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Testing Log K-stability by blowing up formalism

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RÉSUMÉ. — Nous étudions la K-stabilité logarithmique pour des paires, étendant la formule pour les invariants de Donaldson-Futaki au contexte logarithmique. Nous développons également le versant algèbro-géométrique de résultats récents d'existence de métriques Kähler-Einstein à singularités coniques. Nous étudions notamment la relation entre la stabilité logarithmique et les seuils log canoniques globaux.

ABSTRACT. — We study logarithmic K-stability for pairs by extending the formula for Donaldson-Futaki invariants to log setting. We also provide algebro-geometric counterparts of recent results of existence of Kähler-Einstein metrics with cone singularities. In particular, we will study the relation between log K-stability and the global log canonical thresholds.

1. Introduction

One of the central issue of recent developments of Kähler geometry is on the conjectural relationship between the existence of “canonical” Kähler metrics and stability in certain sense. Along that line, K-stability is first defined by Tian [27] and later generalized by Donaldson [7].

The logarithmic K-stability with parameter $\beta \in (0, 1]$ is defined in [8] which conjecturally corresponds to the existence of Kähler-Einstein metrics with cone angle $2\pi\beta$ along the divisor on Fano manifolds.

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The purpose of this paper is to extend most of results for K-stability given in [16], [17], [18], [20], [21] to this logarithmic setting which concerns a pair (X, D) . On the way, we extend [25, Theorem 1.1] purely algebraically, allowing more general anti-canonical divisors. We also recover algebraic counterparts of [3, Theorem 1.8], [5, Theorem 1.1] and [10, Theorem 2]. This provides more evidence for the above logarithmic Yau-Tian-Donaldson conjecture.

We expect that these will have meanings even in the absolute case. On the one hand, Donaldson has recently proposed an approach of constructing Kähler-Einstein metrics on Fano manifolds by deforming Kähler-Einstein metrics with edge singularities along the anti-canonical divisor.¹

On the other hand, the minimal model program (MMP, for short) is nowadays naturally studied in log setting as it is also useful to absolute case study, giving an inductive framework on dimension based on adjunction argument. We expect and partially prove in this paper that the relation with stability and birational geometric framework based on MMP (which first appeared in [16] and developed in [18], [14] etc) fits this expectation.

We make several remarks here. First, about the pathology in [14]; they pointed out the necessity to restrict attention to only test configurations which *satisfy the S_2 condition (or normality, for normal original variety)* for the definition of K-stability. It does not violate our arguments and actually it is compatible with the framework of [17], as we explain later in section 3. Second, although we argue about a variety X with an *integral* divisor D unless otherwise stated, the following argument mostly works to give extension to the case where D can be a \mathbb{Q} -divisor.

We work over \mathbb{C} , the complex number field, although a large part of arguments in this paper works over more general fields as it is purely algebro-geometric. We will use linear equivalence class of Cartier divisor, invertible sheaf, line bundle interchangeably.

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⁽¹⁾ This paper was written in November 2011 and ever since then there has been many developments related to the logarithmic K-stability. In particular, the existence of Kähler-Einstein metrics on smooth K-stable Fano manifolds has been proved in [6] which follows the Donaldson’s idea explained above. Actually the results of this paper entered in part of the algebro-geometric discussion in [6] (as an alternative to Berman’s results [3], [4]). However, we will not discuss these applications in the current paper, and we would like to keep this paper close to its original form. The main purpose of publishing this paper after 4 years is only for the convenience of future applications by the interested readers.

ference” held at the University of Science and Technology of China, Hefei, in August, 2011. Later, the discussion developed and mostly completed when the first author visited the Imperial College London in November, 2011. Both authors are grateful to Professor Xiu-xiong Chen for invitation to the conference and nice hospitality. The first author is also grateful to Professor Simon Donaldson for making the opportunity of visiting the Imperial College London, and to both Professors Simon Donaldson and Richard Thomas for warm hospitality.

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2. Basics of discrepancy

Consult [12] for the details. Along the development of the (log) minimal model program in a few decades, some mild singularities should have been admitted on varieties in concern. On that way, the theory of discrepancy and some mild singularities’ class developed. We review these as the efficiency of that theory in the study of stability turned out in [16] and developed in [17], [18], [14] among others which we follow.

Assume (X, D) to be a pair of a normal variety X and an effective \mathbb{Q} -divisor D in this section. D is usually referred to as a boundary divisor. Let $\pi: X' \rightarrow X$ be a log resolution of D , i.e., π is a proper birational morphism such that X' is smooth and the divisor $\pi^*D + E$ has a simple normal crossing support, where E is the exceptional divisor of π . Let $K_{X'/X} := K_{X'} - \pi^*K_X$. Then, we denote

$$K_{X'} - \pi^*(K_X + D) = \sum a_i E_i,$$

where $a_i \in \mathbb{Q}$ and E_i runs over the set of divisors of X' supported on the exceptional locus or the support $\text{Supp}(\pi_*^{-1}D)$ of the strict transform of D . a_i 's are the so called *discrepancy* (of (X, D) for E_i), which measures the mildness of singularities. We usually write it as $a(E_i; (X, D)) := a_i$. We note that under the above situation, one can also consider the discrepancy of a pair $(X, D + cI)$ attached with additional coherent ideal $I \subset \mathcal{O}_X$ multiplied formally by some real number c as follows:

$$a(E_i; (X, D + cI)) := a(E_i; (X, D)) - c \cdot \text{val}_{E_i}(I)$$

where val_{E_i} means valuation of the ideal I measured by E_i . (In this case, we might get discrepancies which are only *real* numbers.) Under the above notation, we define some classes of mild singularities as follows:

DEFINITION 2.1. — *The pair (X, D) is called log canonical (lc, for short) if and only if $a_i \geq -1$ for any E_i .*

From the definition, for (X, D) to be log canonical, all the coefficients of D should be at most 1.

For a stronger notion, log terminality, we have the following version for pairs.

DEFINITION 2.2. — (i) *The pair (X, D) is called kawamata log terminal (klt, for short) if and only if $a_i > -1$ for any E_i .* (ii) *The pair (X, D) is called purely log terminal (plt, for short) if and only if $a_i > -1$ for any exceptional E_i .*

If we allow negative coefficients for D , these conditions give definitions of *sub kawamata-log-terminality* (resp. *sub log canonicity*). Note these definitions are independent of the choice of log resolution.

One advantage of considering pairs is the following inversion of adjunction, which relates mildness of singularity of pairs to that of their boundary.

