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YUYA KODA

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The Reidemeister-Turaev torsion of standard Spin^c structures on Seifert fibered 3-manifolds

Yuya Koda[1]

Abstract. — The Reidemeister-Turaev torsion is an invariant of 3-manifolds equipped with Spin^c structures. Here, a Spin^c structure of a 3-manifold is a homology class of non-singular vector fields on it. Each Seifert fibered 3-manifold has a standard Spin^c structure, which is represented as a non-singular vector field the set of whose orbits give a Seifert fibration. We provide an algorithm for computing the Reidemeister-Turaev torsion of the standard Spin^c structure on a Seifert fibered 3-manifold. The machinery used to compute the torsion is that of punctured Heegaard diagrams.

Résumé. — La torsion de Reidemeister-Turaev est un invariant des 3-variétés avec structure Spin^c. Ici, une structure Spin^c d'une 3-variété est une classe d'homologie de champ de vecteurs sans singularités sur elle. Chaque variété de Seifert a une structure Spin^c standard, qui est représentée comme un champ de vecteurs sans singularités dont l'ensemble des orbites donne une fibration de Seifert. Nous fournissons un algorithme pour calculer la torsion de Reidemeister-Turaev de la structure Spin^c standard sur une variété de Seifert. La technique utilisée pour calculer la torsion est celle des diagrammes de Heegaard percés.

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(1) Mathematical Institute, Tohoku University, Sendai, 980-8578, Japan
koda@math.tohoku.ac.jp

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Introduction

The Reidemeister-Turaev torsion is an invariant of 3-manifolds equipped with Spin^c structures. This invariant is defined by Turaev [15] as a refinement of the Reidemeister torsion, which is one of the most classical invariant of 3-manifolds. Here a Spin^c structure is a homology class of non-singular vector fields on the ambient 3-manifold. On the other hand, a branched standard spine of a 3-manifold is a transversely oriented branched surface in the 3-manifold on which the 3-manifold collapses. When the ambient 3-manifold is closed, we require that the collape is performed after removing an open 3-ball from it. A branched standard spine of a 3-manifold carries a non-singular vector field, hence a Spin^c structure. The computation of the Reidemeister-Turaev torsion using branched standard spines is first introduced in [3] for the case with non-empty boundary and then in [1] for the closed case. In [6], the author developed the method via Heegaard splittings compatible with the branched standard spines. In [7], the author introduced a Heegaard-type diagram, which we call a punctured Heegaard diagram, to present a branched standard spine and this diagram allows to compute the Reidemeister-Turaev torsion quite easily. In the case of closed 3-manifolds, a punctured Heegaard diagram is exactly a Heegaard diagram with a special connected component of the complement of the slopes; see Section 1.5.

In this paper, we focus on the Reidemeister-Turaev torsions of Spin^c structures on Seifert fibered 3-manifolds. We recall that each Seifert fibered 3-manifold has a standard Spin^c structure, which is represented by a non-singular vector field everywhere tangent to the Seifert fibration. We develop a method for constructing punctured Heegaard diagrams of Seifert fibered 3-manifolds equipped with standard Spin^c structures and then explain how to compute their Reidemeister-Turaev torsions. In fact, such a punctured Heegaard diagram is produced by small elementary pieces.

The theory of Heegaard diagrams and that of surgery presentations are the most common way to present 3-manifolds and our method bases on the former one. We remark that an algorithm for computing Reidemeister-Turaev torsions of any 3-manifold equipped with any Spin^c structure has been described in [11, 19] by means of surgery presentations on links in S^3, and in some cases, it is much easier to compute the invariant using their methods.

In the final section, we observe that the Reidemeister-Turaev torsions of the standard Spin^c structures of a Seifert fibered 3-manifold take particularly simple forms within the set of the Reidemeister-Turaev torsions of all Spin^c structures on the manifold.

Notation. — Let X be a subset of a given topological space or a manifold Y. Throughout this paper, we will denote the interior of X by $\mathrm{Int}\, X$, the closure of X by $\overline{X}$. We will use $\mathrm{Nbd}(X; Y)$ to denote a regular neighborhood of X in Y. If the ambient space Y is clear from the context, we simply denote it by $\mathrm{Nbd}(X)$. By 3-manifold, we always mean a *connected, compact* and *oriented* one, with or without boundary, unless otherwise mentioned.

1. Preliminaries

1.1. Spin^c structures

Let M be a closed smooth 3-manifold. Two non-singular vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ on M are said to be *homologous* if there exists a closed 3-ball $B \subset M$ such that the restrictions of $\mathcal{V}_1$ and $\mathcal{V}_2$ to $M \setminus \mathrm{Int}\, B$ are homotopic as non-singular vector fields. A *Spin^c structure* is a homology class $[\mathcal{V}]$ of non-singular vector fields $\mathcal{V}$. We denote by $\mathrm{Spin}^c(M)$ the set of Spin^c structure on M. The action of $H_1(M)$ to $\mathrm{Spin}^c(M)$ is defined through Reeb surgery; see [19, 11] for details.

1.2. Review of the Reidemeister-Turaev torsion

Let E be an n-dimensional vector space over the field $\mathbb{C}$ of complex numbers. For two ordered bases $b = (b_1, \ldots, b_n)$ and $c = (c_1, \ldots, c_n)$ of E, we write $[b/c] = \det(a_{ij}) \in \mathbb{C}^\times$, where $b_i = \sum_{j=1}^n a_{ij} c_j$. The bases b and c are said to be *equivalent* if $[b/c] = 1$.

Let $C = (0 \overset{\partial_m}{\to} C_m \overset{\partial_{m-1}}{\to} C_{m-1} \to \cdots \to C_1 \overset{\partial_0}{\to} C_0 \overset{\partial_{-1}}{\to} 0)$ be a finite dimensional chain complex over $\mathbb{C}$. For each $0 \leqslant i \leqslant m$, set $B_i = \mathrm{Im}\ \partial_i$, $Z_i = \mathrm{Ker}\ \partial_{i-1}$ and $H_i = Z_i/B_i$. The chain complex is said to be *acyclic* if $H_i = 0$ for all i. Suppose that C is acyclic and C_i is endowed with a distinguished basis c_i for each i. Choose an ordered set of vectors b_i in C_i for each $0 \leqslant i \leqslant m$ such that $\partial_{i-1}(b_i)$ forms a basis of B_{i-1}. By the above construction, $\partial_i(b_{i+1})$ and b_i are combined to be a new basis $\partial_i(b_{i+1})b_i$ of C_i. With this notation, the *torsion* of C is defined by

$$\tau(C) = \prod_{i=0}^{m} [\partial_i(b_{i+1})b_i/c_i]^{(-1)^{i+1}} \in \mathbb{C}^\times.$$

Let M be a compact connected orientable smooth manifold of arbitrary dimension. Let X be a CW-decomposition of M, $\hat{X} \to X$ be its maximal abelian covering. We can equip $\hat{X}$ with the CW-structure naturally induced by that of X, and then we regard $C_*(\hat{X})$ as a left $\mathbb{Z}[\pi_1(X, *)]$-module via the monodromy. Let $\{e_i^k\}$ be the set of all oriented k-cells in X, and $\{\hat{e}_i^k\}$ be a family of their lifts to $\hat{X}$. Give an orientation with each of these cells and order the cells $\{\hat{e}_i^k\}$, for each k, in an arbitrary way. Then this family gives an ordered $\mathbb{Z}[H_1(X)]$-basis of $C_k(\hat{X})$. In this way, we can regard $C_*(\hat{X})$ as an ordered, based chain complex.

Let $\varphi : \mathbb{Z}[H_1(X)] \to \mathbb{C}$ be a ring homomorphism. If the based chain complex $C_*^\varphi(X) = \mathbb{C} \otimes_\varphi C_*(\hat{X})$ over $\mathbb{C}$ is acyclic, the (φ*-twisted*) *Reidemeister torsion* of M is defined as

$$\tau^\varphi(M) = \tau(C_*^\varphi(X)) \in \mathbb{C}^\times / \pm \varphi(H_1(M)).$$

Otherwise, set $\tau^\varphi(M) = 0 \in \mathbb{C}$. The Reidemeister torsion is a topological invariant of smooth manifolds; see e.g. [11, 18, 19].

