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Quadratic forms and singularities of genus one or two

Georges Dloussky[1]

Abstract. — We study singularities obtained by the contraction of the maximal divisor in compact (non-kählerian) surfaces which contain global spherical shells. These singularities are of genus 1 or 2, may be $\mathbb{Q}$-Gorenstein, numerically Gorenstein or Gorenstein. A family of polynomials depending on the configuration of the curves computes the discriminants of the quadratic forms of these singularities. We introduce a multiplicative branch topological invariant which determines the twisting coefficient of a non-vanishing holomorphic 1-form on the complement of the singular point.

Résumé. — On étudie les singularités obtenues en contractant le diviseur maximal des surfaces (non kählerienne) qui contiennent des coquilles sphériques globales. Ces singularités sont de genre 1 ou 2, peuvent être $\mathbb{Q}$-Gorenstein, numériquement Gorenstein ou de Gorenstein. On définit une famille de polynômes qui dépendent de la configuration des courbes rationnelles pour calculer les discriminants des formes quadratiques associées à ces singularités. Un invariant topologique multiplicatif, défini à partir des arbres du graphe détermine le coefficient de torsion des 1-formes holomorphes tordues qui ne s'annulent pas sur le complémentaire du point singulier.

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0. Introduction

We are interested in a large class of singularities which generalize cusps, obtained by the contraction of all the rational curves in compact surfaces S which contain global spherical shells. Particular cases are Inoue-Hirzebruch surfaces with two "dual" cycles of rational curves. The duality can be explained by the construction of these surfaces by sequences of blowing-ups [5]. Several authors have studied cusps [13], [15], [24], [25], [19]. In general, the maximal divisor is composed of a cycle with branches. These (non-kählerian) surfaces contain exactly $n = b_2(S)$ rational curves. The intersection matrices $M(S)$ have been completely classified [23], [3]; they are negative definite in all cases except when the maximal divisor is a cycle D of n rational curves such that $D^2 = 0$. In this article, we study the link between global topological or analytical properties of the surface S and properties of the normal singularities obtained by contracting their maximal compact divisor. These singularities are elliptic or of genus two in which case they are Gorenstein. Using the existence of non-vanishing global sections on S of $-mK_S \otimes L$ for a suitable integer $m \geqslant 1$ and a flat line bundle $L \in H^1(S, \mathbb{C}^\star)$, we show that these singularities are $\mathbb{Q}$-Gorenstein (resp. numerically Gorenstein) if and only if the global property $H^0(S, -mK_S) \neq 0$ (resp. $H^0(S, -K_S \otimes L) \neq 0$) holds. The main part of this article is devoted to the study of the discriminant of the quadratic form associated to the singularity. In [3] the quadratic form has been decomposed into a sum of squares. The intersection matrix is completely determined by the sequence σ of (opposite) self-intersections of the rational curves when taken in the canonical order, i.e. the order in which the curves are obtained in a repeated sequence of blowing-ups. Let $(Y, y) = (Y_\sigma, y)$ be the associated singularity obtained by the contraction of the rational curves. We introduce a family of polynomials P_σ which have integer values on integers, depending on the configuration of the dual graph of the singularity, such that the discriminant is the square of this polynomial. When we fix the sequence σ we introduce an integer Δ_σ which is a multiplicative topological invariant i.e. satisfies $\Delta_{\sigma\sigma'} = \Delta_\sigma \Delta_{\sigma'}$. We show that Δ_σ is equal to the product of the determinants of the intersection matrices of the branches of the maximal divisor. We apply this result to determine the twisting integer of holomorphic 1-forms in a neighbourhood of the singularity. We develop here rather the algebraic point of view, however these singularities have deep relations with properties of compact complex surfaces S containing global spherical shells, the classification of singular contracting germs of mappings and dynamical systems: for instance, the integer Δ_σ is equal to the integer $k = k(S)$ wich appears in the normal form of contracting germs $F(z_1, z_2) = (\lambda z_1 z_2^s + P(z_2), z_2^k)$ which define S [4], [7], [8], [11].

I thank Karl Oeljeklaus for fruitful discussions on that subject.

1. Preliminaries

1.1. Basic results on singularities

Let $D_0, \ldots, D_{n-1}$ be compact curves on a (not necessarilly compact) complex surface X, and $D = D_0 + \cdots + D_{n-1}$ the associated reduced divisor. We assume that D is exceptional i.e. the intersection matrix M of D is negative definite. We denote by $\mathcal{O}_X$ the structural sheaf of X, $K_X = \det T^\star X$ the canonical bundle and by Ω^2_X its sheaf of sections. It is well known by Grauert's theorem that there exists a proper mapping $\Pi : X \to Y$ such that each connected component of $|D| = \cup_i D_i$ is contracted onto a point y which is a normal singularity of Y. For $|D|$ connected, denote by

$$r : H^0(X, \Omega^2_X) \to H^0(Y\backslash\{y\}, \Omega^2_{Y\backslash\{y\}})$$

the canonical morphism induced by Π. We define the **geometric genus** of the singularity (Y, y) by

$$p_g = p_g(Y, y) = h^0(Y, R^1\Pi_*\mathcal{O}_X),$$

where $R^1\Pi_*\mathcal{O}_X$ is the first direct image sheaf of $\mathcal{O}_{\mathcal{X}}$ (see [2] Chap. IV, sections 12 and 13).
When Y is Stein, we have $p_g = \dim \mathrm{H}^0(\mathrm{Y}\backslash\{\mathrm{y}\}, \Omega^2_{\mathrm{Y}\backslash\{\mathrm{y}\}})/\mathrm{r}\mathrm{H}^0(\mathrm{X}, \Omega^2_{\mathrm{X}})$.

A normal singularity (Y, y) is called **rational** (resp. **elliptic**) if $p_g(Y, y) = 0$ (resp. $p_g(Y, y) = 1$). Therefore a singularity is rational if for every holomorphic 2-form ω on $Y\backslash\{y\}$, the 2-form $\Pi^\star\omega$ extends to a 2-form on X.

PROPOSITION 1.1. — *Let $\Pi : X \to Y$ be the proper morphism obtained by the contraction of an exceptional divisor:*

1) The genus $p_g = h^0(Y, R^1\Pi_*\mathcal{O}_X)$ is independent of the choice of the desingularization Π of Y.

2) The following sequence

$$0 \to H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X) \to H^0(Y, R^1\Pi_\star\mathcal{O}_X)$$

$$\to H^2(Y, \mathcal{O}_Y) \to H^2(X, \mathcal{O}_X)$$

is exact.

3) If X is compact and $H^2(X, \mathcal{O}_X) = 0$ then $p_g = \chi(O_Y) - \chi(O_X)$
If X is strictly pseudoconvex (spc) and Y is Stein then $p_g = h^1(X, O_X)$

Proof. — 1) is well-known. 2) is given by the Leray spectral sequence [2] Chap. IV (11.8) and (13.8). Assertion 3) is a consequence of 2) in compact case, and in non compact case is a consequence of 2) with theorem B of Cartan and a theorem of Siu. □

There is the following (necessary but not sufficient) criterion of rationality [27], p. 152:

PROPOSITION 1.2. — *Let $\Pi : X \to Y$ be the minimal resolution of the singularity (Y, y) and denote by D_i the irreducible components of the exceptional divisor D. If (Y, y) is rational, then:*

i) the curves D_i are smooth and rational

ii) for $i \neq j$, $D_i \cap D_j = \emptyset$ or D_i meets D_j tranversally. If D_i, D_j, D_k are distinct irreducible components, $D_i \cap D_j \cap D_k$ is empty

iii) the dual graph of D contains no cycle.

DEFINITION 1.3. — *A normal singularity (Y, y) is called* **Gorenstein** *if the dualizing sheaf ω_Y is trivial, i.e. there exists a small neighbourhood U of y and a non-vanishing holomorphic 2-form on $U \setminus \{y\}$.*

Since there is only a finite number of linearly independent 2-forms in the complement of the exceptional divisor D modulo $H^0(X, \Omega_X^2)$, a 2-form extends meromorphically across D. Therefore we have (see [30])

LEMMA 1.4. — *Let Y be a Gorenstein normal surface and $\Pi : X \to Y$ be the minimal desingularization. Then there is a unique effective divisor D_{-K} on X supported on $D = \Pi^{-1}(Sing(Y))$ such that*

$$\omega_X \simeq \Pi^\star \omega_Y \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(-D_{-K})$$

Moreover, for each singular point $y \in Y$, the part of D_{-K} supported on $\Pi^{-1}(y)$ is an anticanonical divisor of X in the neighbourhood of $\Pi^{-1}(y)$.

1.2. Lattices

Here are recalled some well known facts about lattices (see [31]). We call **lattice**, denoted by $\big(L, <\ .\ ,\ .\ >\big)$, a free **Z**-module L, endowed with an integral non degenerate symmetric bilinear form

$$\begin{array}{rccc} <\ .\,,\,.\ >: & L \times L & \longrightarrow & \mathbf{Z} \\ & (x, y) & \longmapsto & <\ x\ ,\ y\ >. \end{array}$$

If $B = \{e_1, \ldots, e_n\}$ is a basis of L, the determinant of the matrix

$$\big(<\ e_i\ ,\ e_j\ >\big)_{1 \leqslant i,j \leqslant n},$$

is independent of the choice of the basis; this integer, denoted by $d(L)$ is called the **discriminant** of the lattice. A lattice is called **unimodular** if $d(L) = \pm 1$. Let $L^\vee := Hom_{\mathbb{Z}}(L, \mathbb{Z})$ be the dual of L. The mapping

$$\begin{array}{rccc} \phi : & L & \longrightarrow & L^\vee \\ & x & \longmapsto & <\ .\ ,\ x\ > \end{array}$$

identifies L with a sublattice of $L^\vee$ of the same rank, since $d(L) \neq 0$. Moreover, if $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$, it is possible to identify $L^\vee$ with the sub-$\mathbb{Z}$-module

$$\{x \in L_{\mathbb{Q}} \mid \forall y \in L, < x , y > \in \mathbb{Z}\}$$

of $L_{\mathbb{Q}}$. So, we may write $L \subset L^\vee \subset L_{\mathbb{Q}}$, where L and $L^\vee$ have the same rank.

Lemma 1.5. — *1) The index of L in $L^\vee$ is $|d(L)|$.*

2) If M is a submodule of L of the same rank, then the index of M in L satifies

$$[L : M]^2 = d(M)\, d(L)^{-1}.$$

In particular $d(M)$ and $d(L)$ have the same sign.

1.3. Surfaces with global spherical shells

We say that a minimal compact complex surface S belongs to the VII_0 class of Kodaira if its first Betti number and Kodaira dimension satisfy (see [1])

$$b_1(S) = 1, \quad \kappa(S) = -\infty.$$

A large family of surfaces in class VII_0 are surfaces containing global spherical shells which have been first introduced by Ma. Kato [16] and we refer to [3] for details.

Definition 1.6. — *Let S be a compact complex surface. We say that S* **contains a global spherical shell (GSS)***, if there is a biholomorphic map $\varphi : U \to S$ from a neighbourhood $U \subset \mathbb{C}^2 \setminus \{0\}$ of the sphere S^3 into S such that $S \setminus \varphi(S^3)$ is connected.*

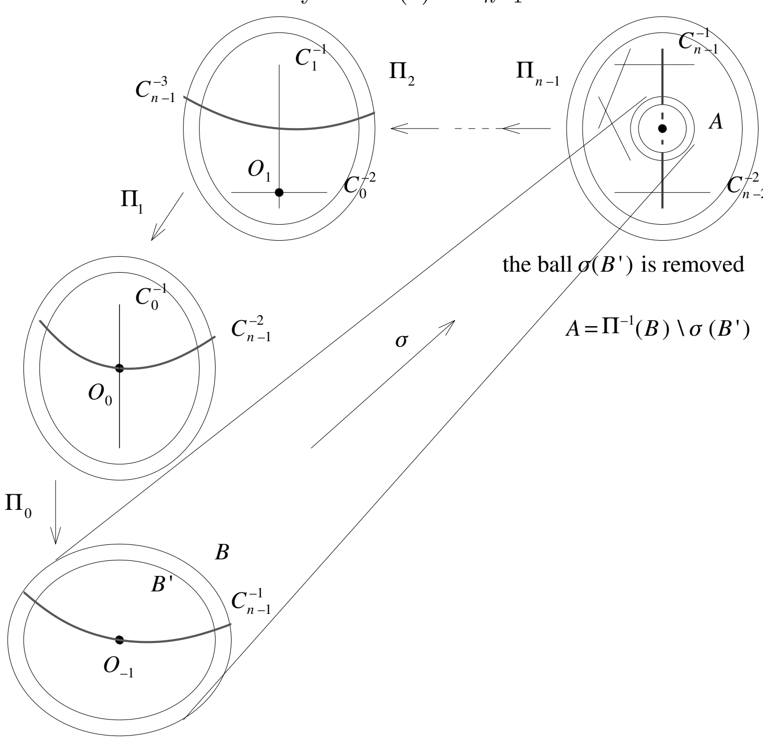
Such surfaces may contain as compact curves only rational or elliptic curves. Hopf surfaces are the simplest examples of surfaces with GSS (see [3]), however they contain no rational curves and their elliptic curves have self-intersection equal to 0, hence no singularity can be obtained by contraction.

Although classification of surfaces of VII_0 class with second Betti number $b_2(S) = 0$ is now well known (see [32] and references there), the classification of surfaces of class VII_0 with $b_2(S) > 0$, called surfaces of class VII_0^+, is still incomplete. The only known surfaces in this class are surfaces containing GSS and they may be characterized by the existence of exactly $b_2(S)$ rational curves [9] or the existence of a non-trivial section in

$H^0(S, -mK_S \otimes L)$ for a suitable integer $m \geqslant 0$ and a suitable topologically trivial line bundle L [6].

Let S be a minimal surface containing a GSS with $n = b_2(S)$. By construction S contains n rational curves. To each choice of such curves it is possible to associate a contracting germ of mapping $F = \Pi\sigma = \Pi_0 \cdots \Pi_{n-1}\sigma : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ where $\Pi = \Pi_0 \cdots \Pi_{n-1} : B^\Pi \to B$ is a sequence of n blowing-ups [3], Prop. 3.9. If we want to obtain a minimal surface, the sequence of blowing-ups has to be done in the following way:

- Π_0 blows up the origin $0 = O_{-1}$ of the two dimensional unit ball B,
- Π_1 blows up a point $O_0 \in C_0 = \Pi_0^{-1}(0)$,...
- Π_{i+1} blows up a point $O_i \in C_i = \Pi_i^{-1}(O_{i-1})$, for $i = 0, \ldots, n-2$, and
- $\sigma : \bar{B} \to B^\Pi$ sends isomorphically a neighbourhood of $\bar{B}$ onto a small ball in B^Π in such a way that $\sigma(0) \in C_{n-1}$.

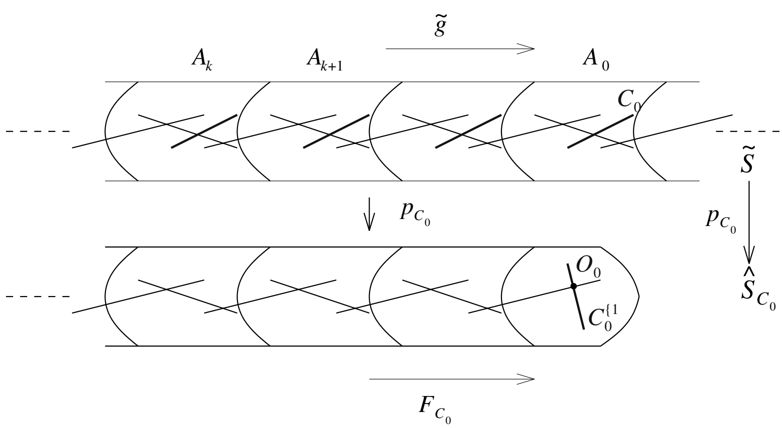


It is easy to see that the homological groups satisfy

$$H_1(S,\mathbb{Z}) \simeq \mathbb{Z}, \quad H_2(S,\mathbb{Z}) \simeq \mathbb{Z}^n$$

In particular, $b_2(S) = n$.