THEOREM 2.3 (Inversion of adjunction [12, section 5.2], [11]). — *Assume D is decomposed as $D = D' + D''$ where D' is an effective integral reduced normal Cartier divisor and D'' is also an effective \mathbb{Q} -divisor which has no common components with D' . Then the followings hold.*

(i) *(X, D) is purely log terminal on some open neighborhood of D' if and only if $(D', D''|_{D'})$ is kawamata log terminal.*

(ii) *(X, D) is log canonical on some open neighborhood of D' if and only if $(D', D''|_{D'})$ is log canonical.*

Note that there are generalizations to the case where D' is not necessarily Cartier nor normal, for which we need to think over the normalization of D' with extra divisors involved. Consult [12, section 5.2], [11], [22, Corollary 1.2] for the details and proofs.

3. A framework to work on log Donaldson-Futaki invariants

In this section, after reviewing the definition of log Donaldson-Futaki invariants [8], we extend the framework of [17] to the logarithmic setting. The definition of logarithmic K-stability is as follows.

DEFINITION 3.1 ([8]). — *Suppose that (X, L) is a n -dimensional polarized variety, and L is an ample line bundle. Also suppose D is an effective integral reduced divisor on X . Then, a log test configuration (resp. log semi-test configuration) of $((X, D), L)$ consists of a pair of test configurations*

(resp. semi-test configurations) $(\mathcal{X}, \mathcal{L})$ for (X, L) and $(\mathcal{D}, \mathcal{L}|_{\mathcal{D}})$ for $(D, L|_D)$ with the same exponent inside $(\mathcal{X}, \mathcal{L})$ and a compatible \mathbb{G}_m -action.

From the definition, we have $\mathcal{D} = \overline{\mathbb{G}_m(D \times \mathbb{A}^1)} \subset \mathcal{X}$. For the definition of (log) Donaldson-Futaki invariants, we prepare the following notation:

- $\chi(X, L^{\otimes m}) = a_0 m^n + a_1 m^{n-1} + O(m^{n-2})$,
- Total weight of \mathbb{G}_m action on $H^0(\mathcal{X}|_{\{0\}}, \mathcal{L}|_{\{0\}}^{\otimes m})$:

$$w_m = b_0 m^{n+1} + b_1 m^n + O(n-1),$$

- $\chi(D, L|_D^{\otimes m}) = \tilde{a}_0 m^{n-1} + \tilde{a}_1 m^{n-2} + O(m^{n-3})$,
- Total weight of \mathbb{G}_m action on $H^0(\mathcal{D}|_{\{0\}}, \mathcal{L}|_{\{0\}}^{\otimes m})$:

$$\tilde{w}_m = \tilde{b}_0 m^n + \tilde{b}_1 m^{n-1} + O(m^{n-2}).$$

Here, $O(-)$ stands for the Landau symbol. Recall that the (usual) Donaldson-Futaki invariant of $(\mathcal{X}, \mathcal{L})$, $DF(\mathcal{X}, \mathcal{L})$ is defined as $2(b_0 a_1 - b_1 a_0)$. Here we deliberately add a constant 2 so that the formula in the following definitions is simplified.

DEFINITION 3.2 ([8]). — *The logarithmic Donaldson-Futaki invariant of a log test configuration $((\mathcal{X}, \mathcal{D}), \mathcal{L})$ with cone angle $2\pi\beta$ ($0 \leq \beta \leq 1$) is*

$$DF_\beta((\mathcal{X}, \mathcal{D}), \mathcal{L}) = DF(\mathcal{X}, \mathcal{L}) + (1 - \beta)(a_0 \tilde{b}_0 - b_0 \tilde{a}_0).$$

Let us call $(1 - \beta)(a_0 \tilde{b}_0 - b_0 \tilde{a}_0)$ the *boundary part*, which does not appear for the absolute case (i.e. if $\beta = 1$).

DEFINITION 3.3 ([8]). — *Assume $0 \leq \beta \leq 1$ and X satisfies Serre's S_2 condition. Recall that this S_2 condition is weaker than X being normal and is equivalent to that, for any closed subset $Z \subset X$ of codimension 2 with the open immersion $j: (X \setminus Z) \hookrightarrow X$, $j_* \mathcal{O}_{(X \setminus Z)} = \mathcal{O}_X$ (recall the Hartogs extension theorem). $((X, D), L)$ is said to be logarithmically K-stable (resp. logarithmically K-semistable) with cone angle $2\pi\beta$ if and only if $DF_\beta((\mathcal{X}, \mathcal{D}), \mathcal{L})$ is positive (resp. non-negative) for any non-trivial (log) test configurations which satisfies S_2 condition.*

$((X, D), L)$ is said to be logarithmically K-polystable with cone angle $2\pi\beta$ when it is logarithmically K-semistable and moreover $DF_\beta((\mathcal{X}, \mathcal{D}), \mathcal{L}) = 0$ for a log test configuration which satisfies the S_2 condition if and only if

geometrically (without action concerned) $(\mathcal{X}, \mathcal{D}) \cong (X, D) \times A^1$. Let us call such log test configurations, product log test configurations.

We also note that, test configurations which are trivial in codimension 2 (in the sense of [24]) are not S_2 which is the point of our S_2 assumption. Please also see the end of this section 3.

Now we recall the formalism of [17]. It is a natural extension of the pioneering [23], which treats $N = 1$ case in the notation below. Let (X, L) be the polarized varieties with the divisor $D \subset X$ in concern. First, let us recall from [17]:

DEFINITION 3.4 ([17]). — Let (X, L) be an n -dimensional polarized variety. A coherent ideal \mathcal{J} of $X \times \mathbb{A}^1$ is called a flag ideal if $\mathcal{J} = I_0 + I_1 t + \dots + I_{N-1} t^{N-1} + (t^N)$, where $I_0 \subseteq I_1 \subseteq \dots \subseteq I_{N-1} \subseteq \mathcal{O}_X$ is a sequence of coherent ideals. (It is equivalent to that the ideal is \mathbb{G}_m -invariant under the natural action of \mathbb{G}_m on $X \times \mathbb{A}^1$.)

