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Monomial ideals with 3-linear resolutions

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Abstract. — In this paper, we study the Castelnuovo-Mumford regularity of square-free monomial ideals generated in degree 3. We define some operations on the clutters associated to such ideals and prove that the regularity is preserved under these operations. We apply these operations to introduce some classes of ideals with linear resolutions and also show that any clutter corresponding to a triangulation of the sphere does not have linear resolution while any proper subclutter of it has a linear resolution.

Résumé. — Dans cet article nous étudions la régularité de Castelnuovo-Mumford des idéaux engendrés par des monômes libres de carré et de degré trois. Nous définissons des opérations sur l'ensemble des clutters associés à ces idéaux et démontrons que la régularité de Castelnuovo-Mumford est conservée par ces opérations. Ces opérations nous permettent d'introduire certaines classes d'idéaux ayant une résolution linéaire. En particulier nous démontrons qu'aucun clutter correspondant à une triangulation de la sphère a une résolution linéaire, mais par contre que tout subclutter propre a une résolution linéaire.

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Article proposé par Marc Spivakovsky.

1. Introduction

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K, with standard grading, and I be a homogeneous ideal of S. Computing the Castelnuovo-Mumford regularity of I or even proving that the ideal I has a linear resolution is difficult in general. It is known that a monomial ideal has a d-linear resolution if and only if its polarization, which is a square-free monomial ideal, has a d-linear resolution. Therefore, classification of monomial ideals with linear resolution is equivalent to the classification of square-free monomial ideals with this property. In this subject, one of the fundamental results is the Eagon-Reiner theorem, which says that the Stanley-Reisner ideal of a simplicial complex has a linear resolution if and only if its Alexander dual is Cohen-Macaulay.

The problem of classifying 2-linear resolutions was completely solved by Fröberg [7] (See also [8]). An ideal of S generated by square-free monomials of degree 2 can be viewed as an edge ideal of a graph. Fröberg proved that the edge ideal of a finite simple graph G has linear resolution if and only if the complementary graph $\bar{G}$ of G is chordal, i.e., every induced cycle in G has length three. Another approach using the same ideas as in this paper is given in [9]. Also, Connon and Faridi in [3] give a necessary and sufficient combinatorial condition for a monomial ideal to have a linear resolution over fields of characteristic 2.

Clutters, a special class of hypergraphs, is another combinatorial object that can be associated to square-free monomial ideals. Let $[n] = \{1, \dots, n\}$. A clutter $\mathcal{C}$ on a vertex set $[n]$ is a set of subsets of $[n]$ (called circuits of $\mathcal{C}$) such that if e_1 and e_2 are distinct circuits, then $e_1 \nsubseteq e_2$. A d-circuit is a circuit with d vertices, and a clutter is called d-uniform if every circuit is a d-circuit. To any subset $T = \{i_1, \dots, i_t\} \subset [n]$ is associated a monomial $\mathbf{x}_T = x_{i_1} \cdots x_{i_t} \in K[x_1, ..., x_n]$. Given a clutter $\mathcal{C}$ with circuits $\{e_1, \dots, e_m\}$, the ideal generated by $\mathbf{x}_{e_j}$ for all $j = 1, \dots, m$ is called the circuit ideal of $\mathcal{C}$ and denoted by $I(\mathcal{C})$. One says that a d-uniform clutter $\mathcal{C}$ has a linear resolution if the circuit ideal of the complementary clutter $\bar{\mathcal{C}}$ has d-linear resolution. Trying to generalize Fröberg's result to d-uniform clutters ($d > 2$), several mathematicians including E. Emtander [6] and R. Woodroofe [13] have defined the notion of chordal clutters and proved that any d-uniform chordal clutter has a linear resolution. These results are one-sided. That is, there are non-chordal d-uniform clutters with a linear resolution.

In the present paper, we introduce some reduction processes on 3-uniform clutters which do not change the regularity of the ideal associated to this clutter. Then a class of 3-uniform clutters which have a linear resolution and a class of 3-uniform clutters which do not have a linear resolution are constructed.

Some of the results of this paper have been conjectured after explicit computations performed by the computer algebra systems SINGULAR [10] and CoCoA [2].

2. Preliminaries

Let K be a field, let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over K with the standard grading, and let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the unique maximal graded ideal of S.

We quote the following well-known results that will be used in this paper.

THEOREM 2.1 (Grothendieck, [11, Theorem 6.3]). — *Let M be a finitely generated S-module. Let $t = \operatorname{depth}(M)$ and $d = \dim(M)$. Then $H^i_{\mathfrak{m}}(M) \neq 0$ for $i = t$ and $i = d$, and $H^i_{\mathfrak{m}}(M) = 0$ for $i < t$ and $i > d$.*

COROLLARY 2.2. — *Let M be a finitely generated S-module. M is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M) = 0$ for $i < \dim M$.*

LEMMA 2.3. — *Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be the polynomial ring and I be an ideal in $K[y_1, \ldots, y_m]$. Then,*

$$\operatorname{depth} \frac{S}{(x_1 \cdots x_n)\, I} = \operatorname{depth} \frac{S}{I}.$$

DEFINITION 2.4 (Alexander duality). — *For a square-free monomial ideal $I = (M_1, \ldots, M_q) \subset K[x_1, \ldots, x_n]$, the Alexander dual of I, denoted $I^\vee$, is defined to be*

$$I^\vee = P_{M_1} \cap \cdots \cap P_{M_q}$$

where P_{M_i} is prime ideal generated by $\{x_j : x_j | M_i\}$.

