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Logarithmic Poisson cohomology: example of calculation and application to prequantization

JOSEPH DONGHO⁽¹⁾

ABSTRACT. — In this paper we introduce the notions of logarithmic Poisson structure and logarithmic principal Poisson structure. We prove that the latter induces a representation by logarithmic derivation of the module of logarithmic Kähler differentials. Therefore it induces a differential complex from which we derive the notion of logarithmic Poisson cohomology. We prove that Poisson cohomology and logarithmic Poisson cohomology are equal when the Poisson structure is log symplectic. We give an example of non log symplectic but logarithmic Poisson structure for which these cohomology spaces are equal. We give an example for which these cohomologies are different. We discuss and modify the K. Saito definition of logarithmic differential forms. This note ends with an application to a prequantization of the logarithmic Poisson algebra: $(\mathbb{C}[x, y], \{x, y\} = x)$.

RÉSUMÉ. — Dans cet article, nous introduisons la notion de structure d'algèbre de Poisson logarithmique et celle de structure d'algèbre de Poisson logarithmique principale. Nous montrons que les structures d'algèbre de Poisson logarithmique principale induisent une représentation du module des différentielles formelles logarithmiques par des dérivations logarithmiques principales. Grâce à cette représentation, nous introduisons la notion de cohomologie de Poisson logarithmique. Nous prouvons que cette cohomologie est isomorphe à la cohomologie de Poisson sous-jacente lorsque la structure d'algèbre de Poisson est log symplectique. Nous donnons un exemple de structure d'algèbre de Poisson logarithmique principale non log symplectique dont les deux cohomologies sont encore isomorphes. Nous montrons sur un exemple qu'en général la cohomologie de

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Poisson et celle de Poisson logarithmique ne sont pas toujours isomorphes. Nous montrons sur un exemple la nécessité d'ajuster les hypothèses du théorème de K. Saito définissant les formes différentielles logarithmiques. Le travail se termine par une application de la cohomologie de Poisson logarithmique à la préquantification de la structure d'algèbre de Poisson logarithmique principale ($\mathbb{C}[x, y], \{x, y\} = x$).

Introduction

The classical Poisson bracket

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) \quad (0.1)$$

defined on the algebra of smooth functions on \mathbb{R}^{2n} plays a fundamental role in the analytical mechanics. It was discovered by D. Poisson in 1809. It was only one century later that A. Lichnerowicz (in [9]) and A. Weinstein (in [16]) extended it in a large theory now known as the Poisson Geometry. It has been remarked by A. Weinstein ([16]) that in fact the theory can be traced back to S. Lie (in [10]). The Poisson bracket (0.1) is derived from a symplectic structure on \mathbb{R}^{2n} and it appears as one of the main ingredients of symplectic geometry.

The basic properties of the bracket (0.1) are that it yields the structure of a Lie algebra on the space of functions and it has a natural compatibility with the usual associative product of functions. These facts are of algebraic nature and it is natural to define an abstract notion of a *Poisson algebra*. Following A. Vinogradov and I. Krasil'shchik in [15], J. Braconnier (in [2]) has developed the algebraic version of Poisson geometry. One of the most important notion related to the Poisson geometry is Poisson cohomology which was introduced by A. Lichnerowicz (in [9]) and in algebraic setting by I. Krasil'shchik (in [8]). Unlike the De Rham cohomology, Poisson cohomology spaces are almost irrelevant to the topology of the manifold and moreover they have bad functorial properties. They are very large and their actual computation is both more complicated and less significant than in the case of the De Rham cohomology. However they are very interesting because they allow us to describe various results concerning Poisson structures in particular one important result about the *geometric quantization* of the manifold. Algebraic aspects of this theory were developed by J. Huebschmann (in [7]) and in the geometrical setting by I. Vaisman (in [14]).

This paper deals with Poisson algebras but Poisson algebras of another kind. More specifically we study the *logarithmic Poisson structures*. If the Poisson structures draw their origins from symplectic structures, logarithmic Poisson structures are inspired by log symplectic structures which are based on the theory of logarithmic differential forms. The logarithmic differential forms was introduced by P. Deligne (in [4]) who defined them in the case of a normal crossings divisor of a given complex manifold. But the theory of logarithmic differential forms along a divisor with not necessarily normal crossings was introduced by K. Saito in [13]. Explicitly if \mathcal{I} is an ideal in a commutative algebra \mathcal{A} over a commutative ring R a derivation D of \mathcal{A} is called logarithmic along \mathcal{I} if $D(\mathcal{I}) \subset \mathcal{I}$. We denote by $Der_{\mathcal{A}}(\log \mathcal{I})$ the \mathcal{A} -module of derivations of \mathcal{A} logarithmic along \mathcal{I} . A Poisson structure $\{.,.\}$ on \mathcal{A} is called logarithmic ¹ along \mathcal{I} if for all $a \in \mathcal{A}$ we have $\{a, .\} \in Der_{\mathcal{A}}(\log \mathcal{I})$. In addition suppose that \mathcal{I} is generated by $\{u_1, \dots, u_p\} \subset \mathcal{A}$ and let $\Omega_{\mathcal{A}}$ be the \mathcal{A} -module of Kähler differential. The \mathcal{A} -module $\Omega_{\mathcal{A}}(\log \mathcal{I})$ generated by $\{\frac{du_1}{u_1}, \dots, \frac{du_p}{u_p}\} \cup \Omega_{\mathcal{A}}$ is called the module of Kähler differentials logarithmic along \mathcal{I} .

With the above definition we point out that the K. Saito definition of logarithmic forms is incomplete if we do not add the hypothesis that the defining function of the divisor is square free. In fact, according to K. Saito (Definition 1.2 in [13]) $\frac{dx}{x^2}$ and $\frac{dy}{x}$ are logarithmic along $D = \{(x, y) \in \mathbb{C}^2, x^2 = h(x, y) = 0\}$. If that is the case the system $(\frac{dx}{x^2}, \frac{dy}{x})$ will be a basis of $\Omega_{\mathbb{C}^2}$. This is a contradiction with Theorem 1.8 in [13] since $\frac{dx \wedge dy}{x^3} \neq \frac{unit}{x^2} dx \wedge dy$.

In the case where \mathcal{I} is generated by $\{u_1, \dots, u_p\} \subset \mathcal{A}$, we say that a Poisson structure $\{.,.\}$ on \mathcal{A} is logarithmic principal along \mathcal{I} if for all $a \in \mathcal{A}$ and $u_i \in \{u_1, \dots, u_p\}$ we have $\frac{1}{u_i} \{a, u_i\} \in \mathcal{A}$.

J. Huebschmann's program of algebraic construction of the Poisson cohomology can be summarized as follows:

Let \mathcal{A} be a commutative algebra over a commutative ring R . A Lie-Rinehart algebra on \mathcal{A} is an \mathcal{A} -module which is an R -Lie algebra acting on \mathcal{A} with suitable compatibly conditions. J. Huebschmann observes that each Poisson structure $\{.,.\}$ gives rise to a structure of Lie-Rinehart algebra in the sense

⁽¹⁾ The statement *the Poisson structure is logarithmic along \mathcal{I}* also expresses as \mathcal{I} is a Poisson ideal of \mathcal{A} . For example any smooth Poisson manifolds is logarithmic along the ideal of the smooth functions which vanish on a given symplectic leaf

of G. Rinehart (in [12]) on the \mathcal{A} -module $\Omega_{\mathcal{A}}$ in natural fashion. But it was proved in [11] that any Lie-Rinehart algebra L on \mathcal{A} gives rise to a complex $Alt_{\mathcal{A}}(L, \mathcal{A})$ of alternating forms which generalizes the usual De Rham complex of manifold and the usual complex computing Chevalley-Eilenberg (in [3]) Lie algebra cohomology. Moreover extending earlier work of Hochschild Kostant and Rosenberg (in [6]). G. Rinehart has shown that when L is projective as an \mathcal{A} -module the homology of the complex $Alt_{\mathcal{A}}(L, \mathcal{A})$ may be identified with $Ext_{U(\mathcal{A}, L)}^*(\mathcal{A}, \mathcal{A})$ over a suitably defined universal algebra $U(\mathcal{A}, L)$ of differential operators. But the latter defines a Lie algebra cohomology $H^*(L, \mathcal{A})$ of L . So, since $\Omega_{\mathcal{A}}$ is free \mathcal{A} -module, it is projective. Therefore the homology of the complex $Alt_{\mathcal{A}}(\Omega_{\mathcal{A}}, \mathcal{A})$ computing the cohomology of the underlying Lie algebra of the Poisson algebra $(\mathcal{A}, \{.,.\})$. Then the Poisson cohomology of $(\mathcal{A}, \{.,.\})$ is the homology of $Alt_{\mathcal{A}}(\Omega_{\mathcal{A}}, \mathcal{A})$.

It follows from the definition of logarithmic Poisson structure that the image of Hamiltonian map of logarithmic principal Poisson structure is submodule of $Der_{\mathcal{A}}(\log \mathcal{I})$. Inspired by this fact we introduce the notion of logarithmic Lie-Rinehart structure. A Lie-Rinehart algebra L on \mathcal{A} is said logarithmic along an ideal \mathcal{I} of \mathcal{A} if it acts by logarithmic derivations on \mathcal{A} .