Let us recall some notation from [17]. We set $\mathcal{L} := p_1^* L$ on $X \times \mathbb{A}^1$ and $\bar{\mathcal{L}}$ on $X \times \mathbb{P}^1$, and denote the i -th projection morphism from $X \times \mathbb{A}^1$ or $X \times \mathbb{P}^1$ by p_i . Let us write the blowing up as $\Pi: \bar{\mathcal{B}}(:= Bl_{\mathcal{J}}(X \times \mathbb{P}^1)) \rightarrow X \times \mathbb{P}^1$ or its restriction $\Pi: \mathcal{B}(:= Bl_{\mathcal{J}}(X \times \mathbb{A}^1)) \rightarrow X \times \mathbb{A}^1$, and the natural exceptional Cartier divisor as E , i.e. $\mathcal{O}(-E) = \Pi^{-1} \mathcal{J}$. Denote $Bl_{\mathcal{J}}|_{(D \times \mathbb{A}^1)}(D \times \mathbb{A}^1)$ (resp. $Bl_{\mathcal{J}}|_{(D \times \mathbb{P}^1)}(D \times \mathbb{P}^1)$) as $\mathcal{B}_{(D \times \mathbb{A}^1)}$ (resp. $\bar{\mathcal{B}}_{(D \times \mathbb{P}^1)}$). We also write $\Pi^* \mathcal{L}$ on \mathcal{B} (resp. $\Pi^* \bar{\mathcal{L}}$ on $\bar{\mathcal{B}}$) simply as \mathcal{L} (resp. $\bar{\mathcal{L}}$). Let us assume $\mathcal{L}^{\otimes r}(-E)$ on \mathcal{B} is (relatively) semi-ample (over \mathbb{A}^1) for $r \in \mathbb{Z}_{>0}$ and consider the Donaldson-Futaki invariant of the blowing up (semi) test configuration $(\mathcal{B}, \mathcal{L}^{\otimes r}(-E))$. Note that it has natural \mathbb{G}_m action as we are dealing with flag ideals i.e. \mathbb{G}_m -invariant ideal on $X \times \mathbb{A}^1$. Actually these “a priori special” semi test configurations are sufficient for the study of log K-stability.

PROPOSITION 3.5 (cf. [17, Proposition 3.8, 3.10]). — For a given test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ of (X, L) with exponent r , we can associate an flag ideal \mathcal{J} such that $(\mathcal{B} := Bl_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{B}_{(D \times \mathbb{A}^1)} := Bl_{\mathcal{J}}|_{D \times \mathbb{A}^1}(D \times \mathbb{A}^1), \mathcal{L}^{\otimes r}(-E))$ with semiample $\mathcal{L}^{\otimes r}(-E)$ with the same (log) Donaldson-Futaki invariants.

Proof. — For the (usual) absolute version, i.e., the case without boundary, this is proved in [17]. As the log Donaldson-Futaki invariant DF_{β} is the usual Donaldson-Futaki invariant plus a boundary part, it suffices to prove that boundary part of \mathcal{X} and \mathcal{B} also coincides. But this is straightforward from the construction of \mathcal{B} starting with given test configuration \mathcal{X} . About the construction, consult [17], or [15, section 2] for details. \square

COROLLARY 3.6 (cf. [17, Corollary 3.11]). — For (X, L) to be logarithmically K -stable (resp. logarithmically K -semistable) with angle $2\pi\beta$, it is necessary and sufficient to check the blow up semi test configurations of the above type, i.e., $(\mathcal{B} := \text{Bl}_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{B}_{(D \times \mathbb{A}^1)} := \text{Bl}_{\mathcal{J}|_{D \times \mathbb{A}^1}}(D \times \mathbb{A}^1), \mathcal{L}^{\otimes r}(-E))$ with \mathcal{B} Gorenstein in codimension 1 with $r \in \mathbb{Z}_{>0}$ such that $\mathcal{L}^{\otimes r}(-E)$ semi-ample.

Proof. — As in [17], recall that we can “partially normalize” \mathcal{B} as follows: take the normalization $\nu: \mathcal{B}^\nu \rightarrow \mathcal{B}$ and take $p\nu: (\mathcal{C} :=) \text{Spec}_{\mathcal{O}_{\mathcal{B}}}(i_* \mathcal{O}_{X \times (\mathbb{A}^1 \setminus \{0\})} \cap \mathcal{O}_{\mathcal{B}^\nu}) \rightarrow \mathcal{B}$, where $i: X \times (\mathbb{A}^1 \setminus \{0\}) \hookrightarrow X \times \mathbb{A}^1$ is the open immersion. Denote the pullback of exceptional Cartier divisor E on \mathcal{B} to $\mathcal{B}^{\nu\nu}$ by $E' := p\nu^*E$. Note that this $\mathcal{B}^{\nu\nu}$ is also a blow up of flag ideal $\mathcal{J}' := (\Pi \circ p\nu)_* \mathcal{O}(-lE')$ for $l \gg 0$ as well and is Gorenstein in codimension 1 by [17, Lemma 3.9].

That does not change boundary part and so this Corollary follows from [23, Proposition 5.1, Remark 5.2] which works for any (not necessarily normal) polarized varieties, although they assumed normality there. \square

Moreover, we have an explicit formula as follows. From now on, we always assume that X is an equi-dimensional reduced algebraic projective scheme, which is \mathbb{Q} -Gorenstein (i.e., K_X is \mathbb{Q} -Cartier), Gorenstein in codimension 1 (i.e., there is an open dense Gorenstein subset U with $\text{codim}(U \subset X) \geq 2$), satisfying Serre’s S_2 condition (which is weaker than normality) and D is an effective integral \mathbb{Q} -Cartier Weil divisor, unless otherwise stated.

THEOREM 3.7. — Let (X, L) , D , β and \mathcal{B} , \mathcal{J} be as above. And we assume that exponent $r = 1$. (It is just to make the formula easier. For general r , put $L^{\otimes r}$ and $\mathcal{L}^{\otimes r}$ to the place of L and \mathcal{L} .) Furthermore, we assume that \mathcal{B} is Gorenstein in codimension 1. Then the corresponding log Donaldson-Futaki invariants (multiplied by a positive constant) can be described as follows:

$$\begin{aligned} (n!)((n+1)!) \text{DF}_\beta(\mathcal{B}, \mathcal{L} - E) &= -n(L^{n-1} \cdot (K_X + (1-\beta)D))(\bar{\mathcal{L}} - E)^{n+1} \\ &+ (n+1)(L^n)((\bar{\mathcal{L}} - E)^n \cdot \Pi^*((K_X + (1-\beta)D) \times \mathbb{P}^1)) \\ &+ (n+1)(L^n)((\bar{\mathcal{L}} - E)^n \cdot (K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)})_{\text{exc}})), \end{aligned}$$

using intersection numbers on $\bar{\mathcal{B}}$ and X , where $K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)}|_{\text{exc}}$ denotes the exceptional parts of $K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)} := K_{\mathcal{B}} - \Pi^*((K_X + (1-\beta)D) \times \mathbb{A}^1)$.

