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Random walks under slowly varying moment conditions on groups of polynomial volume growth

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Résumé. — Soit G un groupe finiment engendré, à croissance polynomiale du volume et muni de la distance des mots associée à un ensemble donné de générateurs. Le but de ce travail est de développer des techniques qui permettent l'étude de marches aléatoires associées à des mesures de probabilité symetriques, μ, telles que, pout tout $\epsilon > 0$, $\sum |\cdot|^{\epsilon}\mu = \infty$. En particulier, nous donnons une borne inférieure optimale pour la probabilité de retour dans le cas où μ a un moment logarithmique de type faible fini.

Abstract. — Let G be a finitely generated group of polynomial volume growth equipped with a word-length $|\cdot|$. The goal of this paper is to develop techniques to study the behavior of random walks driven by symmetric measures μ such that, for any $\epsilon > 0$, $\sum |\cdot|^{\epsilon}\mu = \infty$. In particular, we provide a sharp lower bound for the return probability in the case when μ has a finite weak-logarithmic moment.

1. Introduction

Let G be a finitely generated group. Let $S = (s_1, \ldots, s_k)$ be a generating k-tuple and $S^* = \{e, s_1^{\pm 1}, \cdots, s_k^{\pm 1}\}$ be the associated symmetric generating set. Let $|\cdot|$ be the associated word-length so that $|g|$ is the least integer m such that $g = \sigma_1 \ldots \sigma_m$ with $\sigma_i \in \mathcal{S}^*$ (and the convention that $|e| = 0$).

Given two monotone functions f_1, f_2, write $f_1 \simeq f_2$ if there exists $c_i \in (0, \infty)$ such that $c_1 f_1(c_2 t) \leqslant f_2(t) \leqslant c_3 f_1(c_4 t)$ on the domain of definition of f_1, f_2 (usually, f_1, f_2 are defined on a neighborhood of 0 or infinity and tend

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to 0 or infinity at either 0 or infinity. In some cases, one or both functions are defined only on a countable set such as $\mathbb{N}$).

In [9] it is proved that there exists a function $\Phi_G : \mathbb{N} \to (0,\infty)$ such that, if μ is a symmetric probability measure with generating support and finite second moment, that is $\sum |g|^2\mu(g) < \infty$, then

$$\mu^{(2n)}(e) \simeq \Phi_G(n).$$

In [2], A. Bendikov and the first author considered the question of finding lower bounds for the probability of return $\mu^{(2n)}(e)$ when μ is only known to have a finite moment of some given exponent lower than 2. Very generally, let $\rho : [0,\infty) \to [1,\infty)$ be given.

We say that a measure μ has finite ρ-moment if $\sum \rho(|g|)\mu(g) < \infty$. We say that μ has finite weak-ρ-moment if

$$W(\rho,\mu) := \sup_{s>0} \{s\mu(\{g : \rho(|g|) > s\})\} < \infty. \tag{1.1}$$

Definition 1.1 (Fastest decay under ρ-moment). — *Let G be a countable group. Fix a function $\rho : [0,\infty) \to [1,\infty)$ with $\rho(0) = 1$. Let $\mathcal{S}_{G,\rho}$ be the set of all symmetric probability ϕ on G with the properties that $\sum \rho(|g|)\phi(g) \leqslant 2$. Set*

$$\Phi_{G,\rho} : n \mapsto \Phi_{G,\rho}(n) = \inf\left\{\phi^{(2n)}(e) : \phi \in \mathcal{S}_{G,\rho}\right\}.$$

In words, $\Phi_{G,\rho}$ provides the best lower bound valid for all convolution powers of probability measures in $\mathcal{S}_{G,\rho}$. The following variant deals with finite weak-moments.

Definition 1.2 (Fastest decay under weak-ρ-moment). — *Let G be a countable group. Fix a function $\rho : [0,\infty) \to [1,\infty)$ with $\rho(0) = 1$. Let $\widetilde{\mathcal{S}}_{G,\rho}$ be the set of all symmetric probability ϕ on G with the properties that $W(\rho,\phi) \leqslant 2$. Set*

$$\widetilde{\Phi}_{G,\rho} : n \mapsto \widetilde{\Phi}_{G,\rho}(n) = \inf\left\{\phi^{(2n)}(e) : \phi \in \widetilde{\mathcal{S}}_{G,\rho}\right\}.$$

Remark 1.3. — Since ρ takes values in $[1,\infty)$, it follows that, for any probability measure μ on G, we have $\sum \rho(|g|)\mu(g) \geqslant 1$ and $W(\rho,\mu) \geqslant 1$. In the definitions of $\Phi_{G,\rho}$ (resp. $\widetilde{\Phi}_{G,\rho}$), it is important to impose a uniform bound of the type $\sum \rho(|g|)\mu(g) \leqslant 2$ (resp. $W(\rho,\mu) \leqslant 2$) because relaxing this condition to $\sum \rho(|g|)\mu(g) < \infty$ (resp. $W(\rho,\mu) < \infty$) would lead to a trivial $\Phi_{\rho,G} \equiv 0$ (resp. $\widetilde{\Phi}_{\rho,G} \equiv 0$). The next remark indicates that, under natural circumstances, the choice of the particular constant 2 in these definitions is unimportant.

Remark 1.4.— Assume that ρ has the property that $\rho(x+y) \leqslant C(\rho(x)+\rho(y))$. Under this natural condition [2, Cor 2.3] shows that $\Phi_{G,\rho}$ and $\widetilde{\Phi}_{G,\rho}$ stay strictly positive. Further, [2, Prop 2.4] shows that, for any symmetric probability measure μ on G such that $\sum \rho(|g|)\mu(g) < \infty$ (resp. $W(\rho,\mu) < \infty$), there exist constants c_1, c_2 (depending on μ) such that

$$\mu^{(2n)}(e) \geqslant c_1 \Phi_{G,\rho}(c_2 n)$$

(resp. $\mu^{(2n)}(e) \geqslant c_1 \widetilde{\Phi}_{G,\rho}(c_2 n)$).