Consider for a little smaller ball $B' \subset B$, the "annulus" $A := \Pi^{-1}(B) \setminus \sigma(B')$. Let $(\tilde S, \tilde p, S)$ be the universal covering space of S, where $\tilde p : \tilde S \to S$ is the canonical mapping. Then $\tilde S$ is obtained as a union $\tilde S = \cup_{k\in\mathbb{Z}} A_k$ of copies A_k of the annulus A, $k \in \mathbb{Z}$. The pseudoconcave boundary of A_k is glued with the pseudoconvex boundary of A_{k+1}. The automorphism of the covering $\tilde g : \tilde S \to \tilde S$ sends A_k onto A_{k+1}. At each step we may fill in the hole of any A_k with a ball. If we choose a curve, say $C_0 \subset A_0$ we may obtain a surface $\hat S_{C_0}$ with only one end in which C_0 induces an exceptional curve of the first kind. In fact we fill in an annulus A_k, $k > 0$. We obtain a unique exceptional curve of the first kind, then we blow down successively each exceptional curve which appears till C_0 has itself self-intersection -1. The canonical mapping $p_{C_0} : \tilde S \to \hat S_{C_0}$ blows down all the curves C_i, $i > 0$ onto the point $O_0 \in C_0$.



The universal covering space $\tilde S$ contains only rational curves $(C_i)_{i\in\mathbb{Z}}$ with a canonical order relation, "the order of creation" ([3], p 29). Notice that C_i denote both the curves created by blowing-ups, their strict transforms on the composition of blowing-ups and on the universal covering space $\tilde S$.

Following [3], we can associate to S the following invariants:

- The family of opposite self-intersections of the compact curves in the universal covering space of S, denoted by

$$a(S) := (a_i)_{i\in\mathbb{Z}} = (-C_i^2)_{i\in\mathbb{Z}}.$$

This family is periodic of period n.

- $$\sigma_n(S) := \sum_{i=j}^{j+n-1} a_i = -\sum_{i=0}^{n-1} D_i^2 + 2\ \sharp\{rational\, curves\, with\, nodes\}$$

  where j is any index, and the $D_i = \tilde{p}(C_{i+ln})$, $l \in \mathbb{Z}$, are the rational curves of S. It can be seen that $2n \leqslant \sigma_n(S) \leqslant 3n$ ([3], p 43).

- The intersection matrix of the n rational curves of S,

  $$M(S) := (D_i.D_j).$$

  Important Remark: The essential fact useful to understand the dual graph of D, weighted by the self-intersections of the components D_i, or equivalently the intersection matrix is that
  - if $a_i = -D_i^2 = 2$ then D_i meets D_{i+1},
  - if $a_i = -D_i^2 = 3$ then D_i meets $D_{i+2}, \dots$,
  - if $a_i = -D_i^2 = k+2$ then D_i meets D_{i+k+1},

  the indices being in $\mathbb{Z}/n\mathbb{Z}$, in particular D_i may meet itself: we obtain a rational curve with a double point.

- n classes of contracting holomorphic germs of mappings $F = \Pi\sigma : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, each class corresponding to the initial choice of irreducible component of the maximal compact curve. In fact for every curve C there is a commutative diagram

  $$\begin{array}{ccc} \tilde{S} & \stackrel{\tilde{g}}{\longrightarrow} & \tilde{S} \\ p_C \downarrow & & \downarrow p_C \\ \hat{S}_C & \stackrel{F_C}{\longrightarrow} & \hat{S}_C \end{array}$$

  If we choose the numbering such that $C = C_0$, the germs $F_{C_0}, \dots, F_{C_{n-1}}$ are, in general, not equivalent contracting germs, however F_{C_0} and F_{C_n} are conjugated (see [3], p 30-32 for details).

Proposition 1.7. — *Let S be a surface containing a GSS with $b_2(S) = n$, $D_0, \dots, D_{n-1}$ the n rational curves and $M(S)$ the intersection matrix.*

1) If $\sigma_n(S) = 2n$, then $\det M(S) = 0$.

2) If $\sigma_n(S) > 2n$, then $\sum_{0\leqslant i\leqslant n-1} \mathbf{Z}D_i$ is a sublattice of $H_2(S,\mathbf{Z})$ of maximal rank and its index satisfies

$$\left[H_2(S,\mathbf{Z}) \ : \ \sum_{0\leqslant i\leqslant n-1} \mathbf{Z}D_i\right]^2 = \det\, M(S).$$

In particular, $\det\, M(S)$ *is the square of an integer* $\geqslant 1$.

Proof. — If $\sigma_n(S) = 2n$, S is an Inoue surface; if $\sigma_n(S) > 2n$, $\det M(S) \neq 0$ so the sublattice is of maximal rank and the result is a mere consequence of lemma 1.1.5. □

In order to give a precise description of the intersection matrix we need the following definitions:

DEFINITION 1.8. — *Let $1 \leqslant p \leqslant n$. A p-uple $\sigma = (a_i, \ldots, a_{i+p-1})$ of $a(S)$ is called*

- *a* **singular p-sequence** *of $a(S)$ if*

$$\sigma = (\underbrace{p+2, 2, \ldots, 2}_{p}).$$

  It will be denoted by s_p.

- *a* **regular p-sequence** *of $a(S)$ if*

$$\sigma = (\underbrace{2, 2, \ldots, 2}_{p})$$

  and σ has no common element with a singular sequence. Such a p-uple will be denoted by r_p.

For example $s_1 = (3), s_2 = (4,2), s_3 = (5,2,2), \ldots$ are singular sequences, $r_3 = (2,2,2)$ is a regular sequence if it has no common element with a singular sequence. It is easy to see that if we want to have, for example, a curve C_i with self-intersection -4, necessarily, the curve which follows in the sequence of repeated blowing-ups must have self-intersection -2.

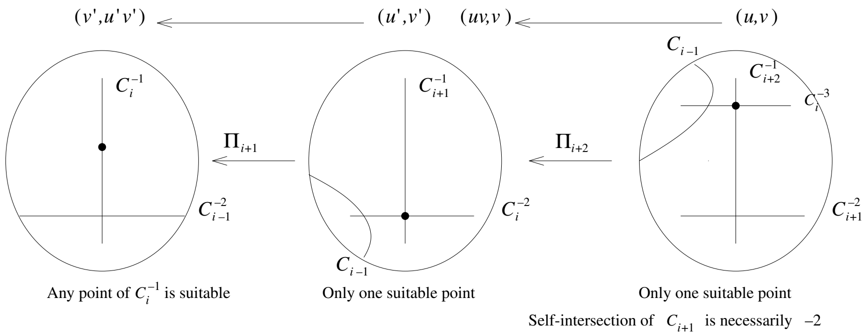
So it is easy to see ([3], p39), that $a(S)$ admits a unique partition by N singular sequences and by $\rho \leqslant N$ regular sequences of maximal length. More precisely, since $a(S)$ is periodic it is possible to find a n-uple σ such that

$$\sigma = \sigma_{p_0} \cdots \sigma_{p_{N+\rho-1}},$$

where σ_{p_i} is a regular or a singular p_i-sequence with

$$\sum_{i=0}^{N+\rho-1} p_i = n$$

and if σ_{p_i} is regular it is between two singular sequences (mod. $N+\rho$).



Notations. — We shall write

$$a(S) = (\overline{\sigma}) = (\overline{\sigma_{p_0}\cdots\sigma_{p_{N+\rho-1}}}).$$

The sequence σ is overlined to indicate that the sequence σ is infinitely repeated to obtain the sequence $a(S) = (a_i)_{i\in\mathbb{Z}}$. The sequence $a(S)$ may be defined by another period. For example

$$a(S) = (\overline{\sigma_{p_1}\cdots\sigma_{p_{N+\rho-1}}\sigma_{p_0}}).$$

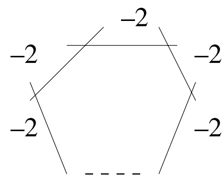
If $\sigma_n(S) = 2n$, $a(S) = (\overline{r}_n)$; if $\sigma_n(S) = 3n$, $a(S)$ is only composed of singular sequences and S is called a Inoue-Hirzebruch surface. Moreover if $a(S)$ is composed by the repetition of an even (resp. odd) number of sequences σ_{p_i}, we shall say that S is an even (resp. odd) Inoue-Hirzebruch surface. An even (resp. odd) Inoue-Hirzebruch surface has exactly 2 cycles (resp. 1 cycle) of rational curves. Another used terminology is respectively hyperbolic Inoue surface and half Inoue surface.

We recall that for any VII_0-class surface without non-constant meromorphic functions, the numerical characters of S are [17], I p755, II p683,

$$h^{0,1} = 1,\ h^{1,0} = h^{2,0} = h^{0,2} = 0,\ -c_1^2 = c_2 = b_2(S),\ b_2^+ = 0,\ b_2^- = b_2(S)$$

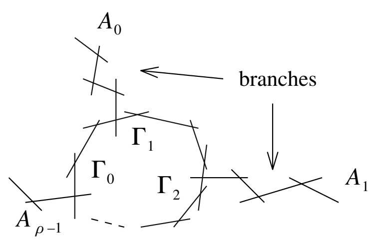
We shall need in the sequel the explicit description of the weighted dual graph which is composed of a cycle with branches in intermediate case. Each branch A_s determines and is determined by a piece Γ_s of the cycle Γ.

Theorem 1.9 ([3] thm 2.39). — *Let S be a minimal surface containing a GSS, $n = b_2(S)$, $D_0, \ldots, D_{n-1}$ its n rational curves and $D = D_0 + \cdots + D_{n-1}$.*
1) If $\sigma_n(S) = 2n$ (Enoki case), then D is a cycle and $D_i^2 = -2$ for $i = 0, \ldots, n-1$.



2) If $2n < \sigma_n(S) < 3n$ (intermediate case), then there are $\rho = \rho(S) \geqslant 1$ branches and

$$D = \sum_{s=0}^{\rho(S)-1} (A_s + \Gamma_s)$$



where

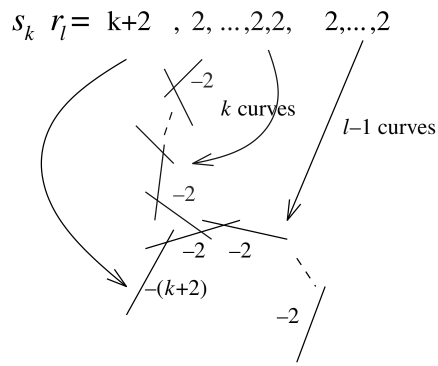
- *i) A_s is a branch for $s = 0, \ldots, \rho(S) - 1$,*
- *ii) $\Gamma = \sum_{s=0}^{\rho(S)-1} \Gamma_s$ is a cycle,*
- *iii) A_s and Γ_s are defined in the following way: For each sequence of integers*

$$(a_{t+1}, \ldots, a_{t+l+k_0+\cdots+k_{p-1}+2}) = (r_l s_{k_0} \cdots s_{k_{p-1}} 2 a_{t+l+k_0+\cdots+k_{p-1}+2})$$

contained in $a(S) = (\overline{\sigma_0 \cdots \sigma_{N+\rho-1}})$, where

- *$l \geqslant 1$ and r_l is a regular l-sequence,*
- *$p \geqslant 1$, $i = 0, \ldots, p-1$, $k_i \geqslant 1$ and s_{k_i}, is a singular k_i-sequence,*

we have the following decomposition into branches A_s and corresponding pieces of cycle Γ_s (where $p = p_s$ to simplify notations):



$$\begin{cases} Selfint(A_s) = (\underbrace{2,\dots,2}_{k_0-1},\ k_1+2,\ \underbrace{2,\dots,2}_{k_2-1},\dots,\ k_{p-2}+2,\ \underbrace{2,\dots,2}_{k_{p-1}-1},\ 2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad If\ p\equiv 1 (\mathrm{mod}\ 2) \\ Selfint(\Gamma_s) = (\underbrace{2,\dots,2}_{l-1},\ k_0+2,\ \underbrace{2,\dots,2}_{k_1-1},\dots,\ k_{p-3}+2,\ \underbrace{2,\dots,2}_{k_{p-2}-1},\ k_{p-1}+2) \end{cases}$$

$$\begin{cases} Selfint(A_s) = (\underbrace{2,\dots,2}_{k_0-1},\ k_1+2,\ \underbrace{2,\dots,2}_{k_2-1},\dots,\ k_{p-3}+2,\ \underbrace{2,\dots,2}_{k_{p-2}-1},\ k_{p-1}+2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad If\ p\equiv 0 (\mathrm{mod}\ 2) \\ Selfint(\Gamma_s) = (\underbrace{2,\dots,2}_{l-1},\ k_0+2,\ \underbrace{2,\dots,2}_{k_1-1},\dots,\ k_{p-2}+2,\ \underbrace{2,\dots,2}_{k_{p-1}-1},\ 2) \end{cases}$$

iv) The top of the branch A_s is its first vertex (or curve); the root of A_s is the first vertex (or curve) of Γ_t where $t = s+1$ (mod $\rho(S)$).

3) If $\sigma_n(S) = 3n$ (Inoue-Hirzebruch case), D has no branch and

i) If $a(S) = (\overline{s_{k_0}\cdots s_{k_{2p-1}}})$ then

$$D = \Gamma + \Gamma'$$

where Γ *and* Γ' *are two disjoint cycles*

$$\left\{\begin{array}{lcl} Selfint(\Gamma) & = & (k_0+2,\underbrace{2,\ldots,2}_{k_1-1},\ k_2+2,\underbrace{2,\ldots,2}_{k_3-1},\ldots,\ k_{2p-2}+2,\underbrace{2,\ldots,2}_{k_{2p-1}-1}) \\ Selfint(\Gamma') & = & (\underbrace{2,\ldots,2}_{k_0-1},\ k_1+2,\underbrace{2,\ldots,2}_{k_2-1},k_3+2,\ldots,\underbrace{2,\ldots,2}_{k_{2p-2}-1},k_{2p-1}+2) \end{array}\right.$$

ii) If $a(S)=(\overline{s_{k_0}\cdots s_{k_{2p}}})$ *then* D *contains only one cycle and*

$$\begin{array}{lcl} Selfint(D) & = & (k_0+2,\underbrace{2,\ldots,2}_{k_1-1},\ k_2+2,\underbrace{2,\ldots,2}_{k_3-1},\ldots,\ k_{2p}+2, \\ & & \underbrace{2,\ldots,2}_{k_0-1},k_1+2,\underbrace{2,\ldots,2}_{k_2-1},\ldots,k_{2p-1}+2,\underbrace{2,\ldots,2}_{k_{2p}-1}) \end{array}$$

1.4. Intersection matrix of the exceptional divisor

Let $\sigma=\sigma_0\cdots\sigma_{N+\rho-1}$ where $\sigma_i=r_{p_i}=(2,2,\ldots,2)$ is a regular sequence of length p_i or $\sigma_i=s_{p_i}=(p_i+2,2,\ldots,2)$ is a singular sequence of length p_i, $i=0,\ldots,N+\rho-1$. We suppose that

- there are N singular sequences and $\rho\leqslant N$ regular sequences if $N\geqslant 1$
- if σ_i is regular and $N\geqslant 1$, then σ_{i-1} and σ_{i+1} are singular, indices being in $\mathbb{Z}/(N+\rho)\mathbb{Z}$.

Let $n=\sum_{i=0}^{N+\rho-1}p_i$ be the number of integers in the sequence σ.