DEFINITION 2.5. — *Let I be a non-zero homogeneous ideal of S. For every $i \in \mathbb{N}$ one defines*

$$t_i^S(I) = \max\{j : \beta^S_{i,j}(I) \neq 0\}$$

where $\beta^S_{i,j}(I)$ is the i,j-th graded Betti number of I as an S-module. The Castelnuovo-Mumford regularity of I, is given by

$$\operatorname{reg}(I) = \sup\{t_i^S(I) - i : i \in \mathbb{Z}\}.$$

We say that the ideal I has a d-linear resolution if I is generated by homogeneous polynomials of degree d and $\beta^S_{i,j}(I) = 0$ for all $j \neq i + d$ and $i \geqslant 0$. For an ideal which has a d-linear resolution, the Castelnuovo-Mumford regularity would be d.

Theorem 2.6 (Eagon-Reiner [4, Theorem 3]). — *Let I be a square-free monomial ideal in $S = K[x_1, \dots, x_n]$. I has a q-linear resolution if and only if $S/I^\vee$ is Cohen-Macaulay of dimension $n - q$.*

Theorem 2.7 ([12, Theorem 2.1]). — *Let I be a square-free monomial ideal in $S = K[x_1, \dots, x_n]$ with $\dim S/I \leqslant n - 2$. Then,*

$$\dim \frac{S}{I^\vee} - \operatorname{depth} \frac{S}{I^\vee} = \operatorname{reg}(I) - \operatorname{indeg}(I),$$

where $\operatorname{indeg}(I)$ denotes the minimum degree of generators of I.

Remark 2.8. — Let I, J be square-free monomial ideals generated by elements of degree $d \geqslant 2$ in $S = K[x_1, \dots, x_n]$. By Theorem 2.7, we have

$$\operatorname{reg}(I) = n - \operatorname{depth} \frac{S}{I^\vee}, \qquad \operatorname{reg}(J) = n - \operatorname{depth} \frac{S}{J^\vee}.$$

Therefore, $\operatorname{reg}(I) = \operatorname{reg}(J)$ if and only if $\operatorname{depth} S/I^\vee = \operatorname{depth} S/J^\vee$.

Definition 2.9 (Clutter). — *Let $[n] = \{1, \dots, n\}$. A clutter $\mathcal{C}$ on a vertex set $[n]$ is a set of subsets of $[n]$ (called circuits of $\mathcal{C}$) such that if e_1 and e_2 are distinct circuits of $\mathcal{C}$ then $e_1 \nsubseteq e_2$. A d-circuit is a circuit consisting of exactly d vertices, and a clutter is d-uniform if every circuit has exactly d vertices. To any subset $T = \{i_1, \dots, i_t\} \subset [n]$ is associated a monomial $\boldsymbol{x}_T = x_{i_1} \cdots x_{i_t} \in K[x_1, ..., x_n]$.*

For a non-empty clutter $\mathcal{C}$ on vertex set $[n]$, we define the ideal $I(\mathcal{C})$, as follows:

$$I(\mathcal{C}) = (\boldsymbol{x}_F : \quad F \in \mathcal{C})$$

and we define $I(\varnothing) = 0$.

Let n, d be positive integers and $d \leqslant n$. We define $\mathcal{C}_{n,d}$, the maximal d-uniform clutter on $[n]$ as follows:

$$\mathcal{C}_{n,d} = \{F \subset [n] : \quad |F| = d\}.$$

If $\mathcal{C}$ is a d-uniform clutter on $[n]$, we define $\bar{\mathcal{C}}$, the complement of $\mathcal{C}$, to be

$$\bar{\mathcal{C}} = \mathcal{C}_{n,d} \setminus \mathcal{C} = \{F \subset [n] : \quad |F| = d, \, F \notin \mathcal{C}\}.$$

Frequently in this paper, we take a d-uniform clutter $\mathcal{C}$ and we consider the square-free ideal $I = I(\bar{\mathcal{C}})$ in the polynomial ring $S = K[x_1, \dots, x_n]$. The ideal I is called the *circuit ideal*.

Definition 2.10 (Clique). — *Let $\mathcal{C}$ be a d-uniform clutter on $[n]$. A subset $G \subset [n]$ is called a clique in $\mathcal{C}$, if all d-subset of G belongs to $\mathcal{C}$.*

Remark 2.11. — Let $\mathcal{C}$ be a d-uniform clutter on $[n]$ and $I = I(\bar{\mathcal{C}})$ be the circuit ideal. If G is a clique in $\mathcal{C}$ and $F \in \bar{\mathcal{C}}$, then $([n] \setminus G) \cap F \neq \varnothing$. So that $\mathbf{x}_{[n]\setminus G} \in P_F$. Hence

$$\mathbf{x}_{[n]\setminus G} \in \bigcap_{F \in \bar{\mathcal{C}}} P_F = I^{\vee}.$$

Example 2.12. — It is well known that $I(\mathcal{C}_{n,d})$ has linear resolution. One way to prove it, is to show that the Alexander dual of $I(\mathcal{C}_{n,d})$ is Cohen-Macaulay by using [1, Exercise 5.1.23]. For a detailed proof we refer the reader to [5, Theorem 3.1].