Before the proof, recall our original Donaldson-Futaki invariants’ formula:

THEOREM 3.8 ([17, Theorem 3.2]). — *Let (X, L) and \mathcal{B}, \mathcal{J} be as above. Then the corresponding Donaldson-Futaki invariant $\text{DF}((\mathcal{B} = \text{Bl}_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{L}(-E)))$ can be described as follows:*

$$\begin{aligned} (n!)((n+1)!) \text{DF}((\mathcal{B}, \mathcal{L}(-E))) &= -n(L^{n-1} \cdot K_X)(\bar{\mathcal{L}}(-E))^{n+1} \\ &\quad + (n+1)(L^n)((\bar{\mathcal{L}}(-E))^n \cdot \Pi^*(p_1^* K_X)) \\ &\quad + (n+1)(L^n)((\bar{\mathcal{L}}(-E))^n \cdot K_{\mathcal{B}/(X \times \mathbb{A}^1)}), \end{aligned}$$

using intersection numbers on $\bar{\mathcal{B}}$ and X , where $K_{\mathcal{B}/(X \times \mathbb{A}^1)} := K_{\mathcal{B}} - \Pi^*(K_X \times \mathbb{A}^1)$.

Proof of Theorem 3.7. — This follows from simple calculation of the boundary part $(1 - \beta)(a_0 \tilde{b}_0 - b_0 \tilde{a}_0)$ combined with Theorem 3.8. More precisely, we can calculate as follows. $b_0 = (\bar{\mathcal{L}} - E)^{n+1}$, $\tilde{b}_0 = (\bar{\mathcal{L}} - E)|_{\bar{\mathcal{B}}_{(D \times \mathbb{P}^1)}^n}$ follows from the following fact in [17]:

Fact 3.9 ([17, formula after Lemma 3.4]). $w(m) = \chi(\bar{\mathcal{B}}, \mathcal{L}^{\otimes m}(-mE)) - \chi(X \times \mathbb{P}^1, \mathcal{L}^{\otimes m}) + O(m^{n-1})$.

(For the estimation of $\tilde{w}(m)$ and calculation of \tilde{b}_0 , simply apply the formula 3.9 to D and $\mathcal{J}|_{(D \times \mathbb{A}^1)}$ instead of X and \mathcal{J} .) $a_0 = \frac{1}{n!}(L^n)$, $\tilde{a}_0 = \frac{1}{(n-1)!}(L|_D^{(n-1)})$ follows from the weak Riemann-Roch theorem (cf. e.g. [17, Lemma 3.5]). Using these descriptions of $\tilde{b}_0, b_0, \tilde{a}_0, a_0$, we can derive our formula 3.7. We also use $\Pi^*(D \times \mathbb{P}^1) = \bar{\mathcal{B}}_{(D \times \mathbb{P}^1)} + (\Pi^*(D \times \mathbb{P}^1))_{\text{exc}}$ as \mathbb{Q} -divisors on $\bar{\mathcal{B}}$ on the way of the calculation. \square

This formula is a natural extension of the intersection formula of Donaldson-Futaki invariants given in [28], [17]. Note that we can extend our formula 3.7 to any log semi-test configurations. It is because the procedure of contracting log semi-test configuration via its semi-ample line bundle on the total space or taking blow up of flag ideal \mathcal{J} of $X \times \mathbb{A}^1$ associated to the log test configuration do not change the (log) Donaldson-Futaki invariants nor the right hand sides of the formula. (The first author talked about this, for the absolute case at CIRM, Luminy, in February of 2011.)

This framework using blow up has advantages such as, we can consider the concepts of *destabilising subschemes* and moreover, the existence of exceptional divisors helps the estimation in some situation as in section 5. In particular, the decomposition of the invariant into two parts are important and useful: “*canonical divisor part*” which means the sum of first two terms, and “*discrepancy term*” which is the last term reflecting the singularity (of the pair $(X, (1 - \beta)D)$).

It is recently explained in [14] that there are certain “pathological test configurations” \mathcal{X} , which are characterized by the following conditions. Their normalizations are trivial though themselves are *not* trivial, and Donaldson-Futaki invariants are vanishing for those. Note that such \mathcal{X} should not satisfy Serre’s S_2 condition nor normality. Thus, if we consider S_2 (or normal) test configurations as in [14], then we do not have problems.

Our arguments work as those pathological test configurations are also characterized by the condition that associated flag ideals are of the form $\mathcal{J} = t^N \mathcal{O}_{X \times \mathbb{A}^1}$ with $m \in \mathbb{Z}_{\geq 0}$, i.e., the case when blow up morphism Π is just the trivial isomorphism. In other words, the first author’s paper [17] was not accurately written in the sense he ignored the case $\mathcal{J} = (t^N)$ there. However, it works for modified K-stability which only concerns S_2 test configurations \mathcal{X} , whose corresponding flag ideal \mathcal{J} should not be of that trivial form. Y. O apologizes for this inaccuracy.

4. Log K-stability of log Calabi-Yau varieties and log canonical models

In this section we extend [18, Theorem 2.6, 2.10] as follows:

THEOREM 4.1. — (i) *Assume $(X, (1 - \beta)D)$ is a log Calabi-Yau pair, i.e., $K_X + (1 - \beta)D$ is numerically equivalent to the zero divisor and it is a semi-log-canonical pair (resp. kawamata-log-terminal pair). Then, $((X, D), L)$ is logarithmically K-semistable (resp. logarithmically K-stable) with cone angle $2\pi\beta$ for any polarization L .*

(ii) *Assume $(X, (1 - \beta)D)$ is a semi-log-canonical model, i.e., $K_X + (1 - \beta)D$ is ample and it is a semi-log-canonical pair. Then, $((X, D), K_X + (1 - \beta)D)$ and $\beta \in \mathbb{Q}_{>0}$ is log K-stable with cone angle $2\pi\beta$.*

Remark 4.2. — Theorem 4.1 (i) extends and algebraically recovers [25, Theorem 1.1], which gave a more differential geometric proof using the existence of Calabi-Yau metrics on D when D is smooth. Also it provides an algebraic counterpart of [5, Theorem 1.1] and [10, Theorem 2], where Kähler-Einstein metrics with cone angle $2\pi\beta$ are constructed on (smooth) log Calabi-Yau and (smooth) log canonical models.

Proof. — It is easy to see that the canonical divisor part vanishes for the case (i) as our log canonical divisor is zero. For the case (ii), as in [18], the canonical divisor part equals to $((\mathcal{L}^{\otimes r}(-E))^n \cdot (\mathcal{L}^{\otimes r}(nE)))$ up to positive constant, and it is proved to be positive in [18, Lemma 2.7, 2.8].