Examples 1.10. — For $0\leqslant N\leqslant 3$ *we have the following possible sequences:*

- *If* $N=0$, $\sigma=r_n$,
- *If* $N=1$, $\sigma=s_n$ *or* $\sigma=s_pr_m$, $p+m=n$,
- *If* $N=2$, $\sigma=s_{p_0}s_{p_1}$, $\sigma=s_{p_0}s_{p_1}r_{m_0}$, $\sigma=s_{p_0}r_{m_0}s_{p_1}$, $\sigma=s_{p_0}r_{m_0}s_{p_1}r_{m_1}$,
- *If* $N=3$, $\sigma=s_{p_0}s_{p_1}s_{p_2}$
  $\sigma=s_{p_0}r_{m_0}s_{p_1}s_{p_2}$, $\sigma=s_{p_0}s_{p_1}r_{m_0}s_{p_2}$, $\sigma=s_{p_0}s_{p_1}s_{p_2}r_{m_0}$,
  $\sigma=s_{p_0}r_{m_0}s_{p_1}r_{m_1}s_{p_2}$, $\sigma=s_{p_0}s_{p_1}r_{m_0}s_{p_2}r_{m_1}$, $\sigma=s_{p_0}r_{m_0}s_{p_1}s_{p_2}r_{m_1}$,
  $\sigma=s_{p_0}r_{m_0}s_{p_1}r_{m_1}s_{p_2}r_{m_2}$.

To a sequence σ we associate a symmetric matrix of type (n,n), $M(\sigma) = (m_{ij})$ "written on a torus", i.e. with indices in $\mathbb{Z}/n\mathbb{Z}$ to express the periodicity of the construction, and defined in the following way: if $\sigma = \sigma_0 \cdots \sigma_{N+\rho-1} = (a_0, \ldots, a_{n-1})$

i) $m_{ii} = \begin{cases} a_i & if \quad a_i \neq n+1 \\ n-1 & if \quad a_i = n+1 \end{cases}$

ii) For $0 \leqslant i < j \leqslant n-1$,

$$m_{ij} = m_{ji} = \begin{cases} -2 & if\ j = i + m_{ii} - 1 \quad \text{and} \quad i = j + m_{jj} - 1 \mod n \\ -1 & if\ j = i + m_{ii} - 1 \quad \text{or else} \quad i = j + m_{jj} - 1 \mod n \\ 0 & \text{in all other cases} \end{cases}$$

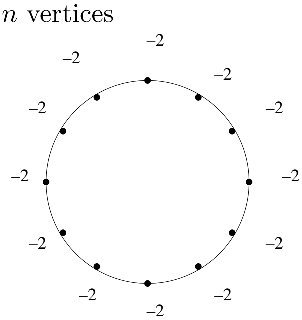
Theorem 1.11 [3, 21]. — *1) Let S be a minimal complex compact surface containing a GSS with $n = b_2(S) > 0$. Then S contains n rational curves $D_0, \ldots, D_{n-1}$ and there exists σ such that the intersection matrix $M(S)$ of the rational curves in S satisfies*

$$M(S) = -M(\sigma).$$

Moreover the curve D_i is non-singular if and only if $a_i \neq n+1$.
Conversely, for any σ there exists a surface S containing a GSS such that $M(S) = -M(\sigma)$.

2) For any $\sigma \neq r_n$, $M(\sigma)$ is positive definite.

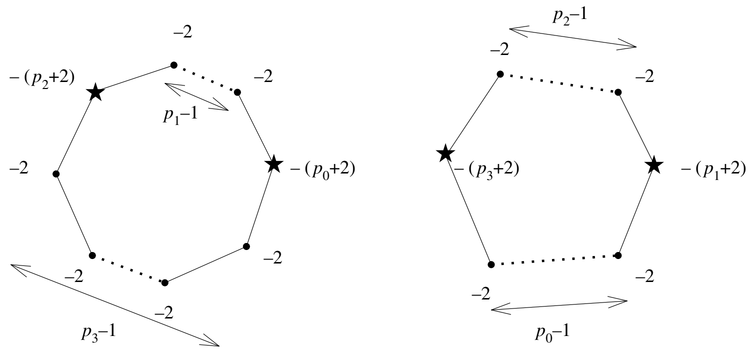
Examples 1.12. — 1) For $\sigma = r_n$, $M(\sigma)$ is not positive definite. The dual graph of the curves has n vertices



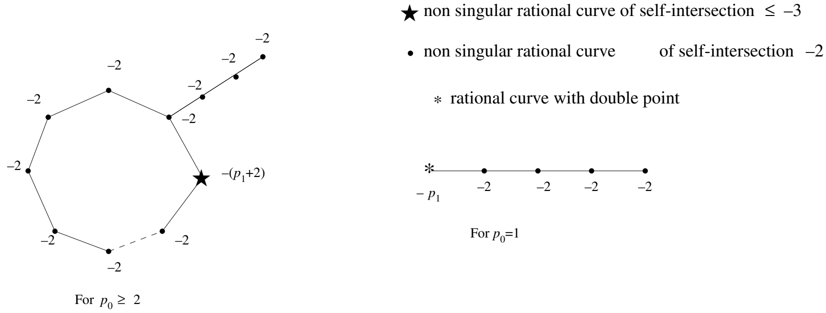
This configuration of curves appears on Enoki surfaces [10], [22], [3].

2) If $\sigma = s_{p_0} \cdots s_{p_{N-1}}$ we obtain respectively one or two cycles if N is odd (resp. even). The singularities are cusps and surfaces are odd (resp. even) Inoue-Hirzebruch surfaces [14, 22, 3]. When there are two cycles, one

of the two cycles determines the other. For example, if $\sigma = s_{p_0}s_{p_1}s_{p_2}s_{p_3}$, we obtain a cycle with p_1+p_3 curves and another with p_0+p_2 curves.



3) The intermediate case [22, 3, 7]. There are branches and the number of branches is equal to the number of regular sequences in σ. For example, if $\sigma = r_{p_0}s_{p_1}$ the dual graph is



2. Normal singularities associated to surfaces with GSS

2.1. Genus of the singularities

If S is a Inoue-Hirzebruch surface we obtain by contraction of a cycle, a singularity called a cusp. They appear also in the compactification of Hilbert modular surfaces [13]. We are interested here in the general situation of any surface containing a GSS.

Proposition 2.1. — *Let S be a compact complex surface of class VII_0 without non-constant meromorphic functions. It is supposed that $n := b_2(S) > 0$, the maximal divisor D is not trivial and the intersection matrix $M(S)$ is negative definite. Denote by $\Pi : S \to \bar{S}$ the contraction of the curves onto isolated singular points. Then the following properties are equivalent:*

i) D contains a cycle of rational curves;

ii) $H^1(\bar{S}, \mathcal{O}_{\bar{S}}) = 0$.

Proof. — **i)** ⇒ **ii)** By Proposition 1.1.1, the sequence

$$(*) \quad 0 \to H^1(\bar{S}, \mathcal{O}_{\bar{S}}) \to H^1(S, \mathcal{O}_S) \to H^0(\bar{S}, R^1\Pi_\star \mathcal{O}_S) \to H^2(\bar{S}, \mathcal{O}_{\bar{S}}) \to 0.$$

is exact. Since $h^1(S, \mathcal{O}_{\mathcal{S}}) = \infty$, we have $h^1(\bar{S}, \mathcal{O}_{\bar{S}}) \leqslant 1$. If D contains a cycle then $h^0(\bar{S}, R^1\Pi_* \mathcal{O}_S) \geqslant 1$. We suppose that $h^1(\bar{S}, \mathcal{O}_{\bar{S}}) = 1$ and we shall derive a contradiction. With these assumptions, Serre-Grothendick duality gives $h^0(\bar{S}, \omega_{\bar{S}}) = h^2(\bar{S}, \mathcal{O}_{\bar{S}}) = 1$ and $h^0(\bar{S}, R^1\Pi_\star \mathcal{O}_S) = 1$ since $\bar{S}$ has no non-constant meromorphic functions. Denote by x_i, $i = 0, \dots, p$ the singular points of $\bar{S}$, $\Gamma_i = \Pi^{-1}(x_i)$ and $p_g(\bar{S}, x_i)$ the geometric genus of $(\bar{S}, x_i)$. Then $\sum p_g(\bar{S}, x_i) = h^0(\bar{S}, R^1\Pi_\star \mathcal{O}_S) = 1$, therefore there are rational singular points and one elliptic singular point. Moreover these singularities are Gorenstein because $h^0(\bar{S}, \omega_{\bar{S}}) = 1$ and a non-trivial section cannot vanish because there are no more curves. Hence there are rational double points with trivial canonical divisor and one minimally elliptic singularity [18] thm 3.10, $(\bar{S}, x_0)$ with canonical divisor Γ_0. This elliptic singularity is a cusp. Since there is a global meromorphic 2-form on S, $n = -K_{\bar{S}}^2 = -\Gamma_0^2$. By [22], S is an odd Inoue-Hirzebruch surface (i.e. with one cycle); but such a surface has no canonical divisor (see for example [5])... a contradiction.

ii) ⇒ **i)** By the exact sequence $(*)$, $h^0(\bar{S}, R^1\Pi_* \mathcal{O}_S) \leqslant 2$ without any assumption and $1 \leqslant h^0(\bar{S}, R^1\Pi_* \mathcal{O}_S)$ by ii). Therefore there is a singular point, say $(\bar{S}, x_0)$ such that $p_g(\bar{S}, x_0) \geqslant 1$. If Γ_0 would be simply connected, then taking a 3-cover space S' of S we would obtain 3 copies of Γ_0 hence $h^0(\bar{S}', R^1\Pi_* \mathcal{O}_{S'}) \geqslant 3$ which is impossible since S' remains in the VII_0-class, has no non-constant meromorphic functions and has to satisfy $h^0(\bar{S}', R^1\Pi_* \mathcal{O}_{S'}) \leqslant 2$. □

LEMMA 2.2. — *Let S be a surface with a GSS and such that $b_2(S) > 0$. Let D be the maximal divisor of S and $\Pi : S \to \bar{S}$ be the contraction of D. Then the sequence*

$$0 \to H^1(S, \mathcal{O}_S) \to H^0(\bar{S}, R^1\Pi_* \mathcal{O}_S) \to H^2(\bar{S}, \mathcal{O}_{\bar{S}}) \to 0$$

is exact and we have

$$(\dagger) \qquad 1 \leqslant h^0(\bar{S}, R^1\Pi_* \mathcal{O}_S) = h^0(\bar{S}, \omega_{\bar{S}}) + 1 \leqslant 2$$

Proof. — By Proposition 2.2.1 we have the desired exact sequence. Since S has no non-constant meromorphic functions, the dimension of $H^0(\bar{S}, \omega_{\bar{S}})$ is 0 or 1. □

The proof of the following theorem follows the arguments of [20] Corollaire.

Theorem 2.3. — *Let S be a surface with a GSS such that $2n < \sigma_n(S) \leqslant 3n$. Let C be a connected component of the maximal divisor D and let $\Pi : S \to \bar{S}$ be the contraction of C, $\{x\} = \Pi(C)$. Then:*

1) $p_g(\bar{S}, x) = 1$ or 2.

2) If $2n < \sigma_n(S) < 3n$ then $|D|$ is connected and the following conditions are equivalent:

i) $p_g(\bar{S}, x) = 2$

ii) the dualizing sheaf of $\bar{S}$ is trivial i.e. $\omega_{\bar{S}} \simeq \mathcal{O}_{\bar{S}}$

iii) the anticanonical bundle $-K$ is defined by an effective divisor Γ i.e. $\omega_S \simeq \mathcal{O}_S(-\Gamma)$ where $\Gamma > 0$.

iv) $(\bar{S}, p)$ is a Gorenstein singularity.

3) If S is an even Inoue-Hirzebruch surface, each cycle gives a minimally elliptic singularity and the dualizing sheaf of $\bar{S}$ is trivial. In particular singularities are Gorenstein.

4) If S is an odd Inoue-Hirzebruch surface the cycle gives a minimally elliptic singularity but the dualizing sheaf of $\bar{S}$ is not trivial. The singularity is still Gorenstein.

Proof. — 1) A connected component contains a cycle and we apply Lemma 2.2.2.

2) $i) \iff ii)$: Notice that a global section of $\omega_{\bar{S}}$ cannot vanish since there is no curve. Therefore by (†) $p_g(\bar{S}, p) = 2$ if and only if $\omega_{\bar{S}}$ is trivial.
$ii) \Rightarrow iii)$ By Lemma 1.1.4.
$iii) \Rightarrow ii)$ Let $\bar{U} = \bar{S} \setminus \{x\}$, $U = \Pi^{-1}(\bar{U})$ and $i : \bar{U} \hookrightarrow \bar{S}$ the inclusion. We have since $\bar{S}$ is normal

$$\omega_{\bar{S}} = i_*\omega_{\bar{U}} \simeq i_*\Pi_*\omega_U \simeq i_*\Pi_*\mathcal{O}_U \simeq i_*\mathcal{O}_{\bar{U}} \simeq \mathcal{O}_{\bar{S}}$$

Trivially $ii) \Rightarrow iv)$, we shall prove $iv) \Rightarrow i)$. In fact, suppose that $p_g(\bar{S}, x) = 1$, then by [18] theorem 3.10, the singularity would be minimally elliptic, but it is impossible since in the case $2n < \sigma_n(S) < 3n$ the maximal divisor contains a cycle with at least one branch [3] p113.

3) Suppose that S is an even Inoue-Hirzebruch surface then the sheaf $R^1\Pi_*\mathcal{O}_S$ is supported by two points. By (†) and Proposition 1.1.2,

$h^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) = 2$, and both singularities are minimally elliptic (see [18] p 1266).

4) It is well known ([14] or [5] Prop.2.14) that the canonical line bundle K of an odd Inoue-Hirzebruch surface is not given by a divisor. The surface S admits a double covering by an even Inoue-Hirzebruch surface. By 3) the singularity is minimally elliptic and Gorenstein. □

Remark 2.4. — Conditions i) and iv) are local conditions, though ii) and iii) are global ones.

2.2. $\mathbb{Q}$-Gorenstein and numerically Gorenstein singularities

DEFINITION 2.5. — *Let D be a connected exceptional divisor in the smooth surface X and $\Pi : X \to \bar{X}$ the contraction onto $x = \Pi(D) \in \bar{X}$. Then $(\bar{X}, x)$ is a* **numerically Gorenstein** *(resp.* $\mathbb{Q}$**-Gorenstein***) singularity if the effective numerically anticanonical $\mathbb{Q}$-divisor D_{-K} is a divisor (resp. there exists an integer m and a spc neighbourhood U of D such that the m-anticanonical bundle K_S^{-m} has a section on U which does not vanish outside D).*

If S contains a GSS, then the fundamental group satisfies $\pi_1(S) = \mathbb{Z}$. Any topologically trivial line bundle is in $H^1(S, \mathbb{C}^\star) \simeq \mathbb{C}^\star$ and given by a representation of $\pi_1(S)$ in $\mathbb{C}^\star$. Therefore we shall denote topologically trivial line bundles by L^α for $\alpha \in \mathbb{C}^\star$.