In the case of logarithmic principal Poisson structure we replace in J. Huebschmann's program $\Omega_{\mathcal{A}}$ by $\Omega_{\mathcal{A}}(\log \mathcal{I})$ and we prove the following:

- every logarithmic principal structure of Poisson algebra induces a structure of Lie-Rinehart algebra on $\Omega_{\mathcal{A}}(\log \mathcal{I})$. The associated cohomology is called *logarithmic Poisson cohomology*,
- Poisson cohomology and logarithmic Poisson cohomology are equal in the case of log symplectic Poisson structure,
- we verify the above result on the example $(\mathcal{A} = \mathbb{C}[x, y], \{x, y\} = x)$. We also show that the logarithmic principal Poisson algebra $(\mathcal{A} = \mathbb{C}[x, y], \{x, y\} = x^2)$ is not log symplectic but its Poisson cohomology is equal to its logarithmic Poisson cohomology,
- we prove that the Poisson structure $(\mathcal{A} = \mathbb{C}[x, y, z], \{x, y\} = 0, \{x, z\} = 0, \{y, z\} = xyz)$ is a logarithmic principal and
 - its 3^{rd} Poisson cohomology is

$$H_P^3 \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x] \oplus xy\mathbb{C}[y] \oplus xy\mathbb{C}[x] \oplus xz\mathbb{C}[x] \oplus xz\mathbb{C}[z] \oplus yz\mathbb{C}[y] \oplus yz\mathbb{C}[z],$$

and

- its 3^{rd} logarithmic Poisson cohomology is

$$H_{PS}^3 \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x].$$

The structure of the paper is as follows:

- Section 1: we introduce the notions of logarithmic principal Poisson structure and logarithmic Poisson cohomology. For this we use the notions of Lie-Rinehart algebra and logarithmic-Lie-Rinehart algebra. The main results of this section are Theorem 1.10 and Corollary 1.13 of Proposition 1.12.
- Section 2: we recall the notion of log symplectic manifold and prove that Poisson structure induced by log symplectic structure is logarithmic principal Poisson structure.
- Section 3: we compute the Poisson cohomology and the logarithmic Poisson cohomology of three logarithmic principal Poisson structures. Thanks to Theorem 3.14, we show that in general these two cohomologies are different.
- Section 4: we apply logarithmic Poisson cohomology to the prequantization of $\{x, y\} = x$.

1. Logarithmic Poisson cohomology

1.1. Notations and conventions

Throughout this paper R denotes a commutative ring, \mathcal{A} is a commutative and unitary R -algebra, $Der_{\mathcal{A}}$ is the \mathcal{A} -module of derivations of \mathcal{A} and $\Omega_{\mathcal{A}}$ is the \mathcal{A} -module of Kähler differentials. An action of a Lie algebra L on \mathcal{A} is a morphism of Lie algebras $\rho : L \rightarrow Der_{\mathcal{A}}$.

1.2. Poisson cohomology

Let L be a Lie algebra over R . A structure of Lie-Rinehart algebra on L (cf [12], [7]) is an action $\rho : L \rightarrow Der_{\mathcal{A}}$ of L on \mathcal{A} satisfying the following compatibility properties:

1. $[\rho(al)](b) = a(\rho(l)(b))$ and
2. $[l_1, al_2] = \rho(l_1)(a)l_2 + a[l_1, l_2]$.

A Lie-Rinehart algebra is a pair (L, ρ) where ρ is a structure of Lie-Rinehart algebra on L . In the sequel, any Lie-Rinehart algebra (L, ρ) will be denoted simply by L if no confusion is possible. Let $Alt_{\mathcal{A}}^p(L, \mathcal{A})$ be the R -module of

the alternating p -forms on a Lie-Rinehart algebra L . The following map d_ρ defined by

$$\begin{aligned} d_\rho(f)(l_1, \dots, l_p) &= \sum_{i=p}^p (-1)^{i+1} \rho(\alpha_i) f(l_1, \dots, \hat{l}_i, \dots, l_p) \\ &+ \sum_{i < j} (-1)^{i+j} f([l_i, l_j], l_1, \dots, \hat{l}_i, \dots, \hat{l}_j, \dots, l_p) \end{aligned}$$

for $f \in \text{Alt}^{(p-1)}(L, \mathcal{A})$ induces a structure of a chain complex on $\text{Alt}_{\mathcal{A}}(L, \mathcal{A}) := p \geq 0 \bigoplus \text{Alt}_{\mathcal{A}}^p(L, \mathcal{A})$ and the associated cohomology is called the *Lie-Rinehart cohomology of L* . It is known that for each Poisson algebra $(\mathcal{A}, \{.,.\})$, the following data:

1. Lie-Poisson bracket $[da, db] := d\{a, b\}$ on $\Omega_{\mathcal{A}}$,
2. Hamiltonian map $H : \Omega_{\mathcal{A}} \rightarrow \text{Der}_{\mathcal{A}}$ defined by $H(da)b := \{a, b\}$.

induce a structure of Lie-Rinehart algebra on $\Omega_{\mathcal{A}}$. The associated Lie-Rinehart cohomology is called Poisson cohomology of $(\mathcal{A}, \{.,.\})$ and the corresponding cohomology spaces are denoted by H_p^* .

1.3. Logarithmic Poisson cohomology

Let \mathcal{I} be a non trivial ideal of \mathcal{A} and let L be a Lie algebra over R who is also an \mathcal{A} -module. For a derivation $\delta \in \text{Der}_{\mathcal{A}}$, we say that:

1. δ is *logarithmic* along \mathcal{I} if $\delta(\mathcal{I}) \subset \mathcal{I}$,
2. δ is *logarithmic principal* along $\{u_1, \dots, u_p\} \subset \mathcal{I}$ if for all $i = 1, \dots, p$ $\delta(u_i) \in u_i \mathcal{A}$.

We denote by $\text{Der}_{\mathcal{A}}(\log \mathcal{I})$ the \mathcal{A} -module of derivations on \mathcal{A} which are logarithmic along \mathcal{I} and $\widehat{\text{Der}}_{\mathcal{A}}(\log \mathcal{I})$ the \mathcal{A} -module of logarithmic principal derivations along \mathcal{I} on \mathcal{A} ; when \mathcal{I} is generated by $\{u_1, \dots, u_p\}$. It is clear that $\text{Der}_{\mathcal{A}}(\log \mathcal{I})$ is a sub-module of $\text{Der}_{\mathcal{A}}$.

DEFINITION 1.1. — *A structure of Lie-Rinehart algebra $\rho : L \rightarrow \text{Der}_{\mathcal{A}}$ on L is called structure of logarithmic-Lie-Rinehart algebra along \mathcal{I} if $\rho(L) \subset \text{Der}_{\mathcal{A}}(\log \mathcal{I})$.*

$\text{Id}_{\text{Der}_{\mathcal{A}}(\log \mathcal{I})} : \text{Der}_{\mathcal{A}}(\log \mathcal{I}) \rightarrow \text{Der}_{\mathcal{A}}(\log \mathcal{I})$ is a structure of logarithmic-Lie-Rinehart algebra on $\text{Der}_{\mathcal{A}}(\log \mathcal{I})$.

DEFINITION 1.2. — *Let L be a logarithmic-Lie-Rinehart algebra. A logarithmic Lie-Rinehart cohomology of L is the Lie-Rinehart cohomology associated to the representation of L by logarithmic derivations along \mathcal{I} .*

For any logarithmic-Lie-Rinehart algebra (L, ρ) we will denote by $(\text{Alt}(L, \mathcal{A}), d_\rho)$ the complex induced by the action of L on \mathcal{A} . As in the case of Lie-Rinehart algebras the notions of logarithmic-Lie-Rinehart-Poisson and logarithmic-Lie-Rinehart-symplectic structures are well defined.

DEFINITION 1.3. — *Let (L, ρ) be a logarithmic Lie-Rinehart algebra. A logarithmic-Lie-Rinehart-Poisson structure on (L, ρ) is a skew-symmetric 2-form*

$$\mu : L \times L \rightarrow \mathcal{A}$$

such that $d_\rho \mu = 0$.

A logarithmic-Lie-Rinehart-Poisson algebra is a triple (L, ρ, μ) where μ is a logarithmic-Lie-Rinehart-Poisson structure on (L, ρ) .

DEFINITION 1.4. — *A logarithmic-Lie-Rinehart-Poisson algebra (L, ρ, μ) is called logarithmic Lie-Rinehart-symplectic if the 2-form μ is nondegenerate. In other words the map*

$$I : L \rightarrow \text{Hom}(L, \mathcal{A}), \quad l \mapsto I(l) = i_l \mu$$

is an isomorphism of \mathcal{A} -modules. Where for all $l \in L$ the map

$$i_l : \text{Alt}(L, \mathcal{A}) \rightarrow \text{Alt}(L, \mathcal{A})$$

is defined by

$$(i_l(f))(l_1, \dots, l_{p-1}) = f(l, l_1, \dots, l_{p-1}).$$

Let $\mathcal{S} := \{u_1, \dots, u_p\} \subset \mathcal{A}$ such that each ideal $u_i \mathcal{A}$ is prime and $u_i \notin u_j \mathcal{A}$ for all $i \neq j, i, j = 1, \dots, p$. We denote by $\Omega_{\mathcal{A}}(\log \mathcal{I})$ the \mathcal{A} -module generated by $\left\{ \frac{du_i}{u_i}; i = 1, \dots, p \right\} \cup \Omega_{\mathcal{A}}$.

DEFINITION 1.5. — *The \mathcal{A} -module $\Omega_{\mathcal{A}}(\log \mathcal{I})$ is called the \mathcal{A} -module of Kähler logarithmic differentials on \mathcal{A} .*

The following Proposition gives the dual of the \mathcal{A} -module $\Omega_{\mathcal{A}}(\log \mathcal{I})$.