Recall that a group G is said to have polynomial (volume) growth of degree D if $V(n) = \#\{g \in G : |g| \leqslant n\} \simeq n^D$. By a celebrated theorem of M. Gromov, a group G such that $V(n) \leqslant Cn^A$ for some fixed constants C, A and all integers n must be of polynomial growth of degree D for some integers $D \in \{0, 1, 2, \dots\}$. In fact, Gromov's theorem states that such a group is virtually nilpotent, i.e., contains a nilpotent subgroup of finite index. See, e.g., [4, 6] and the references therein. One of the most basic results proved in [2] is as follows.

THEOREM 1.5 ([2]). — *Let G have polynomial volume growth of degree D. For any $\alpha \in (0,2)$, let $\rho_\alpha(s) = (1+s)^\alpha$. Then we have*

$$\forall\, n \geqslant 1,\ \ \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq n^{-D/\alpha}.$$

Moreover, if $\rho(s) \simeq [(1+s^2)\ell(1+s^2)]^\gamma$ with $\gamma \in (0,1)$ and ℓ smooth positive slowly varying at infinity with de Bruijn conjugate $\ell^{\#}$ then

$$\forall\, n \geqslant 1,\ \ \widetilde{\Phi}_{G,\rho}(n) \simeq [n^{1/\gamma}\ell^{\#}(n^{1/\gamma})]^{-D/2}.$$

Remark 1.6.— This statement involves the notion of de Bruijn conjugate $\ell^{\#}$ of a positive slowly varying function ℓ. We refer the reader to [3, Theorem 1.5.13] for the definition and existence of the de Bruijn conjugate. Roughly speaking, $\ell^{\#}$ is such that the inverse function of $s \mapsto s\ell(s)$ is $s \mapsto s\ell^{\#}(s)$. When ℓ is so slow that $\ell(s^a) \simeq \ell(s)$ for any $a > 0$, then $\ell^{\#} \simeq 1/\ell$. For further results on de Bruijn conjugate, see [3].

In the case when ρ is slowly varying, [2] provides only partial results. In particular, the techniques of [2] fail to give any kind of lower bound when $\rho(s) = \log(e+s)^\epsilon$ with $\epsilon \in (0,1]$ and for any ρ that varies even slower than these examples. The main goal of this work is to obtain detailed results in such cases including the following theorem.

Theorem 1.7. — *Let G have polynomial volume growth of degree D. For any $\epsilon > 0$ we have*

$$\widetilde{\Phi}_{G,\log^\epsilon}(n) \simeq \exp\left(-n^{1/(1+\epsilon)}\right)$$

where $\log^\epsilon$ *stands for the function* $\rho_\epsilon^{\log}(s) = [1+\log(1+s)]^\epsilon$*. Further, for any* $k \geqslant 2$,

$$\widetilde{\Phi}_{G,(1+\log_{[k]})^\epsilon}(n) \simeq \exp\left(-n/(\log_{[k-1]} n)^\epsilon\right)$$

where $\log_{[k]}(x) = \log(1+\log_{[k-1]} x)$, $\log_{[0]} x = x$.

The upper bound on $\widetilde{\Phi}_{G,\log^\epsilon}$ is contained in [2, 1]. Developing techniques that provide a matching lower bound is the main contribution of this work. In [2], $\widetilde{\Phi}_{G,\log^\epsilon}$ is bounded below by $\exp\left(-n^{1/\epsilon}\right)$ when $\epsilon > 1$ but [2] provides no lower bounds at all when $0 < \epsilon \leqslant 1$. As stated above, the present work provides sharp lower bounds under any iterated logarithmic weak-moment condition.

Remark 1.8.— The proof provided below for the lower bounds included in the statement of Theorem 1.7 provides a much more precise result, namely, it provides some explicit measure μ_ρ which is a witness to the behavior of the infimum $\widetilde{\Phi}_{G,\rho}$ for the given ρ. We note that no such result is known for $\Phi_{G,\rho}$ in general and that even the precise behavior of $\Phi_{\mathbb{Z},\rho_\alpha}$, $\alpha \in (0,2)$ is an open question.

To put our results in perspective, we briefly comment on the classical case when $G = \mathbb{Z}$. Let μ be a symmetric probability measure on $\mathbb{Z}$. The approximate local limit theorem of Griffin, Jain and Pruitt [5] shows that if we set

$$G(x) = \sum_{y:|y|>x} \mu(y),;\ K(x) = x^{-2} \sum_{y:|y|\leqslant x} |y|^2\mu(y) \text{ and } Q(x) = G(x) + K(x)$$

then, under the assumption that $\limsup_{x\to\infty} G(x)/K(x) < \infty$,

$$\mu^{(2n)}(0) \simeq a_n^{-1} \text{ where } Q(a_n) = 1.$$

This of course agrees with Theorem 1.5 but fails to cover laws relevant to Theorem 1.7 such that

$$\mu(y) \simeq \frac{1}{(1+|y|)[\log(e+|y|)]^{1+\epsilon}}$$

because, in such cases, G dominates K. However, basic Fourier transform arguments show that

$$\widetilde{\Phi}_{\mathbb{Z},\log^{\epsilon}}(n) \simeq \exp\left(-n^{1/(1+\epsilon)}\right)$$

with the measure μ above being a witness of this behavior.

We close this introduction with a short description of the content of other sections. The main problem considered in this paper is the construction of explicite measures that satisfy (a) some given moment condition and (b) have a prescribed (optimal) behavior in terms of the probability of return after n steps of the associate random walk. This is done by using subordination techniques based on Bernstein functions.

Section 2 describes how the notion of Bernstein function and the associated subordination techniques lead to a variety of explicit examples of probability measures whose iterated convolutions can be estimated precisely when the underlying group has polynomial volume growth. See Theorems 2.5-2.6.

Section 3 describes assorted results for measures that are supported only on powers of the generators when a given generating set has been chosen.

Section 4 develops a set of Pseudo-Poincaré inequalities adapted to random walks driven by symmetric probability measures that only have very low moments. These Pseudo-Poincaré inequalities are essential to the arguments developed in this paper.

Section 5 contains the main result of this article, Theorem 5.1 of which Theorem 1.7 is an immediate corollary.

The entire paper is written in the natural context of discrete time random walks. Well-known general techniques allow to translate the main results in the context of continuous time random walks.