Let M be a smooth 3-manifold and let X be its CW-decomposition. A family of cells of $\hat{X}$ is said to be *fundamental* if over each cell of X exactly one cell of this family lies. When we choose a fundamental family $\{\hat{e}_i^k\}$ of cells of $\hat{X}$ and orient and order these cells in arbitrary way, this family becomes a free $\mathbb{Z}[H_1(X)]$-basis of $C_k(\hat{X})$. (i.e. $C_k(\hat{X}) = \bigoplus_i \mathbb{Z}[H_1(X)]\hat{e}_i^k$). In this way, we can regard $C_*(\hat{X})$ as a chain complex equipped with basis.

A Spin^c structure $[\mathcal{V}]$ on M *determines* a fundamental family of cells of $\hat{X}$, and hence the Reidemeister torsion is refined to be an invariant $\tau^\varphi(M, [\mathcal{V}]) \in \mathbb{C}/\pm 1$ of Spin^c structures on M; see [15, 16, 18, 19]. In [1, 3, 6, 7], this construction is described via the notion of *branched standard spine.*

Let M be a Seifert fibered 3-manifold. In this paper, all Seifert fibered 3-manifolds are assumed to be closed orientable ones having orientable base surfaces. Recall that a Seifert fibered 3-manifold is said to be *large* if its base surface is different from a sphere with less than four singular points.

We call a non-singular vector field (a Spin^c structure, respectively) on a Seifert fibered 3-manifold *standard* if it is everywhere tangential to a Seifert fibration. In [14], Taniguchi, Tsuboi and Yamashita introduced an algorithm to obtain a *branched spine* of a standard vector field on an arbitrary closed Seifert fibered 3-manifold with the Seifert invariants $(g; b; (p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r))$, where g is the genus of the base surface, b is its obstruction class, and $(p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r)$ are the types of its singular fibers. It is well-known that a large Seifert fibered 3-manifold except the one with the Seifert invariants $(0; 4; (2, 1), (2, 1), (2, -1), (2, -1))$ has a unique (up to isotopy) Seifert fibration; see e.g. [5].

1.3. Branched spines

Let N be a compact orientable 3-manifold. A branched surface $P \subset N$ is a union of finitely many compact smooth surfaces glued together to form a compact subspace locally modeled on one of the three possibilities in Figure 1.

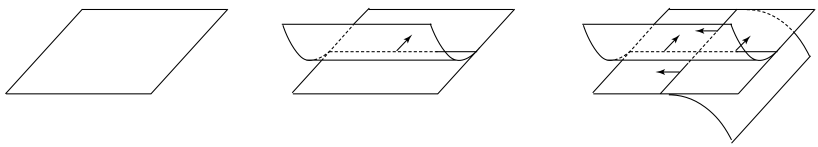

Figure 1. — Local pictures of a branched surface.

We note that the general definition of branched surface allows more sheets than just two on one side and one on the other side, but we only consider this situation (which is generic and stable, i.e. corresponds to an open dense set in the space of branched surfaces).

The *branch locus* $S(P)$ of P is the set of points none of whose neighborhoods in P is a disk. $S(P)$ is a collection of smooth immersed curves in P. Let $V(P)$ be the set of double points of $S(P)$. We associate with every component of $S(P) \setminus V(P)$ a vector (in P) pointing in the locally one-sheeted direction, as shown in Figure 2. We call a component of $P \setminus S(P)$ a *region* of P. Let R be a region of P. If all branch directions along $\partial \overline{R}$ point out from R, then $P \setminus R$ is still a branched surface, see Figure 2 (i).

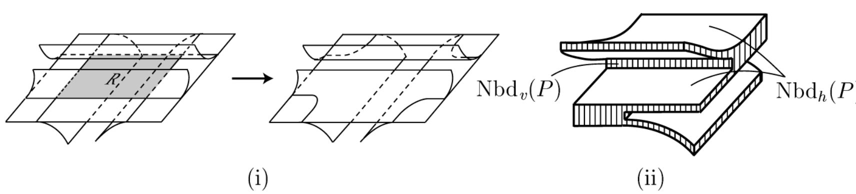

Figure 2. — (i) Removable region; (ii) A regular neighborhood of a branched surface.

One can regard Nbd(P) as an interval bundle over P as drawn in Figure 2 (ii). The boundary ∂Nbd(P) decomposes into two parts: the union ∂_hNbd(P) of endpoints of the fibers, and the rest ∂_vNbd(P). In this paper, all branched surfaces are assumed to be *transversely oriented*, that is, P is equipped with a global orientation on the 1-foliation of Nbd(P) whose leaves are fibers of Nbd(P). Refer to [4, 12] for more details about branched surfaces.

A branched surface $P \subset N$ is called a *branched spine* of N if N collapses onto P. A branched spine P is naturally stratified as $V(P) \subset S(P) \subset P$. A branched spine P is said to be *standard* if this stratification induces a CW decomposition of P, that is, there is no circle components in $S(P)$ and all regions are disks. See [2] for a precise definition. If P is a branched spine of a compact 3-manifold N with $\partial N = S^2$, then P is also called a branched spine of the closed 3-manifold M obtained from N by attaching a 3-ball to the unique 2-sphere boundary. A branched spine of a closed 3-manifold is called a *flow-spine* if $\partial_v \mathrm{Nbd}(P)$ is a single annulus.

In [2], Benedetti and Petronio proved that every orientable 3-manifold admits a branched standard spine and it naturally encodes a homotopy class of so-called *concave traversing fields* on the ambient manifold. We require that the flow intersects P in the same direction as the fixed transverse orientation. In the case where P is a flow-spine of a closed oriented 3-manifold M, one can extend the concave traversing field, whose orbits are the I-fibers of a regular neighborhood of the spine, to the whole of M.

1.4. Oriented, based Heegaard diagrams

Throughout the paper, we only consider closed orientable 3-manifolds.

By a *Heegaard diagram* we mean a triple $(S_g; \alpha, \beta)$, where

1. S_g is a closed, connected, orientable surface of genus $g \in \mathbb{N}$;
2. $\alpha = \bigcup_{i=1}^{g} \alpha_i$ and $\beta = \bigcup_{j=1}^{g} \beta_j$ are compact, mutually transverse 1-manifolds on S_g, each consisting of g components; and
3. both $S \backslash \mathrm{Int}\,\mathrm{Nbd}(\alpha; S)$ and $S \backslash \mathrm{Int}\,\mathrm{Nbd}(\beta; S)$ are 2-spheres with $2g$holes.

A Heegaard diagram gives rise to a closed 3-manifold $M_{(S_g;\alpha,\beta)}$ by adding 2-handles $H_{\alpha_1}, \dots, H_{\alpha_g}$ and $H_{\beta_1}, \dots, H_{\beta_g}$ to $S_g \times [-1, 1]$ along the curves $\alpha_1 \times \{-1\}, \dots, \alpha_g \times \{-1\}$ and $\beta_1 \times \{1\}, \dots, \beta_g \times \{1\}$, respectively, and then adding 3-handles along the resulting 2-sphere boundary components. We will denote the core disk of H_{α_i} (H_{β_i}, respectively) (fairly extended in such a way that its boundary is on S_g) by D_{α_i} (D_{β_i}, respectively) for $1 \leqslant i \leqslant g$. When we consider (and draw in $\mathbb{R}^3$) a Heegaard diagram, we always equip the surface S_g with the positive normal w_p ($x \in S_g$) pointing toward the α side, and with the orientation $(\boldsymbol{u}_p, \boldsymbol{v}_p)$, $\boldsymbol{u}_p, \boldsymbol{v}_p \in T_p S_g$, such that $(\boldsymbol{u}_p, \boldsymbol{v}_p, \boldsymbol{w}_p)$ gives the right-hand orientation on $\mathbb{R}^3$.

A Heegaard diagram is said to be *oriented* if the 1-manifolds α and β are oriented. A Heegaard diagram $(S_g; \alpha, \beta)$ with a fixed point $b_i \in \beta_i \setminus \alpha$ for each β_i is said to be *based*. A Heegaard diagram $(S_g; \alpha, \beta)$ is said to be *standard* if every connected component of $S_g \setminus (\alpha \cup \beta)$ is an open ball. It is clear that we can make any Heegaard diagram standard up to isotopy of

β. In this paper, all Heegaard diagrams are assumed to be standard. We often denote an oriented, based Heegaard diagram by $(S_g; \vec{\alpha}, \vec{\beta}, \{b_k\}_{k=1}^g)$. The set $\{\gamma_1, \gamma_2, \ldots, \gamma_g\}$ of pairwise disjoint, simple, closed, oriented curves on S_g is called a *dual system* of β if each γ_i intersects β_i transversely once at the point b_i in the positive direction shown in Figure 3, where (u_x, v_x) is compatible with the fixed orientation of S_g, and $\gamma_i \cap \beta_j = \emptyset$ when $i \neq j$.