PROPOSITION 1.6. — *The \mathcal{A} -module of the \mathcal{A} -linear maps from $\Omega_{\mathcal{A}}(\log \mathcal{I})$ to \mathcal{A} is isomorphic to the \mathcal{A} -module $\widehat{\text{Der}}_{\mathcal{A}}(\log \mathcal{I})$ of the logarithmic principal derivations.*

Proof. — It follows from the universal property of (Ω, d) that there is an isomorphism σ from $Der_{\mathcal{A}}$ to $Hom(\Omega_{\mathcal{A}}, \mathcal{A})$. Consider

$$\hat{\sigma} : Der_{\widehat{\mathcal{A}}(\log \mathcal{I})} \rightarrow Hom(\Omega_{\mathcal{A}}(\log \mathcal{I}), \mathcal{A})$$

defined by $\hat{\sigma}(\delta)(a \frac{du_i}{u_i} + bdc) = a \frac{1}{u} \sigma(\delta)(du) + b \sigma(\delta)(dc)$. A straightforward computation shows that $\hat{\sigma}$ is an isomorphism. \square

DEFINITION 1.7. — Let $(\mathcal{A}, \{.,.\})$ be a Poisson algebra, let \mathcal{I} be a non trivial ideal of \mathcal{A} and let \mathcal{S} as above. We say that the bracket $\{.,.\}$:

1. defines a structure of logarithmic Poisson algebra along \mathcal{I} if for all $a \in \mathcal{A}$, $\{a, .\} \in Der_{\mathcal{A}}(\log \mathcal{I})$,
2. is a structure of logarithmic principal Poisson structure along \mathcal{S} if for all $a \in \mathcal{A}$, $\{a, .\} \in Der_{\widehat{\mathcal{A}}}(\log \mathcal{I})$.

When \mathcal{A} is endowed with a Poisson structure $\{.,.\}$ which is logarithmic along \mathcal{I} (respectively logarithmic principal along \mathcal{S}) we say that $(\mathcal{A}, \{.,.\})$ is a logarithmic (respectively a logarithmic principal) Poisson algebra.

PROPOSITION 1.8. — Let $(\mathcal{A}, \{.,.\})$ be a Poisson algebra.

1. If $\{.,.\}$ is logarithmic along \mathcal{I} , then $H(\Omega_{\mathcal{A}}) \subset Der_{\mathcal{A}}(\log D)$.
2. If $\{.,.\}$ is logarithmic principal along \mathcal{S} , then $H(\Omega_{\mathcal{A}}) \subset Der_{\widehat{\mathcal{A}}}(\log D)$ and H can be extended to $\Omega_{\mathcal{A}}(\log \mathcal{I})$ by

$$\tilde{H} : \Omega_{\mathcal{A}}(\log \mathcal{I}) \rightarrow Der_{\widehat{\mathcal{A}}}(\log D) \quad \frac{du}{u} \mapsto \frac{1}{u} H(du)$$

for all $u \in \mathcal{S}$.

Proof. — The first item follows from the definition of logarithmic Poisson structure.

To prove item 2 we shall remark that if $\{.,.\}$ is a logarithmic principal Poisson structure on \mathcal{A} then for all $i \neq j$, $\frac{1}{u_i u_j} \{u_i, u_j\} \in \mathcal{A}$. \square

DEFINITION 1.9. — Let $(\mathcal{A}, \{.,.\})$ be a logarithmic principal Poisson algebra. The map \tilde{H} defined in Proposition 1.8 is called the logarithmic Hamiltonian map of $(\mathcal{A}, \{.,.\})$.

We define on $\Omega_{\mathcal{A}}(\log \mathcal{I})$ the following bracket

$$\begin{aligned} \left[a \frac{du_i}{u_i}, bdc \right]_s &= \frac{a}{u_i} \{u_i, b\} dc + b \{a, c\} \frac{du_i}{u_i} + abd \left(\frac{1}{u_i} \{u_i, c\} \right), \\ \left[a \frac{du_i}{u_i}, b \frac{du_j}{u_j} \right]_s &= \frac{a}{u_i} \{u_i, b\} \frac{du_j}{u_j} + \frac{b}{u_j} \{a, u_j\} \frac{du_i}{u_i} + abd \left(\frac{1}{u_i u_j} \{u_i, u_j\} \right), \\ [adc, bde]_s &= a \{c, b\} de + b \{a, e\} dc + abd (\{c, e\}) \end{aligned}$$

for all $u_i, u_j \in \mathcal{S}$ and $a, b, c, e \in \mathcal{A} - \mathcal{S}$.

THEOREM 1.10. — *For all logarithmic principal Poisson algebra $(\mathcal{A}, \{.,.\})$,*

1. $[\cdot, \cdot]_s$ is a Lie bracket,
2. \tilde{H} is a logarithmic Lie-Rinehart structure on $\Omega_{\mathcal{A}}(\log \mathcal{I})$.

COROLLARY 1.11. — *Each logarithmic Poisson structure along \mathcal{I} (respectively logarithmic principal Poisson structure along \mathcal{S}) on \mathcal{A} induces a logarithmic-Lie-Rinehart-Poisson structure μ on $\Omega_{\mathcal{A}}(\log \mathcal{I})$.*

If $\{.,.\}$ is a structure of logarithmic principal Poisson algebra on \mathcal{A} and μ is the associated logarithmic-Lie-Rinehart-Poisson structure then

PROPOSITION 1.12. — *μ is a logarithmic-Lie-Rinehart-symplectic structure if and only if \tilde{H} is an isomorphism.*

Proof. — Suppose that \tilde{H} is an isomorphism. Let $x, y \in \Omega_{\mathcal{A}}(\log \mathcal{I})$ such that $I(x) = I(y)$. Then $-\hat{\sigma}(\tilde{H}(x)) = -\hat{\sigma}(\tilde{H}(y))$. Therefore $x = y$ and we conclude that I is a monomorphism. Let $\psi \in \mathcal{H}om(\Omega_{\mathcal{A}}(\log \mathcal{I}))$, we seek for a $x \in \widehat{\Omega_{\mathcal{A}}(\log \mathcal{I})}$ such that $I(x) = \psi$. Since $\psi \in \mathcal{H}om(\Omega_{\mathcal{A}}(\log \mathcal{I}))$, $\hat{\sigma}^{-1}(\psi) \in \mathcal{D}er_{\mathcal{A}}(\log \mathcal{I})$. Therefore there is a $z \in \Omega_{\mathcal{A}}(\log \mathcal{I})$ such that $\tilde{H}(z) = \sigma^{-1}(\psi)$ i.e; $I(-z) = \hat{\sigma}(\tilde{H}(z)) = \psi$. Just take $x = -z$.

Conversely we suppose that I is an isomorphism and we shall prove that \tilde{H} is also an isomorphism. If $\tilde{H}(x) = \tilde{H}(y)$ then $-\hat{\sigma}(\tilde{H}(x)) = -\hat{\sigma}(\tilde{H}(y))$ i.e; $I(x) = I(y)$. Then $x = y$.

For all $\delta \in \widehat{\mathcal{D}er_{\mathcal{A}}(\log \mathcal{I})}$ there is a $x \in \Omega_{\mathcal{A}}(\log \mathcal{I})$ such that $\hat{\sigma}(\delta) = I(x) = -\hat{\sigma}(\tilde{H}(x))$. \square

Let $f \in \Omega_{\mathcal{A}}^p(\log \mathcal{I})$ we define $\tilde{H}(f) \in \mathcal{A}lt^p(\Omega_{\mathcal{A}}(\log \mathcal{I}), \mathcal{A})$ by

$$\tilde{H}(f)(\alpha_1, \dots, \alpha_p) := (-1)^p f(\tilde{H}(\alpha_1), \dots, \tilde{H}(\alpha_p)).$$

COROLLARY 1.13. — *If $(\mathcal{A}, \{.,.\})$ is a logarithmic principal Poisson algebra then*

$$d_{\tilde{H}} \circ \tilde{H} = -\tilde{H} \circ d$$

DEFINITION 1.14. — *Let $(\mathcal{A}, \{.,.\})$ be a logarithmic principal Poisson algebra along an ideal \mathcal{I} . We call logarithmic Poisson cohomology the Lie-Rinehart logarithmic cohomology associated to the action $\tilde{H} : \Omega_{\mathcal{A}}(\log \mathcal{I}) \rightarrow \text{Der}_{\mathcal{A}}(\log \mathcal{I})$. We will denote by H_{PS}^* the associated cohomology space.*

Let $\mu \in \bigwedge^2 \text{Der}(\log \mathcal{I})$ be a log symplectic structure on \mathcal{A} . According to the definition of a logarithmic-Lie-Rinehart-symplectic structure, the above map I defines an isomorphism which induces an isomorphism between Poisson cohomology H_P^* and logarithmic De Rham cohomology H_{DS}^* .² On the other hand the above proposition proves that \tilde{H} is an isomorphism between logarithmic Poisson cohomology H_{PS}^* ³ and logarithmic De Rham cohomology H_{DS}^* .

Therefore we have the following commutative diagram of chain complex.

$$\begin{array}{ccc}
 (\Omega_{\mathcal{A}}^*(\log \mathcal{I}), d) & \xrightarrow{\cong} & (\text{Der}_{\mathcal{A}}^*(\log \mathcal{I}), d_H) \\
 \searrow \cong & & \downarrow \cong \\
 & & (\text{Der}_{\mathcal{A}}^*(\log \mathcal{I}), d_{\tilde{H}})
 \end{array}$$

We conclude that

COROLLARY 1.15. — *If $\mu \in \bigwedge^2 \text{Der}(\log \mathcal{I})$ is a log symplectic structure on \mathcal{A} then*

$$H_P^* \cong H_{DS}^* \cong H_{PS}^*.$$

2. Log symplectic manifold

It is well known that the first examples of Poisson manifolds are the symplectic manifolds. In this section we recall the notion of log symplectic manifold and we prove that they induce a logarithmic Poisson manifolds. Of course we need to recall the notion of logarithmic forms. In this section X denotes a finite dimensional complex manifold and $h : X \rightarrow \mathbb{C}$ a holomorphic function on X . Recall that h is said *square-free* if for any holomorphic functions g and $k : X \rightarrow \mathbb{C}$ such that $h = g^2k$, g is necessarily a constant.

DEFINITION 2.1. — *Let $h : X \rightarrow \mathbb{C}$ be a holomorphic map on X . Then h is square free if each factor of h is simple.*

Let D be a divisor of X defined by a square free holomorphic function h .