Definition 2.13 (Simplicial submaximal circuit). — *Let $\mathcal{C}$ be a d-uniform clutter on $[n]$. A $(d-1)$-subset $e \subset [n]$ is called a submaximal circuit of $\mathcal{C}$ if there exists $F \in \mathcal{C}$ such that $e \subset F$. The set of all submaximal circuits of $\mathcal{C}$ is denoted by $E(\mathcal{C})$. For $e \in E(\mathcal{C})$, let $N[e] = e \cup \big\{c \in [n] \colon e \cup \{c\} \in \mathcal{C}\big\} \subset [n]$. We say that e is a simplicial submaximal circuit if $N[e]$ is a clique in $\mathcal{C}$. In case of 3-uniform clutters, $E(\mathcal{C})$ is called the edge set and we say simplicial edge instead of simplicial submaximal circuit.*

3. Operations on Clutters

In this section we introduce some operations for a clutter $\mathcal{C}$, such as changing or removing circuits, which do no change the regularity of the circuit ideal. We begin this section with the following well-known results.

Lemma 3.1. — *Let M be an R-module. For any submodules A, B, C of M such that $B \subset C$, one has*

$$(A + B) \cap C = (A \cap C) + B. \tag{3.1}$$

Theorem 3.2 (Mayer-Vietoris sequence). — *For any two ideals I_1, I_2 in the commutative Noetherian local ring $(R, \mathfrak{m})$, the short exact sequence*

$$0 \longrightarrow \frac{R}{I_1 \cap I_2} \longrightarrow \frac{R}{I_1} \oplus \frac{R}{I_2} \longrightarrow \frac{R}{I_1 + I_2} \longrightarrow 0$$

gives rise to the long exact sequence

$$\cdots \to H_{\mathfrak{m}}^{i-1}\left(\tfrac{R}{I_1+I_2}\right) \to H_{\mathfrak{m}}^{i}\left(\tfrac{R}{I_1\cap I_2}\right) \longrightarrow H_{\mathfrak{m}}^{i}\left(\tfrac{R}{I_1}\right) \oplus H_{\mathfrak{m}}^{i}\left(\tfrac{R}{I_2}\right) \longrightarrow H_{\mathfrak{m}}^{i}\left(\tfrac{R}{I_1+I_2}\right) \to$$
$$\to H_{\mathfrak{m}}^{i+1}\left(\tfrac{R}{I_1+I_2}\right) \to \cdots .$$

Lemma 3.3. — *Let I_1, I_2 be ideals in a commutative Noetherian local ring $(R, \mathfrak{m})$ such that*

$$\operatorname{depth} \tfrac{R}{I_1} \geqslant \operatorname{depth} \tfrac{R}{I_2} > \operatorname{depth} \tfrac{R}{I_1+I_2}.$$

Then, $\operatorname{depth} \frac{R}{I_1 \cap I_2} = 1 + \operatorname{depth} \frac{R}{I_1+I_2}$.

Proof. — Let $r := 1 + \operatorname{depth} R/(I_1 + I_2)$. Then, for all $i < r$,

$$H^{i-1}_{\mathfrak{m}}\left(\tfrac{R}{I_1+I_2}\right) = H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1}\right) = H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_2}\right) = 0.$$

Hence by the Mayer-Vietoris exact sequence,

$$\cdots \to H^{i-1}_{\mathfrak{m}}\left(\tfrac{R}{I_1}\right) \oplus H^{i-1}_{\mathfrak{m}}\left(\tfrac{R}{I_2}\right) \to H^{i-1}_{\mathfrak{m}}\left(\tfrac{R}{I_1+I_2}\right) \to H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1 \cap I_2}\right) \to H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1}\right) \oplus H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_2}\right) \to \cdots$$

we have $H^{i}_{\mathfrak{m}}\left(\frac{R}{I_1 \cap I_2}\right) = 0$ for all $i < r$, and $H^{r}_{\mathfrak{m}}\left(\frac{R}{I_1 \cap I_2}\right) \neq 0$. So that

$$\operatorname{depth} \frac{R}{I_1 \cap I_2} = r = 1 + \operatorname{depth} \frac{R}{I_1 + I_2}.$$

□

Lemma 3.4. — *Let I, I_1, I_2 be ideals in a commutative Noetherian local ring $(R, \mathfrak{m})$ such that $I = I_1 + I_2$ and*

$$r := \operatorname{depth} \frac{R}{I_1 \cap I_2} \leqslant \operatorname{depth} \frac{R}{I_2}.$$

Then, for all $i < r - 1$ one has

$$H^{i}_{\mathfrak{m}}\left(\frac{R}{I_1}\right) \cong H^{i}_{\mathfrak{m}}\left(\frac{R}{I}\right).$$

Proof. — For $i < r - 1$, our assumption implies that

$$H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1 \cap I_2}\right) = H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_2}\right) = H^{i+1}_{\mathfrak{m}}\left(\tfrac{R}{I_1 \cap I_2}\right) = 0.$$

Hence, from the Mayer-Vietoris exact sequence

$$\cdots \longrightarrow H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1 \cap I_2}\right) \longrightarrow H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1}\right) \oplus H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_2}\right) \longrightarrow H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I}\right) \longrightarrow H^{i+1}_{\mathfrak{m}}\left(\tfrac{R}{I_1 \cap I_2}\right) \longrightarrow \cdots.$$

we have

$$H^{i}_{\mathfrak{m}}\left(\frac{R}{I_1}\right) \cong H^{i}_{\mathfrak{m}}\left(\frac{R}{I}\right), \qquad \text{for all } i < r - 1,$$

as desired. □

Notation. — For $n > 3$, let $T_{1,n}, T'_{1,n} \subset S = K[x_1, \ldots, x_n]$ denote the ideals

$$T_{1,n} = \bigcap_{2 \leqslant i < j \leqslant n} (x_1, x_i, x_j), \qquad T'_{1,n} = \bigcap_{2 \leqslant i < j \leqslant n} (x_i, x_j).$$