PROPOSITION 2.6. — *Let S be a compact complex surface containing a GSS of intermediate type, i.e $2n < \sigma_n(S) < 3n$, $\Pi : S \to \bar{S}$ the contraction of the maximal divisor and $x = \Pi(D)$ the singular point of $\bar{S}$. Then*

i) $(\bar{S}, x)$ is numerically Gorenstein if and only if there exists a unique $\kappa \in \mathbb{C}^\star$ such that
$$H^0(S, K_S^{-1} \otimes L^\kappa) \neq 0,$$

ii) $(\bar{S}, x)$ is $\mathbb{Q}$-Gorenstein if and only if there exists an integer $m \geqslant 1$ such that
$$H^0(S, K_S^{-m}) \neq 0.$$

Proof. — i) The sufficient condition is evident and the necessary condition derives from [7] thm 4.5.

ii) The sufficient condition is evident. Conversely, suppose that there exists an open neighbourhood U of D with $0 \neq \theta \in H^0(U, K_U^{-m})$, non vanishing outside the exceptional divisor. Since the curves are a basis of $H^2(S,\mathbb{Q})$, K_S^{-m} is numerically equivalent to an effective divisor. The exponential exact sequence for surfaces of class VII_0 ([17] I, p766 and I (14) p756), yields the exact sequence

$$1 \to H^1(S,\mathbb{C}^\star) \to H^1(S,\mathcal{O}_S^\star) \xrightarrow{c_1} H^2(S,\mathbb{Z}) \to 0$$

where $\mathbb{C}^\star \simeq H^1(S,\mathbb{C}^\star)$. Therefore there exists a unique $\kappa \in \mathbb{C}^\star$ such that

$$H^0(S, K_S^{-m} \otimes L^\kappa) \neq 0.$$

Let $0 \neq \omega \in H^0(S, K_S^{-m} \otimes L^\kappa)$. Since in the intermediate case the cycle Γ of rational curves fulfils $H_1(\Gamma,\mathbb{Z}) = H_1(S,\mathbb{Z})$, the restriction $H^1(S,\mathbb{C}^\star) \to H^1(U,\mathbb{C}^\star)$ is an isomorphism. Then $\theta/\omega \in H^0(U, L^{1/\kappa})$ may vanish or may have a pole only on the exceptional divisor. This cannot happen because the intersection matrix is negative definite, therefore $L_{|U}^{1/\kappa}$ is holomorphically trivial and $\kappa = 1$. $\square$

Examples 2.7. — In the example [7] 4.9, there is a family of surfaces with two rational curves, one rational curve with double point D_0 and a non-singular rational curve D_1, $D_0^2 = -1$, $D_1^2 = -2$ and $D_0 D_1 = 1$. The associated singularity is Gorenstein of genus 2 for $\alpha = \pm i$ and is non-Gorenstein numerically Gorenstein elliptic for other values of the parameter α. By [29] Satz 3, we have a family of non-Gorenstein singularities, however in a neighbourhood of $\alpha = \pm i$ there is no global family [29], Satz 5.

3. Discriminants of the singularities

3.1. A family $\mathfrak{P}$ of polynomials

For an integer $N \geqslant 1$, we denote $\mathbb{Z}/N\mathbb{Z} = \{\dot{0}, \dot{1}, \ldots, \dot{\overline{N-1}}\}$. Let

$$A = \{\dot{a}_1, \ldots, \dot{a}_p\} \subset \mathbb{Z}/N\mathbb{Z}$$

a subset with p elements, $0 \leqslant p \leqslant N$. We may suppose that we have

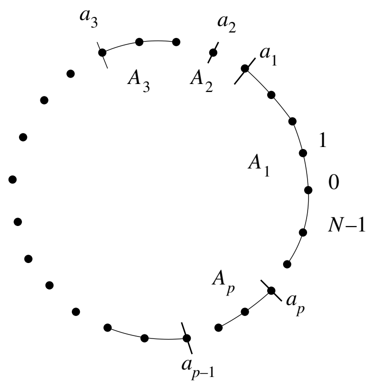
$$0 \leqslant a_1 < a_2 < \ldots < a_p \leqslant N-1$$

which allows to define a partition $\mathcal{A} = (A_i)_{1 \leqslant i \leqslant p}$ of $\mathbb{Z}/N\mathbb{Z}$, where

$$A_1 := \{\dot{k} \in \mathbb{Z}/N\mathbb{Z} \mid 0 \leqslant k \leqslant a_1 \ or\ a_p < k \leqslant N-1\}$$

$$A_i := \{\dot{k} \in \mathbb{Z}/N\mathbb{Z} \mid a_{i-1} < k \leqslant a_i\} \ for\ 2 \leqslant i \leqslant p.$$

When $A = \emptyset$, $\mathcal{A}$ is the trivial partition and $A_1 = \mathbb{Z}/N\mathbb{Z}$.



Definition 3.1. — *Let $N \geqslant 1$, $A \subset \mathbb{Z}/N\mathbb{Z}$ and $B \subsetneq \mathbb{Z}/N\mathbb{Z}$.*

1) We shall say that B is a **generating allowed subset** *relatively to A if B satisfies one of the following conditions:*

i) $B = \{\dot{a}\}$ with $\dot{a} \in A$.

ii) $B = \{\dot{k}, \dot{\overline{k+1}}\}$ and there exists $1 \leqslant i \leqslant p$ such that $B \subset A_i$.

2) We shall say that B is an **allowed subset** *relatively to A if B admits a (possibly empty) partition into generating allowed subsets.*
The set of all allowed subsets will be denoted by $\mathcal{P}_A$.

Definition 3.2. — *For every $N \geqslant 0$, let $\mathfrak{P}_N$ be the family of polynomials defined in the following way: $\mathfrak{P}_0 = \{0\}$.*
If $N \geqslant 1$, $\mathfrak{P}_N \subset \mathbb{Z}[X_0, \ldots, X_{N-1}]$ is the set of polynomials

$$P_A(X_0, \ldots, X_{N-1}) = \sum_{B \in \mathcal{P}_A} \prod_{i \notin B} X_i \quad for \quad A \subset \mathbb{Z}/N\mathbb{Z}$$

We shall denote

$$\mathfrak{P} = \bigcup_{N \geqslant 0} \mathfrak{P}_N$$

the union of all these polynomials.

Examples 3.3. — *For $N = 1$, there is only one polynomial $\mathfrak{P}_1 = \{X\}$. For $N = 2$,*

$$\mathfrak{P}_2 = \Big\{P_\emptyset(X_0, X_1) = X_0X_1,\ P_{\{0\}}(X_0, X_1) = X_0X_1 + X_1,$$

$$P_{\{1\}}(X_0, X_1) = X_0X_1 + X_0, P_{\{0,1\}}(X_0, X_1) = X_0X_1 + X_0 + X_1\Big\}$$

For $N = 3$, $\mathfrak{P}_3$ contains the following polynomials

$$\begin{cases} P_\emptyset(X_0, X_1, X_2) = X_0X_1X_2 + X_0 + X_1 + X_2, \\ P_{\{0\}}(X_0, X_1, X_2) = X_0X_1X_2 + X_1X_2 + X_0 + X_2, \\ P_{\{0,1\}}(X_0, X_1, X_2) = X_0X_1X_2 + X_1X_2 + X_0X_2 + X_1 + X_2, \\ P_{\{0,1,2\}}(X_0, X_1, X_2) = X_0X_1X_2 + X_1X_2 + X_0X_2 + X_0X_1 + X_0 + X_1 + X_2 \end{cases}$$

and those obtained by circular permutation of the variables.

Next proposition 3.3.4 gives the first properties of polynomials of $\mathfrak{P}$, lemma 3.3.8 shows that by vanishing of variables corresponding to an allowed subset, we shall still obtain polynomials of $\mathfrak{P}$, proposition 3.3.9 shows that these polynomials are irreducible, finally proposition 3.3.11 gives a characterization of the family $\mathfrak{P}$.

Proposition 3.4. — *1) If $N \neq N'$, then $\mathfrak{P}_N \bigcap \mathfrak{P}_{N'} = \emptyset$*
2) For $N \geqslant 2$, the mapping

$$\begin{array}{ccc} \mathfrak{P}(\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & \mathfrak{P}_N \\ A & \longmapsto & P_A \end{array}$$

is a bijection from the set $\mathfrak{P}(\mathbb{Z}/N\mathbb{Z})$ of subsets of $\mathbb{Z}/N\mathbb{Z}$ onto $\mathfrak{P}_N$. In particular, if $N \geqslant 2$, $\mathfrak{P}_N$ has 2^N elements.
3) If $A \subset \mathbb{Z}/N\mathbb{Z}$, then:

i) $\deg P_A = N$ *and* $\prod_{i=0}^{N-1} X_i$ *is the only monomial of P_A of degree N.*

ii) For $N \geqslant 2$, the homogeneous part of P_A of degree $N-1$ has $\operatorname{Card} A$ *monomials and these are*

$$\prod_{i \neq a} X_i \qquad \textit{for every } a \in A$$

In particular the homogeneous part of P_A of degree $N-1$ determines A and P_A uniquely.

iii) $P_A(0) = 0$.

4) If $P(X_0, \ldots, X_{N-1}) \in \mathfrak{P}_N$ and α is a circular permutation of $\{0, \ldots, N-1\}$ then $P(X_{\alpha(0)}, \ldots, X_{\alpha(N-1)}) \in \mathfrak{P}_N$.

Proof. — 1) derives from 3) i); 2) from 3) ii). Besides, the only monomial of degree N is obtained for $B = \emptyset \in \mathcal{P}_A$, monomials of degree $N-1$ are obtained for one element subsets $\{a\} \in \mathcal{P}_A$. The integer N being fixed, these monomials determine A and P_A. Finally, an allowed subset is by definition different from $\mathbb{Z}/N\mathbb{Z}$, so we have the assertion 3) iii). Assertion 4) is evident. □

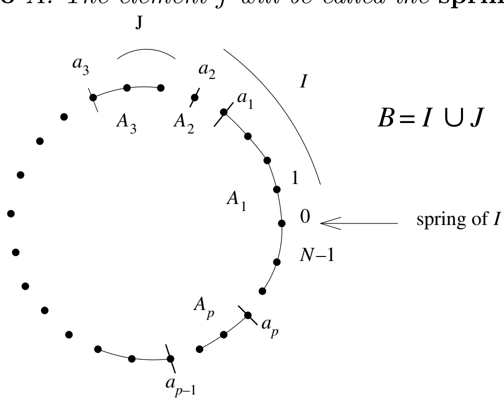
Lemma and Definition 3.5. — *Let $A \subset \mathbb{Z}/N\mathbb{Z}$, $\mathcal{A} = (A_i)_{1 \leqslant i \leqslant p}$ the partition of $\mathbb{Z}/N\mathbb{Z}$ defined by A and let $B \in \mathcal{P}_A$.*

1) Consider subsets of B of the type $I = \{j \dot{+} 1, \dots, j \dot{+} k\}$ such that:

i) $j \dot{+} k \in A$,

ii) $I \subset B$ is maximal for inclusion,

Then I is an allowed subset relatively to A which will be called an **allowed subset fixed to** *A. The element $\dot{j}$ will be called the* **spring** *of I.*



2) Let S_B be the set of springs of allowed subsets fixed to A, then we have $S_B \cap B = \emptyset$.

3) Consider subsets of B of the type $J = \{j \dot{+} 1, \dots, j \dot{+} 2k\}$ such that:

i) there exists i, $1 \leqslant i \leqslant p$ such that $J \subset A_i$,

ii) $J \subset B$ is maximal for inclusion,

iii) For every allowed subset I, fixed to A, we have $J \cap I = \emptyset$

then J is an allowed subset relatively to A, which be called a **wandering allowed subset**.

4) B admits a unique partition by fixed allowed subsets and wandering allowed subsets. This partition will be called the **canonical partition** *of B.*

Proof. — clear. □

Remark 3.6. — If $X \subset \mathbb{Z}/N\mathbb{Z}$ is not empty and $N' = \operatorname{Card} X$, canonical action of $\mathbb{Z}/N\mathbb{Z}$ on itself induces an action of $\mathbb{Z}/N'\mathbb{Z}$ on X, denoted by $+'$, defined in the following way: If $\dot{x} \in X$, let $j \geqslant 1$ be the least integer such that $x \dot{+} j \in X$; we set $\dot{x} +' \dot{1} = x \dot{+} j$.

Lemma 3.7. — *Let $A \subset \mathbb{Z}/N\mathbb{Z}$, and $B \in \mathcal{P}_A$. Denote by B' the complement of B in $\mathbb{Z}/N\mathbb{Z}$, $N' = \operatorname{Card} B'$ and let $\varphi : B' \to \mathbb{Z}/N'\mathbb{Z}$ be a bijection compatible with the actions of $\mathbb{Z}/N'\mathbb{Z}$ on B' and on $\mathbb{Z}/N'\mathbb{Z}$. If*

$$A' = \varphi(A \cap B') \bigcup \varphi(S_B)$$

where S_B is the set of springs of B, then the mapping

$$\begin{array}{rccc} \bar{\varphi}: & \{C \in \mathcal{P}_A \mid C \supset B\} & \longrightarrow & \mathfrak{P}(\mathbb{Z}/N'\mathbb{Z}) \\ & C & \longmapsto & \varphi(C \cap B') \end{array}$$

is a bijection from $\{C \in \mathcal{P}_A \mid C \supset B\}$ on $\mathcal{P}_{A'}$.

Proof. — 1) $\bar{\varphi}$ is clearly injective.

2) Let $C \in \mathcal{P}_{\mathcal{A}}$ such that $C \supset B$; to show that $\bar{\varphi}(C) \in \mathcal{P}_{\mathcal{A}'}$, it is sufficient to show that if $I \subset C$ (resp. $J \subset C$) is an allowed subset fixed to A (resp. a wandering allowed subset) belonging to the canonical partition of C, then $\varphi(I \cap B') \in \mathcal{P}_{\mathcal{A}'}$ (resp. $\varphi(J \cap B') \in \mathcal{P}_{\mathcal{A}'}$). On this purpose, we notice that if the last element of I belongs to $A \cap B$, then $\varphi(I \cap B')$ is an allowed subset with last element in $\varphi(S_B)$; if the last element of I is in $A \cap B'$, $\varphi(I \cap B')$ is an allowed subset with the same last element in $\varphi(A \cap B')$. Therefore in both cases $\varphi(I \cap B')$ is an allowed subset fixed to A'. Besides, $J \cap A = \emptyset$ and $J \cap S_B = \emptyset$, hence $\varphi(J \cap B')$ is contained in an interval of the partition of $\mathbb{Z}/N'\mathbb{Z}$ associated to A'; J has an even number of elements and $J \cap B'$ also. Finally, $J \cap B'$ is a wandering allowed subset.

3) Let $C' \in \mathcal{P}_{\mathcal{A}'}$ and $C = \varphi^{-1}(C') \cup B$. Then $C \in \mathcal{P}_{\mathcal{A}}$, therefore $\bar{\varphi}$ is surjective. □

Lemma 3.8. — *Let $P_A \in \mathfrak{P}_N$, $B \subset \mathbb{Z}/N\mathbb{Z}$ an allowed subset relatively to A, B' the complement of B in $\mathbb{Z}/N\mathbb{Z}$ and $N' = \operatorname{Card} B'$. Then, identifying $\mathbb{Z}[X_i, i \in B']$ with $\mathbb{Z}[X_0, \ldots, X_{N'-1}]$, there exists $A' \subset \mathbb{Z}/N'\mathbb{Z}$ such that*

$$P_A(X_i = 0,\ i \in B) \ = \ P_{A'}.$$

Proof. — In $P_A(X_i = 0,\ i \in B)$ remain only monomials $\prod_{i \notin C} X_i$ of P_A such that $C \supset B$; we then conclude by lemma 3.3.7. □

PROPOSITION 3.9. — *1) If $A = \emptyset$ and N is even (resp. odd), P_A has only monomials of even (resp. odd) degrees.*

2) If $N \geqslant 3$ and $P_A \in \mathfrak{P}_N$, P_A is irreducible in $\mathbb{Z}[X_0, \ldots, X_{N-1}]$.

Proof. — 1) If $B \in \mathcal{P}_{\mathcal{A}}$ then $\operatorname{Card} B = 0 \bmod 2$.

