-Soibelman wallcrossing picture [36] should provide a global framework for this question.

Conjecture 11.1 may be viewed as a geometrical analogue of the first weight-graded piece of the $P = W$ conjecture [7] [18]. That conjecture states that weight filtration W of the mixed Hodge structure on the cohomology of the character variety M_B should be naturally identified with the perverse Leray filtration P induced by the Hitchin fibration. For the case of rank two character varieties on a compact Riemann surface, it was in fact proved by de Cataldo, Hausel and Migliorini [7]. Davison treats a twisted version [6].

It is known [45] that the cohomology of the dual boundary complex is the first weight-graded piece of the cohomology of M_B. Conjecture 11.1 states that this should come from the cohomology of the sphere at infinity in the Hitchin fibration, which looks very much like a Leray piece.

Furthermore, indeed from discussions with L. Migliorini and S. Payne it seems to be the case that the characterization of the cohomology of the dual boundary complex in [45], and the computations [21] [22] [19] [20] of the cohomology ring of M_{Dol} used to prove the $P = W$ conjecture for SL_2 in [7], should serve to show commutativity of the diagram in rational cohomology.

The question of proving the analogue of our Theorem 1.1 for a compact Riemann surface, even in the rank 2 case, is an interesting problem for further study. One may also envision the case of a punctured curve of higher genus. The techniques used here involved a choice of stability condition on each of the pieces of the decomposition, which in the higher genus case would require having at least a certain number of punctures. Weitsman suggests, following [53] and [28], that it might be possible to obtain a similar argument with only at least one puncture. The compact case would seem to be more difficult to handle.

Let us note that Kabaya [29] gives a general discussion of coordinate systems which can be obtained using decompositions, and he treats the problems of indeterminacy of choices of eigenspaces up to permutations.

The other direction which needs to be considered is local systems of higher rank. Here, the first essential case is $P^1 - \{0, 1, \infty\}$, where there is no useful decomposition of the surface into simpler pieces. We could hope that if this basic case could be treated in all ranks, then the reduction techniques we have used above could allow for an extension to the case of many punctures.

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