2. The model case provided by subordination

Recall that a Bernstein function is a function $\psi \in \mathcal{C}^{\infty}((0,\infty))$ such that $\psi \geqslant 0$ and $(-1)^k \frac{d^k\psi}{dt^k} \leqslant 0$. A classical result asserts that a function ψ is a Bernstein function if and only if there are reals $a, b \geqslant 0$ and a measure ν on $(0,\infty)$ satisfying $\int_0^\infty \frac{t d\nu(t)}{1+t} < \infty$ such that $\psi(s) = a + bs + \int_0^\infty (1-e^{-st}) d\nu(t)$. Set

$$c(\psi,1) = b + \int_0^\infty te^{-t}d\nu(t), \;\; c(\psi,n) = \frac{1}{n!}\int_0^\infty t^n e^{-t} d\nu(t), \; n > 1. \quad (2.1)$$

If ψ is a Bernstein function satisfying $\psi(0)=0$, $\psi(1)=1$ and K is a Markov kernel then

$$K_\psi = \sum_1^\infty c(\psi,n)K^n$$

is also a Markov kernel. Further, one can understand K_ψ as given by $K_\psi = I-\psi(I-K)$. See [1] for details. Similarly, if ϕ is a probability measure on a group G, set

$$\phi_\psi = \sum c(\psi,n)\phi^{(n)}.$$

This is a probability measure which we call the ψ-subordinate of ϕ.

Recall that a complete Bernstein function is a function $\psi \in \mathcal{C}^\infty((0,\infty))$ such that

$$\psi(s) = s^2\int_0^\infty e^{-ts}g(t)dt \tag{2.2}$$

where g is a Bernstein function (complete Bernstein functions are Bernstein function). A comprehensive book treatment of the theory of Bernstein functions is [13]. See also [8].

Example 2.1. — The most basic examples of complete Bernstein functions are $\psi(s)=s^\alpha$, $\alpha\in(0,1)$, and $\psi(s)=\log_2(1+s)$. A less trivial example of interest to us is

$$\psi(s) = \frac{1}{[\log_2(1+s^{-1/\alpha})]^{\beta\alpha}},\ 0<\beta\leqslant 1\leqslant\alpha<\infty.$$

The choice of the base 2 logarithm in this definition insures that the additional property $\psi(1)=1$ holds true. If we define $\log_{2,k}$ by setting $\log_{2,1}(s)=\log_2(1+s)$ and $\log_{2,k}(s)=\log_{2,1}(\log_{2,k-1}(s))$, $k>1$, then the function

$$\psi(s) = \frac{1}{[\log_{2,k}(s^{-1/\alpha})]^{\beta\alpha}},\ 0<\beta\leqslant 1\leqslant\alpha<\infty$$

is also a complete Bernstein function.

The following two results from [1] will be very useful for our purpose. For a comprehensive treatment of the theory of function of slow and regular variation, see [3].

Theorem 2.2 ([1, Theorems 2.5–2.6]). — *Assume that $\psi_1:(0,\infty)\to(0,\infty)$ has a positive continuous derivative and satisfies $\psi_1(0^+)=0$. Assume further that $x\mapsto\psi_1(x)$ and $x\mapsto x\psi_1'(x)$ are slowly varying at 0^+ and that*

$$\psi_1'(s)\sim\frac{1}{s\ell(1/s)}$$

where ℓ is slowly varying at infinity. Then there exists a positive constant a and a complete Bernstein function ψ such that $\psi \sim a\psi_1$, $\psi' \sim a\psi_1'$ at 0^+ and $\psi(1) = 1$. Further

$$c(\psi, n) \simeq \frac{1}{(1+n)\ell(1+n)}.$$

THEOREM 2.3 ([1, Theorems 3.3–3.4]). — *Let G be a finitely generated group of polynomial volume growth. Let ϕ be a finitely supported symmetric probability measure with $\phi(e) > 0$ and generating support. Let ψ be a Bernstein function with $\psi(0) = 0$, $\psi(1) = 1$. Assume that*

$$\psi(s) \simeq 1/\theta(1/s)$$

with θ positive increasing slowly varying at infinity.

1. *Assume that the rapidly varying function θ^{-1} (the inverse function of θ) satisfies*
$$\log \theta^{-1}(u) \simeq u^{\gamma}\kappa(u)^{1+\gamma}$$
with $\gamma \in (0, \infty)$ and κ slowly varying at infinity. Then the ψ-subordinate ϕ_ψ of ϕ satisfies
$$-\log(\phi_\psi^{(n)}(e)) \simeq n^{\gamma/(1+\gamma)}/\kappa^{\#}(n^{1/(1+\gamma)})$$
where $\kappa^{\#}$ is the de Bruijn conjugate of κ.

2. *Assume that the function $\kappa = \theta \circ \exp$ is slowly varying and satisfies $s\kappa^{-1}(s) \simeq \kappa^{-1}(s)$ at infinity. Then the ψ-subordinate ϕ_ψ of ϕ satisfies*
$$-\log(\phi_\psi^{(n)}(e)) \simeq n/\kappa(n).$$

Example 2.4. — Fix an integer $k \geqslant 1$ and parameters α, β, $0 < \beta \leqslant 1 \leqslant \alpha < \infty$. Consider the complete Bernstein function

$$\psi(s) = \frac{1}{\theta(1/s)} = \frac{1}{[\log_{2,k}(s^{-1/\alpha})]^{\beta\alpha}}.$$

A simple computation yields

$$\psi'(s) = \frac{\beta}{s(1+s^{1/\alpha})\log_{2,1}(s^{-1/\alpha})\cdots\log_{2,k-1}(s^{-1/\alpha})[\log_{2,k}(s^{-1/\alpha})]^{1+\beta\alpha}}.$$

It follows that

$$c(\psi, n) \simeq \frac{1}{(1+n)(1+\log_{[1]} n)\cdots(1+\log_{[k-1]} n)(1+\log_{[k]} n)^{1+\beta\alpha}}.$$

Here $\log_{[m]} n = \log(1+\log_{[m-1]} n)$ with $\log_{[1]} n = \log(1+n)$. Further if $k=1$, we have $\log\theta^{-1}(u) \simeq u^{1/\alpha\beta}$ and it follows that

$$\phi_\psi^{(n)}(e) \simeq \exp\left(-n^{1/(1+\alpha\beta)}\right).$$

In the case where $k>1$, we have $\kappa(s) = \theta\circ\exp(s) \simeq [\log_{[k-1]}(s)]^{\alpha\beta}$ and we obtain

$$\phi_\psi^{(n)}(e) \simeq \exp\left(-n/[\log_{[k-1]}(n)]^{\alpha\beta}\right).$$

For later purpose, we need the information contained in the following Theorem which is an easy corollary of Theorem 2.2 and the Gaussian bounds of [7].