2) **First case**: $A = \emptyset$. The polynomial $P = P_A$ is invariant by circular permutation of the variables. Suppose that $P = P_1 P_2$, with $P_j \in \mathbb{Q}[X_0, \ldots, X_{N-1}]$, $j = 1, 2$ and P_1 irreducible, $P_2 \notin \mathbb{Q}$. Fix a variable, say X_i, then $\deg_{X_i} P = 1$; therefore the degree of one polynomial is zero and the degree of the other is one. Hence P_1 and P_2 have different variables. Denote by I_j, $j = 1, 2$ the subsets of indices i such that P_j depends on X_i. By proposition 3.3.4 1) and 3),

$$P_j(X_i, i \in I_j) = \lambda_j \prod_{i \in I_j} X_i \mod (X_i, i \in I_j)^{\operatorname{Card} I_j - 2}, \quad \lambda_1 \lambda_2 = 1,$$

and P_j contains only monomials the degree of which have the same parity as $\operatorname{Card} I_j$, because P_1 and P_2 depend on different variables.
We show now that P_1 cannot depend on two consecutive variables: in fact, we could choose X_i and X_{i+1} in such a way that P_1 should not depend on X_{i+2}. However P is stable by circular permutation, then

$$P(X) = P_1(X_i, i \in I_1) P_2(X_i, i \in I_2) = P_1(X_{i+1}, i \in I_1) P_2(X_{i+1}, i \in I_2)$$

where $P_1(X_{i+1}, i \in I_1)$ is irreducible but cannot divide neither $P_1(X_i, i \in I_1)$ neither $P_2(X_i, i \in I_2)$, which is impossible since $\mathbb{Q}[X_0, \ldots, X_{N-1}]$ is factorial.

Finally we fix an allowed subset $\{i, i+1\}$ with $i \in I_1$ and $i + 1 \in I_2$. Then by lemma 3.3.8,

$$P(X_i = X_{i+1} = 0) \in \mathfrak{P}_{N-2},$$

and by proposition 3.3.4 1), $\deg P(X_i = X_{i+1} = 0) = N - 2$. Then

$$\deg P_1(X_i = X_{i+1} = 0) = \deg P_1(X_i = 0) \leqslant \operatorname{Card} I_1 - 2$$

$$\deg P_2(X_i = X_{i+1} = 0) = \deg P_2(X_{i+1} = 0) \leqslant \operatorname{Card} I_2 - 2$$

which yields

$$N-2=\deg\, P_1(X_i=X_{i+1}=0)+\deg\, P_2(X_i=X_{i+1}=0)\leqslant N-4,$$

a contradiction.

Second case: $A\neq\emptyset$. We prove the result by induction on $N\geqslant 3$. The result for $N=3$ is true by example 3.3.3. Let $N\geqslant 4$ and suppose, in order to simplify the notations, that $N-1\in A$. We have

$$\begin{aligned} P_A(X_0,\ldots,X_{N-1}) &= X_{N-1}\big(P_A(X_{N-1}=1)-P_A(X_{N-1}=0)\big)\\ &\quad +P_A(X_{N-1}=0),\\ &= X_{N-1}Q(X_0,\ldots,X_{N-2})+R(X_0,\ldots,X_{N-2}) \end{aligned}$$

with

$$\begin{aligned} Q(X_0,\ldots,X_{N-2}) &:= P_A(X_{N-1}=1)-P_A(X_{N-1}=0)\\ &= \textstyle\prod_{i\neq N-1}X_i \mod (X_0,\ldots,X_{N-2})^{N-2} \end{aligned}$$

$$R(X_0,\ldots,X_{N-2}):=P_A(X_{N-1}=0)=\prod_{i\neq N-1}X_i \mod (X_0,\ldots,X_{N-2})^{N-2}.$$

Since $\{N-1\}$ is an allowed subset for A, $R\in\mathfrak{P}_{N-1}$ by lemma 3.3.8. Now, by induction hypothesis, $R\in\mathbb{Z}[X_0,\ldots,X_{N-2}]$ is irreducible. By Eisenstein criterion, it is sufficient to prove that R does not divide Q. But $\deg R=\deg Q=N-1$ and both polynomials have the same dominant monomial. Therefore we have to check that $R\neq Q$.

- If $A\neq\mathbb{Z}/N\mathbb{Z}$, we may suppose that $N-2\notin A$ and $N-1\in A$, then $\{N-2,N-1\}\in\mathcal{P}_{\mathcal{A}}$ and the monomial $M_{N-3}=\prod_{0\leqslant i\leqslant N-3}X_i$ is in P_A, hence in R, however $M_{N-3}X_{N-1}$ is not in P_A hence M_{N-3} is not in Q.
- If $A=\mathbb{Z}/N\mathbb{Z}$, P_A contains X_{N-1}, therefore $Q(0,\ldots,0)=1$ though $R(0,\ldots,0)=0$. ☐

Remark 3.10. — If $N=2$, the second assertion of the preceeding proposition is wrong as it can be seen in example 3.3.3.

Proposition 3.11. — *Let* $\mathfrak{P}'=\bigcup_{N\geqslant 0}\mathfrak{P}'_N$ *be a family of polynomials where*

$$\mathfrak{P}'_N\subset\mathbb{Q}[X_0,\ldots,X_{N-1}]$$

satisfy the following conditions:

i) For every $0 \leqslant N \leqslant 2$, $\mathfrak{P}'_N = \mathfrak{P}_N$,

ii) For every $N \geqslant 0$, $\operatorname{Card} \mathfrak{P}'_N = \operatorname{Card} \mathfrak{P}_N$,

iii) If $P \in \mathfrak{P}'_N$, *then* $\deg P = N$, *and its homogeneous part of degree* N *is* $\prod_{0 \leqslant i \leqslant N-1} X_i$,

iv) If $N \geqslant 3$ *and* $P \in \mathfrak{P}'_N$, *there exists* $A = A_P \subset \mathbb{Z}/N\mathbb{Z}$ *such that for every generating allowed subset* $B \in \mathcal{P}_A$ *we have*

$$P(X_i = 0,\ i \in B) \ \in\ \mathfrak{P}'_{N - \operatorname{Card} B}.$$

Moreover, for every monomial $\lambda \prod_{i \notin C} X_i$ *of* P, *where* $C \neq \emptyset$ *and* $\lambda \in \mathbb{Q}$, *there exists a generating allowed subset* B *such that* $B \subset C$.

Then, for every $N \geqslant 0$, $\mathfrak{P}'_N = \mathfrak{P}_N$.

Proof. — We show by induction on $N \geqslant 2$ that $\mathfrak{P}'_N = \mathfrak{P}_N$. By i) let $N \geqslant 3$.
Let $P \in \mathfrak{P}'_N$. By condition iv), there exists $A = A_P \subset \mathbb{Z}/N\mathbb{Z}$ such that for every $B \in \mathcal{P}_{\mathcal{A}}$

$$P(X_i = 0, i \in B) \in \mathfrak{P}_{N - \operatorname{Card} B}.$$

We are going to show that $P = P_A$. Both polynomials have the same dominant monomial $\prod_{0 \leqslant i \leqslant N-1} X_i$. Let $B \in \mathcal{P}_{\mathcal{A}}$ and $\prod_{i \notin B} X_i$ one of the monomials of P_A. By iv), induction hypothesis and proposition 3.3.4, 3),

$$P(X_i = 0, i \in B) = \prod_{i \notin B} X_i \mod (X_i, i \notin B)^{N - \operatorname{Card} B - 1}$$

hence this monomial belongs to P and by iii) each monomial of P_A belongs to P. Conversely let $\lambda \prod_{i \notin C} X_i$ be a monomial of P, let $B \in \mathcal{P}_{\mathcal{A}}$ such that $B \subset C$. Denoting by B' the complement of B in $\mathbb{Z}/N\mathbb{Z}$ and $N' = \operatorname{Card} B'$, there exists $A' \subset \mathbb{Z}/N'\mathbb{Z}$ for which

$$P(X_i = 0, i \in B) = P_{A'}.$$

By lemma 3.3.7, $C \in \mathcal{P}_{\mathcal{A}}$ and $\lambda = 1$.
We have now, $\mathfrak{P}'_N \subset \mathfrak{P}_N$. We conclude by ii). □

To end this section we give a property of these polynomials which will allow to compute the discriminant of singularities whose exceptional divisor is associated to concatenation of sequences $\sigma = \sigma' \sigma''$.

Proposition 3.12. — *Let $A' = \{a'_1, \ldots, a'_{p'}\} \subset \mathbb{Z}/N'\mathbb{Z}$, $A'' = \{a''_1, \ldots, a''_{p''}\} \subset \mathbb{Z}/N''\mathbb{Z}$ and $N = N' + N''$. We identify A' (resp. A'') with the subset of $\mathbb{Z}/N\mathbb{Z}$ (denoted in the same way)*

$$A' = \{a'_1, \ldots, a'_{p'}\} \subset \mathbb{Z}/N\mathbb{Z} \quad \big(\text{resp. } A'' = \{a''_1 + N', \ldots, a''_{p''} + N'\} \subset \mathbb{Z}/N\mathbb{Z}\big)$$

i.e.

$$0 \leqslant a'_1 < \cdots < a'_{p'} < N' \leqslant a''_1 + N' < \cdots < a''_{p''} + N' < N.$$

Setting $A = A' \cup A'' \subset \mathbb{Z}/N\mathbb{Z}$ we have

$$\begin{aligned} P_A(X_0, \ldots, X_{N-1}) \;&=\; P_{A'}(X_0, \ldots, X_{N'-1}) P_{A''}(X_{N'}, \ldots, X_{N'+N''-1}) \\ &\quad + P_{A'}(X_0, \ldots, X_{N'-1}) + P_{A''}(X_{N'}, \ldots, X_{N'+N''-1}). \end{aligned}$$

Proof. — With the same identification as in the statement we have

$$\begin{aligned} \mathcal{P}_{\mathcal{A}} \;&=\; \big\{B' \cup B'' \mid B' \in \mathcal{P}_{\mathcal{A}'}, \mathcal{B}'' \in \mathcal{P}_{\mathcal{A}''}\big\} \\ &\quad \bigcup \big\{B' \cup \{N', \ldots, N-1\} \mid B' \in \mathcal{P}_{\mathcal{A}'}\big\} \\ &\quad \bigcup \big\{\{0, \ldots, N'-1\} \cup B'' \mid B'' \in \mathcal{P}_{\mathcal{A}''}\big\} \end{aligned}$$

and this gives the three terms of the decomposition. □

3.2. Main results

Theorem 3.13 (Main theorem). — *Let $\sigma = \sigma_0 \cdots \sigma_i \cdots \sigma_{N+\rho-1}$ be a sequence of integers such that there are $N \geqslant 1$ singular sequences $\sigma_{i_j} = s_{k_j}$, $0 \leqslant j \leqslant N-1$ and $0 \leqslant \rho \leqslant N$ regular sequences r_{m_l}, $0 \leqslant l \leqslant \rho - 1$. Let $A \subset \mathbb{Z}/N\mathbb{Z}$ defined by*

$$A = A(\sigma) := \{0 \leqslant j \leqslant N-1 \mid \sigma_i \text{ is a regular sequence for } i = i_j + 1 \bmod N + \rho\}.$$

Then we have

$$\det\, M(\sigma) = P_A(k_0, \ldots, k_{N-1})^2.$$

Corollary 3.14. — *Let S be a minimal surface containing a GSS with $n = b_2(S) \geqslant 1$. Let $D_0, \ldots, D_{n-1}$ be the rational curves and $M(S) = (D_i D_j) = -M(\sigma)$ the intersection matrix . Then*

i) The index of the sublattice $\sum_{i=0}^{n-1} \mathbb{Z} D_i$ in $H_2(S, \mathbb{Z})$ is

$$\left[H_2(S, \mathbb{Z}) : \sum_{i=0}^{n-1} \mathbb{Z} D_i\right] = P_{A(\sigma)}(k_0, \ldots, k_{N-1});$$

ii) The curves $D_0, \dots, D_{n-1}$ form a basis of $H_2(S,\mathbb{Q})$ if and only if $\sigma \neq r_n$;

iii) The curves $D_0, \dots, D_{n-1}$ form a basis of $H_2(S,\mathbb{Z})$ if and only if $\sigma = s_1 r_{n-1} = (3,2,\dots,2)$ for $n \geqslant 1$ or $\sigma = s_1 s_1 = (3,3)$ if $n = 2$. In these cases we have the following matrices:

- $n = 1$, $M(S) = -1$,
- $n = 2$, $M(S) = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,
- $n \geqslant 3$,

$$\begin{pmatrix} -3 & 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & -2 & 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & -2 & 1 & \ddots & & \vdots \\ 0 & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & -2 \end{pmatrix}$$

The following corollary is a more precise version than [22] (6.9):

Corollary 3.15. — *Let S be an even Inoue-Hirzebruch surface with intersection matrix $M(S) = -M(\sigma)$ and $\sigma = s_{k_0} \cdots s_{k_{2q-1}}$. Let Γ and Γ' be the two cycles with intersection matrices $M(\Gamma)$ and $M(\Gamma')$, then*

$$\begin{aligned}[H^2(\Gamma,\mathbb{Z}) : H_2(\Gamma,\mathbb{Z})] &= |\det\, M(\Gamma)| = P_\emptyset(k_0,\dots,k_{2q-1}) = |\det\, M(\Gamma')| \\ &= [H^2(\Gamma',\mathbb{Z}) : H_2(\Gamma',\mathbb{Z})].\end{aligned}$$

3.3. A multiplicative topological invariant associated to singularities

The following terminology corresponds to the terminology of contracting germs introduced by Oeljeklaus-Toma [26]:

Definition 3.16. — *A* **simple sequence** *σ is a sequence of the form*

$$\sigma = s_{k_0} \cdots s_{k_{N-1}} r_m$$

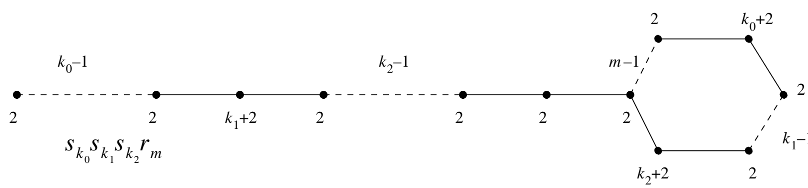
with $N \geqslant 1$. A singularity is called **simple** *if it is obtained by the contraction of a divisor whose weighted dual graph is associated to a simple sequence. Of* *course, up to circular permutation, any sequence* $\sigma = \sigma_0 \cdots \sigma_{N+\rho-1}$*, where* σ_i *is singular or regular, splits into* ρ *simple sequences.*

The polynomial associated to any singularity (X, x) *of type* σ *is defined by*

$$\Delta_\sigma(X_0, \ldots, X_{N-1}) := P_{A(\sigma)}(X_0, \ldots, X_{N-1}) + 1.$$

The integer $k = \Delta_\sigma(k_0, \ldots, k_{N-1})$ *will be called the* **twisting coefficient** *of the singularity.*

The weighted dual graph of the exceptional divisor of a simple singularity has exactly one branch. For example, if $\sigma = s_{k_0} s_{k_1} s_{k_2} r_m$, the dual graph is



Lemma 3.17 *Let* $\sigma' = s_{k'_0} \cdots s_{k'_{N'-1}} r_{m'}$ *and* $\sigma'' = s_{k''_0} \cdots s_{k''_{N''-1}} r_{m''}$ *be two simple sequences. Then, denoting by* $\sigma = \sigma'\sigma''$ *the sequence obtained by concatenation of* σ' *and* σ''*, we have with* $N = N' + N''$

$$\Delta_{\sigma'\sigma''}(X_0, \ldots, X_{N-1}) = \Delta_{\sigma'}(X_0, \ldots, X_{N'-1})\Delta_{\sigma''}(X_{N'}, \ldots, X_{N-1}).$$

Proof. — For $A' = A(\sigma') \subset \mathbb{Z}/N'\mathbb{Z}$, $A'' = A(\sigma'') \subset \mathbb{Z}/N''\mathbb{Z}$, $N = N' + N''$, $A = A' \coprod A'' \subset \mathbb{Z}/N\mathbb{Z}$, and $A = A(\sigma)$, we have by 3.3.12,

$$\begin{aligned}
\Delta_{\sigma'\sigma''}(X_0, \ldots, X_{N'+N''-1}) & \\
&= P_A(X_0, \ldots, X_{N-1}) + 1 \\
&= P_{A'}(X_0, \ldots, X_{N'-1}) P_{A''}(X_{N'}, \ldots, X_{N-1}) \\
&\quad + P_{A'}(X_0, \ldots, X_{N'-1}) + P_{A''}(X_{N'}, \ldots, X_{N-1}) + 1 \\
&= \big(P_{A'}(X_0, \ldots, X_{N'-1}) + 1\big)\big(P_{A''}(X_{N'}, \ldots, X_{N-1}) + 1\big) \\
&= \Delta_{\sigma'}(X_0, \ldots, X_{N'-1})\Delta_{\sigma''}(X_{N'}, \ldots, X_{N-1}).
\end{aligned}$$

□

Now we shall express the invariant Δ_σ for σ simple, thanks to the determinant of the unique branch of its dual graph:

LEMMA 3.18. — *Let σ be a simple sequence with branch B defined by*

$$Selfint(B) = \begin{cases} (\underbrace{2,\dots,2}_{k_0-1},\ k_1+2,\ \underbrace{2,\dots,2}_{k_2-1},\dots,\ k_{p-2}+2,\ \underbrace{2,\dots,2}_{k_{p-1}-1},\ 2) \\ \hfill if\ p \equiv 1 (\mathrm{mod}\ 2) \\ \\ (\underbrace{2,\dots,2}_{k_0-1},\ k_1+2,\ \underbrace{2,\dots,2}_{k_2-1},\dots,\ k_{p-3}+2,\ \underbrace{2,\dots,2}_{k_{p-2}-1},\ k_{p-1}+2) \\ \hfill if\ p \equiv 0 (\mathrm{mod}\ 2) \end{cases}$$

then

$$\Delta_\sigma(k_0,\dots,k_{p-1}) = \det\ B,$$

where det B *is the determinant of the intersection matrix of the curves in* B.