Thus, it is enough to prove that the discrepancy term is positive (resp. non-negative) if $(X, (1 - \beta)D)$ is kawamata-log-terminal (resp. semi-log-canonical). From the inversion of adjunction (Theorem 2.3), it follows that all the coefficients of $(K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)})_{\text{exc}}$ are positive (resp. non-negative) as in [18, proof of Theorems 2.6, 2.10]. Then using semiampleness of $\mathcal{L}^{\otimes r}(-E)$ and the following fact which is proved in [18], we obtain that the discrepancy term is positive (resp. non-negative).

Fact 4.3 ([18, inequality (3) in the Proof of Theorem 2.10]). — The following inequality holds in our setting: $((\mathcal{L}^{\otimes r}(-E))^n \cdot E) > 0$. □

Remark 4.4. — For the case (i), $(\mathcal{B}, \mathcal{L}^{\otimes r}(-E))$ have vanishing log Donaldson-Futaki invariant for \mathcal{J} and $r \in \mathbb{Z}_{>0}$ if and only if all the exceptional prime divisors supported on $\phi_*^{-1}(\mathcal{X}_{\{0\}})$ in \mathcal{B} have coefficients zero in $(K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)})_{\text{exc}}$, where $\phi: \mathcal{B} \rightarrow \mathcal{X} := \text{Proj} \oplus (H^0(X \times \mathbb{A}^1, \mathcal{J}^m(p_1^*L^{\otimes rm})))$ is the natural morphism defined by the semi-ample line bundle $\mathcal{L}^{\otimes r}(-E)$. This follows straightforward from our proof.

In particular, any image of such Π -exceptional prime divisor E_i with $\text{codim}(\phi_*E_i \subset \mathcal{X}) = 1$, $\Pi(E_i)$ is log canonical center of $(X \times \mathbb{A}^1, (1 - \beta)(D \times \mathbb{A}^1) + X \times \{0\})$ which have only finite candidates (cf. [1, Proposition 4.7, 4.8], [9, Theorem 2.4]). For example, if X and D are both smooth, all those E_i have $\Pi(E_i) = D \times \{0\}$.

Remark 4.5. — Concerning the finiteness of automorphism groups of polarized log pair $\text{Aut}((X, D), L) := \{\sigma \in \text{Aut}(X) \mid \sigma^*D = D, \sigma^*L \cong L\}$, as we argued in [18], [21], this follows as a special case of [2, Proposition 4.6]. On the other hand, once we know the reductivity as analogue of Matsushima’s theorem, we can prove the finiteness after Theorem 4.1. However, we allow (semi-)log-canonical singularities to Calabi-Yau pair, from which we can only deduce log *K-semistability*, we do not have finiteness of the automorphism group in general, e.g., \mathbb{P}^1 with two reduced points attached.

5. Log K-stability and alpha invariants

In this section, we extend the result of [21, Theorem 1.4] to results for \mathbb{Q} -Fano varieties with anti-canonical boundaries. On the way, we also recover an algebraic counterpart of [3, Theorem 1.8].

First recall the definition of *global log canonical threshold* (defined in algebro-geometric terms) and the *alpha invariant* (defined in analytic terms), which is known to be equivalent. The definition of the *global log canonical threshold* is the following, which we use.

DEFINITION 5.1. — Assume (X, D) is a log canonical pair with D \mathbb{Q} -Cartier, which we allow to be \mathbb{Q} -divisor in this definition. Set:

$$\text{glct}((X, D); L) := \inf_{m \in \mathbb{Z}_{>0}} \inf_{E \in |mL|} \text{lct}((X, D), \frac{1}{m}E),$$

which we call the global log canonical threshold of the pair (X, D) with respect to the polarization L . If $D = 0$, we simply write $\text{glct}(X; L)$. Here, by the definition of the usual log canonical threshold,

$$\text{lct}((X, D), \frac{1}{m}E) := \sup\{\alpha \mid (X, D + \frac{\alpha}{m}E) \text{ is log canonical}\}.$$

The definition of the *alpha invariant* is the following. It is first defined by [26] and its natural extension to log setting is also discussed in [3, section 6].

DEFINITION 5.2. — Assume X is smooth and (X, D) is a klt pair for an effective \mathbb{Q} -divisor D in this definition. Write D as $D = \sum d_i D_i$ where D_i are prime divisors locally defined by $(f_i = 0)$. Let ω be a fixed Kähler form with Kähler class $c_1(L)$. Let $P(X, \omega)$ be the set of Kähler potentials defined by

$$P(X, \omega) := \{\varphi \in C_{\mathbb{R}}^2(X) \mid \sup \varphi = 0, \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi > 0\},$$

where $C_{\mathbb{R}}^2(X)$ means a space of real valued continuous function of X of class C^2 . The definition of alpha invariant of (X, D) with respect to the polarization L is:

$$\alpha((X, D); L) = \sup\{\alpha > 0 \mid \exists C_\alpha > 0, \text{ s.t. } \int e^{-\alpha\phi} |s_D|^{-2} \omega^n \leq C_\alpha \forall \phi \in P(X, \omega)\},$$

where S_D is the defining section of D and $|s_D|^{-2}$ locally has the form $\prod_i |f_i|^{-2d_i}$.

This is independent of the choice of ω . It is known that these notions are equivalent as follows:

Fact 5.3 ([7, Appendix A], [3, section 6]). — $\text{glct}((X, D); L) = \alpha((X, D); L)$ for klt pair (X, D) with smooth X and polarization L .

It depends on the approximation theory of pluri-subharmonic functions. Consult [7, Appendix A], [3, section 6] for the details.

THEOREM 5.4. — For a \mathbb{Q} -Fano variety X (i.e. $-K_X$ ample) and anti-canonical effective reduced \mathbb{Q} -Cartier divisor D , which form a purely log terminal pair (resp. semi-log-canonical pair) (X, D) , if

$$\text{glct}((X, (1 - \beta)D); -K_X) > (\text{resp. } \geq) (n/n + 1)\beta$$

then $((X, D), -K_X)$ is logarithmically K -stable (resp. logarithmically K -semistable) with cone angle $2\pi\beta$.

We note that $\beta = 0$ case also follows from Theorem 4.1 (i). Using the original alpha invariant, we state a weaker result as follows. This corresponds to the analytic statement of [3, Theorem 1.8].

COROLLARY 5.5. — *For the above setting, we further assume that D' is irreducible (note that we already assumed reduced-ness above) and Cartier pluri-anti-canonical divisor $D' \in |-\mu K_X|$ with some $\mu \in \mathbb{Z}_{>0}$. Then, $((X, D), -K_X)$ is logarithmically K -stable (resp. logarithmically K -semistable) for cone angle $2\pi\beta$ with*

$$\frac{\mu - 1}{\mu} < \beta < \frac{\mu - 1}{\mu} + \frac{n + 1}{n\mu} \min\{\text{glct}(X; -K_X), \text{glct}(D'; -K_X|_D)\}$$

$$(\text{resp.}, \frac{\mu - 1}{\mu} \leq \beta \leq \frac{\mu - 1}{\mu} + \frac{n + 1}{n\mu} \min\{\text{glct}(X; -K_X), \text{glct}(D'; -K_X|_D)\})$$

Please be careful that $-K_X$ (resp. $-K_X|_D$) appeared in the global log canonical threshold above mean *polarization*, but not boundary divisors attached to ambient variety X (resp. D). For the proof, we follow the viewpoint introduced in [20].