Theorem 2.5. — *Let ψ_1 and ψ be as in* Theorem 2.2 *with*

$$\psi_1'(s) \sim \frac{1}{s\ell(1/s)} \quad \text{at } 0^+,$$

where ℓ is slowly varying at infinity. Let G be a finitely generated group of polynomial volume growth of degree D. Let ϕ be a finitely supported symmetric probability measure with $\phi(e)>0$ and generating support. Then there are constants $c, C \in (0,\infty)$ such that the probability measure ϕ_ψ satisfies

$$\forall\, x\in G,\quad \frac{c}{(1+|x|)^D\ell(1+|x|^2)} \leqslant \phi_\psi(x) \leqslant \frac{C}{(1+|x|)^D\ell(1+|x|^2)}.$$

Proof. — By [7], there are constants c_i, $1\leqslant i\leqslant 4$, such that for each x,n such that $\phi^{(2n)}(x)\neq 0$,

$$c_1 n^{-D/2}\exp\left(-c_2\frac{|x|^2}{n}\right) \leqslant \phi^{(n)}(x) \leqslant c_3 n^{-D/2}\exp\left(-c_4\frac{|x|^2}{n}\right).$$

By Definition and Theorem 2.2, $\phi_\psi^{(n)}(x)$ is bounded above and below

$$\sum_1^\infty \frac{c}{(1+n)\ell(1+n)}\phi^{(n)}(x)$$

(with different constants c in the upper and lower bound). Break this sum into two parts S_1, S_2 with S_1 being the sum over $n\geqslant |x|^2$. We have

$$S_1 \simeq \sum_{n\geqslant |x|^2} \frac{1}{(1+n)^{1+D/2}\ell(1+n)} \simeq \frac{1}{(1+|x|^2)^{D/2}\ell(1+|x|^2)}.$$

which already proves the desired lower bound. Similarly, for S_2, note that

$$\frac{(1+|x|)^{2+D}\ell(1+|x|^2)}{(1+n)^{1+D/2}\ell(1+n)} \leqslant C\left(\frac{1+|x|^2}{1+n}\right)^A$$

for some $A>0$. Further, for each k, there are at most $2|x|^2/k^2$ positive integers n such that $k-1<|x|^2/n\leqslant k$. Hence, we obtain

$$\begin{aligned} S_2 &\leqslant \frac{C'}{(1+|x|)^{2+D}\ell(1+|x|^2)}\sum_k (1+|x|^2)(1+k)^{A-2}e^{-c_4(k-1)} \\ &\leqslant \frac{C''}{(1+|x|)^{D}\ell(1+|x|^2)}. \end{aligned}$$

Together with the estimate already obtained for S_1, this gives the desired upper bound on $\phi_\psi^{(n)}(x)$. □

The following statement put together the results gathered above while emphasizing the point of view of the construction of a model with a prescribed behavior.

THEOREM 2.6. — *Let G be a finitely generated group with polynomial volume growth of degree D. Let ϕ be a finitely supported symmetric probability measure with $\phi(e)>0$ and generating support. Let ℓ be a continuous positive slowly varying function at infinity such that $\int_0^1 \frac{ds}{s\ell(1/s)}<\infty$. Then there exists a complete Bernstein function ψ with $\psi(0)=0$, $\psi(1)=1$ such that:*

- *$\psi(s)\sim a\int_0^s \frac{dt}{t\ell(1/t)}$ for some constant $a>0$;*
- *$c(\psi,n)\simeq \frac{1}{(1+n)\ell(1+n)}$;*
- *$\phi_\psi(x)\simeq [(1+|x|)^D\ell(1+|x|^2)]^{-1}$;*

Further, if we set $\theta(s)=1/\int_0^{1/s}\frac{dt}{t\ell(1/t)}$, the following holds:

- *If $\log\theta^{-1}(u)\simeq u^\gamma\kappa(u)^{1+\gamma}$ with $\gamma\in(0,\infty)$ and κ slowly varying at infinity, then we have*

  $$\phi_\psi^{(n)}(e)\simeq \exp\left(-n^{\gamma/(1+\gamma)}/\kappa^{\#}(n^{1/(1+\gamma)})\right)$$

  where $\kappa^{\#}$ is the de Bruijn conjugate of κ.

- *If $\kappa=\theta\circ\exp$ is slowly varying and satisfies $s\kappa^{-1}(s)\simeq\kappa^{-1}(s)$ then*

  $$\phi_\psi^{(n)}(e)\simeq \exp\left(-n/\kappa(n)\right).$$

Example 2.7. — Let G be a finitely generated group with polynomial volume growth of degree D. Then, for any $\delta > 0$, there exists a symmetric probability measure ϕ_δ such that

$$\phi_\delta(x) \simeq \frac{1}{(1+|x|)^D[\log(e+|x|)]^{1+\delta}}$$

and

$$\phi_\delta^{(n)}(e) \simeq \exp\left(-n^{1/(1+\delta)}\right).$$

Also, for any $\delta > 0$ and integer $k \geqslant 1$, there exists a symmetric probability measure $\phi_{k,\delta}$ such that

$$\phi_{k,\delta}(x) \simeq \frac{1}{(1+|x|)(1+\log_{[1]}|x|)\cdots(1+\log_{[k-1]}|x|)(1+\log_{[k]}|x|)^{1+\delta}}$$

and

$$\phi_{k,\delta}^{(n)}(e) \simeq \exp\left(-n/(\log_{[k-1]} n)^{1/\delta}\right).$$

Further, for any $k \geqslant 1$ and $\delta > 0$, the comparison results of [9] imply that any symmetric probability measure φ with the property that

$$\forall\, f \in L^2(G),\ \ \mathcal{E}_\varphi(f,f) \leqslant \mathcal{E}_{\phi_{k,\delta}}(f,f) \leqslant C\mathcal{E}_\varphi(f,f)$$

satisfies $\varphi^{(2n)}(e) \simeq \phi_{k,\delta}^{(2n)}(e)$. See (3.2) for a definition of the Dirichlet form $\mathcal{E}_\mu$ associated with a symmetric probability measure μ.

