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On separately subharmonic functions (Lelong's problem)

A. Sadullaev[1]

Abstract. — The main result of the present paper is : every separately-subharmonic function $u(x, y)$, which is harmonic in y, can be represented locally as a sum two functions, $u = u^* + U$, where U is subharmonic and u^* is harmonic in y , subharmonic in x and harmonic in (x, y) outside of some nowhere dense set S.

Résumé. — Le résultat essentiel de ce papier est le suivant : toute fonction séparément sous-harmonique $u(x, y)$ qui est harmonique en y peut être reprétée localement comme la somme de deux fonctions $u = u^* + U$, où U est sous-harmonique et u^* est harmonique en y, sous-harmonique en x et harmonique en (x, y) en dehors d'une ensemble nulle part dense S.

1. Introduction

We will consider functions $u(x, y)$ of two groups of variables $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. If u is separately harmonic, i.e., harmonic in x for fixed y and harmonic in y for fixed x, then u will be harmonic in both variables (Lelong [2],see also [1]). Lelong investigated also separately subharmonic functions, and proved a series of special results in this area. Here originates the question about subharmonicity of separately subharmonic functions.

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However, Wiegerinck, [3] (see also [4]) has shown that a separately subharmonic function need not to be subharmonic in general. He constructed a separately subharmonic function $u(x,y)$ in the bidisk $U^2 = \{|x|<1\} \times \{|y|<1\} \subset R_x^2 \times R_y^2 \approx \mathbb{C} \times \mathbb{C}$, which is not bounded above near 0.

The problem of subharmonicity of a separately subharmonic function $u(x,y)$ that is in addition harmonic in y, is still open.

In the present paper we will study the class of these functions. Let us begin by recalling the following well-known results:

1. If a separately subharmonic function is bounded above, then it is subharmonic (Lelong [2], Avanissian [5]);
2. If $u^+ \in L^1_{loc}$, then u is subharmonic (Arsove [6]);
3. If $u^+ \in L^p_{loc}, p > 0$, then u is subharmonic(Riihentaus [7]);
4. There are also positive results under weak growth conditions (see [8], [9]).

We note that the conditions in the above results are not separated in x and y. The following results demand separate conditions:

5. Suppose that $u(x,y)$ is defined on the product domain $B = B_1 \times B_2 \subset R_x^n \times R_y^m$. If u is subharmonic in x and harmonic in y, then there are nowhere dense closed sets $S_1 \subset B_1, S_2 \subset B_2$ such that u is subharmonic in $G = (B_1 \times B_2) \setminus (S_1 \times S_2)$ (Cegrell and Sadullaev [10]);
6. If $u(x,y)$ real analytic, subharmonic in x, and harmonic in y, then u is subharmonic (Imomkulov [11]);
7. There exists a separately subharmonic function $u(x,y)$, which is real analytic in x, but which is not subharmonic (Cegrell and Sadullaev [10]);
8. If $u(x,y)$ is C^2 and subharmonic in x, harmonic in y, then u is subharmonic (Kołodziej and Thorbiörnson [12]).

2. Results

Let $u(x,y)$ be a separately subharmonic function in the product domain $B = B_1 \times B_2$, which is harmonic in y. We will assume that u satisfies this condition in a slightly larger domain $\tilde{B} = \tilde{B}_1 \times \tilde{B}_2$ such that $\tilde{B} \supset \overline{B}$. Then $u(x,y)$ is subharmonic in a domain $\left(\tilde{B}_1 \times \tilde{B}_2\right) \setminus (S_1 \times S_2)$, where $S_1 \subset \tilde{B}_1, S_2 \subset \tilde{B}_2$ are closed, nowhere dense sets. Moreover, for every fixed $y \in \tilde{B}_2$ the Laplacian $\Delta_x u\,(x,y)$ defines a positive distribution as follows

$$F\,(\varphi) = \int u\,(x,y)\,\Delta_x \varphi\,(x)\,dx \qquad \varphi \in C_0^\infty,$$

thus for every test function $\varphi\,(x) \in C_0^\infty\,(B_1)$, $\mathrm{supp}\varphi \subset\subset B_1$, $\varphi \geqslant 0$ we have $F\,(\varphi) \geqslant 0$. Hence, $\Delta_x u\,(x,y)$ is a Borel measure, depending on the parameter y.

Theorem 2.1. — *For every test-function $\varphi\,(x) \in C_0^\infty\,(B_1)$ $F\,(\varphi)$ is harmonic in y for $y \in B_2 \backslash S_2$. Moreover, if $\mathrm{supp}\varphi \bigcap S_1 = \emptyset$. then $F(\varphi)$ is harmonic in y for all $y \in B_2$.*

We say that the measure $\Delta_x u\,(x,y)$ has the *harmonic property* with respect to y in the domain $G{=}(B_1 \times B_2) \setminus (S_1 \times S_2)$.