(2) Where DS means De Rham Saito.

(3) Where PS means Poisson Saito

DEFINITION 2.2. — *A meromorphic p-form ω is said logarithmic along D if $h\omega$ and $hd\omega$ are holomorphic forms.*

We denote $\Omega_X^p(\log D)$ the \mathcal{O}_X -module of logarithmic p-forms on D . As in [13] a vector field δ is logarithmic along D if $\delta(h) \in h\mathcal{O}_X$. We denote $Der_X(\log D)$ the module of logarithmic vector fields on X .

Remark 1. — According to our definition of logarithmic forms $\frac{dy}{x^2}$ is not logarithmic along the divisor D defined by the set of zeros of x^2 in \mathbb{C}^2 because the square free defining function of D is x and we have $x d(\frac{dy}{x^2}) = x(\frac{dx \wedge dy}{x^2}) = \frac{dx \wedge dy}{x}$ which is not holomorphic. But following K. Saito's definition of logarithmic forms (see [13] Definition 1.2) and considering x^2 as defining function of D , we have:

$x^2(d(\frac{dy}{x^2}) = x(\frac{dx \wedge dy}{x^2})) = dx \wedge dy \in \Omega_X^2$. And then $\frac{dy}{x}$ is a logarithmic form. Moreover this implies that $\{\frac{dx}{x^2}, \frac{dy}{x}\}$ is a free basis of $\Omega_X(\log D)$. This contradicts item i) of Theorem 1.8 in [13] since $\frac{dx}{x^2} \wedge \frac{dy}{x} = \frac{1}{x^3} dx \wedge dy \neq \frac{unit}{x^2} dx \wedge dy$. Therefore we shall add the hypothesis that h is square free in K. Saito's definition in [13].

In addition we suppose that $dim_{\mathbb{C}} X = 2n$ and that X is compact.

DEFINITION 2.3 [5]. — *A pair (X, D) is a log symplectic manifold if there is a logarithmic 2-form $\omega \in \Omega_X^2(\log D)$ such that $d\omega = 0$, and*

$$\overbrace{\omega \wedge \omega \wedge \dots \wedge \omega}^n \neq 0 \in H^0(X, \Omega^{2n}([D])).$$

From this definition we deduce the following lemma.

LEMMA 2.4. — *Let (X, D) be a log symplectic manifold with log symplectic 2-form ω . The map $\omega^b : Der_X(\log D) \rightarrow \Omega_X(\log D)$ $\delta \mapsto i_\delta \omega$ is a quasi-isomorphism between the Poisson cohomology and the logarithmic De Rham cohomology of X .*

Proof. — It follows from the fact that ω is nondegenerate. □

From this lemma, it follows that for all $f, g \in \mathcal{O}_X$ there are unique $X_f, X_g \in Der_X(\log D)$ such that $\omega^b(X_f) = df$ and $\omega^b(X_g) = dg$. Therefore the following bracket $\{f, g\} := \omega(X_f, X_g)$ is well defined.

PROPOSITION 2.5. — *Let (X, D) be a log symplectic manifold. The bracket*

$$\{f, g\} := \omega(X_f, X_g) \tag{2.1}$$

defines a logarithmic principal Poisson structure on \mathcal{O}_X .

Proof. — It follows from the fact that for all $f \in \mathcal{O}_X$, $\{f, \cdot\} = i_{X_f}\omega \in \text{Der}_X(\log D)$ \square

We have a logarithmic generalization of the Darboux theorem:

LEMMA 2.6 [5]. — *Let (X, D) be a log symplectic manifold with a logarithmic form ω , where D is a reduced divisor of X . In a neighborhood of any smooth point of D , there exists a local holomorphic coordinate system; $(z_0, z_1, \dots, z_{2n-1})$ such that $D = \{z_0 = 0\}$ and ω is given by*

$$\omega = \frac{dz_0}{z_0} \wedge dz_1 + dz_2 \wedge dz_3 + \dots + dz_{2n-2} \wedge dz_{2n-1}.$$

We refer to these coordinates as log Darboux coordinates.

In the next proposition we prove that the logarithmic Poisson cohomology of the logarithmic Poisson structure (2.1) is isomorphic to the associate logarithmic De Rham cohomology of (X, D) .

PROPOSITION 2.7. — *If (X, D) is a log symplectic manifold the logarithmic Hamiltonian map of the associated Poisson structure is an isomorphism.*

Proof. — Let $M_{\tilde{H}}$ (respectively M_H) denote the matrix of \tilde{H} (respectively H). In the log Darboux coordinates we have:

$$M_H = \begin{pmatrix} 0 & -z_0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ z_0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & -1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix}$$

and then

$$M_{\tilde{H}} = \begin{pmatrix} 0 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 0 & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & -1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix};$$

It is obvious that the determinant of $M_{\tilde{H}}$ is different to zero. This ends the proof. \square

3. Computation of some logarithmic Poisson cohomology

In this section we compute the Poisson cohomology and the logarithmic Poisson cohomology of the following logarithmic principal Poisson algebras:

- i- $(\mathcal{A} := \mathbb{C}[x, y], \{x, y\} = x)$,
- ii- $(\mathcal{A} := \mathbb{C}[x, y], \{x, y\} = x^2)$,
- iii- $(\mathcal{A} := \mathbb{C}[x, y, z], \{x, y\} = 0, \{x, z\} = 0, \{y, z\} = xyz)$.

We prove that the first one is a log symplectic Poisson structure; according to Proposition 1.12 our proof implies that Poisson cohomology and logarithmic Poisson cohomology are equal for this structure. We also prove that the second Poisson structure is not log symplectic but we still have the equality between the two cohomologies. Therefore being log symplectic is not a necessary condition to have equality between Poisson and logarithmic Poisson cohomologies. At the end we compute the 3rd groups of Poisson and logarithmic Poisson cohomology of the third Poisson structure. We show that in this case these spaces are different.

3.1. Example 1: $(\mathcal{A} := \mathbb{C}[x, y], \{x, y\} = x)$

Let us define on $\mathcal{A} = \mathbb{C}[x, y]$ the following Poisson bracket

$$(f, g) \mapsto \{f, g\} = x \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right). \tag{3.1}$$

For any $f \in \mathcal{A}$ the derivation $D_f := x \left(\frac{\partial f}{\partial x} \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \right)$ satisfies the relation $D_f(x\mathcal{A}) \subset x\mathcal{A}$. Which means that the bracket $\{.,.\}$ defined by (3.1) is a logarithmic principal Poisson bracket along the ideal $x\mathcal{A}$. The associated Hamiltonian map $H : \Omega_{\mathcal{A}} \rightarrow Der_K(\mathcal{A})$ is defined on generators of $\Omega_{\mathcal{A}}$ by:

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$$H(dx) = D_x = x \frac{\partial}{\partial y} \text{ and } H(dy) = D_y = -x \frac{\partial}{\partial x}.$$

From these relations we deduce the definition of the associated logarithmic Hamiltonian map \tilde{H} on generators of $\Omega_{\mathcal{A}}(\log \mathcal{I})$:

$$\tilde{H}\left(\frac{dx}{x}\right) = \frac{1}{x}H(dx) \text{ and } \tilde{H}(dy) = H(dy).$$

LEMMA 3.1. — *When $(\mathcal{A} := \mathbb{C}[x, y], \{x, y\} = x)$, we have the following description of $\Omega_{\mathcal{A}}(\log \mathcal{I})$:*

$$\Omega_{\mathcal{A}}(\log \mathcal{I}) \cong \mathcal{A} \frac{dx}{x} \oplus \mathcal{A} dy \cong \mathbb{C}[y] \frac{dx}{x} \oplus \Omega_{\mathcal{A}}. \quad (3.2)$$

It follows from this lemma that for any $\alpha \in \Omega_{\mathcal{A}}(\log \mathcal{I})$ there are $a, b \in \mathcal{A}$ such that $\alpha = a \frac{dx}{x} + b dy$. It follows also that \tilde{H} is completely defined by the relation

$$\tilde{H}\left(a \frac{dx}{x} + b dy\right) = -bx \frac{\partial}{\partial x} + a \frac{\partial}{\partial y} \in \text{Der}(\log x \mathcal{A}). \quad (3.3)$$

On the other hand, we have

$$\begin{aligned} & [\alpha_1^0 \frac{dx}{x} + \alpha_1^1 dy, \alpha_2^0 \frac{dx}{x} + \alpha_2^1 dy]_s := \\ & = \left(\frac{\alpha_1^0}{x} \{x, \alpha_2^0\} + \frac{\alpha_2^0}{x} \{\alpha_1^0, x\} + \alpha_2^1 \{\alpha_1^0, y\} + \alpha_1^1 \{y, \alpha_2^0\} \right) \frac{dx}{x} \\ & + \left(\frac{\alpha_1^0}{x} \{x, \alpha_2^1\} + \frac{\alpha_2^1}{x} \{\alpha_1^1, x\} + \alpha_1^1 \{y, \alpha_2^1\} + \alpha_2^1 \{\alpha_1^1, y\} \right) dy \end{aligned} \quad (3.4)$$

LEMMA 3.2. — *$[\cdot, \cdot]_s$ is a Lie bracket on $\Omega_{\mathcal{A}}(\log \mathcal{I})$.*

Proof. — It follows from lemma 3.1 that it suffices to show that this bracket is a Lie bracket on $\mathbb{C}[y] \frac{dx}{x} \oplus \Omega_{\mathcal{A}}$.