3. Measures supported on powers of generators

In [12], the authors introduced the study of random walks driven by measures supported on the powers of given generators. Namely, given a group G equipped with a generating k-tuple $(s_1,\dots,s_k)$, fix a k-tuple of probability measures $(\mu_i)_1^k$, each μ_i being a probability measure on $\mathbb{Z}$, and set

$$\mu(g) = k^{-1}\sum_1^k \sum_{n\in\mathbf{Z}} \mu_i(n)\mathbf{1}_{s_i^n}(g). \tag{3.1}$$

In [12], special attention is given to the case when the μ_i are symmetric power laws. Here, we focus on the case when the μ_i are symmetric, all equal and are of the type

$$\mu_i(n) = \phi(n) \simeq \frac{1}{(1+n)\ell(1+n)}$$

where ℓ is increasing and slowly varying. Obviously, we require here that $\sum_1^\infty [n\ell(n)]^{-1} < \infty$.

The following statement is a special case of [12, Theorem 5.7]. It provides a key comparison between the Dirichlet forms of measures supported on power of generators and associated measures that are radial with respect to the word-length. In this form, this result holds only under the hypothesis that the group G is nilpotent. Recall that the Dirichlet form $\mathcal{E}_\mu$ associated with a symmetric probability measure μ is the quadratic form on $L^2(G)$ given by

$$\mathcal{E}_\mu(f,f) = \frac{1}{2}\sum_{x,y} |f(xy) - f(x)|^2\mu(y). \tag{3.2}$$

Theorem 3.1. — *Let G be a nilpotent group equipped with a generating k-tuple $S = (s_1, \dots, s_k)$. Let $|\cdot|$ be the corresponding word-length and V be the associated volume growth function. Fix a continuous increasing function $\ell : [0,\infty) \to (0,\infty)$ which is slowly varying at infinity. Assume that*

$$\sum_{g\in G} \frac{1}{V(1+|g|)\ell(1+|g|)} < \infty. \tag{3.3}$$

Consider the probability measures ν and μ defined by

$$\nu(g) = \frac{c}{V(1+|g|)\ell(1+|g|)}, \quad c^{-1} = \sum \frac{1}{V(1+|g|)\ell(1+|g|)}, \tag{3.4}$$

and

$$\mu(g) = k^{-1}\sum_{1}^{k}\sum_{n\in\mathbf{Z}} \frac{b\mathbf{1}_{s_i^n}(g)}{(1+n)\ell(1+n)}, \quad b^{-1} = \sum_{\mathbf{Z}} \frac{1}{(1+n)\ell(1+n)}. \tag{3.5}$$

Then there exists a constant C such that

$$\mathcal{E}_\mu \leqslant C\mathcal{E}_\nu.$$

Proof. — We apply [12, Theorem 5.7] with $\phi = \ell$ and $\|\cdot\| = |\cdot|$ (in the notation of [12], this corresponds to having a weight system $\mathfrak{w}$ generated by $w_i = 1$ for all $i = 1, \dots, k$, the simplest case). Referring to the notation used in [12], because of the choice $\|\cdot\| = |\cdot|$, we have $F_{c_1}(r) = r$, $F_{h_i}(r) = r^{m_i}$ where $m_i \geqslant 1$ (m_i is an integer which describes the position of the generator s_i in the lower central series of G, modulo torsion). Having made these observations, the stated result follows from [12, Theorem 5.7] by inspection. $\square$

Remark 3.2. — Note that, in the context of Theorem 3.1 and for any positive function ℓ that is slowly varying at infinity, the conditions

$$(a)\ \sum_{1}^{\infty}\frac{1}{n\ell(n)} < \infty; \quad (b)\ \sum_{1}^{\infty}\frac{1}{n\ell(n^2)} < \infty; \quad (c)\ \sum_{g\in G}\frac{1}{V(|g|)\ell(1+|g|)},$$

are equivalent. To see that (a) and (c) are equivalent, note that

$$\sum_g \frac{1}{V(|g|)\ell(1+|g|)} \simeq \sum_k \frac{V(k)-V(k-1)}{\ell(1+k)(1+k)^D\ell(1+k)}$$

and use Abel summation formula to see that this implies

$$\sum_g \frac{1}{V(|g|)\ell(1+|g|)} \simeq \sum_k \frac{k^D}{(1+k)^{D+1}\ell(1+k)} \simeq \sum_k \frac{1}{k\ell(k)}.$$

The following statement illustrates one of the basic consequences of this comparison theorem.

Theorem 3.3. — *Let $\ell : [0,\infty) \to [0,\infty)$ be continuous increasing, slowly varying at infinity, and such that $\int_0^{1/s} \frac{dt}{t\ell(1/t)} < \infty$. Set $\theta(s) = 1/\int_0^{1/s} \frac{dt}{t\ell(1/t)}$. Let G be a finitely generated nilpotent group equipped with a generating k-tuple $S = (s_1,\dots,s_k)$. Let μ be the symmetric probability measure on G defined by*

$$\mu(g) = k^{-1}\sum_1^k \sum_{n\in\mathbb{Z}} \frac{c\mathbf{1}_{s_i^n}(g)}{(1+|n|)\ell(1+|n|^2)}.$$

- *If $\log\theta^{-1}(u) \simeq u^\gamma\kappa(u)^{1+\gamma}$ with $\gamma\in(0,\infty)$ and κ slowly varying at infinity, then we have*

$$\mu^{(n)}(e) \simeq \exp\left(-n^{\gamma/(1+\gamma)}/\kappa^{\#}(n^{1/(1+\gamma)})\right)$$

where $\kappa^{\#}$ is the de Bruijn conjugate of κ.