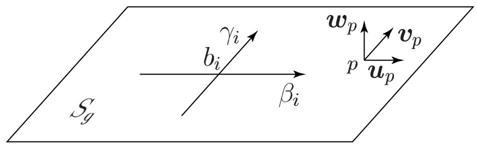

Figure 3. — The positive intersection with a dual loop.

1.5. Punctured Heegaard diagrams

Let $(S_g; \alpha, \beta)$ be a Heegaard diagram. A disk component D of $S_g \setminus (\alpha \cup \beta)$ is called a *joining disk* if it satisfies the following (see Figure 4):

1. $\partial \overline{D}$ is a simple loop, where the closure is taken in the surface S_g; and
2. $\partial \overline{D} \cap \alpha_i$ ($\partial \overline{D} \cap \beta_i$, respectively) is a single connected arc for all $1 \leqslant j \leqslant g$.

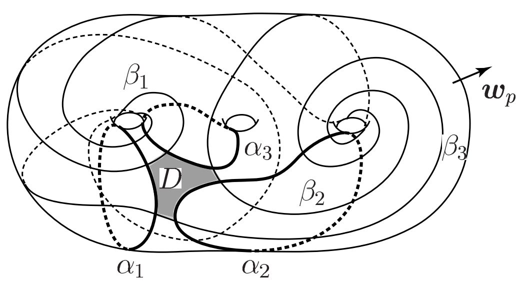

Figure 4. — A punctured Heegaard diagram of genus 3.

We call a Heegaard diagram $(S_g; \alpha, \beta)$ with joining disk D a *punctured Heegaard diagram* and denote it by $(S_g; \alpha, \beta; D)$. Given a punctured Heegaard diagram $(S_g; \alpha, \beta; D)$, we may equip the polyhedron

$$S_g \cup \left(\bigcup_{i=1}^{g} D_{\alpha_i} \right) \cup \left(\bigcup_{i=1}^{g} D_{\beta_i} \right) \subset M_{(S_g; \alpha, \beta)}$$

a structure of branched surface in such a way that the disk D is a removable region. Moreover, for this branch structure

$$P_{(S_g;\alpha,\beta;D)} = \left(S_g \cup \left(\bigcup_{i=1}^{g} D_{\alpha_i} \right) \cup \left(\bigcup_{i=1}^{g} D_{\beta_i} \right) \right) \setminus \operatorname{Int} D \subset M_{(S_g;\alpha,\beta)}$$

becomes an transversely-oriented flow-spine of $M_{(S_g;\alpha,\beta)}$. We denote by $\mathcal{V}_{P_{(S_g;\alpha,\beta;D)}}$ a vector field on $M_{(S_g;\alpha,\beta;D)}$ obtained by extending the concave traversing field on a regular neighborhood of $P_{(S_g;\alpha,\beta;D)}$, see Section 1.3. We note that such a vector field $\mathcal{V}_{P_{(S_g;\alpha,\beta;D)}}$ is uniquely defined up to homotopy.

Each punctured Heegaard diagram $(S_g;\alpha,\beta)$ defines an oriented, based Heegaard diagram as in the following way.

- Since each of the slopes α and β appears on $\partial\overline{D}$ exactly as a single arc, the orientation of $\partial\overline{D}$ determines orientations of all of these slopes. Here, we consider that D inherits the orientation from S_g and we use "outernormal first" convention.
- For each $1 \leqslant i \leqslant g$, take a base point b_i on the interior of the arc $\beta_i \cap \partial\overline{D}$.

Let $(S_g;\vec{\alpha},\vec{\beta};\{b_k\}_{k=1}^{g})$ be an oriented, based Heegaard diagram and set $M = M_{(S_g;\alpha,\beta)}$. Let p be a point on α_i. Then we define the normal vector $\boldsymbol{n}_p \in T_pS_g$ of α_i at p in such a way that $(\boldsymbol{n}_p, \boldsymbol{a}_p)$ is coherent with the fixed orientation of S_g, where $\boldsymbol{a}_p \in T_p\alpha_i$ is coherent with the orientation of α_i. Then α_i determines an element $x_i \in \pi_1(M,*)$ and β_j determines $r_j = r_j(x_1,\ldots,x_g) \in \pi_1(M,*)$ starting at the point b_j and following the oriented loop β_j, for each $i,j = 1,\ldots,g$. Namely, we use the convention such that at each point $p \in \alpha_i \cap \beta_j$ we read x_i (${x_i}^{-1}$, respectively) when the normal vector $\boldsymbol{n}_p \in T_pS_g$ of α_i at p is coherent (not coherent, respectively) with the orientation of β_j at p.

Moreover, if we choose a dual system $\gamma = \bigcup_{i=1}^{g} \gamma_i$ of β, γ_i determines $y_j \in \pi_1(M,*)$ in the same manner. Let $p : \mathbb{Z}[\pi_1(M,*)] \to \mathbb{Z}[H_1(M)]$ be the canonical projection and denote $[z] = p(z)$ for $z \in \pi_1(M,*)$.

Before stating a formula of the Reidemeister-Turaev torion, we recall the notion of *Fox's free differential calclus*. Suppose G is a group presented by $\langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$. For each $1 \leqslant i \leqslant n$, the Fox's free differential calculus with respect to x_i is a map $\frac{\partial}{\partial x_i} : G \to \mathbb{Z}G$ defined by

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij},\ \frac{\partial {x_j}^{-1}}{\partial x_i} = \delta_{ij}{x_j}^{-1},\ \frac{\partial (uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i},$$

where δ_{ij} is the Kronecker delta.

Proposition 1.1. — *Let (S_g,α,β) be a punctured Heegaard diagram and set $M = M((S_g,\alpha,\beta))$. Let $(S_g;\vec{\alpha},\vec{\beta};\{b_j\})$ be an oriented, based Heegaard diagram defined by (S_g,α,β). Let the twisted chain complex $C_*^{\varphi}(M)$ be acyclic. Then there exist two integers $k,l \in \{1,\dots,n\}$ such that*

$$\tau^{\varphi}(M,[\mathcal{V}_{(S_g;\alpha,\beta;D)}]) = \frac{\det B_{k,l}}{(\varphi([x_k])-1)(\varphi([y_l])-1)} \in \mathbb{C}^{\times}/\pm 1,$$

where $B_{k,l}$ is the (k,l)-minor of the matrix $\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1\leqslant i,j\leqslant g}$, namely the matrix obtained by removing k-th row and l-th column from the matrix $\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1\leqslant i,j\leqslant g}$. Here, if $B_{k,l}=\emptyset$, we set $\det B_{k,l}=1$.

The above proposition is obtained in [7]. Indeed, this is obtained from the fact that the presentation matrices of the boundary operators $\partial_2 : C_3^{\varphi}(M) \to C_2^{\varphi}(M)$, $\partial_1 : C_2^{\varphi}(M) \to C_1^{\varphi}(M)$ and $\partial_0 : C_1^{\varphi}(M) \to C_0^{\varphi}(M)$ with respect to the bases on $\{C_i^{\varphi}(M)\}_{i=0}^{3}$ determined by the given oriented, based Heegaard diagram are

$$\begin{pmatrix} \varphi([y_1])-1 \\ \varphi([y_2])-1 \\ \vdots \\ \varphi([y_g])-1 \end{pmatrix}, \quad \left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1\leqslant i,j\leqslant g}$$

$$\text{and } \begin{pmatrix} \varphi([x_1])-1 & \varphi([x_2])-1 & \cdots & \varphi([x_g])-1 \end{pmatrix},$$

respectively. See also Theorem 1.2 in [19] and Section 2.1 of [18] for more basic facts on the computation of the Reidemeister torsion.

1.6. BW-decompositions and DS-diagrams

In this subsection, we review the notion of BW-decomposition of the boundary of a compact 3-mannifold. See [2, Section 3.3] for more details.

Let P be a flow-spine of a closed 3-manifold M. Let N be a regular neighborhood of P. Recall that $\partial N \cong S^2$. Then the collapsing $N \searrow P$ induced a retraction π such that N is the mapping cylinder of $\pi|_{\partial N} : \partial N \to P$. This map satisfies the following.