Since the bracket

$$[dx, dy] := dx \quad (3.5)$$

defines a Lie bracket on $\Omega_{\mathcal{A}}$ we need to put on $\mathbb{C}[y] \frac{dx}{x}$ a Lie bracket such that the following

$$0 \longrightarrow \Omega_{\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}} \oplus \mathbb{C}[y] \frac{dx}{x} \longrightarrow \mathbb{C}[y] \frac{dx}{x} \longrightarrow 0 \quad (3.6)$$

becomes a split short sequence of Lie algebras. According to [1], setting

$$[\gamma_1 + \beta_1, \gamma_2 + \beta_2] = [\gamma_1, \gamma_2] + [\beta_1, \gamma_2] - [\beta_2, \gamma_1] + [\beta_1, \beta_2] \quad (3.7)$$

when $\gamma_i + \beta_i \in \Omega_{\mathcal{A}} \oplus \mathbb{C}[y] \frac{dx}{x}$ for $i = 1, 2$, defines a Lie bracket on $\Omega_{\mathcal{A}} \oplus \mathbb{C}[y] \frac{dx}{x}$. Therefore it is sufficient to prove that the brackets (3.7) and (3.4) are equal. By a simple application of the Jacobi identity to $\{.,.\}$ we have the result. \square

LEMMA 3.3. — For all $\alpha = \alpha_1^0 \frac{dx}{x} + \alpha_1^1 dy, \beta = \beta_1^0 \frac{dx}{x} + \beta_1^1 dy \in \Omega_{\mathcal{A}}(\log \mathcal{I})$ and $a \in \mathcal{A}$ we have

$$[\alpha, a\beta]_s = \tilde{H}(\alpha)(a)\beta + a[\alpha, \beta]_s. \quad (3.8)$$

Proof. — It is a simple application of Jacobi identity of $\{.,.\}$. \square

LEMMA 3.4. — $\tilde{H} : \Omega_{\mathcal{A}}(\log \mathcal{I}) \longrightarrow \text{Der}_{\mathcal{A}}(\log \mathcal{I})$ is a Lie algebra homomorphism.

Proof. — This follows from a direct calculation. \square

We deduce the following Proposition:

PROPOSITION 3.5. — $(\Omega_{\mathcal{A}}(\log \mathcal{I}), [.,.], \tilde{H})$ is a Lie-Rinehart algebra.

In what follows, we will describe explicitly the associated logarithmic Poisson complex. From the above description we can identify in this particular case $\text{Alt}^i(\Omega_{\mathcal{A}}(\log \mathcal{I}), \mathcal{A})$ with $\mathcal{A}^i := \underbrace{\mathcal{A} \times \dots \times \mathcal{A}}_i$. Therefore, the logarithmic Poisson complex is equivalent to

$$0 \longrightarrow \mathcal{A} \xrightarrow{d_{\tilde{H}}^0} \mathcal{A} \times \mathcal{A} \xrightarrow{d_{\tilde{H}}^1} \mathcal{A} \longrightarrow 0.$$

where

$$d_{\tilde{H}}^0(f) = (\partial_y f, -x\partial_x f) \text{ and } d_{\tilde{H}}^1(f_1, f_2) = \partial_y f_2 + x\partial_x f_1.$$

We verify that

$$d_{\tilde{H}}^1(d_{\tilde{H}}^0 f) = x(\partial_{xy}^2 f - \partial_{xy}^2 f) = 0.$$

PROPOSITION 3.6. — The associated Poisson 2-form of $\{x, y\} = x$ is $\mu = x\partial_x \wedge \partial_y$ which is a log symplectic structure.

Proof. — The associated log symplectic 2-form is $\omega = \frac{dx}{x} \wedge dy$. \square

3.1.1. Computation of $H_{PS}^i; i = 0, 1, 2$

These spaces are given by the following Proposition.

PROPOSITION 3.7. — $H_{PS}^0 \cong \mathbb{C}$, $H_{PS}^1 \cong \mathbb{C}$, $H_{PS}^2 \cong 0_{\mathcal{A}}$.

Proof. — According to the above construction of the co-chain spaces of the logarithmic Poisson complex we have:

1. Calculation of H_{PS}^0 .

For all $f \in \mathcal{A}$ $f \in \ker d_{\bar{H}}^0$ iff $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} = 0$. Therefore $f \in \mathbb{C}$,

2. Calculation of H_{PS}^2 .

For all $g \in \mathcal{A}$, $g = d_{\bar{H}}^1(0, \int g dy + k(x))$. Then $d_{\bar{H}}^1$ is an epimorphism,

3. Calculation of H_{PS}^1 .

We have $\mathcal{A}^2 \cong (\mathbb{C}[y] \times \mathbb{C}[x]) \oplus (x\mathcal{A} \times y\mathcal{A})$. Then for all $(f_1, f_2) \in \mathcal{A} \times \mathcal{A}$ there are $g_1 \in \mathbb{C}[y]$, $g_2 \in \mathbb{C}[x]$ and $h_2, h_1 \in \mathcal{A}$ such that $f_1 = g_1(y) + xh_1$ and $f_2 = g_2(x) + yh_2$. But for all $(a(y), b(x)) \in \mathbb{C}[y] \times \mathbb{C}[x]$

$$x \frac{\partial a(y)}{\partial x} + \frac{\partial b(x)}{\partial y} = 0.$$

Then $\mathbb{C}[y] \times \mathbb{C}[x] \subset \ker d_{\bar{H}}^1$. For similar reasons we have

$$\begin{aligned} \ker(d_{\bar{H}}^1) : &= \ker(d_{\bar{H}}^1) \cap \mathcal{A}^2 \\ &= (\mathbb{C}[y] \times \mathbb{C}[x]) \oplus \ker(d_{\bar{H}}^1) \cap (x\mathcal{A} \times y\mathcal{A}) \\ &= (\mathbb{C}[y] \times \mathbb{C}[x]) \oplus \Theta(\mathcal{A}), \end{aligned}$$

where Θ is defined by

$$\mathcal{A} \xrightarrow{\Theta} \mathcal{A}^2, \quad a \mapsto (xa, -\int x \frac{\partial xa}{\partial x} dy).$$

It is easy to verify that $\Theta(\mathcal{A}) \subset \ker(d_{\bar{H}}^1)$.

On the other hand, we have the following decomposition of \mathcal{A} :

$$\mathcal{A} \cong \mathbb{C}[x] \oplus y\mathbb{C}[y] \oplus xy\mathcal{A}.$$

Therefore for any $f \in \mathcal{A}$, there is $(f_1, q, p) \in \mathbb{C}[x] \times \mathbb{C}[y] \times \mathcal{A}$ such that $f = f_1 + yq + xyp$.

Then

$$\begin{aligned} \frac{\partial f}{\partial y} &= q + y \frac{\partial q}{\partial y} + x(p + y \frac{\partial p}{\partial y}) \\ &= (1 + y \frac{\partial}{\partial y})q + x(1 + y \frac{\partial}{\partial y})p \in \mathbb{C}[y] \oplus x(1 + y \frac{\partial}{\partial y})(\mathcal{A}) \end{aligned}$$

and

$$\begin{aligned} -x \frac{\partial f}{\partial x} &= -x \frac{\partial f_1}{\partial x} - xyp - x^2y \frac{\partial p}{\partial x} \\ &= -x \frac{\partial f_1}{\partial x} - xy(1 + x \frac{\partial}{\partial x})p \in x\mathbb{C}[x] \oplus xy(1 + x \frac{\partial}{\partial x})\mathcal{A}. \end{aligned}$$

We consider the map

$$\begin{aligned} \Psi : \mathcal{A} &\rightarrow \mathcal{A}^2 \\ f &\mapsto (x(1 + y \frac{\partial}{\partial y})f, -xy(1 + x \frac{\partial}{\partial x})f). \end{aligned}$$

Since

$$\begin{aligned} (x(1 + y \frac{\partial}{\partial y})f, -xy(1 + x \frac{\partial}{\partial x})f) &= (xf \frac{\partial y}{\partial y} + xy \frac{\partial f}{\partial y}, -x \frac{\partial x}{\partial x}yf - x^2 \frac{\partial yf}{\partial x}) \\ &= (\frac{\partial xyf}{\partial y}, -x \frac{\partial xyf}{\partial x}) \\ &= d_{\tilde{H}}^0(xyf) \end{aligned}$$

and $\Psi(\mathcal{A}) \subset d_{\tilde{H}}^0(\mathcal{A})$.

Then

$$(\frac{\partial f}{\partial y}, -x \frac{\partial f}{\partial x}) \in (\mathbb{C}[y] \times x\mathbb{C}[x]) \oplus \Psi(\mathcal{A}).$$

Conversely, any $F := (f_1(y), xf_2(x)) + \Psi(p)$ is an element of $(\mathbb{C}[y] \times x\mathbb{C}[x]) \oplus \Psi(\mathcal{A})$. Therefore

$$F = d_{\tilde{H}}^0(\int f_1 dy - \int f_2 dx) + d_{\tilde{H}}^0(xyp) = d_{\tilde{H}}^0(\int f_1 dy - \int f_2 dx + xyp) \in d_{\tilde{H}}^0(\mathcal{A}).$$

Then

$$d_{\tilde{H}}^0(\mathcal{A}) \cong (\mathbb{C}[y] \times x\mathbb{C}[x]) \oplus \Psi(\mathcal{A}).$$

On the other hand, due to the fact that $d_{\tilde{H}}^0(\int xady) = (xa, -\int x \frac{\partial xa}{\partial x} dy)$ for all $a \in \mathcal{A}$, we can conclude that $\Theta(\mathcal{A}) \subset d_{\tilde{H}}^0(\mathcal{A})$. Moreover by direct calculation we show that $\Theta(\mathcal{A}) \subset \Psi(\mathcal{A})$.

Since $(\mathbb{C}[y] \times \mathbb{C}[x]) \cong (\mathbb{C}[y] \times x\mathbb{C}) \oplus (0_{\mathcal{A}} \times \mathbb{C})$ and, $x \frac{\partial \mathcal{A}}{\partial x} \cap \mathbb{C} = 0_{\mathcal{A}}$ we have:

$$d_{\tilde{H}}^0(\mathcal{A}) \cap (0_{\mathcal{A}} \times \mathbb{C}) \cong 0_{\mathcal{A}}.$$

Then $H_{PS}^1 \cong \mathbb{C}$.