- *If $\kappa = \theta\circ\exp$ is slowly varying and satisfies $s\kappa^{-1}(s) \simeq \kappa^{-1}(s)$ then*

$$\mu^{(n)}(e) \simeq \exp\left(-n/\kappa(n)\right).$$

Proof. — The lower bounds follow from Theorems 2.6 and 3.1, together with [9, Theorem 2.3]. To prove the upper bounds, forget all but one non-torsion generator, say s_1, and use [9, Theorem] to compare with the corresponding one-dimensional random walk on $\{s_1^n : n\in\mathbb{Z}\}$. □

Remark 3.4. — Assume that μ is given by (3.1) with possibly different μ_i of the form $\mu_i(m) \simeq 1/[(1+|m|)\ell_i(1+|m|^2)]$. Let ℓ,θ be as in Theorem 3.3. If there exists $i\in\{1,\dots,k\}$ such that s_i is not torsion and $\ell_i \leqslant C\ell$ then $\mu^{(2n)}(e)$ can be bounded above by the convolution power $\phi^{(2n)}(0)$ of the one dimensional symmetric probability measure $\phi(m) = c/(1+|m|)\ell(1+|m|^2)$. If we assume that for all $i\in\{1,\dots,k\}$ such that s_i is not torsion we have $\ell_i \geqslant c\ell$ then we obtain a lower bound for $\mu^{(2n)}(e)$ in terms of $\phi^{(2n)}(0)$.

4. Pseudo-Poincaré inequality

In [12], the authors proved and use new (pointwise) pseudo-Poincaré inequalities adapted to spread-out probability measures. These pseudo-Poincaré inequalities are proved for measures of type (3.1) and involve the truncated second moments of the one dimensional probability measures μ_i. More precisely, fix $s \in G$ and let ϕ be a symmetric probability measure on $\mathbb{Z}$. It is proved in [12] that, if we set

$$\mathcal{E}_{s,\phi}(f,f) = \frac{1}{2}\sum_{x\in G}\sum_{n\in\mathbb{Z}} |f(xs^n) - f(x)|^2\phi(n), \;\; \mathcal{G}_\phi(r) = \sum_{|n|\leqslant r} |n|^2\phi(n),$$

and assume that there exists a constant C such that $\phi(n) \leqslant C\phi(m)$ for all $|m| \leqslant |n|$, then it holds that

$$\sum_{x\in G} |f(xs^n) - f(x)|^2 \leqslant C_\phi(\mathcal{G}_\phi(|n|))^{-1}|n|^2\mathcal{E}_{s,\phi}(f,f). \tag{4.6}$$

Under the same notation and hypotheses, set $\mathcal{H}_\phi(r) = \sum_{|n|>r}\phi(n)$. Then we claim that

$$\sum_{x\in G} |f(xs^n) - f(x)|^2 \leqslant C'_\phi(\mathcal{H}_\phi(|n|))^{-1}\mathcal{E}_{s,\phi}(f,f). \tag{4.7}$$

Indeed, write

$$|f(xs^n) - f(x)|^2 \leqslant 2(|f(xs^n) - f(xs^m)|^2 + |f(xs^m) - f(x)|^2).$$

Note note that the set $\{m : |n-m| \leqslant |m|\}$ contains $\{m : m \geqslant n\}$ if n is positive and $\{m : m \leqslant n\}$ if n is negative. Multiply both sides of the displayed inequality above by $\phi(m)$ and sum over $x \in G$ and m such that $|n-m| \leqslant |m|$ to obtain

$$\begin{aligned}
&\left(\sum_{x\in G} |f(xs^n) - f(x)|^2\right)\mathcal{H}_\phi(|n|) \\
&\leqslant \;\; 4\sum_{x\in G}\sum_{|m|\geqslant|n|} (|f(xs^n) - f(xs^m)|^2\phi(m) + |f(xs^m) - f(x)|^2\phi(m)) \\
&\leqslant \;\; 4\sum_{x\in G}\sum_{m\in\mathbb{Z}} (|f(xs^{n-m}) - f(x)|^2\phi(m) + |f(xs^m) - f(x)|^2\phi(m)) \\
&\leqslant \;\; 4C\sum_{x\in G}\sum_{m\in\mathbb{Z}} (|f(xs^{n-m}) - f(x)|^2\phi(n-m) + |f(xs^m) - f(x)|^2\phi(m)) \\
&= \;\; 16C\mathcal{E}_{s,\phi}(f,f).
\end{aligned}$$

Putting together this simple computation and the earlier results from [12], we can state the following theorem.

THEOREM 4.1. — *Let ϕ be a symmetric probability measure on $\mathbb{Z}$ such that there exists a constant C for which, for all $|m| \leqslant |n|$, $\phi(n) \leqslant C\phi(m)$. There exists a constant C_ϕ such that, for any group G and any $s \in G$, we have*

$$\forall\, n, \quad \sum_{x \in G} |f(xs^n) - f(x)|^2 \leqslant C_\phi \min\left\{\frac{1}{\mathcal{H}_\phi(|n|)}, \frac{|n|^2}{\mathcal{G}_\phi(|n|)}\right\} \mathcal{E}_{s,\phi}(f,f).$$

Remark 4.2. — If ϕ is regularly varying of index $\alpha \in (-3,-1)$, then $\mathcal{H}_\phi(r) \simeq r^{-2}\mathcal{G}_\phi(r)$. If ϕ is regularly varying of index $\alpha < -3$ then $\mathcal{H}_\phi(r)$ is much smaller than $r^{-2}\mathcal{G}_\phi(r)$. When ϕ is regularly varying of index -1 then $\mathcal{H}_\phi(r)$ is much larger than $r^{-2}\mathcal{G}_\phi(r)$.

COROLLARY 4.3. — *Let ϕ be a symmetric probability measure on $\mathbb{Z}$ such that there exists a constant C for which, for all $|m| \leqslant |n|$, $\phi(n) \leqslant C\phi(m)$. Let G be a finitely generated nilpotent group equipped with a generating k-tuple $S = (s_1, \dots, s_k)$. Let μ be the symmetric probability measure on G defined by*

$$\mu(g) = k^{-1} \sum_1^k \sum_{n \in \mathbb{Z}} \phi(n) \mathbf{1}_{s_i^n}(g).$$

Then there are constants C_1, C_2 such that, for all $g \in G$, we have

$$\forall\, f \in L^2(G), \sum_{x \in G} |f(xg) - f(x)|^2 \leqslant C_1 \min\left\{\frac{1}{\mathcal{H}_\phi(C_2|g|)}, \frac{|g|^2}{\mathcal{G}_\phi(C_2|g|)}\right\} \mathcal{E}_\mu(f,f).$$