Proposition 3.5. — *For $n \geqslant 3$, let $S = K[x_1, \ldots, x_n]$ be the polynomial ring. Then*

(i) $T'_{1,n} = \left(\prod_{\substack{2 \leqslant i \leqslant n \\ i \neq 2}} x_i, \ldots, \prod_{\substack{2 \leqslant i \leqslant n \\ i \neq n}} x_i \right)$ *and* $T_{1,n} = \left(x_1, \prod_{\substack{2 \leqslant i \leqslant n \\ i \neq 2}} x_i, \ldots, \prod_{\substack{2 \leqslant i \leqslant n \\ i \neq n}} x_i \right)$.

(ii) $\dfrac{S}{T'_{1,n}}$ *(res.* $\dfrac{S}{T_{1,n}}$*) is Cohen-Macaulay of dimension $n-2$ (res. $n-3$).*

Proof. — The assertion is well-known but one can find a direct proof for the primary decomposition of the Alexander dual of $T'_{1,n}$ in [8, Example 7]. □

Let $\mathcal{C}$ be a 3-uniform clutter on the vertex set $[n]$. It is clear that one can also consider $\mathcal{C}$ as a 3-uniform clutter on $[m]$ for any $m \geqslant n$. However, $\bar{\mathcal{C}}$ (and hence $I(\bar{\mathcal{C}})$) will be changed when we consider $\mathcal{C}$ either on $[n]$ or on $[m]$. To be more precise, when we pass from $[n]$ to $[n+1]$, then the new generators $\{x_{n+1}x_ix_j : 1 \leqslant i < j \leqslant n\}$ will be added to $I(\bar{\mathcal{C}})$. Below, we will show that the regularity does not change when we pass from $[n]$ to $[m]$.

Lemma 3.6. — *Let $I \subset K[x_1, \ldots, x_n]$ be a square-free monomial ideal generated in degree 3 such that $x_1x_ix_j \in I$ for all $1 < i < j \leqslant n$. If $J = I \cap K[x_2, \ldots, x_n]$, then* $\operatorname{reg}(I) = \operatorname{reg}(J)$.

Proof. — By our assumption, J is an ideal of $K[x_2, \ldots, x_n]$ and

$$I = J + (x_1x_ix_j : 1 < i < j \leqslant n).$$

It follows that $I^\vee = J^\vee \bigcap T_{1,n}$. By Remark 2.8, it is enough to show that $\operatorname{depth} S/I^\vee = \operatorname{depth} S/J^\vee$.

The ideal $J^\vee$ is intersection of some primes P, such that the set of generators of P is a subset of $\{x_2, \ldots, x_n\}$. So that for all j, $\prod_{\substack{1 < i \leqslant n-1 \\ i \neq j}} x_i \in J^\vee$.

Hence $J^\vee + T_{1,n} = (x_1, J^\vee)$ by Proposition 3.5(i). In particular

$$\operatorname{depth} \frac{S}{J^\vee + T_{1,n}} = \operatorname{depth} \frac{S}{J^\vee} - 1. \tag{3.2}$$

By Proposition 3.5 and (3.2), depth $\frac{S}{T_{1,n}} \geqslant$ depth $\frac{S}{J^\vee} >$ depth $\frac{S}{J^\vee+T_{1,n}}$. Hence by Lemma 3.3 and (3.2), we have

$$\operatorname{depth}\frac{S}{I^\vee} = 1 + \operatorname{depth}\frac{S}{J^\vee + T_{1,n}} = \operatorname{depth}\frac{S}{J^\vee}.$$

□

Sometimes in this paper, where we mention depth of an ideal, we mean the depth of the quotient ring over the ideal.

Theorem 3.7. — *Let $\mathcal{C} \neq \mathcal{C}_{n,d}$ be a d-uniform clutter on $[n]$ and e be a simplicial submaximal circuit. Let*

$$\mathcal{C}' = \mathcal{C} \setminus e = \big\{F \in \mathcal{C} \colon e \nsubseteq F\big\}$$

and $I = I(\bar{\mathcal{C}}), J = I(\bar{\mathcal{C}}')$. Then, $\operatorname{reg}(I) = \operatorname{reg}(J)$.

Proof. — By Remark 2.8, it is enough to show that depth $S/I^\vee$ = depth $S/J^\vee$. Without loss of generality, we may assume that $e = \{1, \ldots, d-1\}$ and $N[e] = \{1, \ldots, r\}$.