Proof. — For $p = 1$, $\sigma = s_{k_0} r_m$,

$$Selfint(B) = (\underbrace{2,\dots,2}_{k_0})$$

and det $B = k_0 + 1 = P_\sigma(k_0) + 1$.
For $p = 2$, $\sigma = s_{k_0} s_{k_1} r_m$,

$$Selfint(B) = (\underbrace{2,\dots,2}_{k_0-1}, k_1+2)$$

and det $B = k_0k_1 + k_0 + 1 = P_\sigma(k_0, k_1) + 1$ (see example 3.3.3). By induction: we suppose that p is odd, i.e. $p = 2q + 1$; the even case is left to the reader. Since there is only one branch we have $\sigma = (s_{k_0} \cdots s_{k_{2q}} r_m)$, $N = 2q + 1$ and

$$B = (\underbrace{2,\dots,2}_{k_0-1},\ k_1+2,\ \underbrace{2,\dots,2}_{k_2-1},\dots,\ \underbrace{2,\dots,2}_{k_{2q-2}-1},\ k_{2q-1}+2,\ \underbrace{2,\dots,2}_{k_{2q}-1},\ 2),$$

For $A = \{2q\} \subset \mathbb{Z}/(2q + 1)\mathbb{Z}$, we denote the allowed subsets by $\mathcal{P}_{2q+1}$. For the sequel we need the following observation: Let $C \in \mathcal{P}_{2q+1}$, then:

- if $2q \notin C$ and $2q - 1 \notin C$, $C \in \mathcal{P}_{2q-1}$ and $\sharp(C)$ is even;
- if $2q \in C$ and $2q - 1 \notin C$, $C = \{2q\} \cup C'$, $C' \in \mathcal{P}_{2q-1}$ and $\sharp(C')$ is even;
- if $2q \notin C$ and $2q - 1 \in C$, $C = \{2q - 1, 2q - 2\} \cup C'$, $C' \in \mathcal{P}_{2q-1}$ and $\sharp(C')$ is even;
- if $2q \in C$ and $2q - 1 \in C$, $C = \{2q, 2q - 1\} \cup C'$, $C' \in \mathcal{P}_{2q-1}$ and $\sharp(C')$ is odd or even.

Denote by Δ_{2q+1} the determinant of the branch when σ contains $2q+1$ singular sequences. Applying lemma 4.4.2 below, we have

$$\begin{aligned}\Delta_{2q+1}(k_0,\ldots,k_{2q}) &= (k_{2q}+1)\Big\{k_{2q-1}\Delta_{2q-1}(k_0,\ldots,k_{2q-2}-1)\\ &\quad+\Delta_{2q-1}(k_0,\ldots,k_{2q-2})\Big\} - k_{2q}\Delta_{2q-1}(k_0,\ldots,k_{2q-2}-1)\\ &= k_{2q}k_{2q-1}\Delta_{2q-1}(k_0,\ldots,k_{2q-2}-1)\\ &\quad+k_{2q}\Big\{\Delta_{2q-1}(k_0,\ldots,k_{2q-2})-\Delta_{2q-1}(k_0,\ldots,k_{2q-2}-1)\Big\}\\ &\quad+k_{2q-1}\Delta_{2q-1}(k_0,\ldots,k_{2q-2}-1)\\ &\quad+\Delta_{2q-1}(k_0,\ldots,k_{2q-2})\end{aligned}$$

In the sequel $\sum_{C'\in\mathcal{P}_{2q-1}}\prod_{i\notin C'}k_i$ is shortened to $\sum_{C'\in\mathcal{P}_{2q-1}}$. Recall that $C'\in\mathcal{P}_{2q-1}$, i.e. $C'\subset\{\dot{0},\ldots,\dot{2q-2}\}=\mathbb{Z}/(2q-1)\mathbb{Z}$. By induction hypothesis,

$$\begin{aligned}\Delta_{2q+1}(k_0,\ldots,k_{2q}) &= k_{2q}k_{2q-1}\left\{\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\in B'}}\prod_{i\notin B'}k_i+\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\notin B'}}\left(\prod_{\substack{i\notin B'\\ i<2q-2}}k_i\right)(k_{2q-2}-1)+1\right\}\\ &+k_{2q}\left\{\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\in B'}}+\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\notin B'}}-\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\in B'}}-\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\notin B'}}\left(\prod_{\substack{i\notin B'\\ i<2q-2}}k_i\right)(k_{2q-2}-1)\right\}\\ &+k_{2q-1}\left\{\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\in B'}}+\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\notin B'}}\left(\prod_{\substack{i\notin B'\\ i<2q-2}}k_i\right)(k_{2q-2}-1)+1\right\}+\sum_{B'\in\mathcal{P}_{2q-1}}+1\end{aligned}$$

$$\begin{aligned}&= k_{2q}k_{2q-1}\left\{\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\in B'}}+\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\notin B'}}-\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\notin B'}}\left(\prod_{i\notin\{2q-2\}\cup B'}k_i\right)+1\right\}+k_{2q}\left\{\sum_{\substack{2q-2\in B'\\ \sharp(B')\ odd}}\right\}\\ &+k_{2q-1}\left\{\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\in B'}}+\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\notin B'}}-\sum_{\substack{B'\in\mathcal{P}_{2q-1}\\ 2q-2\notin B'}}\left(\prod_{i\notin\{2q-2\}\cup B'}k_i\right)+1\right\}+\sum_{B'\in\mathcal{P}_{2q-1}}+1\end{aligned}$$

$$= \quad k_{2q}k_{2q-1}\left\{\sum_{\substack{2q-2\in B'\\ \sharp(B')\ even}}+\sum_{2q-2\notin B'}+1\right\}+k_{2q}\left\{\sum_{\substack{2q-2\in B'\\ \sharp(B')\ odd}}\right\}$$

$$+k_{2q-1}\left\{\sum_{\substack{2q-2\in B'\\ \sharp(B')\ even}} + \sum_{2q-2\notin B'} +1\right\} + \sum_{B'\in\mathcal{P}_{2q-1}} +1$$

$$= \sum_{\substack{B\in\mathcal{P}_{2q+1}\\ 2q\notin B, 2q-1\notin B}} + \sum_{\substack{B\in\mathcal{P}_{2q+1}\\ 2q\notin B, 2q-1\in B}} + \sum_{\substack{B\in\mathcal{P}_{2q+1}\\ 2q\in B, 2q-1\notin B}} + \sum_{\substack{B\in\mathcal{P}_{2q+1}\\ 2q\in B, 2q-1\in B}} +1$$

$$= \sum_{B\in\mathcal{P}_{2q+1}} +1 = P_\sigma(k_0,\dots,k_{2q})+1. \qquad \square$$

PROPOSITION 3.19. — *Let $\sigma=\sigma_0\cdots\sigma_{\rho-1}$ be a decomposition of σ into simple sequences and let $B_0,\dots,B_{\rho-1}$ be the branches of the dual graph, then*

i) $\Delta_\sigma=\prod_{i=0}^{\rho-1}\Delta_{\sigma_i}=\prod_{i=0}^{\rho-1}\det\ B_i$,

ii) $P_{A(\sigma)}=\prod_{i=0}^{\rho-1}(P_{A(\sigma_i)}+1)-1=\prod_{i=0}^{\rho-1}\det\ B_i\ -1$.

(notice that different polynomials depend on different indeterminates).

Proof. — lemmas 3.3.17 and 3.3.18. □

3.4. Twisted holomorphic 1-forms in the complement of the isolated singularity

Let S be a surface containing a GSS such that $b_2(S)=n$, with maximal divisor $D=\sum_{i=0}^{n-1}D_i$. We assume that the intersection matrix $M(S)=-M(\sigma)$ is negative definite. Therefore we have

$$D=\Gamma+\sum_{i=0}^{\rho-1}B_i,$$

where $B_0,\dots,B_{\rho-1}$ denote the branches of the dual graph.

THEOREM 3.20. — *If $2n<\sigma_n(S)<3n$, then there exists a non-vanishing closed twisted logarithmic 1-form*

$$\omega\in H^0(S,\Omega^1(LogD)\otimes L^k)$$

where the integer $k=k(S)\geqslant 2$ satisfies

$$k(S)=\prod_{i=0}^{\rho-1}\det\ B_i.$$

In particular in the complement of the singular point there is a non-trivially twisted non-vanishing holomorphic 1-form.

We recall that in the notation L^α, $\alpha \in \mathbf{C}^\star$ is the defining parameter of the topologically trivial line bundle.

Proof. — By [7] p1537, there exists a global twisted logarithmic 1-form on S which does not vanish. The positive integer $k = k(S)$ is the integer which appears in any contraction F associated to S (see lemma 2.7 and thm 2.8 in [7]). By [11] p480, the germ F is conjugate to a germ of class 4 (conjugate by $(z,w) \mapsto (w,z)$!)

$$F(z,w) = (\mu z w^s + P(w), w^k),$$

and by [12] p35, we have

$$\det\, M(S) = (-1)^n (k-1)^2.$$

With the main theorem 3.3.13 and Proposition 3.3.19 we conclude that

$$k = \prod_{i=0}^{\rho-1} \det\, B_i.$$

We obtain, in the complement of the singularity, a non-vanishing section on $\Omega^1 \otimes \Pi_\star L^k$. The coherent sheaf $\Pi_\star L^k$ is not trivial because the restriction of L^k to any neighbourhood of the exceptional divisor is not holomorphically trivial. □

4. Proof of the main theorem

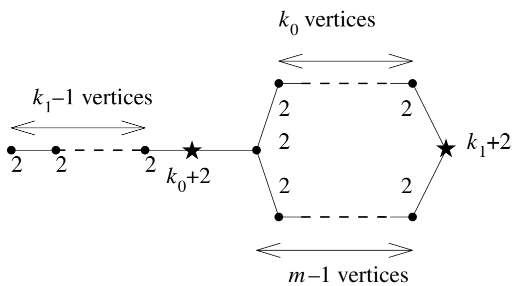
The aim is to compute the discriminant of the quadratic form using the family of polynomials previously introduced.

Sketch of proof. — 1) When we compute the determinant of $M(\sigma)$, a singular sequence $s_k = (k+2, 2, \ldots, 2)$ produces a monomial containing k^2 because k appears two times: one time because of the entry $k+2$ and a second time, according to lemma 4.4.1, due to the sequence

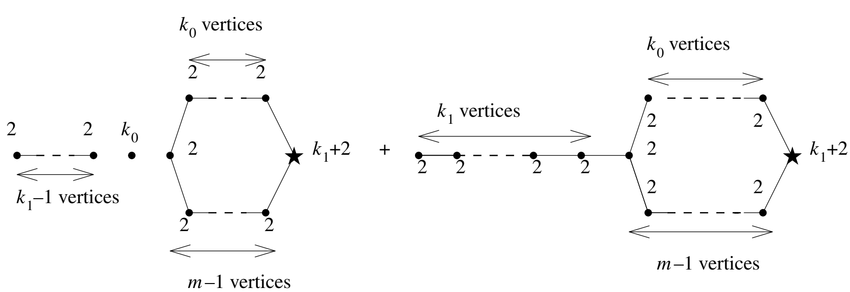
$$(\underbrace{2, \ldots, 2}_{k-1}).$$

By the same lemma, a regular sequence r_m produces the integer m at most at degree one. Therefore the determinant is a polynomial in the variables

$k_0, \dots, k_{N-1}$, and $m_0, \dots, m_{\rho-1}$. The idea is to develop the determinant splitting it into pieces which have a geometrical meaning. For example, consider $M = M(s_{k_0} r_m s_{k_1})$. Its weighted dual graph is



The vertices with weight 2 are represented by a bullet, the vertices with weight $\geqslant 3$ are represented by a star. It splits into



which corresponds to the development of the determinant along the k_1-th column by the splitting

$$\begin{pmatrix} \vdots \\ -1 \\ k_0+2 \\ -1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \\ k_0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ -1 \\ 2 \\ -1 \\ \vdots \end{pmatrix},$$

det $M =$

$$
\left|\begin{array}{ccccccccccc}
2 & -1 & & & & & & & & & \\
-1 & \ddots & \ddots & & & & & & & & \\
 & \ddots & 2 & -1 & & & & & & & \\
 & & -1 & k_0+2 & -1 & & & & & & \\
 & & & -1 & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
 & & & & -1 & \ddots & \ddots & & & & 0 \\
 & & & & 0 & \ddots & 2 & \ddots & & & \vdots \\
 & & & & \vdots & & \ddots & k_1+2 & \ddots & & \vdots \\
 & & & & \vdots & & & \ddots & 2 & \ddots & 0 \\
 & & & & 0 & & & & \ddots & \ddots & -1 \\
 & & & & -1 & 0 & \cdots & \cdots & 0 & -1 & 2
\end{array}\right|
\begin{array}{l}
 \\ {\scriptstyle 0} \\ {\scriptstyle k_1-1} \\ {\scriptstyle k_1} \\ \\ \\ \\ \\ \\ {\scriptstyle k_0+k_1+m-1} \\ \\
\end{array}
$$

$$
= k_0 \left|\begin{array}{cccc}
2 & -1 & & \\
-1 & \ddots & \ddots & \\
 & \ddots & \ddots & -1 \\
 & & -1 & 2
\end{array}\right|
\begin{array}{l}
{\scriptstyle 0} \\ \\ \\ {\scriptstyle k_1-2}
\end{array}
\left|\begin{array}{ccccccc}
2 & -1 & & & & & -1 \\
-1 & \ddots & \ddots & & & & \\
 & \ddots & 2 & \ddots & & & \\
 & & \ddots & k_1+2 & \ddots & & \\
 & & & \ddots & 2 & \ddots & \\
 & & & & \ddots & \ddots & -1 \\
-1 & & & & & -1 & 2
\end{array}\right|
\begin{array}{l}
{\scriptstyle k_1} \\ \\ \\ \\ \\ {\scriptstyle k_0+k_1} \\ {\scriptstyle +m-1}
\end{array}
$$

$$
+ \left|\begin{array}{ccccccccccc}
2 & -1 & & & & & & & & & \\
-1 & \ddots & \ddots & & & & & & & & \\
 & \ddots & 2 & -1 & & & & & & & \\
 & & -1 & 2 & -1 & & & & & & \\
 & & & -1 & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
 & & & & -1 & \ddots & \ddots & & & & 0 \\
 & & & & 0 & \ddots & 2 & \ddots & & & \vdots \\
 & & & & \vdots & & \ddots & k_1+2 & \ddots & & \vdots \\
 & & & & \vdots & & & \ddots & 2 & \ddots & 0 \\
 & & & & 0 & & & & \ddots & \ddots & -1 \\
 & & & & -1 & 0 & \cdots & \cdots & 0 & -1 & 2
\end{array}\right|
\begin{array}{l}
 \\ {\scriptstyle 0} \\ \\ {\scriptstyle k_1-1} \\ {\scriptstyle k_1} \\ \\ \\ \\ \\ {\scriptstyle k_0+k_1+m-1} \\ \\
\end{array}
$$

2) Noticing that such a determinant is always a square of an integer, we prove that the determinant is in fact obtained as a square of a polynomial in $k_0, \dots, k_{N-1}$ and hence *the integers* m *do not appear* in the development.