1. $\pi^{-1}(S(P)) \cap \partial N$ is a trivalent graph;
2. For $x \in P$, $\phi^{-1}(x)$ consists of 2, 3 or 4 points according as $x \in P \setminus S(P)$, $x \in S(P) \setminus V(P)$ or $x \in V(P)$; and
3. There exists a circle e in $\pi^{-1}(S(P)) \cap \partial N$ such that

(a) $\partial N \setminus e$ is the disjoint union of B and W (this is called a *Black and White* (or simply *B-W*) *decomposition*);
(b) Every component of e has B on one side and W on the other side;
(c) π maps $e \setminus \pi^{-1}(V(P))$ bijectively onto $S(P) \setminus V(P)$; and
(d) π maps B (W, respectively) bijectively onto P.

The left-hand side of Figure 5 depicts the B-W decomposition of ∂N. In the figure, the arrows show the concave traversing field on N defined by the branched spine P. We remark that the curve e consists of the concave points on the boundary. The right-hand side shows the trivalent graph $\pi^{-1}(S(P)) \cap \partial N$. In the figure, the arrow shows the retraction π induced by the collapsing.

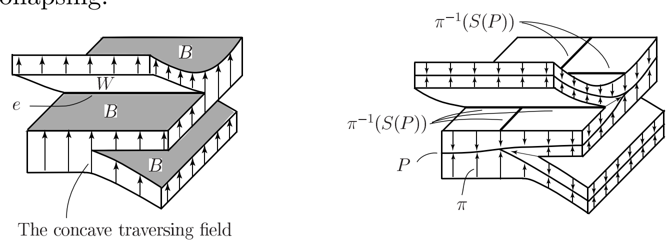

Figure 5. — The B-W decomposition of ∂N.

The above description provides a way to present the flow-spine P by a 3-regular graph $G = \pi^{-1}(S(P)) \cap \partial N \subset \partial N \cong S^2$ and the pairing on S^2 given by π. This presentation is called a *DS-diagram*.

2. The Reidemeister-Tureav torsions of the standard Spin^c structures

In this section, we introduce an algorithmic method for computing the Reidemeister-Turaev torsion of a Seifert fibered 3-manifolds with a standard Spin^c structure. The input datum is Seifert invariants.

2.1. Construction of punctured Heegaard diagrams of standard Spin^c structures

In this subsection, we introduce a way to produce a punctured Heegaard diagram of a standard Spin^c structure of a Seifert fibered 3-manifold. The construction bases on the fact that each Seifert fibered 3-manifold decomposes into finite copies of the pieces (trice-punctured sphere) $\times\, S^1$, (once-punctured torus) $\times$ S^1 and a fibered torus by cutting along tori on which the fibers are tangential. That is, we define the pieces of a punctured Heegaard diagram corresponding to this decomposition and the required punctured Heegaar diagram is obtained by gluing them together.

Let H_R, H_L, $H_{\overline{R}}$, $H_{\overline{L}}$ and H_C be the pieces of a punctured Heegaard diagram shown in Figure 6. In the figure, the curves α are bold and the curves β are thin. In the figure, the holes D^- and D^+ are identified in such a way that the indices of the vertices match, hence H_R, H_L, $H_{\overline{R}}$ and $H_{\overline{L}}$ are diagrams on tori with two holes while H_C is a diagram on a torus with a single hole.

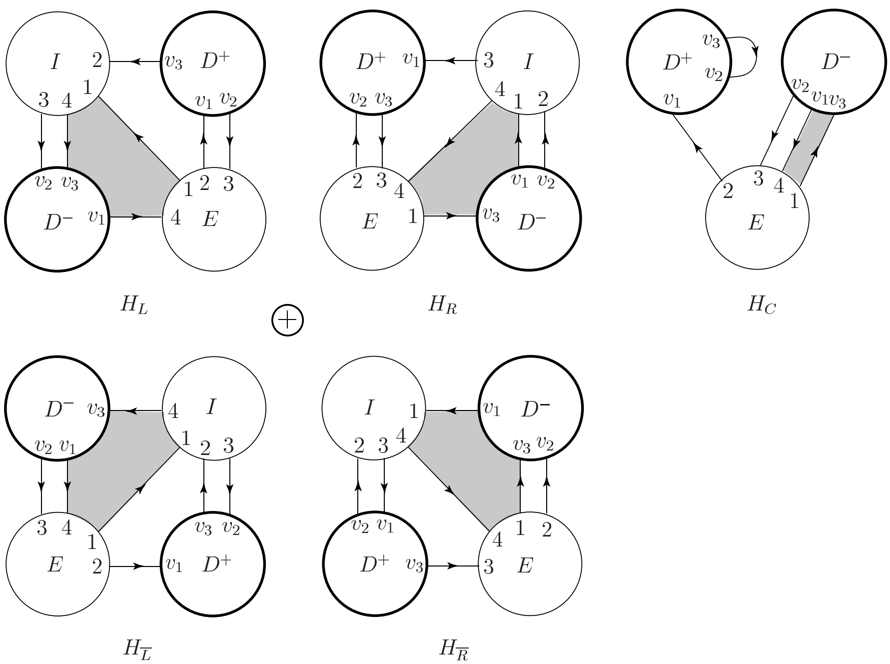

Figure 6. — The pieces H_L, H_R, $H_{\overline{L}}$, $H_{\overline{R}}$ and H_C of a punctured Heegaard diagram.

We use the following notation for a continued fraction

$$[a_1, a_2, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

For a pair (p, q) of relatively prime natural numbers such that $p > q$, we define a word $w(p, q)$ of the letters L and R as follows:

$$w(p,q) = \begin{cases} L^{a_1} R^{a_2} L^{a_3} \cdots L^{a_{n-2}} R^{a_{n-1}} L^{a_n} & \text{(if } n \text{ is odd)} \\ L^{a_1} R^{a_2} L^{a_3} \cdots R^{a_{n-2}} L^{a_{n-1}} R^{a_n} & \text{(if } n \text{ is even),} \end{cases}$$

where $a_1, a_2, \ldots, a_n$ are natural numbers with $q/p = [a_1, a_2, \ldots, a_n, 1]$.

Given a word $w(p, q)$, where $q/p = [a_1, a_2, \ldots, a_n, 1]$, we construct a piece of punctured Heegaard diagram $H_{(p,q)}$, which corresponds to a fibered solid torus of type (p, q), in the following way. We first take a_1 copies of the diagram H_L. We identify the boundary circle ∂E of the diagram H_C with the boundary circle ∂I of the first diagram H_L in such a way that the numbers $1, 2, 3, 4$ on the both boundary circles match. Also we identify the boundary circle ∂E of the i-th diagram H_L with the boundary circle ∂I of the $(i + 1)$-th in such a way that the numbers $1, 2, 3, 4$ on the both boundary circles match, for each $i = 1, 2, \ldots, a_1 - 1$. Now we have a diagram on the genus $1 + a_1$ surface with one boundary component, which is ∂E of the a_1-th H_L. Next, we take a_2 copies of the diagram H_R. We identify the boundary circle ∂E of the above a_1-th diagram H_L with the boundary circle ∂I of the first diagram H_R. Then we indentify the boundary ∂E of the j-th diagram H_R with the boundary circle ∂I of the $j + 1$-th one for each $j = 1, 2, \ldots, a_2 - 1$. Continuing this process, we finally get a diagram by gluing $1 + \sum_{i=1}^{n} a_i$ pieces of H_L, H_R and H_C as shown in Figure 7.

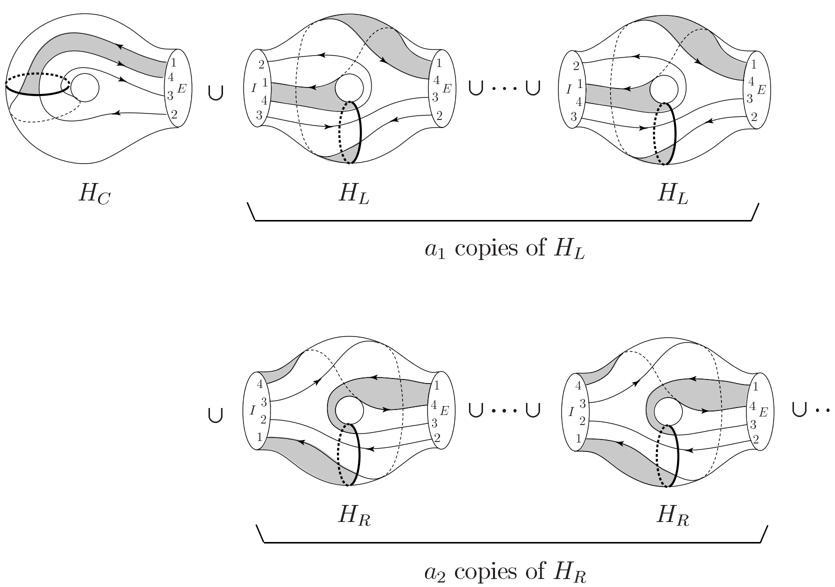

Figure 7. — The piece $H_{(p,q)}$ of a punctured Heegaard diagram.