Since $e = \{1, \ldots, d-1\}$ is a simplicial submaximal circuit, by Remark 2.11 and Lemma 3.1, we have:

$$\begin{aligned}
I^\vee &= (x_1, \ldots, x_{d-1}, x_{r+1}\cdots x_n) \cap \Bigg(\bigcap_{\substack{F\in\bar{\mathcal{C}}\\ \{1,\ldots,d-1\}\nsubseteq F}} P_F\Bigg)\\
&= \Bigg[(x_1, \ldots, x_{d-1}) \cap \Big(\bigcap_{\substack{F\in\bar{\mathcal{C}}\\ \{1,\ldots,d-1\}\nsubseteq F}} P_F\Big)\Bigg] + (x_{r+1}\cdots x_n),\\
J^\vee &= (x_1, \ldots, x_{d-1}, x_d\cdots x_n) \cap \Bigg(\bigcap_{\substack{F\in\bar{\mathcal{C}}\\ \{1,\ldots,d-1\}\nsubseteq F}} P_F\Bigg)\\
&= \Bigg[(x_1, \ldots, x_{d-1}) \cap \Big(\bigcap_{\substack{F\in\bar{\mathcal{C}}\\ \{1,\ldots,d-1\}\nsubseteq F}} P_F\Big)\Bigg] + (x_d\cdots x_n).
\end{aligned}$$

Since

$$\begin{aligned}
&(x_1, \ldots, x_{d-1}) \cap \Bigg(\bigcap_{\substack{F\in\bar{\mathcal{C}}\\ \{1,\ldots,d-1\}\nsubseteq F}} P_F\Bigg) \cap (x_{r+1}\cdots x_n)\\
&\qquad = (x_1x_{r+1}\cdots x_n, \ldots, x_{d-1}x_{r+1}\cdots x_n),\\
&(x_1, \ldots, x_{d-1}) \cap \Bigg(\bigcap_{\substack{F\in\bar{\mathcal{C}}\\ \{1,\ldots,d-1\}\nsubseteq F}} P_F\Bigg) \cap (x_d\cdots x_n) = (x_1x_d\cdots x_n, \ldots, x_{d-1}x_d\cdots x_n)
\end{aligned}$$

have depth equal to $n-(d-1)$, by Lemma 3.4 we have:

$$H_{\mathfrak{m}}^{i}\left(\frac{S}{I^{\vee}}\right) \cong H_{\mathfrak{m}}^{i}\left(\frac{S}{(x_1,\ldots,x_{d-1})\cap(\bigcap_{\substack{F\in\bar{\mathcal{C}}\\ \{1,\ldots,d-1\}\nsubseteq F}} P_F)}\right) \cong H_{\mathfrak{m}}^{i}\left(\frac{S}{J^{\vee}}\right) \quad (3.3)$$

for all $i<n-d$.

Since $\dim S/I^{\vee} = \dim S/J^{\vee} = n-d$, the above equation implies that $\operatorname{depth} S/I^{\vee} = \operatorname{depth} S/J^{\vee}$. □

For a d-uniform clutter $\mathcal{C}$, if there exist only one circuit $F\in\mathcal{C}$ which contains the submaximal circuit $e\in E(\mathcal{C})$, then clearly e is a simplicial submaximal circuit. Hence we have the following result.

Corollary 3.8. — *Let $\mathcal{C}$ be a d-uniform clutter on $[n]$ and $I=I(\bar{\mathcal{C}})$ be the circuit ideal. If F is the only circuit containing the submaximal circuit e, then* $\operatorname{reg}(I)=\operatorname{reg}(I+\mathbf{x}_F)$.

Let $\mathcal{C}$ be 3-uniform clutter on $[n]$ such that $\{1,2,3\},\{1,2,4\},\{1,3,4\}$, $\{2,3,4\}\in\mathcal{C}$. If there exist no other circuit which contains $e=\{1,2\}$, then e is a simplicial edge. Hence by Theorem 3.7 we have the following corollary.

Theorem 3.9. — *Let $\mathcal{C}$ be 3-uniform clutter on $[n]$ and $I=I(\bar{\mathcal{C}})$ be the circuit ideal of $\bar{\mathcal{C}}$. Assume that $\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\in\mathcal{C}$ and there exist no other circuit which contains $\{1,2\}$. If $J=I+(x_1x_2x_3,x_1x_2x_4)$, then* $\operatorname{reg}(I)=\operatorname{reg}(J)$.

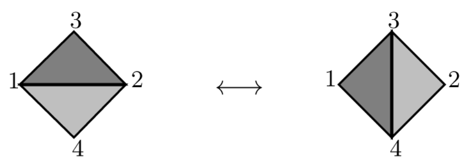
E. Emtander in [6] has introduced a generalized chordal clutter to be a d-uniform clutter, obtained inductively as follows:

- $\mathcal{C}_{n,d}$ is a generalized chordal clutter.
- If $\mathcal{G}$ is generalized chordal clutter, then so is $\mathcal{C}=\mathcal{G}\cup_{\mathcal{C}_{i,d}}\mathcal{C}_{n,d}$ for all $0\leqslant i<n$.
- If $\mathcal{G}$ is a generalized chordal clutter and $V\subset V(\mathcal{G})$ is a finite set with $|V|=d$ and at least one element of $\{F\subset V : |F|=d-1\}$ is not a subset of any element of $\mathcal{G}$, then $\mathcal{G}\cup V$ is generalized chordal.

Also R. Woodroofe in [13] has defined a simplicial vertex in a d-uniform clutter to be a vertex v such that if it belongs to two circuits e_1,e_2, then, there is another circuit in $(e_1\cup e_2)\setminus\{v\}$. He calls a clutter chordal if any minor of the clutter has a simplicial vertex.