Proof of Theorem 5.4. — It follows from the formula 3.7 that, by substituting $-K_X$ by L , our log Donaldson-Futaki invariant $\text{DF}_\beta(\mathcal{B}, \mathcal{L}^{\otimes r}(-E))$ is

$$-\beta(L^n)((\bar{\mathcal{L}} - E)^n \cdot \bar{\mathcal{L}}) + (L^n)((\bar{\mathcal{L}}(-E))^n \cdot (n+1)r(K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)}^{\text{exc}} - nE)), \quad (5.1)$$

where r is the exponent of the log semi-test configuration $(\mathcal{B}, \mathcal{L}^{\otimes r}(-E))$. As [21, Proposition 4.3] proved the first term is always non-negative it is enough to show that all the coefficients of exceptional prime divisor $(n+1)r(K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)}^{\text{exc}}) - nE$ is positive (resp. non-negative) under the assumption of global log canonical threshold that

$$\text{glct}((X, (1 - \beta)D), -K_X) > (n/n + 1)\beta. \quad (5.2)$$

$$(\text{resp.}, \text{glct}((X, (1 - \beta)D), -K_X) \geq (n/n + 1)\beta.)$$

To prove it, we need the following inequalities of discrepancies for any exceptional prime divisor E_i on \mathcal{B} in concern:

$$a(E_i; (X \times \mathbb{A}^1, (1 - \beta)(D \times \mathbb{A}^1) + \frac{n\beta}{r(n + 1)}\mathcal{J} + X \times \{0\}))$$

$$\begin{aligned} &\geq a(E_i; (X \times \mathbb{A}^1, (1 - \beta)(D \times \mathbb{A}^1) + \frac{n\beta}{r(n+1)}I_0 + X \times \{0\})) \\ &\geq a(E_i; (X \times \mathbb{A}^1, (1 - \beta)(D \times \mathbb{A}^1) + \frac{n\beta}{n+1}(F \times \mathbb{A}^1) + X \times \{0\})). \end{aligned}$$

Here, F is taken to be an effective anticanonical \mathbb{Q} -divisor which corresponds to an arbitrary non-zero holomorphic section of $H^0(X, I_0^m(-rmK_X))$. That vector space does not vanish for sufficiently divisible positive integer m as our assumption of semi-ampleness of $\mathcal{L}^{\otimes r}(-E)$ says $H^0(\mathcal{B}, \mathcal{L}^{\otimes rm}(-mE)) = H^0(X \times \mathbb{A}^1, \mathcal{J}^m \mathcal{L}^{\otimes rm})$ generates $\mathcal{J}^m \mathcal{L}^{\otimes rm}$ and $H^0(X, I_0^m(-rmK_X))$ is just the subspace of $H^0(X \times \mathbb{A}^1, \mathcal{J}^m \mathcal{L}^{\otimes rm})$ which is fixed by the \mathbb{G}_m -action. Note that the discrepancy on the first term and second term involve ideal (not necessarily corresponding to divisor) but recall that we can define discrepancy completely similarly in this case as we noted in subsection 2. The comparison between the first term and the second term simply follows from $I_0 \subset \mathcal{J}$ and the last inequality follows from the definition of F . Note that it is enough to show that the first term is bigger than -1 (resp. at least -1) so we only need to prove $(X, (1 - \beta)D + \frac{n\beta}{n+1}F)$ is purely log terminal (resp. log canonical) by the inversion of adjunction of log-terminality and log-canonicity (Theorem 2.3).

On the other hand, the condition (5.2) implies those. This completes the proof of Theorem 5.4. \square

Proof of Corollary 5.5. — Decompose F which appeared in the proof of Theorem 5.4 as $F = aD + F'$ with some $0 \leq a \leq 1$ such that $\text{Supp}(F')$ does not include D . Then, to see the kawamata-log-terminality (resp. log-canonicity) of $(X, (1 - \beta)D + \frac{n}{n+1}\beta F)$, it is sufficient to prove pure-log-terminality (resp. log-canonicity) of $(X, D + \frac{n}{n+1}\beta F')$. Let us assume $\mu = 1$ for simplicity. Indeed, the following argument works once we replace D by D' for $\mu > 1$ case. Note that for log terminal version, we assumed $\beta > 0$.

On the other hand, our assumptions imply the following two.

CLAIM 5.6. — (i) $(X, (\frac{n}{n+1})\beta F')$ is klt (resp. lc).

(ii) $(D, (\frac{n}{n+1})\beta F'|_D)$ is also klt (resp. lc).

Indeed, the condition (i) follows from the condition $\beta < (\frac{n+1}{n})\text{glct}(X; -K_X)$ and the condition (ii) follows from the condition $\beta < (\frac{n+1}{n})\text{glct}(D; -K_X|_D)$.

Claim 5.6 (i) implies that $(X \setminus D, (D + \frac{n}{n+1}\beta F')|_{(X \setminus D)}) = (\frac{n}{n+1}\beta F')|_{(X \setminus D)}$ is klt (resp. lc) and the second condition (ii) implies $(X, (1 - \beta)D + \frac{n}{n+1}\beta F')$

is plt (resp. lc) on an open neighborhood of D , due to the inversion of adjunction 2.3. Combining together, we obtain that $(X, (1 - \beta)D + \frac{n}{n+1}\beta F')$ is plt (resp. lc) as we wanted. \square

Remark 5.7. — In the original version of this paper, we only treated $\mu = 1$ case. For $\mu > 1$ case, the interval where the angle $2\pi\beta$ can take change due to a differential geometric reason that the angle is measured along an integral divisor D' .

Remark 5.8. — If we allow D to be not necessarily Cartier, we obtain similar results by considering pair $(D, \text{Diff}_D(0))$ and associated global log canonical thresholds, instead of those of single D . Here, $\text{Diff}_D(0)$ is a *different*, which is a divisor of D encoding the failure of adjunction (cf. e.g. [11]). Also we can extend to the case where D is not necessarily normal nor Cartier. For that case, we need to think global log canonical threshold $\text{glct}(D, -K_X|_D)$ on the normalization of D^ν with different of conductor divisor $\text{cond}(\nu)$ attached i.e. $\text{glct}((D^\nu, \text{Diff}_{D^\nu}(\text{cond}(\nu))); \nu^*(-K_X|_D))$ instead.