In the sequel we shall associate to M a family of matrices obtained by this type of development. The easy cases are those of a chain or of a cycle with all diagonal entries equal to 2:

LEMMA 4.1. — *Let* δ_m *and* Δ_m *be the determinants of order* $m \geqslant 1$ *defined by*

$$\delta_1 = 2, \quad \delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}, \quad \Delta_1 = 0, \quad \Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}$$

$$\delta_m = \begin{vmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{vmatrix}, \quad \Delta_m = \begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \ddots & & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ 0 & & \ddots & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{vmatrix},$$

$$m \geqslant 3,$$

then

$$\delta_m = m + 1 \quad \text{and} \quad \Delta_m = 0.$$

Proof. — left to the reader. $\square$

LEMMA 4.2. — *Let* $N = (n_{ij})_{0 \leqslant i,j \leqslant p-1}$ *be a matrix of order* $p \geqslant 2$, *of the form*

$$N = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & 2 & -1 & & \\ & & & -1 & n_{mm} & \cdots & n_{m,p-1} \\ & & & & \vdots & & \vdots \\ & & & & n_{p-1,m} & \cdots & n_{p-1,p-1} \end{pmatrix} \begin{matrix} 0 \\ \\ \\ m-1 \\ m \\ \\ p-1 \end{matrix},$$

where $m \leqslant p-2$. *For* $J \subset \{0, \dots, p-1\}$, *we denote by* $N_J = (n_{ij})_{i,j \in J}$ *the submatrix whose entries depend on indices in* J. *Then*

$$\det N = (m+1) \det N_{\{m,\dots,p-1\}} - m \det N_{\{m+1,\dots,p-1\}}.$$

Proof. — The result is trivial if $m = 0$. If $m \geqslant 1$, development along the first column yields with induction hypothesis

$$\begin{aligned}\det N &= 2\det N_{\{1,\dots,p-1\}} - \det N_{\{2,\dots,p-1\}} \\ &= 2\Big(m \det N_{\{m,\dots,p-1\}} - (m-1)\det N_{\{m+1,\dots,p-1\}}\Big) \\ &\quad -\Big((m-1)\det N_{\{m,\dots,p-1\}} - (m-2)\det N_{\{m+1,\dots,p-1\}}\Big) \\ &= (m+1)\det N_{\{m,\dots,p-1\}} - m \det N_{\{m+1,\dots,p-1\}}.\end{aligned}$$

□

4.1. Expression of the determinants by polynomials

Notations 4.3. — Let $N \geqslant 0$ and $\rho \geqslant 0$ be integers such that $\rho = 1$ if $N = 0$ and $\rho \leqslant N$ if $N \geqslant 1$.

Let $M = M(\sigma)$ where $\sigma = \sigma_0 \cdots \sigma_{N+\rho-1} = (a_0, \dots, a_{n-1})$ contains N singular sequences s_{k_i}, $i = 0, \dots, N-1$ and ρ regular sequences r_{m_j}, $j = 0, \dots, \rho - 1$. Let

$$n = \sum_{i=0}^{N-1} k_i + \sum_{j=0}^{\rho-1} m_j$$

be the order of $M = (m_{ij})_{0 \leqslant i,j \leqslant n-1}$ or the number of vertices of the associated dual weighted graph.

We denote by $\mathcal{C}$ the set of subsets $J \subset \{0, \dots, n-1\}$ which satisfy the following condition

$$(C) \qquad \begin{cases} \text{let } 0 \leqslant l \leqslant N + \rho - 1, \text{ and } \sigma_l = (a_r, \dots, a_s). \\ \text{If } \alpha \text{ satisfies } r + 1 \leqslant \alpha \leqslant s \text{ and } \alpha \in J \\ \text{then for all } \beta \text{ such that } r + 1 \leqslant \beta \leqslant s, \text{ we have } \beta \in J. \end{cases}$$

Splitting the graph into some pieces or changing the weights of some vertices, we associate to M a family $\mathcal{M}$ of matrices in the following way: For $J \in \mathcal{C}$, let K_J defined by

$$K_J = \{j \in J \mid m_{jj} > 2\}.$$

For $K \subset K_J$, denote by M_J^K the matrix

$$M_J^K := (m'_{ij})_{i,j \in J}$$

where

$$\begin{cases} m'_{kk} & = & 2 & \text{if } k \in K \\ m'_{ij} & = & m_{ij} & \text{in other cases} \end{cases}$$

The family $\mathcal{M}$ is

$$\mathcal{M} = \left\{ \mathcal{M}_{\mathcal{J}}^{\mathcal{K}} \mid \mathcal{J} \in \mathcal{C}, \mathcal{K} \subset \mathcal{K}_{\mathcal{J}} \right\}.$$

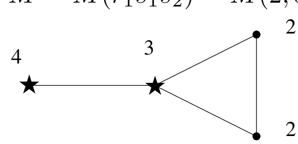
Now, for a fixed matrix M_J^K, we consider

- a partition $J = J' \cup J''$ of J, where J' (resp. J'') is the subset of indices of vertices of the cycle (resp. of the branches), and
- another partition of J' and of J'' depending on K, composed of subsets of the following two types:
  (1) singletons $\{i\}$ such that $m_{ii} > 2$,
  (2) when elements of type (1) are removed, connected components of vertices j with weight $m_{jj} = 2$

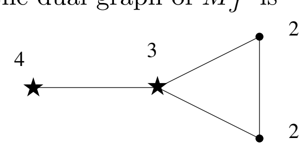
To end, denote by $\nu_1(M_J^K)$ (resp. $\nu_2(M_J^K)$) the total number of subsets of type (1) (resp. type (2)) in the partitions of J' and J'' and we set

$$\nu(M_J^K) = \nu_1(M_J^K) + \nu_2(M_J^K).$$

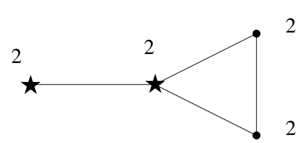
Examples 4.4. — Let $M = M(r_1 s_1 s_2) = M(2, 3, 42)$. Its dual graph is



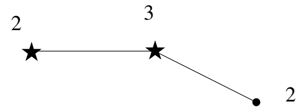
- If $J = \{0, 1, 2, 3\}$ and $K = \emptyset$ then $J' = \{0, 1, 3\}$, $J'' = \{2\}$ and $\nu(M_J^K) = 3$ and the dual graph of M_J^K is



- If $J = \{0, 1, 2, 3\}$ and $K = \{1, 2\}$ then $\nu(M_J^K) = 2$ and the dual graph of M_J^K is



- If $J=\{1,2,3\}$ and $K=\{2\}$ then $J'=\{1,3\}$, $J''=\{2\}$, $\nu(M_J^K)=3$,



Lemma 4.5. — *Let $M_J^K\in\mathcal{M}$, $\nu_1=\nu_1(M_J^K)$ and $\nu_2=\nu_2(M_J^K)$. Then there exists a polynomial*

$$Q\in\mathbb{Z}[X_0,\dots,X_{\nu_1-1},Y_0,\dots,Y_{\nu_2-1}]$$

of degree 1 *respectively each variable, such that*

$$\det\ M_J^K=Q(k_{i_0},\dots,k_{i_{\nu_1-1}},m_0,\dots,m_{\nu_2-1})$$

where subsets $\{i_j\}$ are of type (1) *and m_j are the cardinals of the subsets of type* (2) *which compose the partition of J.*

Proof. — By induction on $\nu=\nu_1+\nu_2\geqslant 1$. We have $\nu_1\leqslant N$ and $\nu_2\leqslant N+\rho$ by condition (C).

If $\nu=1$, either $\nu_1=1$, i.e. the determinant is of order 1 and the result is clear, either $\nu_2=1$ and the results derives from lemma 4.4.1.

If $\nu\geqslant 2$, we may suppose that $M_J^K=(m'_{ij})$ is irreducible because reducible case is an immediate consequence of the induction hypothesis. Several cases may happen:

1) M_J^K is a matrix of a cycle: Since $\nu\geqslant 2$ there exists an index $j\in J$ such that $m'_{jj}=k_{i_j}+2$. The decomposition of the j-th column

$$\begin{pmatrix}\vdots\\0\\-1\\k_{i_j}+2\\-1\\0\\\vdots\end{pmatrix}=\begin{pmatrix}\vdots\\0\\0\\k_{i_j}\\0\\0\\\vdots\end{pmatrix}+\begin{pmatrix}\vdots\\0\\-1\\2\\-1\\0\\\vdots\end{pmatrix}$$

yields the relation

$$(\dagger)\qquad \det\ M_J^K=k_{i_j}\ \det\ M^K_{J\setminus\{j\}}+\det\ M_J^{K\cup\{j\}}$$

where $M^K_{J\setminus\{j\}}$ (resp. $M_J^{K\cup\{j\}}$) is a matrix of a chain (resp. of a cycle). Setting

$$\nu'_i=\nu_i(M^K_{J\setminus\{j\}}),\quad \nu''_i=\nu_i(M_J^{K\cup\{j\}}),\qquad i=1,2,$$

we have

$$\begin{cases} \nu_1' = \nu_1 - 1 & \nu_2' = \nu_2 \\ \\ \nu_1'' = \nu_1 - 1 & \nu_2'' \leqslant \nu_2 \end{cases}$$

(there is one exception : when all entries of the cycle are $\geqslant 3$. We have $\nu_1'' = \nu_1 - 1$ but $\nu_2'' = \nu_2 + 1$ but then we repete the procedure).

By induction hypothesis there exist polynomials

$$Q \in \mathbb{Z}[X_0, \ldots, \widehat{X_j}, \ldots, X_{\nu_1 - 1}, Y_0, \ldots, Y_{\nu_2 - 1}]$$

$$R \in \mathbb{Z}[X_0, \ldots, \widehat{X_j}, \ldots, X_{\nu_1 - 1}, Y_0, \ldots, Y_l, \widehat{Y_{l+1}}, \ldots, Y_{\nu_2 - 1}]$$

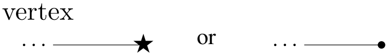
such that, by a suitable numbering of the indices

$$\det\, M^K_{J \setminus \{j\}} = Q(k_{i_0}, \ldots, \widehat{k_{i_j}}, \ldots, k_{\nu_1 - 1}, m_0, \ldots, m_{\nu_2 - 1})$$

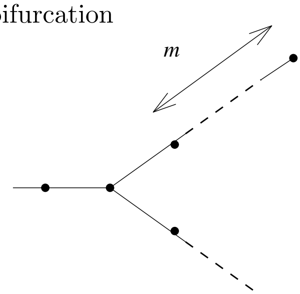
$$\det\, M^{K \cup \{j\}}_J = $$
$$R(k_{i_0}, \ldots, \widehat{k_{i_j}}, \ldots, k_{\nu_1 - 1}, m_0, \ldots, \ldots, m_{l-1}, m_l + m_{l+1} + 1, m_{l+2}, \ldots, m_{\nu_2 - 1}).$$

We conclude replacing in (†).

2) M^K_J is not the matrix of a cycle: then the dual graph is a part of a cycle or contains bits of branches of M. In any cases, the dual graph contains a terminal vertex



- If in this chain there is a vertex with weight > 2, we develop as before,
- If not, all vertices have a weight equal to 2, but since $\nu \geqslant 2$, this chain leads to a bifurcation



Either the vertex of bifurcation has a weight > 2 and we develop as before, either we apply lemma 4.4.2 with appropriate numbering of entries of M^K_J:

(‡) $\det\, M^K_J = (m+1) \det\, M^K_{J \setminus \{0, \ldots, m-1\}} - m \det\, (M^K_J)_{J \setminus \{0, \ldots, m\}}$.

The matrix $(M_J^K)_{J\setminus\{0,\dots,m\}}$ obtained by deletion of the branch with its root may not be in $\mathcal{M}$, however applying once again lemma 4.4.2, we obtain a matrix in $\mathcal{M}$ thanks to the explicit description of M given by the theorem 1.1.9. We apply then induction hypothesis and (‡). □

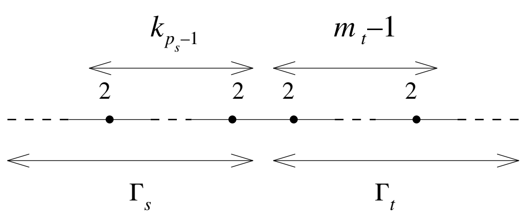
Lemma 4.6. — *Let $M = M(\sigma_0 \cdots \sigma_{N+\rho-1})$ be a matrix satisfying notations 4.4.3. Then, there exists a polynomial*

$$Q \in \mathbb{Z}[X_0, \dots, X_{N-1}, Y_0, \dots, Y_{\rho-1}]$$

of degree at most 2 (resp. 1) relatively X_i, $i = 0, \dots, N-1$ (resp. Y_j, $j = 0, \dots, \rho-1$) which satisfies

$$\det M = Q(k_0, \dots, k_{N-1}, m_0, \dots, m_{\rho-1}).$$

Proof. — We have $M = M^{\emptyset}_{\{0,\dots,n-1\}} \in \mathcal{M}$ and by theorem 1.1.9, $\nu_1 = N$, $\nu_2 \leqslant N+\rho$ (with $\rho = \rho(S)$). More precisely (with notations of 1.1.9), if there exists an integer s such that $p_s = 0 \bmod 2$, we have for $t = s+1 \bmod N+\rho$, the chain



Let $\mathfrak{S} = \{\mathfrak{t} \mid \mathfrak{p}_\mathfrak{s} = 0 \bmod 2,\ \text{for } \mathfrak{s} = \mathfrak{t} - 1\}$. Then

$$\nu_2 = N + \rho - \operatorname{Card} \mathfrak{S}.$$

Lemma 4.4.5 gives a polynomial in $2N + \rho - \operatorname{Card}\mathfrak{S}$ indeterminates

$$Q \in \mathbb{Z}[X_0, \dots, X_{N-1}, Y_0, \dots, Y_{N-1}, Y_N, \dots, \widehat{Y_{N+t}}, \dots, Y_{N+\rho-1} \mid t \in \mathfrak{S}]$$

such that for suitable indices $i(t) \leqslant N-1$,

$$\det M(\sigma_0 \cdots \sigma_{N+\rho-1}) = Q(k_0, \dots, k_{N-1}, k_0, \dots, k_{i(t)} + m_t, \dots, k_{N-1}, m_0, \dots, \widehat{m_t}, \dots, m_{\rho-1})$$

Setting $Y_i = X_i$ for $i \leqslant N-1$, $i \neq i(t)$, $t \in S$, and substituing $X_{i(t)} + m_t$ in $Y_{i(t)}$ for $t \in S$, the wished polynomial is obtained.
If for every s, $p_s \equiv 1 \bmod 2$, we develop starting from an entry $k_{p_s-1} + 2$. □

Here is the key lemma for the reduction lemma of the following section:

LEMMA 4.7. — *1) Let P, Q be two polynomials in $\mathbb{Q}[X_0, \ldots, X_{n-1}]$. Suppose that there exists an integer N such that for $k_0 \geqslant N, \ldots, k_{n-1} \geqslant N$ the following equality*

$$P(k_0, \ldots, k_{n-1}) = \pm Q(k_0, \ldots, k_{n-1})$$

holds. Then $P = Q$ or $P = -Q$.