Remark 3.10. — Let $\mathscr{C}$ be the class of 3-uniforms clutters which can be transformed to the empty set after a sequence of deletions of simplicial edges. Using Theorem 3.7, it is clear that if $\mathcal{C} \in \mathscr{C}$, then the ideal $I(\bar{\mathcal{C}})$ has a linear resolution over any field K. It is easy to see that generalized 3-uniform chordal clutters are contained in this class, so they have linear resolution over any field K. This generalizes Theorem 5.1 of [6]. It is worth mentioning that $\mathscr{C}$ strictly contains the set of generalized chordal clutters. For example, $\mathcal{C} = \{123, 124, 134, 234, 125, 126, 156, 256\}$ is in $\mathscr{C}$ but it is not a generalized chordal clutter. Also it is easy to see that any 3-uniform clutter which is chordal in the sense of [13] has simplicial edges.

DEFINITION 3.11 (Flip). — *Let $\mathcal{C}$ be 3-uniform clutter on $[n]$. Assume that $\{1,2,3\}, \{1,2,4\} \in \mathcal{C}$ are the only circuits containing $\{1,2\}$ and there is no circuit in $\mathcal{C}$ containing $\{3,4\}$. Let $\mathcal{C}' = \mathcal{C} \cup \big\{\{1,3,4\},\{2,3,4\}\big\} \setminus \big\{\{1,2,3\},\{1,2,4\}\big\}$. Then $\mathcal{C}'$ is called a flip of $\mathcal{C}$. Clearly, if $\mathcal{C}'$ is a flip of $\mathcal{C}$, then $\mathcal{C}$ is a flip of $\mathcal{C}'$ too (see the following illustration).*



$\mathcal{C}$ $\qquad\qquad$ $\mathcal{C}'$

COROLLARY 3.12. — *Let $\mathcal{C}$ be 3-uniform clutter on $[n]$ and $\mathcal{C}'$ be a flip of $\mathcal{C}$. Then,* $\operatorname{reg} I(\bar{\mathcal{C}}) = \operatorname{reg} I(\bar{\mathcal{C}}')$.

Proof. — With the same notation as in the above definition, let $\mathcal{C}'' = \mathcal{C} \cup \big\{\{1,3,4\},\{2,3,4\}\big\}$. Theorem 3.9 applied to $\{3,4\}$, shows that $\operatorname{reg} I(\bar{\mathcal{C}}'') = \operatorname{reg} I(\bar{\mathcal{C}}')$. Using Theorem 3.9 again applied to $\{1,2\}$, we conclude that $\operatorname{reg} I(\bar{\mathcal{C}}'') = \operatorname{reg} I(\bar{\mathcal{C}})$. So that $\operatorname{reg} I(\bar{\mathcal{C}}) = \operatorname{reg} I(\bar{\mathcal{C}}')$, as desired. □

For our next theorem, we use the following lemmas.

LEMMA 3.13. — *Let $n \geqslant 4, S = K[x_1, \ldots, x_n]$ be the polynomial ring and T_n be the ideal*

$$T_n = (x_4 \cdots x_n, x_1x_2x_3\, \hat{x}_4 \cdots x_n, \ldots, x_1x_2x_3\, x_4 \cdots \hat{x}_n).$$

Then, we have:

(i) $T_n = (T_{n-1} \cap (x_n)) + (x_1x_2x_3\, x_4 \cdots \hat{x}_n)$.

(ii) $\operatorname{depth} \dfrac{S}{T_n} = n - 2$.

Proof. — (i) This is an easy computation.

(ii) The proof is by induction on n. For $n = 4$, every thing is clear. Let $n > 4$ and suppose (ii) is true for $n-1$.
Clearly, $(T_{n-1} \cap (x_n)) \cap (x_1x_2x_3\ x_4 \cdots \hat{x}_n) = (x_1x_2x_3\ x_4 \cdots x_n)$, and the ring $S/(x_1x_2x_3\ x_4 \cdots x_n)$ has depth $n-1$. So by Lemma 3.4, 2.3 and the induction hypothesis, we have:

$$\operatorname{depth} \frac{S}{T_n} = \operatorname{depth} \frac{S}{T_{n-1}} = n-2.$$

□

Lemma 3.14 *Let $\mathcal{C}$ be a 3-uniform clutter on $[n]$ such that $F = \{1,2,3\} \in \mathcal{C}$ and for all $r > 3$,*

$$\big\{\{1,2,r\},\{1,3,r\},\{2,3,r\}\big\} \nsubseteq \mathcal{C}. \tag{3.4}$$

Let $\mathcal{C}_1 = \mathcal{C} \setminus F$ and $I = I(\bar{\mathcal{C}}), I_1 = I(\bar{\mathcal{C}}_1)$. Then,

(i) $\operatorname{depth} \dfrac{S}{I^\vee + (x_1,x_2,x_3)} \geqslant \operatorname{depth} \dfrac{S}{I^\vee} - 1.$

(ii) $\operatorname{depth} \dfrac{S}{I_1^\vee} \geqslant \operatorname{depth} \dfrac{S}{I^\vee}.$

Proof. — Let $t := \operatorname{depth} S/I^\vee \leqslant \dim S/I^\vee = n-3$.
(i) One can easily check that condition (3.4), is equivalent to saying that:

for all $r > 3$, there exists $F \in \bar{\mathcal{C}}$ such that $P_F \subset (x_1,x_2,x_3,x_r)$.