Remark 5.9. — Assume $(X, -K_X)$ is K-stable in the absolute sense, then if we allow $\beta > 1$ and consider logarithmic K-stability (resp. logarithmic K-semistability) in the same way as in Definitions 3.2, 3.3, $\beta < (\frac{n+1}{n})\text{glct}(X)$ (resp., $\beta \leq (\frac{n+1}{n})\text{glct}(X)$) simply implies log K-stability (resp. log K-semistability) with cone angle $2\pi\beta$. This is because sub kawamata-log-terminality (resp. sub log-canonicity) condition of $(X, (1 - \beta)D + (\frac{n}{n+1})\beta E)$ implies log K-stability (resp. log K-semistability) as in the proof of Corollary 5.5 and $(1 - \beta)D < 0$ so that we can simply ignore that term. It is interesting that this bound does not depend on D .

An easy consequences of Theorem 5.4 is

COROLLARY 5.10. — *There is no algebraic subgroup of $\text{Aut}(X, D)$ isomorphic to \mathbb{G}_m .*

Here, $\text{Aut}(X, D) := \{\sigma \in \text{Aut}(X) \mid \sigma^*(D) = D\} \subset \text{Aut}(X)$ is the automorphism group of the pair (cf. [2, Proposition 4.6]).

Proof. — If there is such a subgroup and consider one corresponding non-trivial one parameter subgroup $\lambda: \mathbb{G}_m \rightarrow \text{Aut}(X, D)$, then at least one of log Donaldson-Futaki invariants of product log test configurations coming from λ or λ^{-1} should be negative as the sum of two is zero. \square

Now given Theorem 5.4, one can define for any pair (X, D) an invariant

$$\beta(X, D) := \sup\{\beta > 0 \mid (X, D) \text{ is log K-stable with cone angle } 2\pi\beta\}.$$

It is well defined as, if we take \mathcal{J} which corresponds to maximal ideal of $(p, 0) \in D \times \{0\}$, then we have $\text{DF}_\beta < 0$ for $\beta \gg 0$ as first part of formula (5.1) $-\beta(L^n)((\tilde{\mathcal{L}}^{\otimes r}(-E)^n \cdot \mathcal{L})$ vanishes and the second part of formula (5.1) goes to $-\infty$ as $\beta \rightarrow \infty$. We have also proved that

$$\beta(X, D) \geq \left(\frac{n+1}{n} \right) \min\{\text{glct}(X; -K_X), \text{glct}(D; -K_X|_D)\}.$$

In particular, we proved $\beta(X, D)$ is a positive number.

The following corollary is a simple application of definition of log-K-stability.

COROLLARY 5.11. — *Under the same assumptions as in Theorem 5.4, the pair $((X, D), -K_X)$ is logarithmically K-stable for $\beta \in (0, \beta(X, D))$, and logarithmically K-unstable for $\beta > \beta(X, D)$.*

We expect the pair (X, D) is logarithmically K-semi-stable for $\beta = \beta(X, D)$. This fits in with the conjecture of Donaldson [8], in terms of existence of Kähler-Einstein metrics on X with cone singularities along D . More precisely, when X and D are smooth, one can also define an invariant

$$R(X, D) := \sup\{\beta > 0 \mid \exists \text{ a KE metric on } X \text{ with cone angle } 2\pi\beta \text{ along } D\}.$$

The logarithmic version of the Yau-Tian-Donaldson conjecture would suggest that $\beta(X, D) = R(X, D)$.

6. Log K-stability and semi-log-canonicity

In this section, we generalize [16, Theorem 1.1, 1.2] as follows.

THEOREM 6.1. — (i) *If a log polarized variety $((X, D), L)$ is logarithmically K-semistable with cone angle $2\pi\beta$, then $(X, (1-\beta)D)$ is semi-log-canonical pair.*

(ii) *If a log \mathbb{Q} -Fano anti-(pluri-)canonically polarized variety $((X, D), L)$ is logarithmically K-semistable with cone angle $2\pi\beta$ and $L = \mathcal{O}_X(-m(K_X + (1-\beta)D))$ with $m \in \mathbb{Z}_{>0}$, then $(X, (1-\beta)D)$ is kawamata-log-terminal pair with $\beta > 0$.*

Proof. — We prove it in completely similar way as in [16]. Assume the contrary. First, we argue for version (i). As a first step of the proof, take the semi-log-canonical model $\pi: B = \text{Bl}_I(X) \rightarrow (X, (1-\beta)D)$ of $(X, (1-\beta)D)$, which is possible by [22]. Then, all the coefficients of $(K_{B/((X, (1-\beta)D))}_{\text{exc}})$ is less than -1 by the negativity lemma (cf. [12, Lemma 3.38]). Second, if

X is normal we take $\mathcal{J} := \overline{((I + (t^l))^N)}$ where we have taken the integral closure of the ideal, l is sufficiently divisible positive integer and $N \gg 0$. If X is non-normal, as similarly as in [16, section 5] or Corollary 3.6, we first take partial normalization $\mathcal{B}^{\rho\nu}$ of $\text{Bl}_{I+(t^l)}(X \times \mathbb{A}^1)$ with sufficiently divisible $l \in \mathbb{Z}_{>0}$ and take corresponding flag ideal \mathcal{J} whose blow up is $\mathcal{B}^{\rho\nu}$.

Then, as in [16], all the coefficients of $(K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)})_{\text{exc}}$ are negative for this \mathcal{J} so that $DF_{\beta}(\mathcal{B}, \mathcal{L}^{\otimes r}(-E)) < 0$ for $r \gg 0$ by the formula 3.7. Hence, this implies logarithmic K-unstability of $((X, D), L)$ with cone angle $2\pi\beta$. This completes the proof for the general case (i).

In the case (ii), if $(X, (1-\beta)D)$ is semi-log-canonical but not klt in codimension 1 or not normal, then we can take a flag ideal \mathcal{J} with $\text{Supp}(\mathcal{O}/\mathcal{J})$ has dimension $n-1$ which is included in non-kawamata-log-terminal locus or non-normal locus, such that \mathcal{B} is Gorenstein in codimension 1 (otherwise, take partial normalization) the coefficients of an exceptional prime divisor E_i of $(K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)})_{\text{exc}}$ is 0 if $\dim(\Pi(E_i)) = n-1$. (Recall that we did similar procedure in [16, section 6]). In this case, the leading coefficient of $DF_{\beta}(\mathcal{B}, \mathcal{L}^{\otimes r}(-E))$ with respect to the variable r is $(\mathcal{L}^{n-1} \cdot E^2) < 0$.

