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MASAHIKO YOSHINAGA

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Freeness of hyperplane arrangements and related topics

MASAHIKO YOSHINAGA⁽¹⁾

ABSTRACT. — These are the expanded notes of the lecture by the author in “Arrangements in Pyrénées”, June 2012. We are discussing relations of freeness and splitting problems of vector bundles, several techniques proving freeness of hyperplane arrangements, K. Saito’s theory of primitive derivations for Coxeter arrangements, their application to combinatorial problems and related conjectures.

RÉSUMÉ. — Cet article est un développement des notes de l’exposé donnée par l’auteur à la conférence « Arrangements en Pyrénées », en juin 2012. Nous discutons les relations entre les problèmes de liberté et ceux de décomposabilité pour les fibrés vectoriels, plusieurs techniques qui prouvent la liberté pour des arrangements d’hyperplans, la théorie de K. Saito des dérivations primitives pour les arrangements de Coxeter, leur application à des problèmes combinatoires et quelques conjectures liées.

0. Introduction

Roughly speaking, there are two kind of objects in mathematics: general objects and specialized objects. In the study of general objects, individual objects are not so important, the totality of general objects is rather interesting (e.g. stable algebraic curves and moduli spaces). On the other hand, specialized objects are isolated, tend to be studied individually.

Let us fix a manifold (algebraic, complex analytic, whatever) X . Then the divisors on X are general objects. In 1970’s Kyoji Saito [24] introduced

⁽¹⁾ Department of Mathematics, Hokkaido University, N 10 W 8 Sapporo 060-0810, Japan
yoshinaga@math.sci.hokudai.ac.jp

the notion of *free divisors* with the motivation to compute Gauss-Manin connections for universal unfolding of isolated singularities. It was proved that the discriminant in the parameter space of the universal unfolding is a free divisor. Free divisors are specialized objects. The discriminant for a simple singularity is obtained as a quotient of the union of hyperplanes of finite reflection group, which implies that the union of reflecting hyperplanes (Coxeter arrangement) is also a free divisor (free arrangement). He also studied Coxeter arrangements in terms of invariant theory and found deep structures related to freeness. This has made deep impact on combinatorics of Coxeter arrangements (which is summarized in §2). Subsequently Terao developed basic techniques and laid the foundations of the theory of free arrangements. Now this becomes a rich area which is related to combinatorics and algebraic geometry.

The purpose of this article is to survey the aspects of free arrangements. §1 is devoted to describe techniques proving freeness. In early days, the freeness of arrangements was studied mainly from algebraic and combinatorial view point. It was pointed out by Silvotti [32] and Schenck [28, 19] that the freeness is equivalent to splitting of a reflexive sheaf on the projective space \mathbb{P}^n into sum of line bundles (“splitting problem”). This point of view has been a source of ideas of several recent studies on freeness of arrangements. The importance of multiarrangements emerged in these researches, and the general theory of free multiarrangements has been developed in the last decade. We are trying to depict them in §1.

In §2, we summarize the theory of primitive derivation for Coxeter arrangements. I recommend [27] for full details.

§3 is devoted to the problems concerning truncated affine Weyl arrangements. As an application of results in previous sections, it is proved that the so-called extended Catalan and extended Shi arrangements are free, which has also combinatorial consequences via Terao’s factorization theorem [39] and Solomon-Terao’s formula [34]. This settled the conjecture by Edelman and Reiner [12]. We also try to convince that some open problems (including “Riemann Hypothesis” by Postnikov and Stanley [23]) seems to be related to the algebraic structures studied in §2.

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1. Splitting v.s. Freeness

In this section, we discuss relations of freeness of divisors (especially hyperplane arrangements) and splitting problems of vector bundles. We emphasize parallelism and subtle differences.

1.1. Splitting problems

First let us recall the correspondence between graded modules and coherent sheaves on the projective space. (Basic reference is [17, Chap II §5]) Let $S = \mathbb{C}[x_1, x_2, \dots, x_\ell]$ be the polynomial ring and $\mathbb{P}_{\mathbb{C}}^{\ell-1} = \text{Proj} S$ the projective $(\ell - 1)$ -space (denote $\mathbb{P}^{\ell-1}$ for simplicity). $\mathbb{P}^{\ell-1}$ is covered by open subsets U_{x_i} ($i = 1, \dots, \ell$), where U_{x_i} is an open subset defined by $\{x_i \neq 0\}$. Let M be a graded S -module. Then M induces a sheaf \widetilde{M} on $\mathbb{P}^{\ell-1}$, with sections

$$\Gamma(U_{x_i}, \widetilde{M}) = (M_{x_i})_0,$$

where $M_{x_i} = M \otimes_S S[\frac{1}{x_i}]$ is the localization by x_i and $(-)_d$ denotes the degree d component of the graded module. For $k \in \mathbb{Z}$, define the graded module $M[k]$ by shifting degrees by k , namely, $M[k]_d = M_{d+k}$. Denote $\mathcal{O} = \widetilde{S}$. The sheaf $\widetilde{S[k]}$ is a rank one module over \mathcal{O} , which is denoted by $\mathcal{O}(k)$.

Using the natural map $\Gamma(\mathbb{P}^{\ell-1}, \mathcal{E}) \times \Gamma(\mathbb{P}^{\ell-1}, \mathcal{F}) \longrightarrow \Gamma(\mathbb{P}^{\ell-1}, \mathcal{E} \otimes \mathcal{F})$, we can define a graded ring structure on $\Gamma_*(\mathcal{O}) := \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathbb{P}^{\ell-1}, \mathcal{O}(d))$, which is isomorphic to S . More generally, for any sheaf (\mathcal{O} -module) \mathcal{F} on $\mathbb{P}^{\ell-1}$,

$$\Gamma_*(\mathcal{F}) := \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathbb{P}^{\ell-1}, \mathcal{F} \otimes \mathcal{O}(d))$$

has a graded S -module structure. For a graded S -module M , $\Gamma_*(\widetilde{M})$ is expressed as

$$\begin{aligned} \Gamma_*(\widetilde{M}) &= \{(f_1, \dots, f_\ell) \mid f_i \in M_{x_i}, f_i = f_j \text{ in } M_{x_i x_j}\} \\ &= \{f \in M_{x_1 x_2 \dots x_\ell} \mid \exists N \gg 0, x_i^N f \in M, \forall i = 1, \dots, \ell\}. \end{aligned}$$

Hence there is a natural homomorphism $\alpha : M \longrightarrow \Gamma_*(\widetilde{M})$. The above map α is not necessarily isomorphic.

DEFINITION 1.1. — A sheaf of \mathcal{O} -modules \mathcal{F} on \mathbb{P}^n is said to be *splitting* if there exist integers $d_1, \dots, d_r \in \mathbb{Z}$ such that

$$\mathcal{F} \simeq \mathcal{O}(d_1) \oplus \dots \oplus \mathcal{O}(d_r).$$

(Note that if we pose $d_1 \geq d_2 \geq \dots \geq d_r$, the degrees are uniquely determined.)

Let \mathcal{E} be an \mathcal{O} -module. Denote by $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O})$ the dual module of \mathcal{E} . An \mathcal{O} -module \mathcal{E} is called *reflexive* if the natural map $\mathcal{E} \rightarrow \mathcal{E}^{\vee\vee}$ is an isomorphism. \mathcal{E} is called a *vector bundle* if it is locally free.

A torsion free \mathcal{O} -module on \mathbb{P}^1 is always splitting.

THEOREM 1.2 (*Grothendieck's splitting theorem*). — *Let \mathcal{E} be a vector bundle on \mathbb{P}^1 . Then \mathcal{E} is splitting.*

A vector bundle \mathcal{E} on \mathbb{P}^n , with $n \geq 2$ is non-splitting in general. For example, the tangent bundle $T_{\mathbb{P}^n}$ is irreducible rank n vector bundle on \mathbb{P}^n for $n \geq 2$, i.e. not a sum of proper sub-bundles.

Let \mathcal{E} be a torsion free sheaf. Let H be a hyperplane defined by a linear form α . Since $\alpha \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$, we have the following short exact sequence

$$0 \rightarrow \mathcal{E}(-1) \xrightarrow{\alpha} \mathcal{E} \rightarrow \mathcal{E}|_H \rightarrow 0. \quad (1.1)$$

The short exact sequence (1.1) plays a crucial role in splitting problems.

Let \mathcal{E} be a rank r vector bundle on \mathbb{P}^n . Then $\det \mathcal{E} := \bigwedge^r \mathcal{E}$ is a line bundle and is called the *determinant bundle*. The *first Chern number* of \mathcal{E} is the integer $c_1 \in \mathbb{Z}$ satisfying $\det \mathcal{E} = \mathcal{O}(c_1)$.

PROPOSITION 1.3. — *Let \mathcal{E} be a rank r vector bundle on \mathbb{P}^n with $n \geq 2$.*

- (i) *Let $\delta_i \in \Gamma(\mathcal{E} \otimes \mathcal{O}(-d_i))$ for certain $d_i \in \mathbb{Z}$, $i = 1, \dots, r$. Assume that $\delta_1, \dots, \delta_r$ are linearly independent over rational function field (or equivalently, $\delta_1 \wedge \dots \wedge \delta_r \in \Gamma(\det \mathcal{E} \otimes \mathcal{O}(-d_1 - \dots - d_r))$ is non-zero) and $\sum_{i=1}^r d_i = c_1(\mathcal{E})$. Then \mathcal{E} is splitting and $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$.*
- (ii) *Let $H \subset \mathbb{P}^n$ be a hyperplane. If the restriction $\mathcal{E}|_H$ to H is splitting and the induced map*

$$\Gamma_*(\mathcal{E}) \rightarrow \Gamma_*(\mathcal{E}|_H)$$

is surjective, then \mathcal{E} is also splitting.

Proof. — (i) Let $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$. Then $(\delta_1, \dots, \delta_r)$ determines a homomorphism

$$\mathcal{F} \rightarrow \mathcal{E} : (f_1, \dots, f_r) \mapsto f_1\delta_1 + \dots + f_r\delta_r.$$

The Jacobian of this map is an element of $\Gamma(\mathcal{H}om(\mathcal{O}(d_1 + \dots + d_r), \mathcal{O}(c_1))) = \Gamma(\mathcal{O}) = \mathbb{C}$. By assumption, the Jacobian is nowhere vanishing, hence $\mathcal{F} \simeq \mathcal{E}$.