So that

$$\begin{aligned} I^\vee = \bigcap_{F \in \bar{\mathcal{C}}} P_F &= \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F\right) \cap ((x_1,x_2,x_3,x_4) \cap \cdots \cap (x_1,x_2,x_3,x_n)) \\ &= \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F\right) \cap (x_1,x_2,x_3,x_4 \cdots x_n) = I^\vee \cap (x_1,x_2,x_3,x_4 \cdots x_n). \end{aligned}$$

Clearly, $x_4 \cdots x_n \in I^\vee$. So, from the Mayer-Vietoris long exact sequence

$$\cdots \to H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{I^\vee}\right) \oplus H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{(x_1,x_2,x_3,x_4 \cdots x_n)}\right) \to H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{I^\vee + (x_1,x_2,x_3)}\right) \to H_{\mathfrak{m}}^{i}\left(\tfrac{S}{I^\vee}\right) \to \cdots$$

we have:

$$H_{\mathfrak{m}}^{i-1}\left(\frac{S}{I^{\vee}+(x_1,x_2,x_3)}\right)=0, \qquad \text{for all } i<t\leqslant n-3. \tag{3.5}$$

This proves inequality (i).
(ii) Clearly, $I_1^{\vee}=I^{\vee}\cap(x_1,x_2,x_3)$. So from Mayer-Vietoris long exact sequence

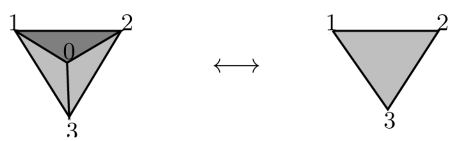
$$\cdots\to H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{I^{\vee}+(x_1,x_2,x_3)}\right)\to H_{\mathfrak{m}}^{i}\left(\tfrac{S}{I_1^{\vee}}\right)\to H_{\mathfrak{m}}^{i}\left(\tfrac{S}{I^{\vee}}\right)\oplus H_{\mathfrak{m}}^{i}\left(\tfrac{S}{(x_1,x_2,x_3)}\right)\to\cdots$$

and (3.5), we have:

$$H_{\mathfrak{m}}^{i}\left(\frac{S}{I_1^{\vee}}\right)=0, \qquad \text{for all } i<t\leqslant n-3.$$

□

Theorem 3.15. — *Let $\mathcal{C}$ be a 3-uniform clutter on $[n]$ such that $F=\{1,2,3\}\in\mathcal{C}$ and for all $r>3$, $\big\{\{1,2,r\},\{1,3,r\},\{2,3,r\}\big\}\nsubseteq\mathcal{C}$. Let $\mathcal{C}_1=\mathcal{C}\setminus F$, $\mathcal{C}'=\mathcal{C}_1\cup\big\{\{0,1,2\},\{0,1,3\},\{0,2,3\}\big\}$ and $I=I(\bar{\mathcal{C}})$, $J=I(\bar{\mathcal{C}}')$ be the circuit ideals in the polynomial ring $S=K[x_0,x_1,\ldots,x_n]$. Then,* $\operatorname{reg}(I)=\operatorname{reg}(J)$.



Proof. — By Remark 2.8, it is enough to show that $\operatorname{depth} S/I^{\vee}=\operatorname{depth} S/J^{\vee}$.

Let $I_1=I(\bar{\mathcal{C}}_1)$. Clearly, $I_1^{\vee}=(x_1,x_2,x_3)\cap I^{\vee}$ and

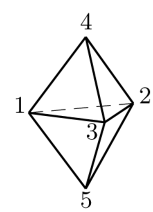
$$\begin{aligned} J^{\vee}&=I_1^{\vee}\cap\left(\bigcap_{i=4}^{n}(x_0,x_1,x_i)\right)\cap\left(\bigcap_{i=4}^{n}(x_0,x_2,x_i)\right)\cap\left(\bigcap_{3\leqslant i<j\leqslant n}(x_0,x_i,x_j)\right)\\ &=(x_0,x_4\cdots x_n,x_1x_2x_3\,\hat{x}_4\cdots x_n,\ldots,x_1x_2x_3\,x_4\cdots\hat{x}_n)\cap I_1^{\vee}.\end{aligned}$$

Let T be the ideal $T=(x_0,x_4\cdots x_n,x_1x_2x_3\,\hat{x}_4\cdots x_n,\ldots,x_1x_2x_3\,x_4\cdots\hat{x}_n)$. Then, $J^{\vee}=I_1^{\vee}\cap T$ and by Lemma 3.13, $\operatorname{depth}\frac{S}{T}=n-2$. Moreover, our assumption implies that for all $i>4$, there exists $F\in\bar{\mathcal{C}}$ such that $P_F\subset(x_1,x_2,x_3,x_r)$. So that

$$\begin{aligned} I_1^\vee + T &= (x_0, x_4 \cdots x_n, I_1^\vee) \\ &= (x_0) + \left(x_4 \cdots x_n, \left[(x_1, x_2, x_3) \cap \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F \right) \right] \right) \\ &= (x_0) + \left((x_1, x_2, x_3, x_4) \cap \cdots \cap (x_1, x_2, x_3, x_n) \cap \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F \right) \right) \\ &= (x_0) + \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F \right) = (x_0, I^\vee). \end{aligned} \tag{3.6}$$