Proof. — Apply [12, Theorem 2.10] in the simplest case when the weight system $\mathfrak{w}$ is generated by constant weights $w_i = 1$, $1 \leqslant i \leqslant k$, so that the corresponding length function on G is just the word-length $g \mapsto |g|$. This result yields the existence of a constant C_0, an integer M and a sequence $(i_1, \dots i_M) \in \{1, \dots, k\}^M$ such that any element $g \in G$ can be written in the form

$$g = \prod_{j=1}^M s_{i_j}^{x_j} \text{ with } |x_j| \leqslant C_0 |g|.$$

Further, by construction, for each $i \in \{1, \dots, k\}$,

$$\mathcal{E}_{s_i,\phi} \leqslant k \mathcal{E}_\mu.$$

Hence, the stated Corollary follows easily from a finite telescoping sum argument and Theorem 4.1. $\square$

THEOREM 4.4. — *Let $\ell : [0,\infty) \to [0,\infty)$ be continuous increasing, slowly varying at infinity, and such that $\int_0^{1/s} \frac{dt}{t\ell(1/t)} < \infty$. Set $\theta(s) = 1/\int_0^{1/s} \frac{dt}{t\ell(1/t)}$.*

Let G be a finitely generated group with word-length $|\cdot|$ and polynomial volume growth of degree D. Let φ be a symmetric probability measure on G such that

$$\varphi(g) \simeq \frac{1}{(1+|g|)^D \ell(1+|g|^2)}.$$

Then, there exists a constant C such that for any $g \in G$ and any $f \in L^2(G)$,

$$\sum_{x \in G} |f(xg) - f(x)|^2 \leqslant C\theta(1+|g|^2)\mathcal{E}_\varphi(f,f).$$

Proof. — As a key first step in the proof of this theorem, consider the special case when G is a finitely generated nilpotent group equipped with a generating k-tuple $S = (s_1, \dots, s_k)$. In this case, the theorem follows from Corollary 4.3 and Theorem3.1 by inspection after noting that

$$\mathcal{H}_\phi(r) \simeq 1/\theta(r^2).$$

Next, consider the general case when G has polynomial volume growth of degree D. Then, by Gromov's theorem [6], G contains a finitely generated nilpotent group G_0 of finite index in G. Fix finite symmetric generating sets in G and G_0. Let $|\cdot|$ be the word-length in G and $\|\cdot\|$ be the word-length in G_0. It is well-known that, for any $g_0 \in G_0 \subset G$, we have $\|g_0\| \simeq |g_0|$.

Let A, B be finite sets of coset representatives for $G_0\backslash G$ and G/G_0, respectively. Fix $g \in G$ and write $g = g_0 b$, $g_0 \in G_0$, $b \in B$. Observe that $G = \{x = ax_0 : a \in A, x_0 \in G_0\}$. Hence, for any $f \in L^2(G)$, we can write

$$\sum_{x \in G} |f(xg) - f(x)|^2 = \sum_{a \in A} \sum_{x_0 \in G_0} |f(ax_0 g_0 b) - f(ax_0)|^2.$$

Applying the result already proved for nilpotent groups to G_0 and the functions $f_a : G_0 \to \mathbb{R}$, $f_a(x_0) = f(ax_0)$, $a \in A$, we obtain

$$\sum_{x_0 \in G_0} |f(ax_0 g_0) - f(ax_0)|^2 \leqslant \frac{C\theta(1+\|g_0\|^2)}{2} \sum_{x_0, y_0 \in G_0} \frac{|f(ax_0 y_0) - f(ax_0)|^2}{\ell(1+\|y_0\|^2)(1+\|y_0\|)^D}.$$

Summing over $a \in A$ and using the fact that $\|g_0\| \simeq |g_0|$ easily yield

$$\sum_{x \in G} |f(xg_0) - f(x)|^2 \leqslant C\theta(1+|g_0|^2)\mathcal{E}_\varphi(f,f).$$

Since we trivially have

$$\forall\, b \in B, \ \sum_{x \in G} |f(xb) - f(x)|^2 \leqslant C\mathcal{E}_\varphi(f,f),$$

the desired result follows. □

5. Probability of return lower bounds under weak-moment conditions

In this section we use the results obtained in earlier Sections together with [2, Theorem 2.10] to prove our main theorem, Theorem 5.1. Note that Theorem 1.7 stated in the introduction is an immediate corollary of this more general result.

Theorem 5.1. — *Let G be a finitely generated group with word-length $|\cdot|$ and polynomial volume growth of degree D. Let $\ell : [0,\infty) \to [0,\infty)$ be a positive continuous increasing function which is slowly varying at infinity and satisfies $\int_1^\infty \frac{dt}{t\ell(t)} < \infty$. Set $\theta(s) = 1/\int_0^{1/s} \frac{dt}{t\ell(1/t)}$ and $\theta_2(s) = \frac{1}{2}\theta(s^2)$. Let φ be a symmetric probability measure such that*

$$\varphi(g) \simeq \frac{1}{(1+|g|)^D \ell(1+|g|^2)}.$$

Then we have

$$\widetilde{\Phi}_{G,\theta_2}(n) \simeq \varphi^{(n)}(e).$$

The proof of this result is based on a simple special case of [2, Theorem 2.10]. For clarity and the convenience of the reader, we state the precise statement we need. Abusing notation, if φ_1 is a probability measure and $c_\alpha(n)$ is defined by $1-(1-x)^\alpha = \sum c_\alpha(n)x^n$, $x \in [-1,1]$, $\alpha \in (0,1)$, we call $\varphi_\alpha = \sum c_\alpha(n)\varphi_1^{(n)}$ the α-subordinate of φ_1.

Theorem 5.2 (See [2, Theorem 2.10]). — *Let G be a finitely generated group with word-length $|\cdot|$. Let φ_1 be a symmetric probability measure on a group G and δ be a positive increasing function with $\delta(0) = 1$. Assume that, for any $g \in G$ and $f \in L^2(G)$,*

$$\sum_{x\in G} |f(xg) - f(x)|^2 \leqslant C\delta(|g|)^2 \mathcal{E}_{\varphi_1}(f,f).$$

Fix $\alpha \in (0,1)$. Let μ be a symmetric measure on G satisfying the weak moment condition

$$W(\delta^{2\alpha},\mu) = \sup_{s>0}\left\{s\mu(\{g : \delta(|g|)^{2\alpha} > s\})\right\} < \infty.$$