(ii) Suppose that $\mathcal{E}|_H = \bigoplus_{i=1}^r F_i$ and $F_i \simeq \mathcal{O}_H(d_i)$. Then by the surjectivity assumption, there is $\delta_i \in \Gamma(\mathcal{E} \otimes \mathcal{O}(-d_i))$ such that $\delta_i|_H$ is a nowhere vanishing section of $\Gamma(H, \mathcal{F}_i \otimes \mathcal{O}(-d_i)) = \mathbb{C}$. Since $\delta_1|_H, \dots, \delta_r|_H$ are linearly independent, so are $\delta_1, \dots, \delta_r$. Then by (i), \mathcal{E} is also splitting. \square

Remark 1.4. — Comments to those who are already familiar with free arrangements: (i) and (ii) in Proposition 1.3 are analogies of Saito’s and Ziegler’s criteria respectively. See Theorem 1.14 and Corollary 1.35.

Here we present some criteria for splitting.

THEOREM 1.5. — *Let \mathcal{F} be a vector bundle on \mathbb{P}^n .*

- (1) (Horrocks) *Assume that $n \geq 2$. \mathcal{F} is splitting $\iff H^i(\mathcal{F}(d)) = 0$, for any $0 < i < n$ and $d \in \mathbb{Z} \iff H^1(\mathcal{F}(d)) = 0$, for any $d \in \mathbb{Z}$.*
- (2) (Horrocks) *Assume that $n \geq 3$. Fix a hyperplane $H \subset \mathbb{P}^n$. Then \mathcal{F} is splitting $\iff \mathcal{F}|_H$ is splitting.*
- (3) (Elencwajg-Forster, [14]) *Assume that $n \geq 2$. Let $L \subset \mathbb{P}^n$ be a generic line and set $\mathcal{E}|_L = \mathcal{O}_L(d_1) \oplus \dots \oplus \mathcal{O}_L(d_r)$. Then*

$$c_2(\mathcal{E}) \geq \sum_{i < j} d_i d_j,$$

and \mathcal{E} is splitting if and only if the equality holds.

Proof. — Here we give the proof for (1). The direction \implies is well-known ($H^i(\mathbb{P}^n, \mathcal{O}(d)) = 0$, for $0 < i < n$ and $\forall d \in \mathbb{Z}$). Let us assume that $H^1(\mathbb{P}^n, \mathcal{F}(d)) = 0$ for $d \in \mathbb{Z}$. Let us first consider the case $n = 2$. By Grothendieck’s splitting theorem, $\mathcal{F}|_H$ is splitting. By the long exact sequence associated with (1.1), we have the surjectivity of $\Gamma_*(\mathcal{F}) \rightarrow \Gamma_*(\mathcal{F}|_H)$. Hence by Proposition 1.3 (i), \mathcal{F} is splitting. For $n \geq 3$, it is proved by induction. \square

Remark 1.6. — Horrocks’ restriction criterion (2) is generalized to reflexive sheaves ([4]). Later we will give a refinement of (3) for $n = r = 2$ (see Theorem 1.45).

1.2. Basics of arrangements

Let V be an ℓ -dimensional vector space. A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \dots, H_n\}$ is called a *hyperplane arrangement*. For each hyperplane H_i we fix a defining equation α_i such that $H_i = \alpha_i^{-1}(0)$. An arrangement

\mathcal{A} is called *central* if each H_i passes the origin $0 \in V$. In this case, the defining equation $\alpha_i \in V^*$ is linear homogeneous. Let $L(\mathcal{A})$ be the set of non-empty intersections of elements of \mathcal{A} . Define a partial order on $L(\mathcal{A})$ by $X \leq Y \iff Y \subseteq X$ for $X, Y \in L(\mathcal{A})$. Note that this is reverse inclusion.

Define a *rank function* on $L(\mathcal{A})$ by $r(X) = \text{codim} X$. Denote $L^p(\mathcal{A}) = \{X \in L(\mathcal{A}) \mid r(X) = p\}$. We call \mathcal{A} *essential* if $L^\ell(\mathcal{A}) \neq \emptyset$.

Let $\mu : L(\mathcal{A}) \rightarrow \mathbb{Z}$ be the *Möbius function* of $L(\mathcal{A})$ defined by

$$\mu(X) = \begin{cases} 1 & \text{for } X = V \\ -\sum_{Y < X} \mu(Y), & \text{for } X > V. \end{cases}$$

The *characteristic polynomial* of \mathcal{A} is $\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}$. The characteristic polynomial is characterized by the following recursive relations.

PROPOSITION 1.7. — *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in V . Let $\mathcal{A}' = \{H_1, \dots, H_{n-1}\}$ and $\mathcal{A}'' = H_n \cap \mathcal{A}'$ the induced arrangement on H_n . Then*

- *in case \mathcal{A} is empty, $\chi(\emptyset, t) = t^{\dim V}$, and*
- $\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t)$.

We also define the i -th Betti number $b_i(\mathcal{A})$ ($i = 1, \dots, \ell$) by the formula

$$\chi(\mathcal{A}, t) = \sum_{i=0}^{\ell} (-1)^i b_i(\mathcal{A}) t^{\ell-i}.$$

This naming and the importance of the characteristic polynomial in combinatorics would be justified by the following result.

THEOREM 1.8. — (1) *If \mathcal{A} is an arrangement in \mathbb{F}_q^ℓ (vector space over a finite field \mathbb{F}_q), then $|\mathbb{F}_q^\ell \setminus \bigcup_{H \in \mathcal{A}} H| = \chi(\mathcal{A}, q)$.*

(2) *If \mathcal{A} is an arrangement in \mathbb{C}^ℓ , then the topological i -th Betti number of the complement is $b_i(\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H) = b_i(\mathcal{A})$.*

(3) *If \mathcal{A} is an arrangement in \mathbb{R}^ℓ , then $|\chi(\mathcal{A}, -1)|$ is the number of chambers and $|\chi(\mathcal{A}, 1)|$ is the number of bounded chambers.*

(1) of the above theorem can be used for the computation of $\chi(\mathcal{A}, t)$ for \mathcal{A} defined over \mathbb{Q} . It is sometimes called “Finite field method”. Athanasiadis pointed out that we may drop the assumption “field”.

THEOREM 1.9. — *Let \mathcal{A} be a hyperplane arrangement such that each $H \in \mathcal{A}$ is defined by a linear form α_H of \mathbb{Z} -coefficients. For a positive integer $m > 0$, consider $\overline{H} = \{x \in (\mathbb{Z}/m\mathbb{Z})^\ell \mid \alpha_H(x) \equiv 0m\}$. There exists a positive integer N which depends only on \mathcal{A} such that if $m > N$ and m is coprime to N , then*

$$|(\mathbb{Z}/m\mathbb{Z})^n \setminus \bigcup_{H \in \mathcal{A}} \overline{H}| = \chi(\mathcal{A}, m).$$

Athanasiadis systematically used this result to compute characteristic polynomials. (See [8, 9].)

1.3. Basics of free arrangements

Let $V = \mathbb{C}^\ell$ be a complex vector space with coordinate (x_1, \dots, x_ℓ) , $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central hyperplane arrangement, namely, $0 \in H_i$ for all $i = 1, \dots, n$. We denote by $\text{Der}_V = \bigoplus_{i=1}^\ell S \frac{\partial}{\partial x_i}$ the set of polynomial vector fields on V (or S -derivations) and by $\Omega_V^p = \bigoplus_{i_1 < \dots < i_p} S dx_{i_1} \wedge \dots \wedge dx_{i_p}$ the set of polynomial differential p -forms.

DEFINITION 1.10. — *Let $\theta = \sum_{i=1}^\ell f_i \partial_{x_i}$ be a polynomial vector field. θ is said to be homogeneous of polynomial degree d when f_1, \dots, f_ℓ are homogeneous polynomial of degree d . It is denoted by $\text{pdeg } \theta = d$.*

Remark 1.11. — *Usually the degree of θ is considered to be $\deg f_i - 1$ which is one less than $\text{pdeg } \theta$. To avoid confusion, we use the notation $\text{pdeg } \theta$.*

Let us denote by $S = S(V^*) = \mathbb{C}[x_1, \dots, x_\ell]$ the polynomial ring and fix $\alpha_i \in V^*$ a defining equation of H_i , i.e., $H_i = \alpha_i^{-1}(0)$.

DEFINITION 1.12. — *A multiarrangement is a pair $(\mathcal{A}, \mathbf{m})$ of an arrangement \mathcal{A} with a map $\mathbf{m} : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$, called the multiplicity.*

An arrangement \mathcal{A} can be identified with a multiarrangement with constant multiplicity $m \equiv 1$, which is sometimes called a *simple arrangement*. With this notation, the main object is the following module of S -derivations which has contact to each hyperplane of order m . We also put $Q = Q(\mathcal{A}, \mathbf{m}) = \prod_{i=1}^n \alpha_i^{\mathbf{m}(H_i)}$ and $|\mathbf{m}| = \sum_{H \in \mathcal{A}} \mathbf{m}(H)$.

DEFINITION 1.13. — *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement, and define the module of vector fields logarithmic tangent to \mathcal{A} with multiplicity \mathbf{m} (logarithmic vector field) by*

$$D(\mathcal{A}, \mathbf{m}) = \{\delta \in \text{Der}_V \mid \delta \alpha_i \in (\alpha_i)^{\mathbf{m}(H_i)}, \forall i\},$$

and differential forms with logarithmic poles along \mathcal{A} (logarithmic forms) by

$$\Omega^p(\mathcal{A}, \mathbf{m}) = \left\{ \omega \in \frac{1}{Q} \Omega_V^p \mid d\alpha_i \wedge \omega \text{ does not have pole along } H_i, \forall i \right\}.$$

The module $D(\mathcal{A}, \mathbf{m})$ is obviously a graded S -module. It is proved in [24] that $D(\mathcal{A}, \mathbf{m})$ and $\Omega^1(\mathcal{A}, \mathbf{m})$ are dual modules to each other. Therefore, they are reflexive modules. A multiarrangement $(\mathcal{A}, \mathbf{m})$ is said to be *free* with exponents (e_1, \dots, e_ℓ) if and only if $D(\mathcal{A}, \mathbf{m})$ is an S -free module and there exists a basis $\delta_1, \dots, \delta_\ell \in D(\mathcal{A}, \mathbf{m})$ such that $\text{pdeg } \delta_i = e_i$. When $\mathbf{m} \equiv 1$, $D(\mathcal{A}, 1)$ and $\Omega^p(\mathcal{A}, 1)$ is denoted by $D(\mathcal{A})$ and $\Omega^p(\mathcal{A})$ for simplicity. An arrangement \mathcal{A} is said to be free if $(\mathcal{A}, 1)$ is free. The Euler vector field $\theta_E = \sum_{i=1}^{\ell} x_i \partial_i$ is always contained in $D(\mathcal{A})$ for simple case.