□

3.1.2. Computation of $H_{DS}^i, i = 0, 1, 2$

By definition, the logarithmic De Rham complex associated to the ideal $x\mathcal{A}$ is:

$$0 \longrightarrow \mathcal{A} \xrightarrow{d^0} \Omega_{\mathcal{A}}^1(\log x\mathcal{A}) \xrightarrow{d^1} \Omega_{\mathcal{A}}^2(\log x\mathcal{A}) \longrightarrow 0. \quad (3.9)$$

where

$$d^0(a) := x\partial_x(a)\frac{dx}{x} + \partial_y(a)dy \text{ and}$$

$$d^1(a\frac{dx}{x} + bdy) := (x\partial_x(b) - \partial_y(a))\frac{dx}{x} \wedge dy.$$

PROPOSITION 3.8. — *The following diagram of \mathcal{A} -modules is commutative*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A} & \xrightarrow{d^0} & \Omega_{\mathcal{A}}(\log x\mathcal{A}) & \xrightarrow{d^1} & \Omega_{\mathcal{A}}^2(\log x\mathcal{A}) & \longrightarrow & 0. \\ & & \downarrow & & \downarrow -\tilde{H} & & \downarrow -\tilde{H} & & \\ 0 & \longrightarrow & \mathcal{A} & \xrightarrow{d_{\tilde{H}}^0} & \mathcal{A}^2 & \xrightarrow{d_{\tilde{H}}^1} & \mathcal{A} & \longrightarrow & 0 \end{array}$$

Proof. — For any $a \in \mathcal{A}$ we have

$$\begin{aligned} \tilde{H}(da) &= \tilde{H}(x\partial_x(a)\frac{dx}{x} + \partial_y(a)dy) \\ &= -\partial_y(a)x\partial_x + x\partial_x(a)\partial_y \\ &\cong (-\partial_y(a), x\partial_x(a)) \end{aligned}$$

and

$$\begin{aligned} d_{\tilde{H}}^0(a) &\cong (\partial_y(a), -x\partial_x(a)) \\ &= -\tilde{H}(da). \end{aligned}$$

Moreover for any $\alpha = f\frac{dx}{x} + gdy \in \Omega_{\mathcal{A}}(\log \mathcal{I})$ we have

$$d^1(\alpha) = (x\partial_x(g) - \partial_y(f))\frac{dx}{x} \wedge dy, \quad \text{and} \quad -\tilde{H}(d^1(\alpha)) \cong x\partial_x(g) - \partial_y(f).$$

However

$$\begin{aligned} -\tilde{H}(\alpha) &= gx\partial_x - f\partial_y \\ &\cong (g, -f) \end{aligned}$$

we have

$$\begin{aligned} d_{\tilde{H}}^1(-\tilde{H}) &= d_{\tilde{H}}^1(gx\partial_x - f\partial_y) \\ &\cong x\partial_x(g) - \partial_y(f). \end{aligned}$$

This ends the proof. \square

The following Proposition gives the logarithmic De Rham cohomology spaces.

PROPOSITION 3.9. — $H_{DS}^0 \cong \mathbb{C}$, $H_{DS}^1 \cong \mathbb{C}$, $H_{DS}^2 \cong 0_{\mathcal{A}}$.

Proof. — For simplicity we adopt the following notations

$$\begin{aligned} \Omega_{\mathcal{A}}^1(\log x\mathcal{A}) &\cong \rightarrow \mathcal{A} \times \mathcal{A} & \Omega_{\mathcal{A}}^2(\log x\mathcal{A}) &\cong \rightarrow \mathcal{A} \\ a\frac{dx}{x} + bdy &\mapsto (a, b) & a\frac{dx}{x} \wedge dy &\mapsto a. \end{aligned}$$

With these notations, the complex (3.9) becomes

$$0 \longrightarrow \mathcal{A} \xrightarrow{d^0} \mathcal{A} \times \mathcal{A} \xrightarrow{d^1} \mathcal{A} \longrightarrow 0 \quad (3.10)$$

where $d^0(f) = (x\partial_x f, \partial_y f)$ and $d^1(f_1, f_2) = x\partial_x f_2 - \partial_y f_1$.

For all $f \in \mathcal{A}$, $f = d^1(-\int f dy, 0)$. Then $\mathcal{A} \cong d^1(\mathcal{A} \times \mathcal{A})$ and therefore $H_{DS}^2 \cong 0$. It is easy to see that $H_{DS}^0 \cong \mathbb{C}$.

Let $(f^1, f^2) \in \mathcal{A} \times \mathcal{A}$. Then $(f^1, f^2) \in \ker(d^1)$ iff $f^1 = x \int \partial_x f^2 dy + k(x)$. Therefore

$$\ker(d^1) \cong \left\{ \left(x \int \partial_x u dy, u \right), u \in \mathcal{A} \right\} \oplus x\mathbb{C} \oplus \mathbb{C}.$$

The following map is a monomorphism of vector spaces:

$$\begin{aligned} \theta: \mathcal{A} &\rightarrow x\mathcal{A} \times \mathcal{A} \\ u &\mapsto \left(x \int \partial_x u dy, u \right) \end{aligned}$$

and $\ker(d^1) \cong \theta(\mathcal{A}) \oplus (x\mathbb{C} \times 0_{\mathcal{A}}) \cong \theta(\mathcal{A}) \oplus (x\mathbb{C} \oplus \mathbb{C})$.

Moreover for any $u \in \mathcal{A}$ and $a \in \mathbb{C}[x]$ we have:

$$\begin{aligned} d^0\left(\int u dy + \int a dx\right) &= \left(x \int \partial_x u dy + xa, u \right) \\ &= \left(x \int \partial_x u dy, u \right) + (xa, 0) \\ &= \theta(u) + (xa, 0) \in \theta(\mathcal{A}) \oplus (x\mathbb{C}). \end{aligned}$$

Then

$$\theta(\mathcal{A}) \oplus (x\mathbb{C}) \subset d^0(\mathcal{A}).$$

Since $\mathbb{C} \cap d^0(\mathcal{A}) = 0_{\mathcal{A}}$ we have $d^0(\mathcal{A}) = d^0(\mathcal{A}) \cap (\ker(d^1)) \cong \theta(\mathcal{A}) \oplus (x\mathbb{C})$.

Therefore $\ker(d^1) \cong d^0(\mathcal{A}) \oplus \mathbb{C}$. And then $H_{DS}^1 \cong \mathbb{C}$. \square

3.1.3. Computation of Poisson cohomology of $\{x, y\} = x$

By a direct calculation we show that the Poisson complex of $\{x, y\} = x$ is given by

$$0 \longrightarrow \mathcal{A} \xrightarrow{d_H^0} \mathcal{A} \times \mathcal{A} \xrightarrow{d_H^1} \mathcal{A} \longrightarrow 0. \quad (3.11)$$

where $d_H^0(f) = (x\partial_y f, -x\partial_x f)$ and $d_H^1(f_1, f_2) = x\partial_y f_2 + x\partial_x f_1 - f_1$

PROPOSITION 3.10. — $H_P^0 \cong \mathbb{C}$, $H_P^1 \cong \mathbb{C}$ and $H_P^2 \cong 0_{\mathcal{A}}$.

Proof. — It is shown without difficulty that $H_P^0 \cong \mathbb{C}$ and $H_P^2 \cong 0_{\mathcal{A}}$. So we have to prove that $H_P^1 \cong \mathbb{C}$. For all $(f_1, f_2) \in \mathcal{A} \times \mathcal{A}$ $(f_1, f_2) \in \ker(d_H^1)$ iff there is $u \in \mathcal{A}$ and $a(x) \in \mathbb{C}[x]$ such that

$$(f_1, f_2) = (xu, -x \int \partial_x u dy) + (0, a(x)).$$

We set

$$\beta : \mathcal{A} \rightarrow x\mathcal{A} \times \mathcal{A}, u \mapsto (xu, -x \int \partial_x u dy).$$

Clearly β is a monomorphism $\ker(d_H^1) \cong \beta(\mathcal{A}) \oplus x\mathbb{C}[x] \oplus \mathbb{C}$ and $\beta(\mathcal{A}) \oplus x\mathbb{C}[x] \subset d_H^0(\mathcal{A})$. In addition there is no $f \in \mathcal{A}$ such that $x\partial_x f \in \mathbb{C}^*$. Then $\ker(d_H^1) \cong d_H^0(\mathcal{A}) \oplus \mathbb{C}$. Therefore $H_P^1 \cong \mathbb{C}$. \square

3.2. Example 2: $(\mathcal{A} := \mathbb{C}[x, y], \{x, y\} = x^2)$

Let us consider on $\mathcal{A} = \mathbb{C}[x, y]$ the Poisson bracket defined on the variables x, y by $\{x, y\} = x^2$.