Proof. — The result 5) above states that $u\,(x,y)$ is subharmonic and therefore u is locally bounded above in $G{=}(B_1 \times B_2) \setminus (S_1 \times S_2)$. Hence the integral

$$F\,(\varphi)\,(y) = \int_{B_1} \varphi\,(x)\,\Delta_x u\,(x,y) = \int_{B_1} u\,(x,y)\,\Delta_x \varphi\,(x)$$

is harmonic in $B_2 \backslash S_2$. If $\mathrm{supp}\varphi \bigcap S_1 = \emptyset$, then this integral is harmonic in all B_2. □

Corollary 2.2 *The measure $F_E\,(y) = \int_E \Delta_x u\,(x,y)$ is harmonic in B_2 for any $E \subset\subset B_1 \backslash S_1$.*

Corollary 2.3. — *The total measure $\|\Delta_x u\,(x,y)\|_{B_1} = \int_{B_1} \Delta_x u\,(x,y)$ is finite ($\neq \infty$) for every fixed $y \in B_2$ and is harmonic function in $B_2 \backslash S_2$.*

Theorem 2.4. — *The function $F_{B_1 \backslash S_1}\,(y) = \int_{B_1 \backslash S_1} \Delta_x u\,(x,y)$ is bounded and positive harmonic in B_2.*

Proof. — Let us take an increasing sequence of compacts $E_j \subset E_{j+1} \subset\subset B_1 \backslash S_1$ such that $\bigcup_j E_j = B_1 \backslash S_1$. Then the functions $F_{E_j}\,(y) = \int_{E_j} \Delta_x u\,(x,y)$

are harmonic in B_2 and form an increasing sequence in j. By Harnacks' theorem either $F_{E_j}(y) \nearrow +\infty$ or $(F_{E_j})_j$ converges to a harmonic function. The first possibility is ruled out, because Corollary 2.3 provides a bound on the $F_{E_j}(y)$ for every $y \in B_2 \backslash S_2$.

Thus $\lim_{j\to\infty} F_{E_j}(y) = \int_{B_1\backslash S_1} \Delta_x u(x,y)$ is harmonic in B_2, which completes the proof. □

Now we consider the potential

$$U(x,y) = \int_{B_1\backslash S_1} K(x-w)\,\Delta_w u(w,y),$$

where K is the Newtonian kernel,

$$K(w) = \begin{cases} \dfrac{1}{2\pi}\ln|w|, & \text{if } n=2 \\ -\dfrac{1}{(n-2)\sigma_n\,|w|^{n-2}}, & \text{if } n>2. \end{cases}$$

The measure $\Delta_x u(x,y)$ has the harmonic property in $(B_1\backslash S_1)\times B_2$. Moreover, for some constant C the total measure $\int_{B_1\backslash S_1} \Delta_x u(x,y) \leqslant C$, $y \in B_2$. It follows that the integral $\int_{B_1\backslash S_1} \varphi(w)\,\Delta_w u(w,y)$ is harmonic in y for every continuous function $\varphi \in C\left(\bar{B}_1\right)$. Let $K_j(w) \in C^\infty(\mathbb{R}^n)$ approximate K from above, $K_j(w) \downarrow K(w)$. Then for every fixed $x \in B_1$ we have

$$\int_{B_1\backslash S_1} K_j(x-w)\,\Delta_w u(w,y) \downarrow \int_{B_1\backslash S_1} K(x-w)\,\Delta_w u(w,y)$$

for $j\to\infty$, hence $U(x,y)$ is harmonic in y for fixed $x \in B_1$. Moreover, U is subharmonic in x and bounded above in $B_1\times B_2$. It follows by the theorem of Lelong and Avanissian (1), that U is subharmonic in $B_1\times B_2$.

Now we take the difference $u^*(x,y) = u(x,y) - U(x,y)$. The function $u^*(x,y)$ is separately subharmonic and is harmonic in y. Moreover, $u^*(x,y)$ is harmonic in x outside S_1. Thus we have

THEOREM 2.5. — *Every separately subharmonic function, which is harmonic in y, can locally be represented as a sum of two functions:*

$$u(x,y) = u^*(x,y) + U(x,y),$$

where U is a subharmonic function and u^* is separately subharmonic and harmonic in y, such that the associated measure $\Delta_x u^*(x,y)$ is supported on S_1 for every fixed $y \in B_2$.

Problem 2.6. — *We finish our discussion by recalling an open problem on the definition of plurisubharmonic functions: in this definition one demands two conditions.*

a. *The function $u(z)$ is upper semicontinuous;*

b. *For each complex line l the restriction $u|_l$ is subharmonic.*

The above results on separately subharmonic functions seem to indicate, that the condition a. may be implied by b. But this is still open.

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