Let $\delta_1, \dots, \delta_\ell \in D(\mathcal{A}, \mathbf{m})$. Then $\delta_1 \wedge \dots \wedge \delta_\ell$ is divisible by $Q(\mathcal{A}, \mathbf{m}) \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_\ell}$. The determinant of coefficient matrix of $\delta_1, \dots, \delta_\ell$ can be used to characterize freeness.

THEOREM 1.14 (*Saito's criterion, [24]*). — *Let $\delta_1, \dots, \delta_\ell \in D(\mathcal{A}, \mathbf{m})$. Then the following are equivalent:*

- (i) $D(\mathcal{A}, \mathbf{m})$ is free with basis $\delta_1, \dots, \delta_\ell$, i. e., $D(\mathcal{A}, \mathbf{m}) = S \cdot \delta_1 \oplus \dots \oplus S \cdot \delta_\ell$.
- (ii) $\delta_1 \wedge \dots \wedge \delta_\ell = c \cdot Q(\mathcal{A}, \mathbf{m}) \cdot \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_\ell}$, where $c \in \mathbb{C}^*$.
- (iii) $\delta_1, \dots, \delta_\ell$ are linearly independent over S and $\sum_{i=1}^{\ell} \text{pdeg } \delta_i = |\mathbf{m}| = \sum_{H \in \mathcal{A}} \mathbf{m}(H)$.

From Saito's criterion, we also obtain that if a multiarrangement $(\mathcal{A}, \mathbf{m})$ is free with exponents (e_1, \dots, e_ℓ) , then $|\mathbf{m}| = \sum_{i=1}^{\ell} e_i$.

PROPOSITION 1.15. — *If \mathcal{A} is free, then \mathcal{A} is locally free, i.e., $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}$ is free for any $X \in L(\mathcal{A})$, $X \neq 0$.*

For simple arrangement case, there is a good connection of these modules with the characteristic polynomial. The following result shows that the graded module structure of $D(\mathcal{A})$ determines the characteristic polynomial $\chi(\mathcal{A}, t)$.

THEOREM 1.16 (*Solomon-Terao's formula [34]*). — *Denote by $\text{Hilb}(\Omega^p(\mathcal{A}), x) \in \mathbb{Z}[[x]][x^{-1}]$ the Hilbert series of the graded module $\Omega^p(\mathcal{A})$. Define*

$$\Phi(\mathcal{A}; x, y) = \sum_{p=0}^{\ell} \text{Hilb}(\Omega^p(\mathcal{A}), x) y^p. \tag{1.2}$$

Then

$$\chi(\mathcal{A}, t) = \lim_{x \rightarrow 1} \Phi(\mathcal{A}; x, t(1-x) - 1). \quad (1.3)$$

In particular, for free arrangements, we have the following beautiful formula, which is known as Terao's factorization theorem.

THEOREM 1.17 ([39]). — *Suppose that \mathcal{A} is a free arrangement with exponents (e_1, \dots, e_ℓ) . Then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - e_i). \quad (1.4)$$

Remark 1.18. — *There is a notion of the characteristic polynomial of a multiarrangement $(\mathcal{A}, \mathbf{m})$ [3]. However it can not be defined combinatorially, rather by the Solomon-Terao's formula for $\Omega^p(\mathcal{A}, \mathbf{m})$.*

Example 1.19. — (Braid arrangement or A_{n-1} -type arrangement) Let $H_{ij} = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i = x_j\}$. Consider the arrangement $\mathcal{A} = \{H_{ij} \mid 1 \leq i < j \leq n\}$. In other words $Q(\mathcal{A}) = \prod_{i < j} (x_i - x_j)$.

The characteristic polynomial of this arrangement is easily computed by the finite field method. For the complement with $\otimes \mathbb{F}_q$ is expressed as

$$\{(x_1, \dots, x_n) \in \mathbb{F}_q^n \mid x_i \neq x_j, \text{ for } i \neq j\}.$$

It is naturally bijective to (ordered) choices of n distinct elements from \mathbb{F}_q . Hence the cardinality is

$$|\mathbb{F}_q^n \setminus \bigcup_{i < j} H_{ij}| = q(q-1) \dots (q-n+1),$$

then we have $\chi(\mathcal{A}, t) = t(t-1)(t-2) \dots (t-n+1)$.

Furthermore, \mathcal{A} is a free arrangement. Indeed set

$$\begin{aligned} \delta_0 &= \partial_{x_1} + \partial_{x_2} + \dots + \partial_{x_n}, \\ \delta_1 &= x_1 \partial_{x_1} + x_2 \partial_{x_2} + \dots + x_n \partial_{x_n}, \\ \delta_2 &= x_1^2 \partial_{x_1} + x_2^2 \partial_{x_2} + \dots + x_n^2 \partial_{x_n}, \\ &\dots \\ \delta_{n-1} &= x_1^{n-1} \partial_{x_1} + x_2^{n-1} \partial_{x_2} + \dots + x_n^{n-1} \partial_{x_n}. \end{aligned}$$

Then $\delta_k(x_i - x_j) = x_i^k - x_j^k$, which is divisible by $(x_i - x_j)$. Hence $\delta_k \in D(\mathcal{A})$.

Furthermore, by Vandermonde's formula

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i),$$

and by Saito's criterion, we may conclude that $\delta_0, \dots, \delta_{n-1}$ is a basis of $D(\mathcal{A})$. Hence \mathcal{A} is free with exponents $(0, 1, \dots, n-1)$.

To conclude this section, we note that the module of logarithmic vector fields is recovered from the sheafification:

$$D(\mathcal{A}, \mathfrak{m}) \xrightarrow{\simeq} \Gamma_*(D(\widetilde{\mathcal{A}}, \mathfrak{m})). \tag{1.5}$$

Therefore freeness of $(\mathcal{A}, \mathfrak{m})$ is equivalent to the splitting of $D(\mathcal{A}, \mathfrak{m})$.

PROPOSITION 1.20. — *$(\mathcal{A}, \mathfrak{m})$ is free with exponents (d_1, \dots, d_ℓ) if and only if $D(\widetilde{\mathcal{A}}, \mathfrak{m}) \simeq \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \oplus \dots \oplus \mathcal{O}(-d_\ell)$.*

1.4. 2-multiarrangements

A simple arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ in dimension two is always free with exponents $(1, n-1)$. We can construct an example of basis explicitly as follows:

$$\delta_1 = x\partial_x + y\partial_y, \quad \delta_2 = (\partial_y Q)\partial_x - (\partial_x Q)\partial_y.$$

The multiarrangement $(\mathcal{A}, \mathfrak{m})$ in dimension two is also always free. There are two ways to prove it. First idea is based on $D(\mathcal{A}, \mathfrak{m})$ being a reflexive module. Then 2-dimensional and reflexivity implies freeness. Another idea is based on the isomorphism

$$D(\mathcal{A}, \mathfrak{m}) \xrightarrow{\simeq} \Gamma_*(D(\widetilde{\mathcal{A}}, \mathfrak{m})).$$

If \mathcal{A} is in dimension two, the sheafification $D(\widetilde{\mathcal{A}}, \mathfrak{m})$ is a torsion free sheaf on \mathbb{P}^1 . By Grothendieck splitting theorem, we conclude $D(\mathcal{A}, \mathfrak{m})$ is free. We have the following.

PROPOSITION 1.21. — *Let $(\mathcal{A}, \mathfrak{m})$ be a 2-multiarrangement. Then it is free and the exponents (d_1, d_2) satisfy $d_1 + d_2 = |\mathfrak{m}|$.*

The determination of exponents of 2-multiarrangements is difficult, but it is an important problem because it is related to the freeness of 3-arrangements (see §1.5). The following lemma is useful for the computation of the exponents.

LEMMA 1.22. — Let $(\mathcal{A}, \mathbf{m})$ be a 2-multiarrangement. Let $\delta \in D(\mathcal{A}, \mathbf{m})$. Assume that $d = \text{pdeg } \delta \leq \frac{|\mathbf{m}|}{2}$ and no nontrivial divisor of δ is contained in $D(\mathcal{A}, \mathbf{m})$. Then $\exp(\mathcal{A}, \mathbf{m}) = (d, |\mathbf{m}| - d)$.

Proof. — Suppose that $\exp(\mathcal{A}, \mathbf{m}) = (d_1, d_2)$ with $d_1 \leq d_2$. Then clearly $d_1 \leq d$. There exists δ_1 of $\text{pdeg } \delta_1 = d_1$. Since $d \leq \frac{|\mathbf{m}|}{2} = \frac{d_1 + d_2}{2}$, we have $d_1 \leq d \leq d_2$. If $d_1 < d$, then we have $d_1 < d < d_2$. Hence δ can be expressed as $\delta = F \cdot \delta_1$ with some polynomial F of $\deg F > 0$. But this contradicts the assumption that no nontrivial divisor of δ is contained in $D(\mathcal{A}, \mathbf{m})$. So $\deg F = 0$ and we have $d_1 = d, d_2 = |\mathbf{m}| - d$. \square

For the following cases we can determine the exponents combinatorially.

PROPOSITION 1.23. — Let $(\mathcal{A}, \mathbf{m})$ be a 2-multiarrangement. We may assume that $m_i = \mathbf{m}(H_i)$ satisfies $m_1 \geq m_2 \geq \dots \geq m_n > 0$. Set $m = \sum_{i=1}^n m_i$.

- (i) If $m_1 \geq \frac{m}{2}$, then the exponents are $\exp(\mathcal{A}, \mathbf{m}) = (m_1, m - m_1)$.
- (ii) if $n \geq \frac{m}{2} + 1$, then $\exp(\mathcal{A}, \mathbf{m}) = (m - n + 1, n - 1)$.
- (iii) If $m_1 = m_2 = \dots = m_n = 2$, then $\exp(\mathcal{A}, \mathbf{m}) = (n, n)$.
- (iv) If $n = 3$ and $m_1 \leq m_2 + m_3$, then

$$\exp(\mathcal{A}, \mathbf{m}) = \begin{cases} (k, k), & \text{if } |\mathbf{m}| = 2k, \\ (k, k + 1), & \text{if } |\mathbf{m}| = 2k + 1. \end{cases}$$

Proof. — (i) We can set coordinates (x, y) such that $H_1 = \{x = 0\}$, in other words, $\alpha_1 = x$. Set $\delta = (\prod_{i=2}^n \alpha_i^{m_i}) \cdot \partial_y$. Then $\delta x = 0$ and $\delta \alpha_i \in (\alpha_i)^{m_i}$ for $i \geq 2$. Hence $\delta \in D(\mathcal{A}, \mathbf{m})$. We also have

$$\text{pdeg } \delta = m_2 + \dots + m_n = |\mathbf{m}| - m_1 \leq \frac{|\mathbf{m}|}{2},$$

and no divisor of δ is not contained in $D(\mathcal{A}, \mathbf{m})$. From Lemma 1.22, $\exp(\mathcal{A}, \mathbf{m}) = (m_1, |\mathbf{m}| - m_1)$.