Note that $\Omega_{\mathcal{A}}(\log x^2 \mathcal{A})$ is isomorphic to the \mathcal{A} -module generated by

$\{\frac{dx}{x} \cup \Omega_{\mathcal{A}}\}$ since $\frac{dx^2}{x^2} = 2\frac{dx}{x}$. It is easy to see that the bracket $\{x, y\} = x^2$ is

a logarithmic principal Poisson bracket along the ideal $x^2 \mathcal{A}$. The associated logarithmic Hamiltonian map is defined on the generators of $\Omega_{\mathcal{A}}(\log x^2 \mathcal{A})$

by $\tilde{H}(\frac{dx}{x}) = x\partial_y$ and $\tilde{H}(dy) = -x^2\partial_x$. We deduce that the associated logarithmic Poisson complex is defined by

$$d_H^0(f) = (x\partial_y f, -x^2\partial_x f), d_H^1(f_1, f_2) = x\partial_y f_2 + x^2\partial_x f_1 - x f_1;$$

where we consider the following identification

$$\begin{aligned} Der_{\mathcal{A}}(\log x^2 \mathcal{A}) &\xrightarrow{\cong} \mathcal{A} \times \mathcal{A} \\ ax\partial_x + b\partial_y &\mapsto (a, b) \end{aligned}$$

$$\begin{aligned} Der_{\mathcal{A}}(\log x^2 \mathcal{A}) \wedge Der_{\mathcal{A}}(\log x^2 \mathcal{A}) &\xrightarrow{\cong} \mathcal{A} \\ ax\partial_x \wedge \partial_y &\mapsto a \end{aligned}$$

3.2.1. Computation of H_{PS}^2

Since $\mathcal{A} \cong \mathbb{C}[y] \oplus x\mathcal{A}$ for all $g \in \mathcal{A}$ there are $g_1, g_2 \in \mathcal{A}$ such that $g = g_1 + xg_2$. Therefore for all $g \in \mathcal{A}, g \in d_{\bar{H}}^1(\mathcal{A})$ iff $g = xg_2 = x\partial_y f_2 + x^2\partial_x f_1 - x f_1$. But $xg_2 = x\partial_y(x \int \partial_x g_2 dy) - x^2\partial_x g_2 - xg_2$ and the equation $x(\partial_y v + x\partial_x u - u) = g(y) \in \mathbb{C}[y]^*$ has no solutions in $\mathcal{A} \times \mathcal{A}$. Then $\mathcal{A} \cong d_{\bar{H}}^1(\mathcal{A} \times \mathcal{A}) \oplus \mathbb{C}[y]$. It follows that

$$H_{PS}^2 \cong \mathbb{C}[y].$$

3.2.2. Computation of H_{PS}^1

To compute H_{PS}^1 we first recall the following fact.

LEMMA 3.11. — *Let $\varphi : E \rightarrow F$ be a monomorphism of vector spaces. Then for any linear subspaces A and B of $E, \varphi(A \oplus B) = \varphi(A) \oplus \varphi(B)$*

Proof. — It is clear that $\varphi(A \oplus B) = \varphi(A) + \varphi(B)$. If $z \in \varphi(A) \cap \varphi(B)$ then $z \in \varphi(A \oplus B) = 0_E$. Therefore $\varphi(A \oplus B) = \varphi(A) \oplus \varphi(B)$. \square

Let $(f_1, f_2) \in \mathcal{A} \times \mathcal{A}$. Then $(f_1, f_2) \in \ker(d_{\bar{H}}^1)$ iff there is $k \in \mathbb{C}[x]$ such that $f_2 = \int(1 - x\partial_x)f_1 dy + k(x)$. So

$$\ker(d_{\bar{H}}^1) \cong \left\{ (u, \int(1 - x\partial_x)udy), u\mathcal{A} \right\} \oplus \mathbb{C}[x].$$

We put for all $u \in \mathcal{A}; \eta(u) = (u, \int(1 - x\partial_x)udy)$. Then $\eta : \mathcal{A} \rightarrow \mathcal{A} \times \mathcal{A}$ is a monomorphism of vector spaces and

$$\ker(d_{\bar{H}}^1) \cong \eta(\mathcal{A}) \oplus \mathbb{C}[x] \cong \eta(\mathbb{C}[y]) \oplus \eta(x\mathcal{A}) \oplus \mathbb{C}[x]$$

since $\mathcal{A} \cong \mathbb{C}[y] \oplus x\mathcal{A}$. On the other hand for all $g \in \eta(x\mathcal{A}) \oplus (0_{\mathcal{A}}, x^2\mathbb{C}[x])$ there is $u \in \mathcal{A}$ and $v \in \mathbb{C}[x]$ such that

$$g = (xu, -x^2 \int \partial_x dy + x^2 v(x)) = d_{\bar{H}}^0 \left(\int u dy - \int v(x) dx \right).$$

Moreover for all $u(y) \in \mathbb{C}[y]$ and $a_0, a_1 \in \mathbb{C}$ the partial differential equation

$$\begin{cases} x f_y & = & u(y) \\ -x^2 f_x & = & \int u(y) dy + a_0 + a_1 x \end{cases}$$

has no solutions in \mathcal{A} . Then $\ker(d_{\bar{H}}^1) \cong \eta(\mathbb{C}[y]) \oplus \mathbb{C}_1[x] \oplus d_{\bar{H}}^0(\mathcal{A})$. Therefore

$$H_{PS}^1 \cong \eta(\mathbb{C}[y]) \oplus \mathbb{C}_1[x].$$

where $\mathbb{C}_1[x] := \{a_0 + a_1x; a_0, a_1 \in \mathbb{C}\}$. On the other hand since η is a monomorphism, $\eta(\mathbb{C}[y]) \cong \mathbb{C}[y]$. Then $H_{PS}^1 \cong \mathbb{C}[y] \oplus \mathbb{C}_1[x]$. This ends the proof of the following Proposition.

PROPOSITION 3.12. — *The logarithmic Poisson cohomology spaces of $\{x, y\} = x^2$ are*

$$H_{PS}^1 \cong \mathbb{C}[y] \oplus \mathbb{C}_1[x], H_{PS}^2 \cong \mathbb{C}[y] \quad \text{and} \quad H_{PS}^0 \cong \mathbb{C}.$$

3.2.3. Poisson cohomology of $(\mathcal{A} = \mathbb{C}[x, y], \{x, y\} = x^2)$

The action of the Hamiltonian map associated to this Poisson structure on generators of $\Omega_{\mathcal{A}}$ is $H(dx) = x^2\partial_y$ and $H(dy) = -x^2\partial_x$.

For the sake of simplicity we shall use the following isomorphisms:

$$\begin{array}{ccc} Der_{\mathcal{A}} & \cong \rightarrow & \mathcal{A} \times \mathcal{A} \\ a\partial_x + b\partial_y & \mapsto & (a, b) \end{array} \quad \begin{array}{ccc} Der_{\mathcal{A}} \wedge Der_{\mathcal{A}} & \cong \rightarrow & \mathcal{A} \\ a\partial_x \wedge \partial_y & \mapsto & a. \end{array}$$

With these isomorphisms the associated Poisson complex is given by

$$d_H^0(f) = (x^2\partial_y f, -x^2\partial_x f) \quad \text{and} \quad d_H^1(f_1, f_2) = x^2\partial_x f_1 + x^2\partial_y f_2 - 2xf_1.$$

For all $g \in \mathcal{A}$ we have $xg = -2x(-\frac{1}{2}g) + x^2(\frac{1}{2})(-\partial_x g + \partial_y(\int \partial_x g dy))$. Then $\mathcal{A} \cong d_H^1(\mathcal{A} \times \mathcal{A}) \oplus \mathbb{C}[y]$. Therefore

$$H_P^2 \cong \mathbb{C}[y].$$

Let $(f_1, f_2) \in \mathcal{A} \times \mathcal{A}$. Then $(f_1, f_2) \in \ker(d_H^1)$ iff there is $u \in \mathcal{A}, a \in \mathbb{C}[x]$ such that $f_1 = xu$ and $f_2 = \int(1 - x\partial_x)udy + a(x)$.

So $\ker(d_H^1) = \{(xu, \int(1 - x\partial_x)udy + a(x)), \quad u \in \mathcal{A}, a(x) \in \mathbb{C}[x]\}$. We put $\varphi(u) = (xu, \int(1 - x\partial_x)udy)$ for all $u \in \mathcal{A}$. Then $\varphi : \mathcal{A} \rightarrow x\mathcal{A} \times \mathcal{A}$ is a isomorphisms of vector spaces and

$$\ker(d_H^1) \cong \varphi(\mathcal{A}) \oplus \mathbb{C}[x].$$

On the other hand since $\mathcal{A} \cong \mathbb{C}[y] \oplus x\mathcal{A}$ then $\varphi(\mathcal{A}) \cong \varphi(\mathbb{C}[y]) \oplus \varphi(x\mathcal{A})$. It is easy to prove that $\varphi(x\mathcal{A}) \oplus x^2\mathbb{C}[x] \subset d_H^0(\mathcal{A})$ and

$$d_H^0(\mathcal{A}) \cap \varphi(\mathbb{C}[y]) \oplus \mathbb{C}_1[x] = \{0_{\mathcal{A}}\}.$$

Therefore

$$\ker(d_H^1) \cong \varphi(\mathbb{C}[y]) \oplus \mathbb{C}_1[x] \oplus d_H^0(\mathcal{A}) \cong \mathbb{C}[y] \oplus \mathbb{C}_1[x] \oplus d_H^0(\mathcal{A}).$$

Then,

$$H_P^1 \cong \mathbb{C}[y] \oplus \mathbb{C}_1[x].$$

This ends the proof of the following Proposition.

PROPOSITION 3.13. — *The Poisson cohomology spaces of $\{x, y\} = x^2$ are:*

$$H_P^1 \cong \mathbb{C}[y] \oplus \mathbb{C}_1[x], H_P^2 \cong \mathbb{C}[y] \quad \text{and} \quad H_P^0 \cong \mathbb{C}.$$

Remark 2. — It follows from Propositions 3.13 and 3.12 that the Poisson cohomology and the logarithmic Poisson cohomology of the Poisson bracket $\{x, y\} = x^2$ on $\mathbb{C}[x, y]$ are equal although the latter is not log symplectic. Consequently it can be concluded that being log symplectic is not a necessary condition for the equality between the Poisson cohomology spaces and the logarithmic Poisson cohomology spaces. In the next section we give an example in which the two concepts are different.