Then, for all $f \in L^2(G)$,

$$\mathcal{E}_\mu(f,f) \leqslant C_\alpha C W(\delta^{2\alpha},\mu)\mathcal{E}_{\varphi_\alpha}(f,f)$$

where φ_α is the α-subordinate of φ_1. In particular,

$$\mu^{(2n)}(e) \geqslant c\varphi_\alpha^{(2Nn)}(e).$$

Proof. — This is a special case of [2, Theorem 2.10]. Referring to the notation used in [2, Theorem 2.10], the operator A is taken to be $Af = f*(\delta_e - \varphi_1)$, the function ψ is simply $\psi(s) = s^\alpha$ so that $\omega(s) = \Gamma(2-\alpha)^{-1}s^{1-\alpha}$. It follows that the function ρ satisfies $\rho(s) \simeq 1 + s^{2\alpha}$. Note that, by definition, $\|\psi(A)^{1/2}f\|_2^2 = \mathcal{E}_{\phi_\alpha}(f,f)$. The last statement in the theorem follows from [9]. □

Proof of Theorem 5.1. — To prove that $n \mapsto \widetilde{\Phi}_{G,\theta_2}(n)$ is controlled from above by $n \mapsto \varphi^{(2n)}(e)$, it suffices to show that φ has a finite weak-θ_2-moment. For $s \geqslant 1$, write

$$\begin{aligned}
\varphi(\{g : \theta_2(|g|) > s\}) &= \sum_{|g| \geqslant \theta_2^{-1}(s)} \frac{1}{(1+|g|)^D \ell(1+|g|^2)} \\
&\simeq \sum_{k \geqslant \theta_2^{-1}(s)} \frac{V(k) - V(k-1)}{(1+k)^D \ell(1+k^2)} \\
&\simeq \sum_{k \geqslant \theta_2^{-1}(s)} \frac{1}{(1+k)\ell(1+k^2)} \\
&\simeq 1/\theta_2(\theta_2^{-1}(s)) \simeq 1/s.
\end{aligned}$$

This shows that $W(\theta_2, \varphi) < +\infty$. By [2, Proposition 2.4], this implies that there exist N, C such that, for all n, $\widetilde{\Phi}_{G,\theta_2}(Nn) \leqslant C\varphi^{(2n)}(e)$.

The more interesting statement is the bound

$$\widetilde{\Phi}_{G,\theta_2}(n) \geqslant c\varphi^{(2Nn)}(e).$$

Let ϕ be a symmetric finitely supported probability measure on G with generating support and $\phi(e) > 0$. Using the basic hypothesis regarding the function ℓ and Theorems 2.2 and 2.5, we can find a complete Bernstein function ψ_0 such that $\psi_0'(s) \sim \frac{a}{s\ell(1/s)}$, $\psi_0 \sim \frac{a}{\theta(1/s)}$ at 0^+ (for some $a > 0$) and

$$\phi_{\psi_0}(g) \simeq \frac{1}{(1+|g|)^D \ell(1+|g|^2)}.$$

This implies $\phi_{\psi_0}^{(n)}(e) \simeq \varphi^{(n)}(e)$.

Next, we claim that for any $\alpha \in (0,1)$, we can find a complete Bernstein function $\psi = \psi_\alpha$ such that $\psi \sim b\psi_0^{1/\alpha}$, $\psi' \sim (b/\alpha)\psi_0'\psi_0^{-1+(1/\alpha)}$. If we set $\bar{\psi} = (\psi)^\alpha$ it then follows that $\bar{\psi} \sim \bar{a}\psi_0$ and $(\bar{\psi})' \sim \bar{a}\psi_0'$. If such a function exists, then we have:

(a) By construction and Theorem 4.4, for all $g \in G$ and $f \in L^2(G)$, we have
$$\sum_{x \in G} |f(xg) - f(x)|^2 \leqslant C\theta_2(|g|)^{1/\alpha}\mathcal{E}_{\phi_\psi}(f, f).$$

(b) By construction, $\phi_{\bar{\psi}}$ is the α-subordinate of ϕ_ψ.

(c) Since $(\bar{\psi})' \sim \bar{a}\psi_0'$, we have
$$\phi_{\bar{\psi}}(g) \simeq \frac{1}{(1+|g|)^D \ell(1+|g|^2)} \simeq \phi_{\psi_0}(g) \simeq \varphi(g)$$
and, by [9], $\phi_{\bar{\psi}}^{(2n)}(e) \simeq \phi_{\psi_0}^{(2n)}(e) \simeq \varphi^{(2n)}(e)$.

Using (a)-(b) and Theorem 5.2, we obtain that $\widetilde{\Phi}_{G,\theta_2}(n) \geqslant c\phi_{\bar{\psi}}^{(2Nn)}(e)$. Then (c) gives the desired inequality, $\widetilde{\Phi}_{G,\theta_2}(n) \geqslant c\varphi^{(2Nn)}(e)$.

We are left with the task of constructing the appropriate complete Bernstein function $\psi = \psi_\alpha$, for each $\alpha \in (0,1)$. Since we want that $(\psi)^\alpha \simeq \psi_0$, the simple minded choice is to try $\psi = \psi_0^{1/\alpha}$. Unfortunately, this is not always a complete Bernstein function (because $1/\alpha > 1$). However, in the present case, $\psi_1 = \psi_0^{1/\alpha}$ has derivative $\psi_1' = \alpha^{-1}\psi_0'\psi_0^{-1+(1/\alpha)}$. Hence
$$\psi_1'(s) \sim \frac{a^{-1+(1/\alpha)}}{\alpha s \ell(1/s)\theta_2^{(1/\alpha)-1}(1/s)}.$$

Since $t \mapsto \ell(t)\theta_2^{(1/\alpha)-1}(t)$ is a continuous increasing slowly varying function, the desired complete Bernstein function ψ is provided by Theorem 2.2. $\square$

Together, Theorem 1.5 and Theorem 5.1 provide sharp results for a wide variety of regularly varying moment conditions ranging through the entire index range $[0,2)$ in the context of groups of polynomial volume growth (see [10] for sharp results regarding the special case $\alpha = 2$). The results of [2] also provide sharp result in the case $\alpha \in (0,2)$ for groups of exponential volume growth such that $\Phi_G(n) \simeq \exp(-n^{1/3})$ (this covers all polycyclic groups with exponential volume growth).

Results regarding slowly varying moment conditions for a variety of classes of groups with super-polymonial volume growth require different techniques and will be discussed elsewhere. See [11].

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