- (ii) Let us define δ as

$$\delta = \frac{\prod_{i=1}^n \alpha_i^{m_i}}{\prod_{i=1}^n \alpha_i} \cdot \theta_E,$$

where $\theta_E = x\partial_x + y\partial_y$ is the Euler vector field. Then since $\theta_E \alpha = \alpha$ for any linear form α , $\delta \in D(\mathcal{A}, \mathbf{m})$. From the assumption, we have

$$\text{pdeg } \delta = |\mathbf{m}| - n + 1 \leq n - 1.$$

Since $(|\mathbf{m}| - n + 1) + (n - 1) = |\mathbf{m}|$, we have $|\mathbf{m}| - n + 1 = \text{pdeg } \delta \leq \frac{|\mathbf{m}|}{2}$. It is also easily checked that δ does not have non trivial divisor which is contained in $D(\mathcal{A}, \mathbf{m})$. Hence we have $\exp(\mathcal{A}, \mathbf{m}) = (|\mathbf{m}| - n + 1, n - 1)$.

(iii) and (iv) are proved by explicit constructions of basis. See [44, Exapmple 2.2] and [43] respectively. (Both are highly nontrivial.) In §1.8 we present an alternative proof of (iii) given by T. Abe. \square

Thus if either $\max\{\mathbf{m}(H) \mid H \in \mathcal{A}\}$ is large (not less than the half of $|\mathbf{m}| = \sum \mathbf{m}(H)$) or the number of lines $n = |\mathcal{A}|$ is large (not less than $\frac{|\mathbf{m}|}{2} + 1$), then the exponents are combinatorially determined. This motivates us to give the following definition.

DEFINITION 1.24. — *The multiplicity $\mathbf{m} : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$ is said to be balanced if $\mathbf{m}(H) \leq \frac{\sum_{H \in \mathcal{A}} \mathbf{m}(H)}{2}$ for all $H \in \mathcal{A}$.*

As we have seen, if the multiplicity is not balanced, then the exponents are combinatorially determined. However the exponents are not combinatorially determined for balanced cases in general.

Example 1.25. — Let $(\mathcal{A}_t, \mathbf{m})$ be a multiarrangement defined by

$$Q(\mathcal{A}_t, \mathbf{m}) = x^3 y^3 (x + y)^1 (tx - y)^1,$$

where $t \in \mathbb{C} \setminus \{0, -1\}$. Then exponents are

$$\exp(\mathcal{A}_t, \mathbf{m}) = \begin{cases} (3, 5), & \text{if } t = 1, \\ (4, 4), & \text{if } t \neq 1. \end{cases}$$

Indeed, it is easily seen that

$$\begin{aligned} \delta_1 &= x^3 \partial_x + y^3 \partial_y, \\ \delta_2 &= x^5 \partial_x + y^5 \partial_y, \end{aligned}$$

form a basis of $D(\mathcal{A}_1, \mathbf{m})$. For $t \neq 1$ (and $t \neq 0, -1$), $\delta_1 \notin D(\mathcal{A}_t, \mathbf{m})$. But $(tx - y)\delta_1 \in D(\mathcal{A}_t, \mathbf{m})$ with $\text{pdeg} = 4$. If there exists an element of $D(\mathcal{A}_t, \mathbf{m})$ of $\text{pdeg} = 3$, it should be a divisor of $(tx - y)\delta_1$. It is impossible. Thus exponents for other cases are $(4, 4)$.

We may observe that any 4-lines can be moved by $PGL_2(\mathbb{C})$ -action to $xy(x + y)(tx - y)$ with $t \in \mathbb{C} \setminus \{0, -1\}$. On a Zariski open subset of the parameter space $\mathbb{C} \setminus \{0, 1, -1\} \subset \mathbb{C} \setminus \{0, -1\}$, the exponents are $(4, 4)$ and at $t = 1$, they become $(3, 5)$. This generally happens. We shall prove the upper-semicontinuity on the parameter space of the following function.

DEFINITION 1.26. — Put $\exp(\mathcal{A}, \mathbf{m}) = (d_1, d_2)$. Then we denote

$$\Delta(\mathcal{A}, \mathbf{m}) = |d_1 - d_2|.$$

The difference of exponents $\Delta(\mathcal{A}, \mathbf{m})$ is a function on \mathcal{A} and \mathbf{m} . We first fix the multiplicity \mathbf{m} . The parameter space of \mathcal{A} can be described as

$$\mathcal{M}_n = \{(H_1, \dots, H_n) \in (\mathbb{P}^{1*})^n \mid H_i \neq H_j, \text{ for } i \neq j\}$$

PROPOSITION 1.27. — Fix the multiplicity $\mathbf{m} : \{1, \dots, n\} \rightarrow \mathbb{Z}_{>0}$. Then

$$\Delta : \mathcal{M}_n \longrightarrow \mathbb{Z}_{>0}, (\mathcal{A} \longmapsto \Delta(\mathcal{A}, \mathbf{m}))$$

is upper semi-continuous, i. e., the subset $\{\Delta < k\} \subset \mathcal{M}_n$ is a Zariski open subset for any $k \in \mathbb{R}$.

Proof. — It suffices to prove that $\{\Delta \geq k\}$ is Zariski closed in \mathcal{M}_n . Since $d_1 + d_2 = |\mathbf{m}|$, $\Delta(\mathcal{A}, \mathbf{m}) \geq k$ if and only if there exists $\delta \in D(\mathcal{A}, \mathbf{m})$ such that $\text{pdeg } \delta \leq \lfloor \frac{|\mathbf{m}|}{2} - \frac{k}{2} \rfloor$. Thus we consider when $\delta \in D(\mathcal{A}, \mathbf{m})$ of $\text{pdeg } \delta = \lfloor \frac{|\mathbf{m}|}{2} - \frac{k}{2} \rfloor$ exists. Put $d = \lfloor \frac{|\mathbf{m}|}{2} - \frac{k}{2} \rfloor$, $\alpha_i = p_i x + q_i y$ and

$$\delta = (a_0 x^d + a_1 x^{d-1} y + \dots + a_d y^d) \partial_x + (b_0 x^d + b_1 x^{d-1} y + \dots + b_d y^d) \partial_y.$$

The assertion $\delta \alpha_i \in (\alpha_i^{m_i})$ is equivalent to

$$\delta \alpha_i = (p_i x + q_i y)^{m_i} (c_0 x^{d-m_i} + c_1 x^{d-m_i-1} y + \dots + c_{d-m_i} y^{d-m_i}), \quad (1.6)$$

for some c_0, c_1, \dots . Hence the existence of $\delta \in D(\mathcal{A}, \mathbf{m})$ of degree d is equivalent to the existence of the solution to the system (1.6) of linear equations on a_i, b_i and c_i . It is a Zariski closed condition on the parameters p_i and q_i . \square

The following are the two fundamental results on exponents of 2-multi-arrangements.

THEOREM 1.28. — Let $\mathbf{m} : \{1, \dots, n\} \rightarrow \mathbb{Z}_{>0}$ be a balanced multiplicity and $\mathcal{A} = \{H_1, \dots, H_n\}$ a 2-arrangement.

- (i) (Wakefield-Yuzvinsky [44]) For generic \mathcal{A} , $\Delta(\mathcal{A}, \mathbf{m}) \leq 1$.
- (ii) (Abe [1]) $\Delta(\mathcal{A}, \mathbf{m}) \leq n - 2$.

The proof of (i) is a careful extension of that of upper semicontinuity (Proposition 1.27). See cited papers for proof. The proof of (ii) is of a very

different nature. Abe ([1] and Abe-Numata [2]) first fix \mathcal{A} and then consider Δ as a function from the set of multiplicities $\mathbb{Z}_{>0}^n$ to $\mathbb{Z}_{\geq 0}$,

$$\Delta : \mathbb{Z}_{>0}^n \longrightarrow \mathbb{Z}_{\geq 0}, \mathbf{m} \longmapsto \Delta(\mathcal{A}, \mathbf{m}).$$

They studied the structure of this function in great detail. The proof of (ii) is based on this.

(i) tells the generic behavior of the function Δ . (ii) tells the upper bound of Δ for balanced multiplicities. As far as the author knows, the examples $(\mathcal{A}, \mathbf{m})$ attaining the upper bound of Δ are related to interesting free arrangements of rank 3. Abe found a class of free arrangements which are combinatorially characterizable [1]. See Example 1.42.

PROBLEM 1.29. — Give a unified proof for Theorem 1.28 (i) and (ii).

1.5. Multiarrangements and free arrangements

Multiarrangements appear as restrictions of simple arrangements. Namely, let \mathcal{A} be an arrangement in V of $\dim V = \ell$. For $H \in \mathcal{A}$ let us denote by \mathcal{A}^H the induced arrangement on H .

DEFINITION 1.30. — Define the function $\mathbf{m}^H : \mathcal{A}^H \longrightarrow \mathbb{Z}_{>0}$ by

$$X \in \mathcal{A}^H \longmapsto \#\{H' \in \mathcal{A} \mid H' \supset X\} - 1.$$

We call $(\mathcal{A}^H, \mathbf{m}^H)$ the Ziegler's multirestriction.