3.3. Example 3 $\mathcal{A} = \mathbb{C}[x, y, z]$ and $\{x, y\} = 0, \{x, z\} = 0, \{y, z\} = xyz$

It is easy to prove that this Poisson structure is logarithmic principal along the ideal $xyz\mathcal{A}$ and the associated logarithmic Poisson differential is defined by

$$\begin{aligned} d_H^0(f) &= (0, xz \frac{\partial f}{\partial z}, -xy \frac{\partial f}{\partial y}) \\ d_H^1(f_1, f_2, f_3) &= (xz \frac{\partial f_3}{\partial z} + xy \frac{\partial f_2}{\partial y} - xf_1, -xy \frac{\partial f_1}{\partial y}, -xz \frac{\partial f_1}{\partial z}) \\ d_H^2(f_1, f_2, f_3) &= xz \frac{\partial f_2}{\partial z} + xy \frac{\partial f_3}{\partial y}. \end{aligned} \quad (3.12)$$

By definition we have the following expressions of the associated Poisson differential:

$$\begin{aligned} \delta^0(f) &= xyz(0, \frac{\partial f}{\partial z}, -\frac{\partial f}{\partial y}) \\ \delta^1(f_1, f_2, f_3) &= (xyz \frac{\partial f_3}{\partial z} + xyz \frac{\partial f_2}{\partial y} - yzf_1 - xzf_2 - xyf_3, -xyz \frac{\partial f_1}{\partial y}, -xyz \frac{\partial f_1}{\partial z}) \\ \delta^2(f_1, f_2, f_3) &= xyz(\frac{\partial f_2}{\partial z} + \frac{\partial f_3}{\partial y}) \end{aligned} \quad (3.13)$$

3.3.1. Computation of H_{PS}^3

We deduce from equations (3.12) that $d_H^2(\mathcal{A}^3) \subset x\mathcal{A}$.

But

$$\mathcal{A} \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathcal{A} \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x] \oplus xy\mathbb{C}[y] \oplus xz\mathbb{C}[z] \oplus x^2y\mathcal{A} \oplus x^2z\mathcal{A} \oplus xyz\mathcal{A}.$$

On the other hand, for all $xg(x) \in x\mathbb{C}[x]$ the partial differential equation

$$z \frac{\partial u}{\partial z} + y \frac{\partial v}{\partial y} = g(x)$$

has no solutions in $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$. Moreover for all

$$g \in xy\mathbb{C}[y] \oplus xz\mathbb{C}[z] \oplus x^2y\mathcal{A} \oplus x^2z\mathcal{A} \oplus xyz\mathcal{A},$$

there are $g_1(y), g_2(z), g_3(x, y, z), g_4(x, y, z), g_5(x, y, z) \in \mathcal{A}$ such that

$$g = xyg_1(y) + xzg_2(z) + x^2yg_3(x, y, z) + x^2zg_4(x, y, z) + xyzg_5(x, y, z).$$

Therefore 2 co-boundary are given by

$$z \frac{\partial f_2}{\partial z} + y \frac{\partial f_3}{\partial y} = yg_1(y) + zg_2(z) + xyg_3(x, y, z) + xzg_4(x, y, z) + yzg_5(x, y, z), \quad (3.14)$$

which is equivalent to

$$z \left(\frac{\partial f_2}{\partial z} - g_2(z) - xg_4(x, y, z) \right) + y \left(\frac{\partial f_3}{\partial y} - g_1(y) - xg_3(x, y, z) - zg_5(x, y, z) \right) = 0. \quad (3.15)$$

So just take

$$f_2 = \int g_2(z) + xg_4(x, y, z) dz; \quad f_3 = \int g_1(y) + xg_3(x, y, z) + zg_5(x, y, z) dy. \quad (3.16)$$

This proves that

$$d_H^2(\mathcal{A}^3) \cong xy\mathbb{C}[y] \oplus xz\mathbb{C}[z] \oplus x^2y\mathcal{A} \oplus x^2z\mathcal{A} \oplus xyz\mathcal{A}.$$

Therefore we deduce that

$$H_{PS}^3 \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x]. \quad (3.17)$$

3.3.2. Computation of H_P^3

It follows from equation (3.13) that $\delta^2(\mathcal{A}^3) \subset xyz\mathcal{A}$. But

$$\mathcal{A} \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x] \oplus xy\mathbb{C}[y] \oplus xy\mathbb{C}[x] \oplus xz\mathbb{C}[x] \oplus xz\mathbb{C}[z] \oplus yz\mathbb{C}[y] \oplus yz\mathbb{C}[z] \oplus xyz\mathcal{A} \quad (3.18)$$

and

$$\delta^2(\mathcal{A}^3) \cap \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x] \oplus xy\mathbb{C}[y] \oplus xy\mathbb{C}[x] \oplus xz\mathbb{C}[x] \oplus xz\mathbb{C}[z] \oplus yz\mathbb{C}[y] \oplus yz\mathbb{C}[z] \cong 0_{\mathcal{A}}.$$

Since the map

$$\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}, (u, v) \mapsto \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \tag{3.19}$$

is surjective $\delta^3(\mathcal{A}^3) \cong xyz\mathcal{A}$.

Therefore

$$H_P^3 \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x] \oplus xy\mathbb{C}[y] \oplus xy\mathbb{C}[x] \oplus xz\mathbb{C}[x] \oplus xz\mathbb{C}[z] \oplus yz\mathbb{C}[y] \oplus yz\mathbb{C}[z].$$

In conclusion we have proved the following.

THEOREM 3.14. —

1. The 3rd Poisson cohomology of $(\mathcal{A} = \mathbb{C}[x, y, z], \{x, y\} = 0, \{x, z\} = 0, \{y, z\} = xyz)$ is

$$H_P^3 \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x] \oplus xy\mathbb{C}[y] \oplus xy\mathbb{C}[x] \oplus xz\mathbb{C}[x] \oplus xz\mathbb{C}[z] \oplus yz\mathbb{C}[y] \oplus yz\mathbb{C}[z].$$

2. The 3rd logarithmic Poisson cohomology of $(\mathcal{A} = \mathbb{C}[x, y, z], \{x, y\} = 0, \{x, z\} = 0, \{y, z\} = xyz)$ is

$$H_{PS}^3 \cong \mathbb{C}[y] \oplus z\mathbb{C}[z] \oplus x\mathbb{C}[x]. \tag{3.20}$$

Remark 3. — It has to be noticed that $H_{PS}^3 \neq H_P^3$.

4. Application to prequantization of $\{x, y\} = x$

The problem of geometric quantization is based on the Dirac principle in which we represent the underlying algebra of a Poisson algebra into a Hilbert space \mathcal{H} . In other words one shall build the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{R} & \longrightarrow & (\mathcal{A}, \{ \cdot, \cdot \}) & \longrightarrow & (d\mathcal{A}, [\cdot, \cdot]_{LP}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{B} \longrightarrow 0 \end{array}$$

where the first line is an extension of Lie algebras and the second is an extension of Lie-Rinehart algebras. In this diagram, \mathcal{H} is a simplified notation for the algebra $(\mathcal{O}_{\mathcal{H}}, [\cdot, \cdot])$ where $\mathcal{O}_{\mathcal{H}}$ is the set of the quantum observable operators (which in the symplectic case are the self-adjoint operators on \mathcal{H}) and $[\cdot, \cdot]$ is the commutator. But according to ([17]) the following bracket

$$[a + \alpha, b + \beta] := \{a, b\} + \pi(\alpha, \beta) + [\alpha, \beta] + \tilde{H}(\alpha)b - \tilde{H}(\beta)a$$

is a Lie structure on $\mathcal{A} \oplus \Omega_{\mathcal{A}}(\log x\mathcal{A})$ such that the following is an extension of Lie-Rinehart algebras

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A} \oplus \Omega_{\mathcal{A}}(\log x\mathcal{A}) \longrightarrow \Omega_{\mathcal{A}}(\log x\mathcal{A}) \longrightarrow 0$$

Where $\pi = x\partial_x \wedge \partial_y$ is the Poisson bivector of $\{x, y\} = x$. By construction, π is the associated class of this extension.

We consider the map $r : \mathcal{A} \rightarrow \mathcal{A} \oplus \Omega_{\mathcal{A}}(\log x\mathcal{A})$ defined by

$$r(a) = a + x\partial_x(a)\frac{dx}{x} + \partial_y(a)dy.$$

By definition, r is Lie algebra homomorphism and the following diagram commute.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & (\mathcal{A}, \{ \cdot, \cdot \}) & \longrightarrow & (d\mathcal{A}, [\cdot, \cdot]_{LP}) \longrightarrow 0 \\ & & \downarrow & & \downarrow r & & \downarrow \\ 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A} \oplus \Omega_{\mathcal{A}}(\log x\mathcal{A}) & \longrightarrow & \Omega_{\mathcal{A}}(\log x\mathcal{A}) \longrightarrow 0. \end{array}$$

We adopte the following definition.

DEFINITION 4.1. — *A Poisson structure, logarithmic along an ideal \mathcal{I} of \mathcal{A} is said log prequantizable if there is a projective \mathcal{A} -module M of rank 1 with an $\Omega_{\mathcal{A}}(\log \mathcal{I})$ -connection with curvature π .*

THEOREM 4.2. — *([7]) Let $Pic(\mathcal{A})$ be the group of projective rank one \mathcal{A} -modules. For any Lie-Rinehart algebra L , the correspondence*

$$i : Pic(\mathcal{A}) \rightarrow H^2(Alt_{\mathcal{A}}(L, \mathcal{A}))$$

which associates to any class $[M]$ of projective \mathcal{A} -module of rang 1 the class $[\Omega_M] \in H^2(Alt_{\mathcal{A}}(L, \mathcal{A}))$ of the curvature of the associated L -connection of M is an homomorphism of \mathcal{R} -modules.

It follows from this theorem that the logarithmic Poisson structure $\{x, y\} = x$ is log prequantizable if and only if the logarithmic Poisson cohomology class of π is an element of the image of i .

But according to lemma 3.7, we have $[\pi] \in H_{PS}^2 \cong 0$. Therefore $\{x, y\} = x$ is a log prequantizable Poisson structure.

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