Thus, we can assume that $(X, (1-\beta)D)$ is klt in codimension 1. Assume that it is not klt. Then, we can take non-trivial flag ideal \mathcal{J} with $(K_{\mathcal{B}/((X, (1-\beta)D) \times \mathbb{A}^1)})_{\text{exc}} = 0$ in the same way as for the case (i). On the other hand, in this log \mathbb{Q} -Fano case (ii), the canonical divisor part is always negative so that the whole log Donaldson-Futaki invariant is also negative. This completes the proof of the log \mathbb{Q} -Fano case (ii). \square

Remark 6.2. — Note that the case (ii) discussed above corresponds to log \mathbb{Q} -Fano case, which are more general than the pair we discussed in section 5 i.e., \mathbb{Q} -Fano varieties with anti-canonical boundaries.

By combining Theorem 4.1 and Theorem 6.1 (i), we get the following.

COROLLARY 6.3. — (i) *Assume $(X, (1-\beta)D)$ is a log Calabi-Yau pair, i.e., $K_X + (1-\beta)D$ is numerically equivalent to zero divisor with a polarization L . Then, $((X, D), L)$ is logarithmically K-semistable with cone angle $2\pi\beta$ if and only if $(X, (1-\beta)D)$ is a semi-log-canonical pair.*

(ii) *Assume $(X, (1-\beta)D)$ is (pluri-)log canonically polarized, i.e., $K_X + (1-\beta)D$ is ample and $L = \mathcal{O}_X(m(K_X + (1-\beta)D))$ for $m \in \mathbb{Z}_{>0}$. Then, the following three conditions are equivalent:*

- (a) $((X, D), L)$ is log K-stable with cone angle $2\pi\beta$,
- (b) $((X, D), L)$ is log K-semistable with cone angle $2\pi\beta$,
- (c) $(X, (1-\beta)D)$ is semi-log-canonical.

Bibliography

- [1] AMBRO (F.). — Quasi-log varieties, Tr. Mat. Inst. Steklova vol. 240 p. 220-239 (2003); English translation in Proc. Steklov Inst. Math. vol. 240, p. 214-233 (2003).
- [2] AMBRO (F.). — The moduli b -divisor of an lc-trivial fibration. (English summary) Compos. Math. vol. 141, p. 385-403 (2005).
- [3] BERMAN (R. J.). — A thermodynamical formalism for Monge-Ampere equations, Moser-Trudinger inequalities and Kahler-Einstein metrics. Adv. Math. 248, p. 1254-1297 (2013).
- [4] BERMAN (R.). — K-polystability of \mathbb{Q} -Fano varieties admitting Kähler-Einstein metrics, to appear in Invent. Math.
- [5] BRENDLE (S.). — Ricci-flat Kähler metrics with edge singularities, Int. Math. Res. Not. IMRN, no. 24, p. 5727-5766 (2013).
- [6] CHEN (X.-X.), DONALDSON (S.), SUN (S.). — Kähler-Einstein metrics on Fano manifolds, I, II, III. J. Amer. Math. Soc. 28, p. 183-278 (2015).
- [7] CHELTSOV (I. A.), SHRAMOV (K. A.). — Log canonical thresholds of smooth Fano threefolds, With an appendix by Jean-Pierre Demailly, English translation: Russian Mathematical Surveys, vol. 63, no. 5, p. 859-958 (2008).
- [7] DONALDSON (S.). — Scalar curvature and stability of toric varieties, J. Diff. Geom. (2002).
- [8] DONALDSON (S. K.). — Kähler metrics with cone singularities along a divisor. Essays in mathematics and its applications, 49-79, Springer, Heidelberg (2012).
- [9] FUJINO (O.). — Non-vanishing theorems for log canonical pairs, J. Alg. Geom. vol. 20, p. 771-783 (2011).
- [10] JEFFRES (T.), MAZZEO (R.), RUBINSTEIN (Y.). — Kähler-Einstein metrics with edge singularities, arxiv: 1105.5216.
- [11] KAWAKITA (M.). — Inversion of adjunction on log canonicity, Invent. Math. vol. 167, p. 129-133, (2007).
- [12] KOLLÁR (J.), MORI (S.). — Birational geometry of algebraic varieties, Cambridge University Press (1998).
- [13] LI (C.). — Remarks on logarithmic K-stability, Communications in Contemporary Mathematics Vol. 17, No. 2 (2015).
- [14] LI (C.), XU (C.). — Special test configurations and K-stability of Fano varieties, Ann. of Math. (2) 180, no. 1, p. 197-232 (2014).
- [15] MUMFORD (D.). — Stability of Projective Varieties, Enseignement Math. vol. 23 (1977).
- [16] ODAKA (Y.). — On the GIT stability of polarized varieties via Discrepancy, Ann. of Math. (2) 177, no. 2, p. 645-661 (2013).
- [17] ODAKA (Y.). — A generalization of Ross-Thomas slope theory, Osaka J. Math. 50, no. 1, p. 171-185 (2013).
- [18] ODAKA (Y.). — The Calabi conjecture and K-stability, Int. Math. Res. Not. IMRN, no. 10, p. 2272-2288 (2012).
- [20] ODAKA (Y.). — On the K-stability of $(\mathbb{Q}$ -)Fano varieties, slide of the talk at Kyoto Symposium, Complex analysis, July (2011). available at <http://www.kurims.kyoto-u.ac.jp/~yodaka/Kyoto.Hayama.pdf>
- [21] ODAKA (Y.), Y. Sano. — Alpha invariants and K-stability of \mathbb{Q} -Fano varieties, Adv. Math. 229, no. 5, p. 2818-2834 (2012).
- [22] ODAKA (Y.), XU (C.). — Log canonical models of singular pairs and its applications, Math. Res. Notices. vol. 19 (2012)

- [23] ROSS (J.), THOMAS (R.). — A study of Hilbert-Mumford criterion of stability of projective varieties, *J. Alg. Geom.* (2007).
- [24] STOPPA (J.). — A note on the definition of K-stability, arXiv:1111.5826 (2011).
- [25] SUN (S.). — Note on K-stability of pairs, *Math. Ann.* 355, no. 1, p. 259-272 (2013).
- [26] TIAN (G.). — On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, *Invent. Math.* vol. 89, p. 225-246 (1987).
- [27] TIAN (G.). — Kähler-Einstein metrics with positive scalar curvature, *Invent. Math.* vol. 130, p. 1-37, (1997).
- [28] WANG (X.). — Heights and GIT weights, *Math. Res. Notices* vol. 19 (2012).