Example 1.31. — Let $V = \mathbb{C}^3$ with coordinates x, y, z . Put $H_1 = \{z = 0\}$, $H_2 = \{x = 0\}$, $H_3 = \{y = 0\}$, $H_4 = \{x - z = 0\}$, $H_5 = \{x + z = 0\}$, $H_6 = \{y - z = 0\}$, $H_7 = \{y + z = 0\}$, $H_8 = \{x - y = 0\}$, $H_9 = \{x + y = 0\}$. Then $\mathcal{A} = \{H_1, \dots, H_9\}$ is free with exponents $(1, 3, 5)$. Ziegler's multirestriction to $(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$ is $x^3y^3(x - y)(x + y)$. (See Figure 1)

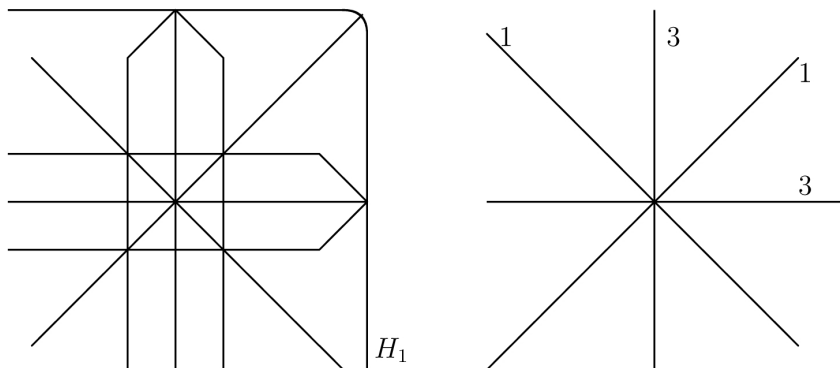


Figure 1. — $\mathcal{A} = \{H_1, \dots, H_9\}$ and $(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$

DEFINITION 1.32. — Fix a hyperplane $H_1 \in \mathcal{A}$. Then we define a submodule $D_1(\mathcal{A})$ of $D(\mathcal{A})$ by

$$D_1(\mathcal{A}) = \{\delta \in D(\mathcal{A}) \mid \delta\alpha_{H_1} = 0\}.$$

LEMMA 1.33. — Under the above notations, $D(\mathcal{A}) = S \cdot \theta_E \oplus D_1(\mathcal{A})$.

Proof. — Let $\delta \in D(\mathcal{A})$. Since $\delta - \frac{\delta\alpha_{H_1}}{\alpha_{H_1}} \cdot \theta_E$ is in $D_1(\mathcal{A})$, $\delta = \frac{\delta\alpha_{H_1}}{\alpha_{H_1}} \cdot \theta_E + \left(\delta - \frac{\delta\alpha_{H_1}}{\alpha_{H_1}} \cdot \theta_E\right)$ gives the desired decomposition. \square

THEOREM 1.34 (Ziegler [54]). — Notations as above.

- (i) If $\delta \in D_1(\mathcal{A})$, then $\delta|_{H_1} \in D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$.
- (ii) If \mathcal{A} is free with exponents $(1, d_2, \dots, d_\ell)$, then $(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$ is free with exponents (d_2, \dots, d_ℓ) .

Proof. — We can choose coordinates x_1, \dots, x_ℓ in such a way that $x_1 = \alpha_{H_1}$. Let $X \in \mathcal{A}^{H_1}$ and put

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supset X\} = \{H_1, H_{i_1}, H_{i_2}, \dots, H_{i_m}\}.$$

Since $H \cap H_{i_1} = \dots = H \cap H_{i_m} = X$, the restriction $\alpha_{i_p}|_{x_1=0}$ determines the same hyperplane. Thus we may assume that α_{i_p} have the form

$$\begin{aligned} \alpha_{i_1}(x_1, \dots, x_\ell) &= c_1x_1 + \alpha'(x_2, \dots, x_\ell) \\ \alpha_{i_2}(x_1, \dots, x_\ell) &= c_2x_1 + \alpha'(x_2, \dots, x_\ell) \\ &\dots \dots \\ \alpha_{i_m}(x_1, \dots, x_\ell) &= c_mx_1 + \alpha'(x_2, \dots, x_\ell), \end{aligned}$$

where c_1, \dots, c_m are mutually distinct. Let $\delta \in D_1(\mathcal{A})$. By definition,

$$\delta(c_kx_1 + \alpha'(x_2, \dots, x_\ell)) \in (c_kx_1 + \alpha'(x_2, \dots, x_\ell)).$$

Then since $\delta x_1 = 0$, $\delta\alpha'(x_2, \dots, x_\ell)$ is divisible by $c_kx_1 + \alpha'(x_2, \dots, x_\ell)$ for all $k = 1, \dots, m$. Hence it is divisible by

$$\prod_{k=1}^m (c_kx_1 + \alpha'(x_2, \dots, x_\ell)).$$

Now we restrict to $x_1 = 0$. Then $\delta|_{x_1=0}\alpha'$ is divisible by $(\alpha')^m$. Thus (i) is proved.

(ii) Let $\delta_1 = \theta_E, \delta_2, \dots, \delta_\ell$ be a basis of $D(\mathcal{A})$ such that $\delta_2, \dots, \delta_\ell \in D_1(\mathcal{A})$. Let us set $\delta_i = \sum_{j=2}^\ell f_{ij}\partial_{x_j}$. We will prove that $\delta_2|_{x_1=0}, \dots, \delta_\ell|_{x_1=0}$

are linearly independent over $S/x_1S = \mathbb{C}[x_2, \dots, x_\ell]$. Indeed by Saito's criterion, the determinant

$$\det \begin{pmatrix} x_1 & x_2 & \cdots & x_\ell \\ 0 & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & f_{n2} & \cdots & f_{nn} \end{pmatrix} = x_1 \cdot \det \begin{pmatrix} f_{22} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{n2} & \cdots & f_{nn} \end{pmatrix}$$

is divisible by x_1 just once. Hence $\det(f_{ij})$ is not divisible by x_1 , which implies that $\delta_2|_{x_1=0}, \dots, \delta_\ell|_{x_1=0}$ are linearly independent over $S/x_1S = \mathbb{C}[x_2, \dots, x_\ell]$. Furthermore, we have

$$\sum_{i=2}^{\ell} \text{pdeg } \delta_i|_{x_1=0} = |\mathcal{A}| - 1 = \sum_{X \in \mathcal{A}^{H_1}} \mathbf{m}^{H_1}(X).$$

By Saito's criterion, they form a free basis of $D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$. □

It seems natural to pay attention to the exact sequence

$$0 \longrightarrow D_1(\mathcal{A}) \xrightarrow{x_1} D_1(\mathcal{A}) \xrightarrow{\rho} D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}). \quad (1.7)$$

From the above proof, we know that if \mathcal{A} is free, then the restriction map ρ is surjective.

COROLLARY 1.35. — *If the restriction map ρ is surjective, and $D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$ is free with exponents (d_2, \dots, d_ℓ) , then \mathcal{A} is free with exponents $(1, d_2, \dots, d_\ell)$.*

Proof. — By the assumption, there exists $\delta_2, \dots, \delta_\ell \in D_1(\mathcal{A})$ such that $\rho(\delta_2) = \delta_2|_{x_1=0}, \dots, \rho(\delta_\ell) = \delta_\ell|_{x_1=0}$ are basis of $D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$. As in the previous proof, $\delta_2, \dots, \delta_\ell$ and θ_E are linearly independent and the sum of pdeg is $|\mathcal{A}|$. Hence by Saito's criterion, $(\theta_E, \delta_2, \dots, \delta_\ell)$ is a basis of $D(\mathcal{A})$. □

Generally, ρ is not surjective. However, local freeness implies local surjectivity.

DEFINITION 1.36. — *Let \mathcal{A} be an arrangement and $H_1 \in \mathcal{A}$. Then \mathcal{A} is said to be locally free along H_1 if $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}$ is free for all $X \in L(\mathcal{A})$ with $X \subset H_1$ and $X \neq 0$.*

Local freeness along H_1 implies

$$0 \longrightarrow D_1(\mathcal{A}_X) \xrightarrow{x_1} D_1(\mathcal{A}_X) \xrightarrow{\rho} D(\mathcal{A}_X^{H_1}, \mathbf{m}_X^{H_1}) \longrightarrow 0$$

for all $X \in L(\mathcal{A})$, $X \neq 0$ with $X \subset H_1$. Thus we have an exact sequence of sheaves over $\mathbb{P}^{\ell-1}$.

$$0 \longrightarrow \widetilde{D_1(\mathcal{A})}(-1) \xrightarrow{x_1} \widetilde{D_1(\mathcal{A})} \xrightarrow{\rho} D(\mathcal{A}^{\widetilde{H_1}}, \mathbf{m}^{H_1}) \longrightarrow 0. \quad (1.8)$$

Thus we obtain a relation between Ziegler's multirestriction and restriction of the sheaf $D_1(\mathcal{A})$.

PROPOSITION 1.37. — *If \mathcal{A} is locally free along H_1 , then*

$$\widetilde{D_1(\mathcal{A})}|_{H_1} = D(\mathcal{A}^{\widetilde{H_1}}, \mathbf{m}^{H_1}).$$

By the above proposition combined with Proposition 1.20 and Horrocks criterion (Theorem 1.5 (2), see also subsequent Remark 1.6), we have the following criterion for freeness.

THEOREM 1.38 ([47]). — *Assume that $\ell \geq 4$. Then \mathcal{A} is free with exponents $(1, d_2, \dots, d_\ell)$ if and only if the following conditions are satisfied.*

- \mathcal{A} is locally free along H_1 ,
- Ziegler's multirestriction $(\mathcal{A}^{H_1}, m^{H_1})$ is free with exponents (d_2, \dots, d_ℓ) .

The above criterion is not valid for $\ell = 3$. Indeed for $\ell = 3$, both conditions are automatically satisfied, however, there exist non free 3-arrangements. For characterizing freeness of 3-arrangements, we need characteristic polynomials.

THEOREM 1.39 ([48]). — *Let \mathcal{A} be a 3-arrangement. Set $\chi(\mathcal{A}, t) = (t-1)(t^2 - b_1t + b_2)$ and $\exp(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}) = (d_1, d_2)$. Then*

- (i) $b_2 \geq d_1d_2$, furthermore
 $b_2 - d_1d_2 = \dim \text{Coker}(\rho : D_1(\mathcal{A}) \rightarrow D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}))$.
- (ii) If $b_2 = d_1d_2$, then \mathcal{A} is free with exponents $(1, d_1, d_2)$.

The proof is based on an analysis of Solomon-Terao's formula. Theorem 1.39 is also a corollary of a result in the next section (Theorem 1.45).

By combining Theorem 1.38 and 1.39, we recently obtained the following criterion for $\ell \geq 4$.

THEOREM 1.40 (Abe-Yoshinaga [6]). — *Assume that $\ell \geq 4$ and the multirestriction is free with $\exp(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}) = (d_2, \dots, d_\ell)$. Put $\chi(\mathcal{A}, t) = (t-1)(t^{\ell-1} - b_1t^{\ell-2} + b_2t^{\ell-3} - \dots)$. Then*

$$b_2 \geq \sum_{2 \leq i < j \leq \ell} d_i d_j,$$

and \mathcal{A} is free if and only if the equality holds.