We denote the resulting piece of a punctured Heegaard diagram by $H_{(p,q)}$.

We define H_b ($b \in \mathbb{Z}$) to be another piece of a punctured Heegaard diagram constructed following the same argument using the word $LR^{b+1}\overline{L}$, where $R^n = \bar{R}^{-n}$ when n is negative.

Let H_S and H_T be the pieces of a punctured Heegaard diagram shown in Figure 8 and 9, respectively. These pieces correspond to either thrice-punctured sphere) $\times S^1$ and (once-punctured torus)$\times S^1$, respectively. Again, we consider that the curves α are bold and the curves β are thin in the figure.

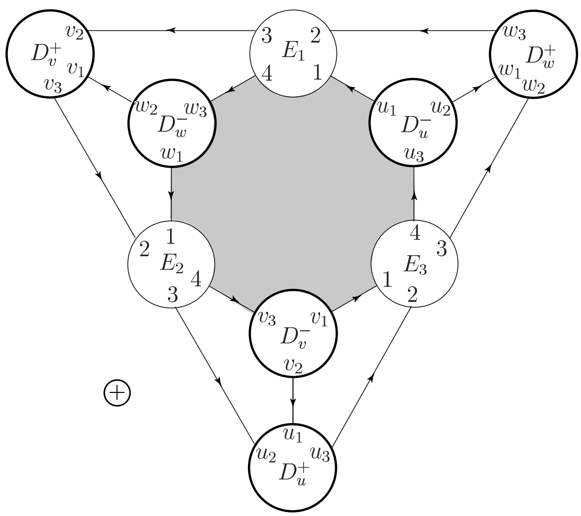

Figure 8. — The piece H_S of a punctured Heegaard diagram.

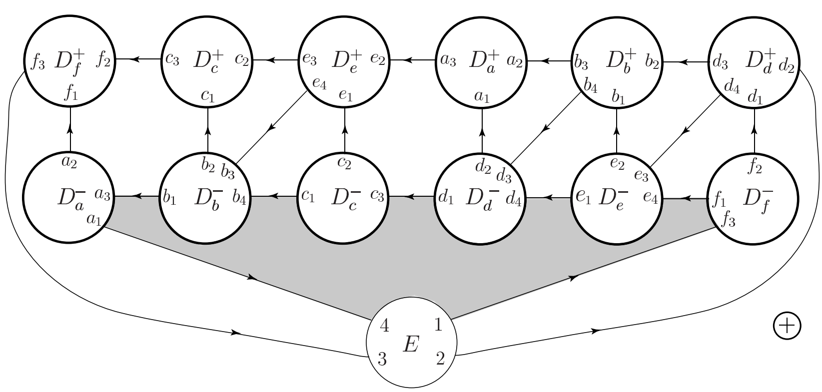

Figure 9. — The piece H_T of a punctured Heegaard diagram.

Let g be a non-negative integer and b be an integer. Let $(p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r)$ be pairs of relatively prime integers such that $1 < p_i$ and $0 < q_i < p_i$ $(i = 1, 2, \ldots, r)$. For each $i = 1, 2, \ldots, r$, let q'_i be an integer satisfying $q_i q'_i \equiv 1 \pmod p)_i$ and $0 < q'_i < p_i$.

Assume that $g + r \geqslant 2$. Prepare $g + r - 1$ copies $H_S^1, H_S^2, \ldots, H_S^{g+r-1}$ of the piece H_S and g copies $H_T^1, H_T^2, \ldots, H_T^g$ of the piece H_T. We recall that the piece H_S is a diagram on a pair of pants. First, we identify the boundary circle ∂E_3 of H_S^i with the boundary circle ∂E_1 of H_S^{i+1} in such a way that the numbers $1, 2, 3, 4$ on the both boundary circles match. Now we have a diagram on the 2-sphere with $(g + r + 1)$ holes. Then we attach the totally $g+r+1$ boundary circles of the diagrams H_b, H_{p_i, q'_i} $(1 = 1, 2, \ldots, r)$ and $H_T^1, H_T^2, \ldots, H_T^g$ to the $g+r+1$ boundary circles of the above diagram in such a way that the numbers $1, 2, 3, 4$ on the boundary circles match. If $g+r < 2$, we use just only a single H_S and we attach the copies of H_C to all the remaining boundary circles of H_S. We denote the resulting punctured Heegaard diagram by $H_{(g;b;(p_1,q_1),(p_2,q_2),\ldots,(p_r,q_r))}$.

Theorem 2.1. — *The punctured Heegaard diagram* $H_{(g;b;(p_1,q_1),(p_2,q_2),\ldots,}$ $_{(p_r,q_r))}$ *corresponds to the Seifert fibered* 3*-manifold with the Seifert invariants* $(g; b; (p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r))$ *with a standard Spin*^{c} *structure.*

Proof. — The idea of the proof is to construct the pieces of the punctured Heegaard diagram corresponding to the pieces of the DS-diagram constructed in [14] following the proof of Theorem 5.5 in [7].

Let π, B, W and e be as described in Section 1.6. Set $A = \mathrm{Nbd}(e; \partial\mathrm{Nbd}(P))$. Recall that e has the B part on one side and the W one on the other side. The key idea is to draw a simple closed curve C in A such that

1. C is isotopic to e in A;
2. $C \cap e \neq \emptyset$ and C intersects e transversely; and
3. $C \cap \pi^{-1}(S(P)) \subset e \setminus \pi^{-1}(V(P))$.

Let $\mathcal{H}_L$ be a piece of DS-diagram (on the annulus) shown in Figure 10 (i). This diagram was constructed in [14]. The curve e lies horizontally in the middle part of the diagram and it separates the diagram into B-part, on the upper side, and W-part, on the lower side.

Then the intersection $C \cap \mathcal{H}_R$ is depicted by the two bold curves in Figure 10 (ii). The curves $C \cap \mathcal{H}_R$ cut the annulus into two disks, the under piece of which corresponds to the joining disk. We note that the D^- shown in the figure is identified via the projection π with D^+. Figure 10 (iii) illustrates the same diagram as (ii) but after smoothing the corners. Now we get a piece H_L of a punctured Heegaard diagram as shown in Figure 11 by gluing the two disks D^+ and D^-.

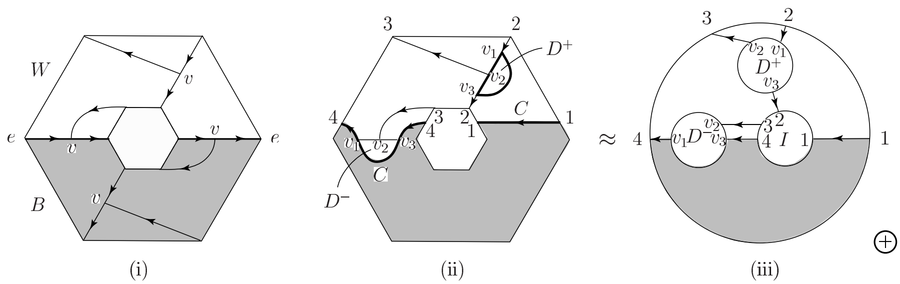

Figure 10. — From $\mathcal{H}_L$ to H_L.

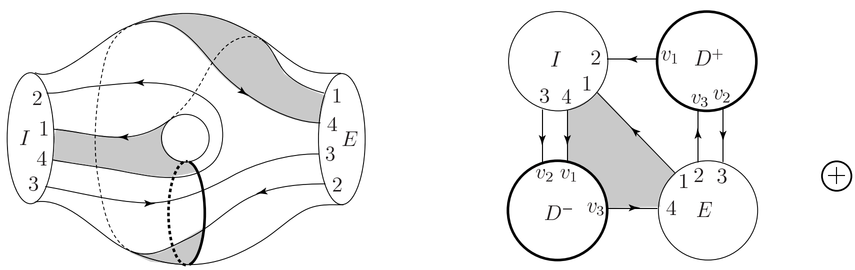

Figure 11. — The piece H_L of a punctured Heegaard diagram.