Hence, by Lemma 3.14(ii), $\operatorname{depth} \frac{S}{I_1^\vee + T} = \operatorname{depth} \frac{S}{I^\vee} - 1 \leqslant \operatorname{depth} \frac{S}{I_1^\vee} - 1$. Thus, $\operatorname{depth} \frac{S}{T} \geqslant \operatorname{depth} \frac{S}{I_1^\vee} > \operatorname{depth} \frac{S}{I_1^\vee + T}$. Using Lemma 3.3 and (3.6), $\operatorname{depth} \frac{S}{J^\vee} = 1 + \operatorname{depth} \frac{S}{I_1^\vee + T} = \operatorname{depth} \frac{S}{I^\vee}$. □

Lemma 3.16. — *Let $\mathfrak{T}$ be a hexahedron. Then, the circuit ideal of $\overline{\mathfrak{T}}$ does not have linear resolution. If $\mathfrak{T}'$ be the hexahedron without one or more circuits, then the circuit ideal of $\overline{\mathfrak{T}'}$ has a linear resolution.*



Proof. — Let $I = I(\bar{\mathfrak{T}})$. We know that $\bar{\mathfrak{T}} = \{145, 245, 345, 123\}$. So that

$$I^\vee = (x_1x_2x_3, x_4, x_5) \cap (x_1, x_2, x_3) \subset S := K[x_1, \ldots, x_5].$$

It follows from Theorem 3.2 that $H^1_{\mathfrak{m}}\left(\frac{S}{I^\vee}\right) \neq 0$. Since $\dim S/I^\vee = 5 - 3 = 2$, we conclude that $S/I^\vee$ is not Cohen-Macaulay. So that the ideal I does not have linear resolution by Theorem 2.6.
The second part of the theorem, is a direct conclusion of Theorem 3.8. □

Let $\mathcal{S}^2$ be a sphere in $\mathbb{R}^3$. A *triangulation* of $\mathcal{S}^2$ is a finite simple graph embedded on $\mathcal{S}^2$ such that each face is triangular and any two faces share at most one edge. Note that if $\mathcal{C}$ is a triangulation of a surface, then $\mathcal{C}$ defines a 3-uniform clutter which we denote this again by $\mathcal{C}$. Moreover, any proper subclutter $\mathcal{C}' \subset \mathcal{C}$ has an edge $e \in E(\mathcal{C}')$ such that e is contained in only one circuit of $\mathcal{C}'$.

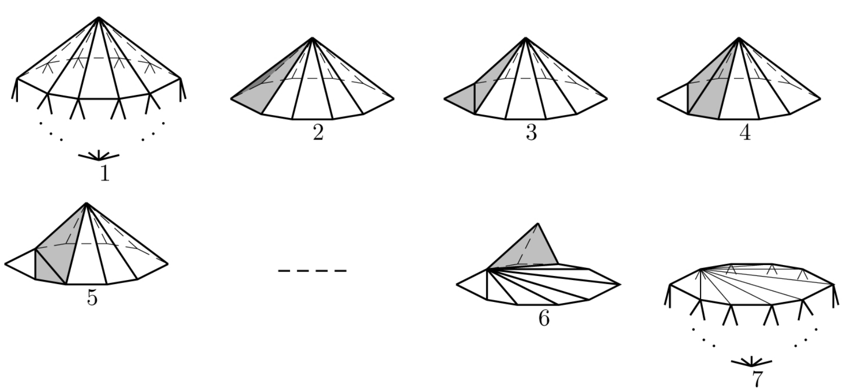
Corollary 3.17 *Let $S = K[x_1, \dots, x_n]$. Let $\mathfrak{P}_n$ be the clutter defined by a triangulation of the sphere with $n \geqslant 5$ vertices, and let $I \subset S$ be the circuit ideal of $\overline{\mathfrak{P}}_n$. Then,*

(i) *For any proper subclutter $\mathcal{C}_1 \subset \mathfrak{P}_n$, the ideal $I(\bar{\mathcal{C}}_1)$ has a linear resolution.*

(ii) *S/I does not have linear resolution.*

Proof. — (i) If $\mathcal{C}_1$ is a proper subclutter of $\mathfrak{P}_n$, then $\mathcal{C}_1$ has an edge e such that e is contained in only one circuit of $\mathcal{C}_1$ and can be deleted without changing the regularity by Corollary 3.8. Continuing this process proves the assertion.

(ii) The proof is by induction on n, the number of vertices. First step of induction is Lemma 3.16. Let $n > 5$. If there is a vertex of degree 3 (the number of edges passing through the vertex is 3), then by Theorem 3.15, we can remove the vertex and three circuits containing it and add a new circuit instead. Then, we have a clutter with fewer vertices and by the induction hypothesis, S/I does not have linear resolution. Now, assume that there are no vertices of degree three, and take a vertex u of degree > 3 and all circuits containing u (see the following illustrations). Using several flips and Corollary 3.12, we can reduce our triangulation to another one such that there are only 3 circuits containing u. Now, using Theorem 3.15, we get a triangulation of the sphere with $n - 1$ vertices which does not have linear resolution by the induction hypothesis.



□

Remark 3.18. — Let $\mathfrak{P}_n$ be the 3-uniform clutter as in Corollary 3.17. Let I be the circuit ideal of $\bar{\mathfrak{P}}_n$ and Δ be a simplicial complex such that the Stanley-Reisner ideal of Δ is I. In this case, $\Delta^\vee$, the Alexander dual of Δ, is a pure simplicial complex of dimension $n-4$ which is not Cohen-Macaulay, but adding any new facet to $\Delta^\vee$ makes it Cohen-Macaulay.

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