Remark 1.41. — At a glance, this result is similar to that of Elenčwaĭg-Forster (Theorem 1.5 (3) and see Bertone-Roggero [10] for torsion free case). However at this moment, we can not find any (simple) logical implications.

Example 1.42. — Let $\mathcal{A} = \{H_0, H_1, \dots, H_{18}\}$ be the cone of the G_2 -Catalan arrangement $G_2^{[-1,1]}$ (see Figure 1.42), where H_0 is corresponding to the line at infinity. Using Abe's inequality (Theorem 1.28 (ii)) and Theorem 1.39, we can prove the freeness of \mathcal{A} as follows. First the characteristic polynomial is

$$\chi(\mathcal{A}, t) = (t - 1)(t - 7)(t - 11).$$

Let us consider the multirestriction $(\mathcal{A}^{H_0}, \mathbf{m}^{H_0})$. Put the exponents $\exp(\mathcal{A}^{H_0}, \mathbf{m}^{H_0}) = (d_1, d_2)$. Then by Theorem 1.39,

$$d_1 d_2 \leq 77.$$

Since the multirestriction is balanced, by Abe's inequality, we have

$$|d_1 - d_2| \leq 6 - 2 = 4.$$

Combining these two inequalities, we have $d_1 d_2 = 77$ hence \mathcal{A} is free with exponents $(1, 7, 11)$.

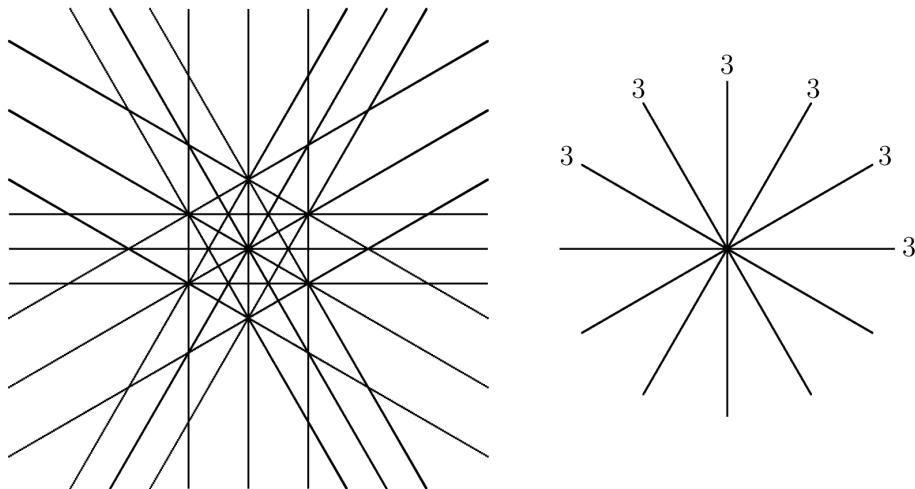


Figure 2. — $G_2^{[-1,1]}$ and restriction of its cone $cG_2^{[-1,1]}$ to H_0

We emphasise that in the above example, only the computation of characteristic polynomial is enough to prove freeness.

1.6. Characteristic polynomials and Chern polynomials

Let \mathcal{A} be an arrangement in V of $\dim V = \ell$. By Terao's factorization theorem, if \mathcal{A} is free with exponents (d_1, \dots, d_ℓ) , then

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - d_i).$$

On the other hand, the sheafification splits $\widetilde{D}(\mathcal{A}) = \mathcal{O}_{\mathbb{P}^{\ell-1}}(-d_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^{\ell-1}}(-d_\ell)$. The Chern polynomial of this sheaf is

$$\begin{aligned} c_t(\widetilde{D}(\mathcal{A})) &= \sum_{i=1}^{\ell-1} c_i(\widetilde{D}(\mathcal{A}))t^i \\ &\equiv \prod_{i=1}^{\ell} (1 - d_i t)t^\ell, \end{aligned} \tag{1.9}$$

where $c_i(-)$ is i -th Chern number. It is easily seen that these two polynomials are related by the following formula

$$t^\ell \cdot \chi(\mathcal{A}, \frac{1}{t}) = c_t(\widetilde{D}(\mathcal{A}))t^\ell. \tag{1.10}$$

Note that the left hand side of (1.10) is computed by Solomon-Terao's formula (Theorem 1.16). Mustařă and Schenck proved that a similar formula computes the Chern polynomial for arbitrary vector bundle on the projective space.

THEOREM 1.43 (*Mustařă and Schenck [19]*). — *Let \mathcal{E} be a vector bundle over \mathbb{P}^n of rank r . Then*

$$c_t(\mathcal{E}) = \lim_{x \rightarrow 1} (-t)^r (1-x)^{n+1-r} \sum_{i=0}^r \text{Hilb}(\Gamma_*(\bigwedge^i \mathcal{E}), x) \left(\frac{x-1}{t} - 1 \right)^i.$$

As a corollary, we have:

COROLLARY 1.44. — *Let \mathcal{A} be a locally free arrangement. Then the formula (1.10) holds.*

Using Mustařă-Schenck, we can prove the following.

THEOREM 1.45. — *Let \mathcal{E} be a rank two vector bundle on \mathbb{P}^2 . Let $L \subset \mathbb{P}^2$ be a line. Put $\mathcal{E}|_L = \mathcal{O}_L(d_1) \oplus \mathcal{O}_L(d_2)$. Then $c_2(\mathcal{E}) \geq d_1 d_2$, furthermore*

$$c_2(\mathcal{E}) - d_1 d_2 = \dim \text{Coker}(\Gamma_*(\mathcal{E}) \rightarrow \Gamma_*(\mathcal{E}|_L)).$$

\mathcal{E} is splitting if and only if $c_2(\mathcal{E}) = d_1 d_2$.

Proof. — By Theorem 1.43, the second Chern class is

$$c_2(\mathcal{E}) = \lim_{x \rightarrow 1} \left(\frac{1}{(1-x)^2} - (1-x)\text{Hilb}(\Gamma_*(\mathcal{E}), x) + \frac{x^{-c_1(\mathcal{E})}}{(1-x)^2} \right).$$

On the other hand, $c_1(\mathcal{E}) = d_1 + d_2$ and

$$d_1 d_2 = \lim_{x \rightarrow 1} \left(\frac{1}{(1-x)^2} - \frac{x^{-d_1} + x^{-d_2}}{(1-x)^2} + \frac{x^{-d_1-d_2}}{(1-x)^2} \right).$$

Hence

$$\begin{aligned} c_2(\mathcal{E}) - d_1 d_2 &= \lim_{x \rightarrow 1} \left(\frac{x^{-d_1} + x^{-d_2}}{(1-x)^2} - (1-x)\text{Hilb}(\Gamma_*(\mathcal{E}), x) \right) \\ &= \lim_{x \rightarrow 1} (\text{Hilb}(\Gamma_*(\mathcal{E}|_L), x) - \text{Hilb}(\text{Im}(\Gamma_*(\mathcal{E}) \rightarrow \Gamma_*(\mathcal{E}|_L)), x)) \\ &= \dim \text{Coker}(\Gamma_*(\mathcal{E}) \rightarrow \Gamma_*(\mathcal{E}|_L)). \end{aligned}$$

□

1.7. Around Terao Conjecture

In [40], Terao posed the following problem.

PROBLEM 1.46. — *Let $\mathcal{A}_1, \mathcal{A}_2$ be arrangements in V s. t. $L(\mathcal{A}_1) \simeq L(\mathcal{A}_2)$. Assume that \mathcal{A}_1 is free. Then is \mathcal{A}_2 also free?*

It is obviously true in dimension 2. However the cases $\ell \geq 3$ are still open. In view of Theorem 1.39, if the exponents of multirestriction were determined combinatorially, the freeness is also determined combinatorially.

PROPOSITION 1.47. — *Let $\mathcal{A}_1, \mathcal{A}_2$ be in V of $\dim V = \ell = 3$ such that $L(\mathcal{A}_1) \simeq L(\mathcal{A}_2)$. Assume that \mathcal{A}_1 is free. If there exists a hyperplane $H \in \mathcal{A}$ such that the multirestriction $(\mathcal{A}^H, \mathbf{m}^H)$ satisfies one of conditions in Proposition 1.23, then \mathcal{A}_2 is also free.*

Thus the difficulty of Terao's conjecture for $\ell = 3$ is equivalent to the difficulty of determining exponents of 2-multiarrangements.

A possible approach to Terao's conjecture is to look at the set of arrangements which have prescribed intersection lattice, and then analyze the freeness on the set. We first introduce such a set, the parameter space of arrangements having the fixed lattice. Let $\ell \geq 3, n \geq 1$. Fix a poset L . Then define the set $\mathcal{M}_{\ell, n}(L)$ of arrangements with lattice L by

$$\mathcal{M}_{\ell, n}(L) = \{ \mathcal{A} = (H_1, \dots, H_n) \in (\mathbb{P}^{\ell-1})^n \mid H_i \neq H_j, L(\mathcal{A}) \simeq L \}.$$

Terao's conjecture is equivalent to the preservation of the freeness/nonfreeness on $\mathcal{M}_{\ell,n}(L)$. Yuzvinsky proved that free arrangements form a Zariski open subset in $\mathcal{M}_{\ell,n}(L)$.

THEOREM 1.48 (Yuzvinsky [51, 52, 53]). —

$$\mathcal{M}_{\ell,n}(L)^{\text{free}} = \{ \mathcal{A} \in \mathcal{M}_{\ell,n}(L) \mid \mathcal{A} \text{ is free} \}$$

is a Zariski open subset of $\mathcal{M}_{\ell,n}(L)$.

In his proof, Yuzvinsky defines lattice cohomology using the structure of $L(\mathcal{A})$ and $D(\mathcal{A})$. Then he characterizes the freeness of \mathcal{A} via vanishings of these cohomology groups. The statement looks very similar to that of Horrocks (Theorem 1.5 (1)).

PROBLEM 1.49. — *Establish the relation between Yuzvinsky's and Horrocks' criteria for freeness. (More precisely, establish the relation between Yuzvinsky's lattice cohomology and sheaf cohomology on \mathbb{P}^n .)*

Here we recover (slightly modified version of) Yuzvinsky's openness result for $\ell = 3$ by using upper semicontinuity of exponents of 2-multiarrangements. Similar to $\mathcal{M}_{\ell,n}(L)$, we introduce the following set of arrangements which have prescribed characteristic polynomial. Let $f(t) \in \mathbb{Z}[t]$.

$$\mathcal{C}_{\ell,n}(f) = \{ \mathcal{A} = (H_1, \dots, H_n) \in (\mathbb{P}^{\ell-1*})^n \mid H_i \neq H_j, \chi(\mathcal{A}, t) = f(t) \}.$$

THEOREM 1.50. — *The set*

$$\mathcal{C}_{\ell,n}(f)^{\text{free}} = \{ \mathcal{A} \in \mathcal{C}_{\ell,n}(f) \mid \mathcal{A} \text{ is free} \}$$

is a Zariski open subset of $\mathcal{C}_{\ell,n}(f)$.