We can apply the same argument for the other pieces shown in [14]. Consequently, we get the assertion. □

Forgetting the joining disk of the diagram $H_{(g;b;(p_1,q_1),(p_2,q_2),\dots,(p_r,q_r))}$, one has a Heegaard diagram of the Seifert fibered 3-manifold with the Seifert invariants $(g;\ b;\ (p_1,q_1),\ (p_2,q_2),\dots,(p_r,q_r))$. We note that each piece $H_{(p_i,q_i')}$ of the Heegaard diagram $H_{(g;b;(p_1,q_1),(p_2,q_2),\dots,(p_r,q_r))}$ corresponding to a singular fiber of type (p_i,q_i) can be destabilized so that it is a diagram on the torus with a single hole.

2.2. Algorithm

Let M be a Seifert fibered 3-manifold with the Seifert invariants $(g;b;$ $(p_1,q_1),(p_2,q_2),\dots,(p_r,q_r))$. Set $(S_g;\alpha,\beta,D) = H_{(S(g;b;(p_1,q_1),(p_2,q_2),\dots,(p_r,q_r))}$. Let $\varphi : \mathbb{Z}[H_1(M_{(S;\alpha,\beta;D)})] \to \mathbb{C}$ be a ring homomorphism. We can calculate the Reidemeister-Turaev torsion of the standard Spin^c structure of M in the following algorithmic way (cf. [7]):

Step 1 Orient α and β, and take base points of β following the rule prescribed in Section 1.

Step 2 Get a presentation $\langle x_1, \ldots, x_g \mid r_1, \ldots, r_g \rangle$ of $\pi_1(M, *)$ using the punctured Heegaard diagram $(S; \alpha, \beta; D)$ as in the rule of Section 1.5.

Step 3 Find an arbitrary dual system γ of β in the diagram $(S; \alpha, \beta; D)$ and relate a word y_i of $x_1, \ldots, x_g$ to each loop γ_i in γ as explained in Section 1.5.

Step 4 If there exist two integers $k, l \in \{1, \ldots, g\}$ such that all of $\det B_{k,l}$, $\varphi([y_l]) - 1$ and $\varphi([y_l]) - 1$ are nonzero, then we have

$$\tau^{\varphi}(M, \mathcal{V}_{st}) = \pm \frac{\det B_{k,l}}{(\varphi([x_k]) - 1)(\varphi([y_l]) - 1)} \in \mathbb{C}^{\times} / \pm 1,$$

where $B_{k,l}$ is the (k, l)-minor of the matrix $\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1 \leqslant i,j \leqslant g}$ due to Proposition 1.1. If there are not such integers k and l, then it turns out that the twisted chain complex $C^{\varphi}(M)$ is not acyclic, hence we have $\tau^{\varphi}(M, \mathcal{V}_{st}) = 0$ by definition.

We remark that due to [8] and [17], the above also gives a purely combinatorial algorithm to compute the Seiberg-Witten invariant of a standard Spin^c structure for a Seifert fibered 3-manifold with positive first Betti number.

3. Examples and observations

3.1. Lens spaces

Recall that the Reidemeister torsion of lens spaces is first computed in [13] by the pioneer of the invariant. Using the algorithm in Section 2.2 for a lens space $L(p, q)$, a presentation of $\pi_1(L(p, q))$ corresponding to a standard Spin^c structure is $\langle x \mid x^p \rangle$ after simplifying the generators and relators. Then for a representation $\varphi : H_1(L(p, q); \mathbb{Z}) \to \mathbb{C}^{\times}$, $[x] \mapsto \zeta$, we have $\tau^{\varphi}(L(p, q), [\mathcal{V}_{st}]) = \pm 1/(\zeta - 1)(\zeta^r - 1)$; see [7] for more details of this computation. We remark that even for lens spaces, finding the *combinatorial Euler structures* (see [18, 19]) corresponding to their standard Spin^c structures is not straightforward without punctured Heegaard diagrams.

Let us focus on the lens space $L(11,1)$. The set of the Reidemeister-Turaev torsions of the Spin^c structures of $L(11,1)$ is

$$\left\{ \pm\frac{\zeta^i}{(\zeta-1)^2} \in \mathbb{C}^\times/\pm 1 \;\middle|\; 0 \leqslant i < 11 \right\}.$$

In this set, only the two values $\pm 1/(\zeta-1)^2$ and $\pm\zeta^2/(\zeta-1)^2$ can be modified so that the numerator is ± 1 and the denominator are the form of $(\zeta^{m_1}-1)(\zeta^{m_2}-1)$ for some $m_1, m_2 \in \mathbb{Z}$. In fact, we have $\pm\zeta^2/(\zeta-1)^2 = \pm 1/(\zeta^{10}-1)^2$. Here we note that the value $\pm 1/(\zeta-1)^2$ is the torsion of the Spin^c structure derived from the standard Seifert fibration of $(L(11,1))$ and $\pm\zeta^2/(\zeta-1)^2$ is that of the Spin^c structure derived from the standard Seifert fibration of $(L(11,10))$.

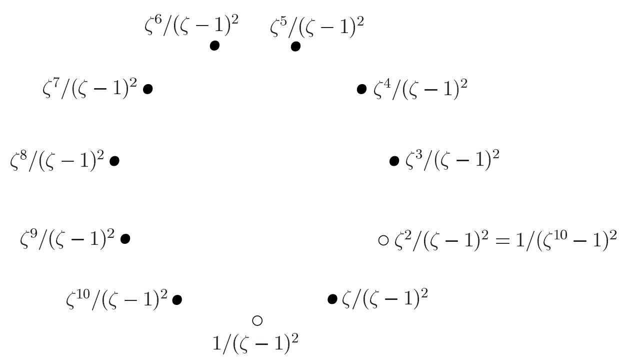

Figure 12. — The set of Spin^c structures on $L(11,1)$ and their Reidemeister-Turaev torsions (the signs $\pm$ are omitted). The white dots are the standard Spin^c structures.

Next, consider the lens space $L(11,2)$. For this manifold, the set of the Reidemeister-Turaev torsions of the Spin^c structures is

$$\left\{ \pm\frac{\zeta^i}{(\zeta-1)(\zeta^6-1)} \in \mathbb{C}^\times/\pm 1 \;\middle|\; 0 \leqslant i < 11 \right\}.$$

In this set, exactly the four values $\pm 1/(\zeta-1)(\zeta^6-1)$, $\pm\zeta/(\zeta-1)(\zeta^6-1)$, $\pm\zeta^6/(\zeta-1)(\zeta^6-1)$ and $\pm\zeta^7/(\zeta-1)(\zeta^6-1)$ can be modified so that the numerator is ± 1 and the denominator are the form of $(\zeta^a-1)(\zeta^b-1)$ for some $a, b \in \mathbb{Z}$. In fact, we have $\pm\zeta/(\zeta-1)(\zeta^6-1) = \pm 1/(\zeta^6-1)(\zeta^{10}-1)$, $\pm\zeta^6/(\zeta-1)(\zeta^6-1) = \pm 1/(\zeta-1)(\zeta^5-1)$ and $\pm\zeta^7/(\zeta-1)(\zeta^6-1) = \pm 1/(\zeta^5-1)(\zeta^{10}-1)$.

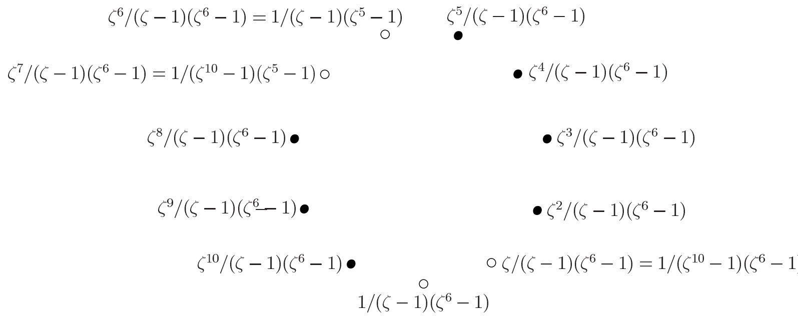

Figure 13. — The set of Spin^c structures on $L(11,2)$ and their Reidemeister-Turaev torsions (the signs $\pm$ are omitted). The white dots are the standard Spin^c structures.

Observation 3.1. — The Reidemeister-Turaev torsion of a Spin^c structure of a lens space is of the form $\pm 1/(\zeta^{m_1}-1)(\zeta^{m_2}-1)$ with $m_1, m_2 \in \mathbb{Z}$ if and only if the Spin^c structure is standard.