Proof. — By Terao's factorization theorem, if $f(t)$ is not split, then $\mathcal{C}_{\ell,n}(f)^{\text{free}}$ is empty. We may assume that $f(t) = (t-1)(t-d_1)(t-d_2)$. Fix $H_1 \in \mathcal{A}$ and set $\exp(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}) = (d_1^{H_1}, d_2^{H_1})$. Then by Theorem 1.39, $|d_1^{H_1} - d_2^{H_1}| \geq |d_1 - d_2|$ and \mathcal{A} is free if and only if the equality holds. By the upper semicontinuity (Proposition 1.27) of the difference $\Delta(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}) = |d_1^{H_1} - d_2^{H_1}|$, the free locus $\{ \mathcal{A} \mid \Delta < |d_1 - d_2| + \frac{1}{2} \}$ is a Zariski open subset of $\mathcal{C}_{\ell,n}(f)$. \square

Let L be a poset, and $f(t)$ be the corresponding characteristic polynomial. Then $\mathcal{M}_{\ell,n}(L) \subset \mathcal{C}_{\ell,n}(f)$. Since $\mathcal{M}_{\ell,n}(L)^{\text{free}} = \mathcal{M}_{\ell,n}(L) \cap \mathcal{C}_{\ell,n}(f)^{\text{free}}$. We have obtained Yuzvinsky's openness result for $\ell = 3$.

We conclude this section with an observation. Lots of free arrangements which are not inductively free are rigid. It seems natural to ask whether or not the following (which is stronger than Terao conjecture) holds:

$$\{\text{Free arrangements}\} \subset \{\text{Inductively free}\} \cup \{\text{Rigid}\}. \quad (1.11)$$

1.8. Affine connection ∇

DEFINITION 1.51. — Let $\delta, \theta \in \text{Der}_V$. Set $\theta = \sum_i f_i \partial_{x_i}$. Define $\nabla_\delta \theta \in \text{Der}_V$ by

$$\nabla_\delta \theta = \sum_i (\delta f_i) \partial_{x_i}.$$

It is easily seen that for any linear form $\alpha \in V^*$,

$$(\nabla_\delta \theta)\alpha = \delta(\theta\alpha). \tag{1.12}$$

Using this we have the following.

PROPOSITION 1.52. — Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement with $\mathbf{m}(H) > 0, \forall H \in \mathcal{A}$. Let $\delta \in D(\mathcal{A}, \mathbf{m})$ and $\eta \in \text{Der}_V$. Then $\nabla_\eta \delta \in D(\mathcal{A}, \mathbf{m} - 1)$.

Proof. — By the assumption, we may write $\delta\alpha_H = \alpha_H^{\mathbf{m}(H)} F$. Then applying (1.12) we have

$$(\nabla_\eta \delta)\alpha_H = \eta(\alpha_H^{\mathbf{m}(H)} F) = \mathbf{m}(H)\alpha_H^{\mathbf{m}(H)-1}\eta(\alpha_H)F + \alpha_H^{\mathbf{m}(H)}\eta(F),$$

which is divisible by $\alpha_H^{\mathbf{m}(H)-1}$. □

The use of the connection ∇ goes back to K. Saito [26, 27]. He studied discriminant in the Coxeter group quotient V/W . The space V/W admits a degenerate metric induced from the W -invariant metric I on V . The connection ∇ is originally defined as the Levi-Civita connection for the degenerate metric. Since I is flat on V , it is nothing but the connection above (see also §2). It has been gradually recognized that ∇ is useful for the construction of various vector fields [1, 5, 35, 41, 42, 46].

Here we give a proof of Proposition 1.23 (iii).

PROPOSITION 1.53. — Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a 2-arrangement. Then the multiarrangement $(\mathcal{A}, 2)$ is free with exponents $(d_1, d_2) = (n, n)$.

Proof. — Since $d_1 + d_2 = 2n$, it is sufficient to show that there does not exist $\delta \in D(\mathcal{A}, 2)$ with $\text{pdeg } \delta = n - 1$. Suppose that it exists. Then by Proposition 1.52, $\nabla_{\partial_{x_1}} \delta, \nabla_{\partial_{x_2}} \delta \in D(\mathcal{A}, 1)$ and $\text{pdeg } \nabla_{\partial_{x_1}} \delta = \text{pdeg } \nabla_{\partial_{x_2}} \delta = n - 2$. Since $(\mathcal{A}, 1)$ is free with exponents $(1, n - 1)$ and the degrees of $\nabla_{\partial_{x_1}} \delta$ and $\nabla_{\partial_{x_2}} \delta$ are smaller than $n - 1$, they are multiples of the Euler vector field θ_E (Lemma 1.22). We have expressions $\nabla_{\partial_{x_1}} \delta = F_1 \cdot \theta_E, \nabla_{\partial_{x_2}} \delta = F_2 \cdot \theta_E$ with $\deg F_1 = \deg F_2 = n - 3$. On the other hand,

$$(\text{pdeg } \delta) \cdot \delta = \nabla_{\theta_E} \delta = x_1 \nabla_{\partial_{x_1}} \delta + x_2 \nabla_{\partial_{x_2}} \delta = (x_1 F_1 + x_2 F_2) \theta_E.$$

Hence $(x_1F_1 + x_2F_2)\theta_E\alpha_H = (x_1F_1 + x_2F_2)\alpha_H$ is divisible by α_H^2 for all $H \in \mathcal{A}$, equivalently, $x_1F_1 + x_2F_2$ is divisible by $\prod_{i=1}^n \alpha_{H_i}$. However it contradicts $\deg(x_1F_1 + x_2F_2) = n - 2$. \square

2. K. Saito's theory of primitive derivation

Let $V = \mathbb{R}^\ell$. Let W be a finite reflection group which is generated by reflections in V and acts irreducibly on V . The set of reflecting hyperplanes $\mathcal{A}(W)$ is called the Coxeter arrangement. There exists, unique up to a constant factor, a W -invariant symmetric bilinear form $I : V \times V \rightarrow \mathbb{R}$. The bilinear form I induces a linear isomorphism $I : V^* \rightarrow V$. Let $S = S(V^*)$ be the symmetric product. Since $\text{Der}_V = S \otimes V$ and $\Omega_V = S \otimes V^*$, the map I can be extended to an S -isomorphism $I : \Omega_V \rightarrow \text{Der}_V$.

We first observe that a W -invariant vector field $\delta \in \text{Der}_V^W$ is logarithmically tangent to \mathcal{A} . Indeed, let $\alpha_H \in V^*$ be a defining linear form of $H \in \mathcal{A}$ and $r_H \in W$ be the reflection with respect to H . Then $r_H(\alpha_H) = -\alpha_H$ and we have $r_H(\delta\alpha_H) = -\delta\alpha_H$. It is easily seen that if a polynomial $f \in S$ satisfies $r_H(f) = -f$, then f is divisible by α_H . Therefore $\delta\alpha_H$ is divisible by α_H . Hence $\text{Der}_V^W \subset D(\mathcal{A})^W$.

The ring S^W of invariant polynomials is known to be isomorphic to a polynomial ring $\mathbb{R}[P_1, P_2, \dots, P_\ell]$ (Chevalley [11]). We can choose the polynomials P_1, \dots, P_ℓ to be homogeneous, with degrees $2 = \deg P_1 < \deg P_2 \leq \dots \leq \deg P_{\ell-1} < \deg P_\ell$. The numbers $e_i = \deg P_i - 1$, $i = 1, \dots, \ell$ are called the exponents and $h = \deg P_\ell$ the Coxeter number. The Coxeter arrangement \mathcal{A} is free. Furthermore, the basis of $D(\mathcal{A})$ can be constructed explicitly by using basic invariants P_1, \dots, P_ℓ .

THEOREM 2.1 ([24, 26, 27]). — $D(\mathcal{A})^W = \text{Der}_V^W = \bigoplus_{i=1}^\ell S^W \cdot I(dP_i)$
 $D(\mathcal{A}) = \text{Der}_V^W \otimes_{S^W} S = \bigoplus_{i=1}^\ell S \cdot I(dP_i)$.

In particular, the Coxeter arrangement \mathcal{A} is free with $\exp(\mathcal{A}) = (e_1, \dots, e_\ell)$.

Proof. — We shall give the proof of the second equality. From the above remarks, the inclusions

$$D(\mathcal{A}) \supset \text{Der}_V^W \otimes_{S^W} S \supset \bigoplus_{i=1}^\ell S \cdot I(dP_i), \tag{2.1}$$

are clear. Fix a coordinate system (x_1, \dots, x_ℓ) . Recall that the Jacobian of the basic invariant

$$\Delta := \frac{\partial(P_1, \dots, P_\ell)}{\partial(x_1, \dots, x_\ell)} = \prod_{H \in \mathcal{A}} \alpha_H,$$

is the product of linear forms of reflecting hyperplanes up to non-zero constant factors. Hence by Saito's criterion (Theorem 1.14), $I(dP_1), \dots, I(dP_\ell)$ form a basis of $D(\mathcal{A})$. Thus the left hand side and right hand side in (2.1) are equal. \square

Fix a system of basic invariants P_1, \dots, P_ℓ and a coordinate system x_1, \dots, x_ℓ . Since $\deg P_i < \deg P_\ell$ ($i = 1, \dots, \ell - 1$), the rational vector field

$$D = \frac{\partial}{\partial P_\ell} = \frac{1}{\Delta} \det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \frac{\partial}{\partial x_1} \\ \frac{\partial P_1}{\partial x_2} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_2} & \frac{\partial}{\partial x_2} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \frac{\partial}{\partial x_\ell} \end{pmatrix} \quad (2.2)$$

is uniquely determined up to constant factor, and it is also characterized by

$$DP_i = \begin{cases} 1 & i = \ell, \\ 0 & i \neq \ell. \end{cases}$$

The vector field D is called the primitive vector field.

THEOREM 2.2 ([26, 27]). — *For every W -invariant vector field $\delta \in D(\mathcal{A})^W$, there exists a unique vector field $\theta \in D(\mathcal{A})^W$ such that*

$$\nabla_D \theta = \delta.$$

We denote $\theta = \nabla_D^{-1} \delta$.