3.2. The product space $S_g \times S^1$

Let S_g be a closed orientable surface of genus $g > 1$ and consider the Seifert fibered 3-manifold $S_g \times S^1$, whose Seifert invariants are $(g; 0; \emptyset)$. Using the algorithm in Section 2.2 for $S_g \times S^1$, we get a Spin^c structure $\mathcal{V}_{st}$ on $S_g \times S^1$ and a presentation of $\pi_1(S_g \times S^1)$ corresponding to the Spin^c structure is

$$\langle x_1, x_2, \ldots, x_{2g}, y \mid x_i y {x_i}^{-1} y^{-1}, i = 1, 2, \ldots, 2g, \prod_{i=1}^{g} (x_{2i-1} x_{2i} {x_{2i-1}}^{-1} {x_{2i}}^{-1}) \rangle,$$

and its abelianization is

$$H_1(S_g \times S^1) = \left(\bigoplus_{i=1}^{2g} \mathbb{Z}\langle [x_i] \rangle \right) \oplus \mathbb{Z}\langle [y] \rangle.$$

Let $\varphi : \mathbb{Z}[H_1(S_g \times S^1; \mathbb{Z})] \to \mathbb{C}^\times$ be a ring homomorphism such that each of $\zeta_i = \varphi([x_i])$ and $\zeta = \varphi([y])$ has an infinite order. Then we have

$$\tau^\varphi(S_g \times S^1, [\mathcal{V}_{st}]) = \pm(\zeta - 1)^{2g-2}.$$

The set of the Reidemeister-Turaev torsions of the Spin^c structures of $S_g \times S^1$ is

$$\left\{ \pm \zeta_1^{i_1} \cdots \zeta_{2g}^{i_{2g}} \zeta^i (\zeta - 1)^{2g-2} \in \mathbb{C}^\times / \pm 1 \;\middle|\; i_1, \ldots, i_{2g}, i \in \mathbb{Z} \right\}.$$

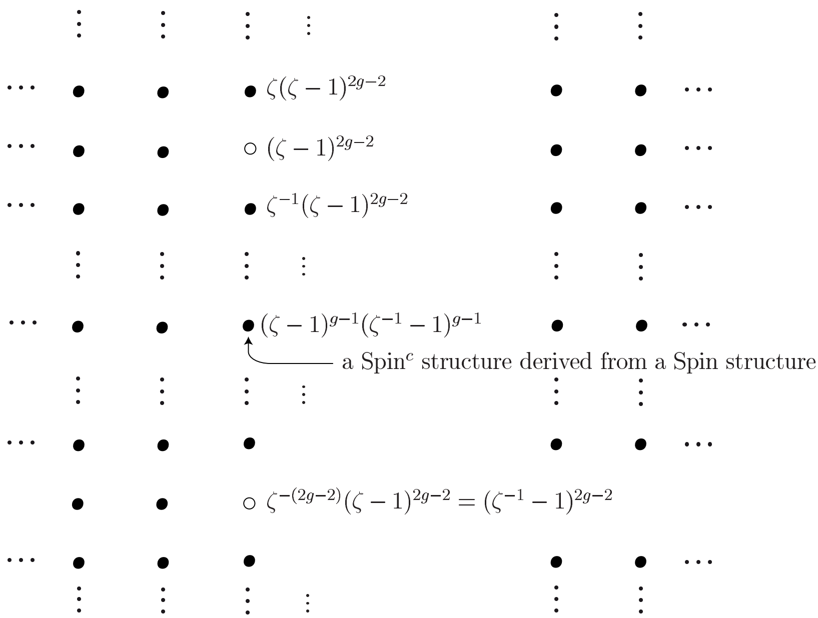

Figure 14. — The set of Spin^c structures on $S_g \times S^1$ and their Reidemeister-Turaev torsions (the signs $\pm$ are omitted). The white dots are the standard Spin^c structures.

Observation 3.2. — The Reidemeister-Turaev torsion of a Spin^c structure of $S_g \times S^1$ is of the form $\pm(\zeta^m - 1)^{2g-2}$ with $m \in \mathbb{Z}$ if and only if the Spin^c structure is standard.

3.3. Brieskorn 3-manifolds

The Brieskorn manifold $\Sigma(p,q,r)$ of type (p,q,r) is a closed 3-manifold defined by

$$\Sigma(p,q,r) = \{(x,y,z) \in \mathbb{C}^3 \mid |x|^2 + |y|^2 + |z|^2 = 1, x^p + y^q + z^r = 0\},$$

where p,q and r are integers greater than 1. The Seifert invariants of $\Sigma(p,q,r)$ can be obtained as described in [10]. We also recall that $\Sigma(p,q,r)$ is the r-fold branched covering of the 3-sphere S^3 branched along a torus knot or link of type; see [9]. The first integral homology group of $\Sigma(2,3,n)$ is

$$H_1(\Sigma(2,3,n);\mathbb{Z}) = \begin{cases} 1 & n = \pm 1 \pmod 6 \\ \mathbb{Z}/3\mathbb{Z} & n = \pm 2 \pmod 6 \\ \mathbb{Z}\ 2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 3 \pmod 6 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 0 \pmod 6 \end{cases}.$$

Using the algorithm in Section 2.2 for $\Sigma(2,3,6n)$, a presentation of $\pi_1(\Sigma(2,3,6n))$ corresponding to a standard Spin^c structure is

$$\langle x_1, x_2, \ldots, x_{6n} \mid x_i {x_{i+6n-1}}^{-1} {x_{i+1}}^{-1}, 1 \leqslant i \leqslant 6n \rangle$$

and its abelianization is

$$H_1(\Sigma(2,3,6n);\mathbb{Z}) = \mathbb{Z}\langle [x_1] \rangle \oplus \mathbb{Z}\langle [x_2] \rangle.$$

Let $\varphi : \mathbb{Z}[H_1(\Sigma(2,3,6n);\mathbb{Z})] \to \mathbb{C}^\times$ be a ring homomorphism such that each of $\zeta_1 = \varphi([x_1])$ and $\zeta_2 = \varphi([x_2])$ has an infinite order. Then we have

$$\tau^\varphi(\Sigma(2,3,6n), [\mathcal{V}_{st}]) = \pm \frac{\det\left(\varphi([\frac{\partial x_i {x_{i+6n-1}}^{-1} {x_{i+1}}^{-1}}{\partial x_j}])\right)_{1,1}}{({\zeta_1}^{-1}-1)(\zeta_1 - 1)} = \pm n.$$

The set of the Reidemeister-Turaev torsions of the Spin^c structures of $\Sigma(2,3,6n)$ is

$$\left\{ \pm n {\zeta_1}^{i_1} {\zeta_2}^{i_2} \in \mathbb{C}^\times / \pm 1 \;\middle|\; i_1, i_2 \in \mathbb{Z} \right\}.$$

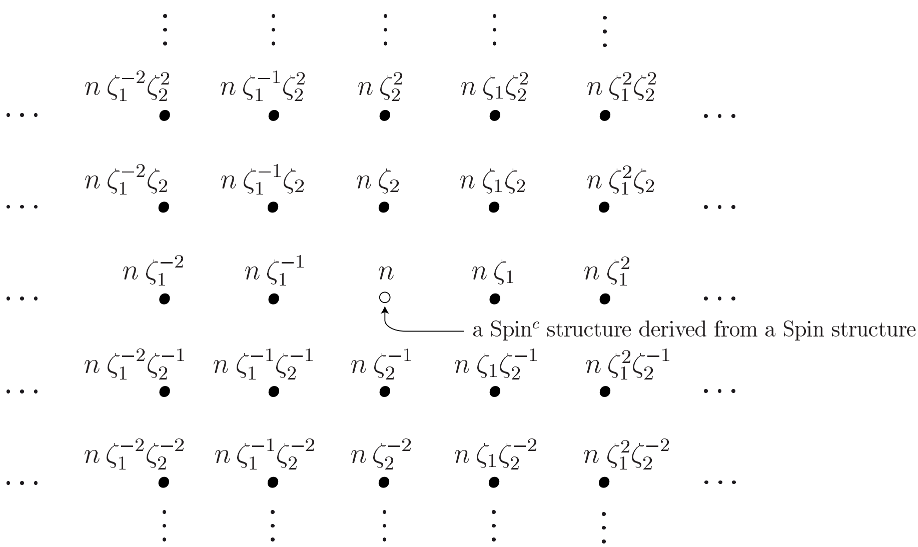

Figure 15. — The set of Spin^c structures on $\Sigma(2,3,6n)$ and their Reidemeister-Turaev torsions (the signs $\pm$ are omitted). The white dot is the standard Spin^c structure.