Thus the operator ∇_D^{-1} acts on $D(\mathcal{A})^W$. It induces a filtration, the so-called “Hodge filtration”,

$$\cdots \nabla_D^{-2} D(\mathcal{A})^W \subset \nabla_D^{-1} D(\mathcal{A})^W \subset D(\mathcal{A})^W. \quad (2.3)$$

The operator increases the contact order of the vector fields.

THEOREM 2.3 ([5, 35, 41, 42, 46]). — *Let \mathcal{A} be a Coxeter arrangement with exponents (e_1, \dots, e_ℓ) and Coxeter number h . Let $\mathbf{m} : \mathcal{A} \rightarrow \{0, 1\}$ be a $\{0, 1\}$ -valued multiplicity.*

(i) *For a positive integer k , we have*

$$\begin{aligned} D(\mathcal{A}, 2k + \mathbf{m}) &\simeq D(\mathcal{A}, \mathbf{m})[-kh], \\ D(\mathcal{A}, 2k - \mathbf{m}) &\simeq (D(\mathcal{A}, \mathbf{m})^\vee)[-kh] \simeq \Omega^1(\mathcal{A}, \mathbf{m})[-kh]. \end{aligned}$$

(ii) *($\mathbf{m} \equiv 1$) The multiarrangement $(\mathcal{A}, 2k + 1)$ is free with $\exp(\mathcal{A}, 2k + 1) = (e_1 + kh, \dots, e_\ell + kh)$.*

- (iii) ($\mathbf{m} \equiv 0$) The multiarrangement $(\mathcal{A}, 2k)$ is free with $\exp(\mathcal{A}, 2k) = (kh, kh, \dots, kh)$.

In particular, the filtration (2.3) is equivalent to the following.

$$\dots \subset D(\mathcal{A}, 5)^W \subset D(\mathcal{A}, 3)^W \subset D(\mathcal{A})^W. \quad (2.4)$$

3. Weyl, Catalan and Shi arrangements

In this section we consider a crystallographic Coxeter group (Weyl group) W . The reflecting hyperplanes are determined by a root system $\Phi \subset V^*$. We fix a positive system $\Phi^+ \subset \Phi$. For a given $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, define an affine hyperplane $H_{\alpha, k}$ by

$$H_{\alpha, k} = \{x \in V \mid \alpha(x) = k\}.$$

We consider the following type of arrangement

$$\mathcal{A}_{\Phi}^{[a, b]} = \{H_{\alpha, k} \mid \alpha \in \Phi^+, k \in \mathbb{Z}, a \leq k \leq b\},$$

where $a \leq b$ are integers. (For example, see Figure 2 for $\mathcal{A}_{G_2}^{[-1, 1]}$.)

3.1. Freeness of Extended Catalan and Shi arrangements

The next result was originally conjectured by Edelman-Reiner [12].

THEOREM 3.1 ([147]). — *Let k be a nonnegative integer.*

- (i) *The cone $c\mathcal{A}_{\Phi}^{[-k, k]}$ of the extended Catalan arrangement $\mathcal{A}_{\Phi}^{[-k, k]}$ is free with exponents $(1, e_1 + kh, \dots, e_{\ell} + kh)$.*
- (ii) *The cone $c\mathcal{A}_{\Phi}^{[1-k, k]}$ of the extended Shi arrangement $\mathcal{A}_{\Phi}^{[1-k, k]}$ is free with exponents $(1, kh, kh, \dots, kh)$.*

Proof. — The proof is done by induction on the rank ℓ of the root system Φ . First one can check for the case $\ell = 2$, $\Phi = A_2, B_2, G_2$ (using Theorem 1.39 and Theorem 2.3, or proving inductive freeness). For $\ell \geq 3$, consider the restriction of $c\mathcal{A}_{\Phi}^{[-k, k]}$ (resp. $c\mathcal{A}_{\Phi}^{[1-k, k]}$) to the hyperplane at infinity H_0 and apply Theorem 1.38. The multirestriction $((c\mathcal{A}_{\Phi}^{[-k, k]})^{H_0}, \mathbf{m}^{H_0})$ (resp. $((c\mathcal{A}_{\Phi}^{[1-k, k]})^{H_0}, \mathbf{m}^{H_0})$) is equal to the multiarrangement $(\mathcal{A}, 2k + 1)$ (resp. $(\mathcal{A}, 2k)$). Thus the second condition in Theorem 1.38 is verified by Theorem 2.3 (ii) (resp. (iii)). The localization of $c\mathcal{A}_{\Phi}^{[-k, k]}$ at $x \in H_0 \setminus \{0\}$ is a direct sum of Coxeter arrangements of lower ranks. Hence the first condition in Theorem 1.38 is verified by the inductive assumption. \square

Using Terao's factorization theorem (Theorem 1.17), we have the following.

COROLLARY 3.2. —

$$(i) \quad \chi(\mathcal{A}_{\Phi}^{[-k,k]}, t) = \prod_{i=1}^{\ell} (t - e_i - kh).$$

$$(ii) \quad \chi(\mathcal{A}_{\Phi}^{[1-k,k]}, t) = (t - kh)^{\ell}.$$

In the above corollary, (i) has been proved by Athanasiadis [9] by a purely combinatorial method. Edelman-Reiner [12] and Headley [18] proved (ii) for some special cases (type A and the case $k = 1$). However as far as we know, the combinatorial proof for (ii) is not known.

3.2. Beyond free arrangements

There are several conjectures on the characteristic polynomials $\chi(\mathcal{A}_{\Phi}^{[a,b]}, t)$.

CONJECTURE 3.3 (“*Riemann hypothesis*” by Postnikov-Stanley, [23]). — *If $0 \leq a < b$ are integers, then all roots of the characteristic polynomial $\chi(\mathcal{A}_{\Phi}^{[-a,b]}, t)$ have the same real part $\frac{(a+b+1)h}{2}$.*

This conjecture has been verified for types ABC and D by Athanasiadis [8]. We also note that for the parameters $b = a + 1$, it is a special case of Theorem 3.1 (ii). Generally it is still an open problem. Conjecture 3.3 implies that the roots of the characteristic polynomial $\chi(\mathcal{A}_{\Phi}^{[-a,b]}, t)$ sit on the line of complex numbers whose real part is $\text{Re} = \frac{(a+b+1)h}{2}$, which concludes the following nontrivial property of the characteristic polynomial.

CONJECTURE 3.4 (“*Functional Equation*” by Postnikov-Stanley, [23]). — *If a, b are integers such that $-1 \leq a \leq b$ (except for $(a, b) = (-1, 0)$ and $(-1, -1)$), then the characteristic polynomial satisfies*

$$\chi(\mathcal{A}_{\Phi}^{[-a,b]}, (a+b+1)h - t) = (-1)^{\ell} \chi(\mathcal{A}_{\Phi}^{[-a,b]}, t). \quad (3.1)$$

Note that the “Functional Equation” is true when $a = b \geq 0$. Indeed, in this case $\chi(\mathcal{A}_{\Phi}^{[-a,a]}, t) = \prod_{i=1}^{\ell} (t - e_i - ah)$. The relation (3.1) is equivalent to the relation so-called duality of exponents:

$$e_i + e_{\ell+1-i} = h, \quad (3.2)$$

$i = 1, \dots, \ell$. Thus the “Functional Equation” can be considered as a generalization of the duality of exponents.

The following is also observed in the work by Athanasiadis [8, 9].

CONJECTURE 3.5. — *If a, b are integers such that $-1 \leq a \leq b$ (except for $(a, b) = (-1, 0)$ and $(-1, -1)$), then the characteristic polynomial satisfies*

$$\chi(\mathcal{A}_{\Phi}^{[-a-1, b+1]}, t) = \chi(\mathcal{A}_{\Phi}^{[-a, b]}, t - h). \quad (3.3)$$

Except for $[a, b] = [-k, k]$ and $[1 - k, k]$, the characteristic polynomial $\chi(\mathcal{A}_{\Phi}^{[a, b]}, t)$ can not be decomposed into linear terms. So the cone $c\mathcal{A}_{\Phi}^{[a, b]}$ is no more free. The simplest such example may be $\Phi = A_3$ ($h = 4$) with $(a, b) = (-1, 1)$. More explicitly, after change of coordinates,

$$Q(\mathcal{A}_{A_3}^{[1, 1]}) = (x - 1)(y - 1)(z - 1)(x + y - 1)(y + z - 1)(x + y + z - 1)$$

$$Q(\mathcal{A}_{A_3}^{[0, 2]}) = \prod_{k=0}^2 (x - k)(y - k)(z - k)(x + y - k)(y + z - k)(x + y + z - k).$$

Then $\chi(\mathcal{A}_{A_3}^{[1, 1]}, t) = (t - 2)(t^2 - 4t + 7)$ and $\chi(\mathcal{A}_{A_3}^{[0, 2]}, t) = (t - 6)(t^2 - 12t + 39)$. It is easily seen that roots have the real part 2, respectively 6, and $\chi(\mathcal{A}_{A_3}^{[0, 2]}, t) = \chi(\mathcal{A}_{A_3}^{[1, 1]}, t - 4)$.

It seems to be interesting to investigate these conjectures through the module $D(\mathcal{A})$ of logarithmic vector fields.

PROBLEM 3.6. — *Prove the above conjectures by using $D(\mathcal{A})$. Is it possible to refine these conjectures in terms of algebraic/geometric structures of the module of logarithmic vector fields?*

Remark 3.7. — Recall that $D_0(\mathcal{A}_{\Phi}^{[a, b]}) \simeq D(\mathcal{A}_{\Phi}^{[a, b]})/S \cdot \theta_E$. In the lecture in Pau (June 2012), the author asked whether or not if the following isomorphisms hold

$$D_0(c\mathcal{A}_{\Phi}^{[-a, b]})^{\vee} \simeq D_0(c\mathcal{A}_{\Phi}^{[-a, b]})[(a + b + 1)h]$$

$$D_0(c\mathcal{A}_{\Phi}^{[-a-1, b+1]}) \simeq D_0(c\mathcal{A}_{\Phi}^{[-a, b]})[-h],$$

which induce Conjecture 3.4 and Conjecture 3.5, respectively via Solomon-Terao's formula Theorem 1.16 (see [5] for details). These seem to be strongly supported by Theorem 2.3. However Professor D. Faenzi pointed out to us (October 2012) that the first isomorphism $D_0(c\mathcal{A}_{\Phi}^{[-a, b]})^{\vee} \simeq D_0(c\mathcal{A}_{\Phi}^{[-a, b]})[(a + b + 1)h]$ does not hold at least for some cases.

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