Observation 3.3. — The Reidemeister-Turaev torsion of a Spin^c structure of the Brieskorn 3-manifolds $\Sigma(2,3,6n)$ $(n \in \mathbb{N})$ is of the form $\pm n$ if and only if the Spin^c structure is standard.

3.4. A few more examples

Let M_1 be a closed orientable 3-manifold with the Seifert invariants $(0; -1; (3,1), (5,1), (7,1))$. A presentation of $\pi_1(\Sigma(M_1))$ corresponding to a standard Spin^c structure is

$$\pi_1(M_1) = \langle x_1, x_2, x_3 \mid x_1 x_3 {x_2}^{-4}, x_2 x_1 {x_3}^{-6}, x_3 x_2 {x_1}^{-2} \rangle$$

and its abelianization is

$$H_1(M_1;\mathbb{Z}) = (\mathbb{Z}/34\mathbb{Z})\,\langle [x_2] \rangle,$$

where $[x_1] = 13[x_2]$ and $[x_3] = 25[x_2]$. Let $\varphi : \mathbb{Z}[H_1(M_1;\mathbb{Z})] \to \mathbb{C}^\times$ be a ring homomorphism and set $\zeta = \varphi([x_2])$. Then we have

$$\begin{aligned} \tau^\varphi(M_1, [\mathcal{V}_{st}]) &= \pm \frac{\det \begin{pmatrix} 1 & \zeta \\ -(1+\zeta+\zeta^2+\zeta^3) & 1 \end{pmatrix}}{(\zeta^{13}-1)(\zeta^{25}-1)} = \pm \frac{1+\zeta+\zeta^2+\zeta^3+\zeta^4}{(\zeta^{13}-1)(\zeta^{25}-1)} \\ &= \pm \frac{\zeta^5 - 1}{(\zeta^{13}-1)(\zeta^{25}-1)(\zeta-1)}. \end{aligned}$$

Let M_2 be a closed orientable 3-manifold with the Seifert invariants $(0;\ -1;\ (5,1),\ (6,1),\ (7,1))$. A presentation of $\pi_1(M_2)$ corresponding to a standard Spin^c structure is

$$\pi_1(M_2) = \langle x_1, x_2, x_3 \mid x_1 x_3 {x_2}^{-5}, x_2 x_1 {x_3}^{-6}, x_3 x_2 {x_1}^{-4} \rangle$$

and its abelianization is

$$H_1(M_1;\mathbb{Z}) = (\mathbb{Z}/103\mathbb{Z})\,\langle [x_3] \rangle,$$

where $[x_1] = 22[x_3]$ and $[x_2] = 87[x_3]$. Let $\varphi : \mathbb{Z}[H_1(M_2;\mathbb{Z})] \to \mathbb{C}^\times$ be a ring homomorphism and set $\zeta = \varphi([x_3])$. Then we have

$$\begin{aligned} \tau^\varphi(M_2, [\mathcal{V}_{st}]) &= \pm \frac{\det \begin{pmatrix} 1 & \zeta \\ -(1+\zeta+\zeta^2+\zeta^3+\zeta^4+\zeta^5) & 1 \end{pmatrix}}{(\zeta^{87}-1)(\zeta^{22}-1)} \\ &= \pm \frac{1+\zeta+\zeta^2+\zeta^3+\zeta^4+\zeta^5+\zeta^6}{(\zeta^{87}-1)(\zeta^{22}-1)} \\ &= \pm \frac{\zeta^7 - 1}{(\zeta^{87}-1)(\zeta^{22}-1)(\zeta-1)}. \end{aligned}$$

Finally, let M_3 be a closed orientable 3-manifold with the Seifert invariants $(0;\ -1;\ (5,1),\ (5,1),\ (5,3))$. A presentation of $\pi_1(M_3)$ corresponding to a standard Spin^c structure is

$$\pi_1(M_3) = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1x_3{x_2}^{-4}, x_2x_1{x_3}^{-4}, x_3x_2{x_4}^{-1}x_4x_5^{-2}x_5x_4{x_1}^{-1}\rangle$$

and its abelianization is

$$H_1(M_1;\mathbb{Z}) = \mathbb{Z}\langle[x_5]\rangle \oplus (\mathbb{Z}/5\mathbb{Z})\,\langle[x_3]-[x_5]\rangle,$$

where $[x_1] = 3[x_5]$, $[x_2] = [x_5] - ([x_3]-[x_5])$, $[x_3] = [x_5] + ([x_3]-[x_5])$ and $[x_4] = 2[x_5]$. Let $\varphi : \mathbb{Z}[H_1(M_3;\mathbb{Z})] \to \mathbb{C}^\times$ be a ring homomorphism, and set $\zeta_1 = \varphi([x_5])$ and $\zeta_2 = \varphi([x_3]-[x_5])$. Then we have

$$\begin{aligned}\tau^\varphi(M_3,[\mathcal{V}_{st}]) &= \pm\frac{\det\begin{pmatrix} 1 & {\zeta_1}^3 & 0 & 0 \\ 0 & 1 & -\zeta_1 & 0 \\ 0 & 0 & 1 & -\zeta_1(1+{\zeta_1}^{-1}) \\ -1 & 0 & \zeta_1 & 1 \end{pmatrix}}{(\zeta_1\zeta_2-1)(\zeta_1{\zeta_2}^{-1}-1)} \\ &= \pm\frac{1+\zeta_1+{\zeta_1}^2+{\zeta_1}^4+{\zeta_1}^5}{(\zeta_1\zeta_2-1)(\zeta_1{\zeta_2}^{-1}-1)}.\end{aligned}$$

Observation 3.4. — The Reidemeister-Turaev torsions of Spin^c structures of the above Seifert fibered 3-manifolds M_1, M_2 and M_3 are still rather simple within the set of Reidemeister-Turaev torsions of all Spin^c structures on the manifolds. Thought we may not get complete conditions on the values of the Reidemeister-torsion to detect standard Spin^c-structures on them as in Observations 3.1–3.3, we see that the numerators of the Reidemeister-Turaev torsions of standard Spin^c structures on M_1, M_2 and M_3 have non-zero constant terms when we describe them as in Proposition 1.1.

3.5. Summary of observations

As we have seen in all of the above computations and Observations 3.1–3.4, the Reidemeister-Turaev torsion of a standard Spin^c structure of a Seifert fibered 3-manifold seems to take a particularly simple form within the set of the Reidemeister-Turaev torsions of all Spin^c structures on the manifold. More precisely, let M be a Seifert fibered 3-manifold and $[\mathrm{V}_{st}]$ be a standard Spin^c structure on M. Let $[x_1], [x_2], \ldots, [x_n]$ be a set of generators of $H_1(M;\mathbb{Z})$. Let $\varphi : \mathbb{Z}[H_1(M;\mathbb{Z})] \to \mathbb{C}^\times$ be a ring homomorphism and set $\zeta_i = \varphi([x_i])$ for $i = 1, 2, \ldots, n$. Suppose that the φ-twisted chain complex $C_*^\varphi(M)$ is acyclic. Then we might expect that the Reidemeister-Turaev torsion of $(M, [\mathcal{V}])$ is of the form

$$\pm\frac{f(\zeta_1,\zeta_2,\ldots,\zeta_n)}{({\zeta_i}^{m_1}-1)({\zeta_j}^{m_2}-1)},$$

where $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial function satisfying $f(0) \neq 0$, $i, j \in \{1, 2, \ldots, n\}$ and $m_1, m_2 \in \mathbb{Z} \setminus \{0\}$. Here we note that, by Proposition 1.1, the Reidemeister-Turaev torsions of an arbitrary Spin^c structure on M is in general of the form

$$\pm\zeta_1^{k_1}\zeta_2^{k_2}\cdots\zeta_n^{k_n}\frac{f(\zeta_1,\zeta_2,\ldots,\zeta_n)}{({\zeta_i}^{m_1}-1)({\zeta_j}^{m_2}-1)},\ k_1,k_2,\ldots,k_n\in\mathbb{Z},$$

with the above f.

The above expectation implies in partiular that when the *maximal abelian torsion* $\tau(M)$ of a Seifert fibered 3-manifold M with positive first Betti number is non-zero, the Seiberg-Witten invariant of a standard Spin^c structure of M is non-zero; see [18, 19] for the definition of the maximal abelian torsion and its relation with the Seiberg-Witten invariant. At the present time, there is no theoretical support for the expectation, but it would be quite natural to expect that the Reidemeister-Turaev torsion of a standard Spin^c structure of a Seifert fibered 3-manifold takes a special form in a certain sense.

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