2) Let $P \in \mathbb{Q}[X_0, \ldots, X_{n-1}]$ of degree at most 2 relatively to each indeterminate. Suppose that there exists an integer N such that for $k_0 \geqslant N, \ldots, k_{n-1} \geqslant N$, $P(k_0, \ldots, k_{n-1})$ is the square of a rational. Then there exists $Q \in \mathbb{Q}[X_0, \ldots, X_{n-1}]$ satisfying

$$P = Q^2.$$

In particular, if $\deg_{X_i} P \leqslant 1$, P does not depend on X_i.

Proof. — 1) By induction on $n \geqslant 1$.

2) The statement is true for $n = 1$ without condition on the power by [28]. Then by induction: suppose $n \geqslant 2$ and fix $k_0, \ldots, k_{n-2} \geqslant N$. Set

$$\begin{aligned} A(X_{n-1}) &= P(k_0, \ldots, k_{n-2}, X_{n-1}) \\ &= X_{n-1}^2 P_2(k_0, \ldots, k_{n-2}) + X_{n-1} P_1(k_0, \ldots, k_{n-2}) \\ &\quad + P_0(k_0, \ldots, k_{n-2}). \end{aligned}$$

For each $k_{n-1} \geqslant N$, $A(k_{n-1})$ is the square of a rational, hence by the one indeterminate case, $P_0(k_0, \ldots, k_{n-2})$ and $P_2(k_0, \ldots, k_{n-2})$ are squares of rationals. Induction hypothesis shows that there exist polynomials $Q_0, Q_1 \in \mathbb{Q}[X_0, \ldots, X_{n-2}]$, unique up to sign, which satisfy

$$P_0 = Q_0^2, \quad \text{and} \quad P_2 = Q_1^2.$$

Replacing, one obtains

$$P_1(k_0, \ldots, k_{n-2}) = \pm 2 Q_0(k_0, \ldots, k_{n-2}) Q_1(k_0, \ldots, k_{n-2}).$$

By 1), one concludes that

$$P = (X_{n-1} Q_1 \pm Q_0)^2.$$

□

4.2. The reduction lemma

In this section we shall prove that the polynomial which gives the value of a determinant depends on the positions of the regular sequences in σ, but not on their lengths.

LEMMA 4.8 (REDUCTION LEMMA). — *Let $M = M(\sigma_0 \cdots \sigma_{N+\rho-1})$ be a matrix which fulfils conditions 4.4.3. Then, there exists a polynomial $P_\sigma \in \mathbb{Q}[X_0, \ldots, X_{N-1}]$ of degree at most 1 relatively to each indeterminate X_i, $i = 0, \ldots, N-1$ such that*

$$\det M(\sigma) = P_\sigma(k_0, \ldots, k_{N-1})^2.$$

In particular the determinant of M does not depend on the lengths of the regular sequences.

Proof. — By lemma 4.4.6 there exists a polynomial $Q \in \mathbb{Q}[X_0, \ldots, X_{N-1}, Y_0, \ldots, Y_{\rho-1}]$ of degree at most 2 in X_i and at most 1 in Y_j, such that when $k_i \geqslant 1$, $i = 0, \ldots, N-1$ and $m_j \geqslant 1$, $j = 0, \ldots, \rho$

$$\det M = Q(k_0, \ldots, k_{N-1}, m_0, \ldots, m_{\rho-1}).$$

The matrix $-M$ is an intersection matrix hence det M is the square of an integer by proposition 1.1.7. Then lemma 4.4.7 implies the existence of a polynomial

$$P_\sigma \in \mathbb{Q}[X_0, \ldots, X_{N-1}, Y_0, \ldots, Y_\rho]$$

which satisfies $Q = P^2$. But $\deg_{Y_j} Q \leqslant 1$, therefore P and Q do not depend on Y_j. $\square$

4.3. Relation between determinants and polynomials of $\mathfrak{P}$

The next step is to prove that the polynomials P_σ of the reduction lemma 4.4.8 belong in fact in the family $\mathfrak{P}$ previously defined. We shall apply the caracteristic properties of $\mathfrak{P}$ given in 3.3.11. We start with examples.

Examples 4.9. — 1) Case $N = 0$: then $M = M(\sigma) = M(r_m)$ and $\det M = 0$. Therefore $P_\sigma = 0$.

2) Case $N = 1$: If $M = M(s_k) = \begin{pmatrix} k+2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$, then

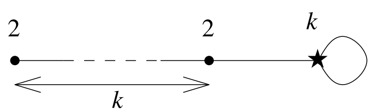
$$\det M = k\delta_{k-1} + \Delta_k = k^2, \quad \text{and} \quad P_\sigma(X) = X.$$

If $M = M(s_k r_m)$ we have by the reduction lemma 4.4.8

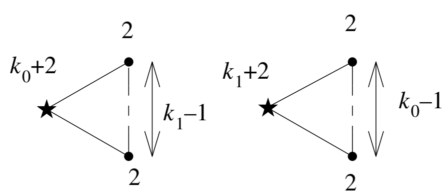
$$\det M = \det M(s_k r_1) = \begin{vmatrix} k & & & & -1 \\ & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{vmatrix}$$

$$= k\delta_k - \delta_{k-1} = k(k+1) - k = k^2,$$

and $P_\sigma(X) = X$.

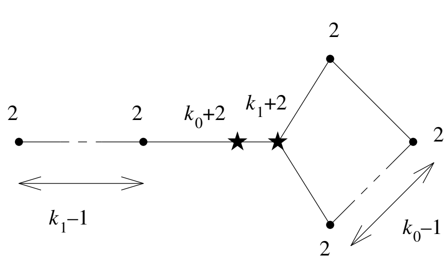


3) Case $N = 2$: If $M = M(s_{k_0} s_{k_1})$ the matrix is reducible and $\det M = (k_0 k_1)^2$.



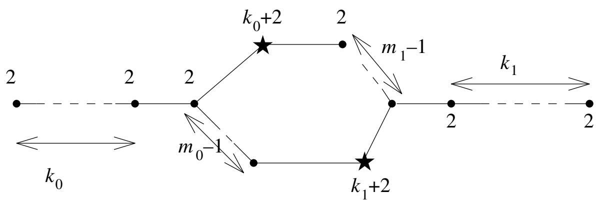
If $M = M(s_{k_0} r_m s_{k_1})$ we have by 4.4.8,

$$\det M = \det M(s_{k_0} r_1 s_{k_1}) = (k_0 k_1 + k_1)^2$$



If $M = M(s_{k_0} r_{m_0} s_{k_1} r_{m_1})$ we have

$$\det M = \det M(s_{k_0} r_1 s_{k_1} r_1) = (k_0 k_1 + k_0 + k_1)^2.$$



Proposition 4.10. — *Let $\mathfrak{P}'_N$ be the family of polynomials $P_\sigma \in \mathbb{Q}[X_0, \ldots, X_{N-1}]$ such that*

$$\det\ M(\sigma) = \det\ M(\sigma_0 \cdots \sigma_{N+\rho-1}) = P_\sigma(k_0, \ldots, k_{N-1})^2,$$

1) For any $N \geqslant 0$, $\mathfrak{P}_N = \mathfrak{P}'_N$,

2) Let $\sigma_{i_j} = s_{k_j}$, $0 \leqslant i_j \leqslant N + \rho - 1$, $0 \leqslant j \leqslant N - 1$ be the singular sequences in σ and let $A \subset \mathbb{Z}/N\mathbb{Z}$ be the subset of indices j such that σ_{i_j+1} is a regular sequence, then

$$P_\sigma = P_A.$$

Proof. — To prove 1) it is sufficient to check conditions i) to iv) of proposition 3.3.11.

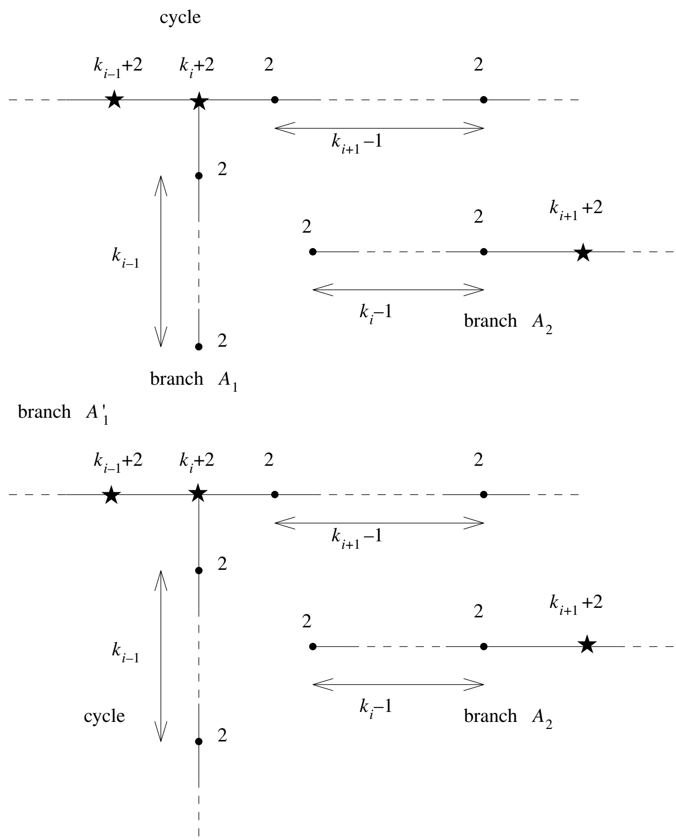
a) Condition i) has been checked in examples 3.3.3 and 4.4.9. It is not possible to have two adjacent regular sequences, hence there are 2^N ways to insert regular sequences among N singular sequences, therefore we have ii).

b) We suppose now that $N \geqslant 3$. Let A be the subset (perhaps empty) of indices j in $\mathbb{Z}/N\mathbb{Z}$ such that the singular sequence $\sigma_{i_j} = s_{k_j}$ is followed by a regular sequence. Let $\lambda \prod_{j\in J} X_j$, $\lambda \in \mathbb{Q}$, $J \subset \{0, \ldots, N-1\}$ be a monomial of P_σ. We prove first that:

$$\text{If } i \notin J \text{ but } i-1 \in J \text{ and } i+1 \in J, \text{ then } i \in A.$$

Suppose that $i \notin A$, $i-1 \in J$ and $i+1 \in J$. Since $\det M = P_\sigma(k_0, \ldots, k_{N-1})^2$, it is sufficient to show that in the development of $\det M$, any term which contains the factor $(k_{i-1}k_{i+1})^2$ must also contain the factor k_i^2, or more simply the factor k_i. In view of the reduction lemma 4.4.8 we may suppose that all regular sequences are of the type r_1. Since $i \notin A$, s_{k_i} is followed by a singular sequence and there are two possible cases:

- σ contains the sequence $s_{k_{i-1}} r_1 s_{k_i} s_{k_{i+1}}$: By theorem 1.1.9, the dual graph of M contains one of the two subgraphs



Notice that to obtain the factor $(k_{i-1}k_{i+1})^2$ one has to develop the determinant relatively to the branch A_2, then each term containing the factor k_{i+1}^2 has to contain $k_i k_{i+1}^2$.

- σ contains the sequence $s_{k_{i-1}} s_{k_i} s_{k_{i+1}}$: a similar argument gives the result.

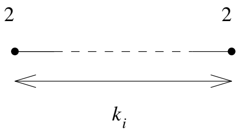
c) Any monomial of P_σ may be written as $\lambda \prod_{j \notin C} X_j$, $\lambda \in \mathbb{Q}$ (C is the complement of J !). Suppose that $C \neq \emptyset$. Then, either C contains an element of A, either C doesn't, however by b), C contains a pair $\{j, j+1\}$. We have proved that in all cases C contains a generating allowed subset, hence we have the second part of iv).

d) In order to see that for each allowed subset $B \in \mathcal{P}_A$ we have

$$P_\sigma(X_i = 0,\ i \in B) \in \mathfrak{P}'_{N-\mathrm{Card}\,B}$$

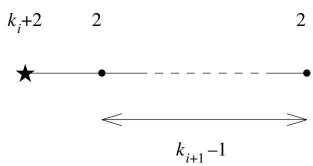
it is sufficient to check this property for generating allowed subsets B. By theorem 1.1.9,

- If $i \in A$, then the weighted dual graph of M contains the subgraph

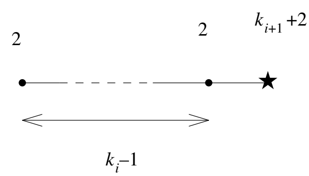


  with k_i vertices (and not $k_i - 1$!). Vanishing of k_i yields a configuration of a branch A_s and part of cycle Γ_s whose parity are changed (see 1.1.9).

- If $\{i, i+1\}$ is generating allowed pair, the dual graph contains the subgraphs



and



  Vanishing of k_i and k_{i+1} yields the graph of $M(\sigma')$, where σ' is obtained from σ deleting the sequences s_{k_i} and $s_{k_{i+1}}$.

e) To end we have to compute the homogeneous parts of P_σ of degrees N and $N-1$. We shall derive from proposition 3.3.4 that $P\sigma = P_A$.

By reduction lemma, $\deg_{X_i} P_\sigma \leqslant 1$, hence if we show that P_σ contains the monomial $\prod_{i=0}^{N-1} X_i$, it is necessarily its homogeneous part of highest degree.

If $A = \emptyset$, the dual graph contains one or two cycles without branches. To obtain in the development of det M the term $(\prod_{i=0}^{N-1} k_i)^2$, it is sufficient to develop successively relatively each vertex of weight > 2. By b), P_σ contains no monomials of degree $N-1$, which gives the result in this case.

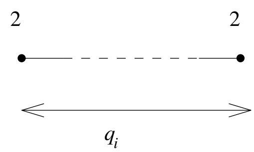
If $A \neq \emptyset$, we may suppose by reduction lemma, that all regular sequences are equal to r_1. By theorem 1.1.9, all roots of the branches have weight > 2. If we develop successively relatively to each column corresponding to a vertex of weight > 2, we obtain:

$$\det M(\sigma) = \prod_{i=0}^{N-1} k_i \det B + \sum_{i=0}^{N-1} \prod_{j \neq i} k_j \det B_i \mod (k_0, \ldots, k_{N-1})^{2N-2},$$

where

- the dual graph of B is obtained from the one of $M(\sigma)$ by deletion of all the vertices of weights > 2, and
- the dual graph of B_i by deletion all the vertices of weights > 2 but the one of weight $k_i + 2$ and setting it equal to 2.

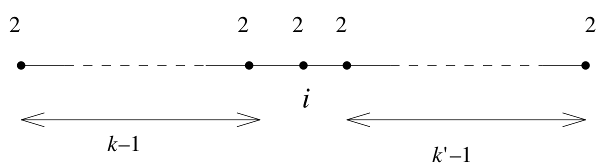
Now the graph of B is composed of connected components which are chains of the form



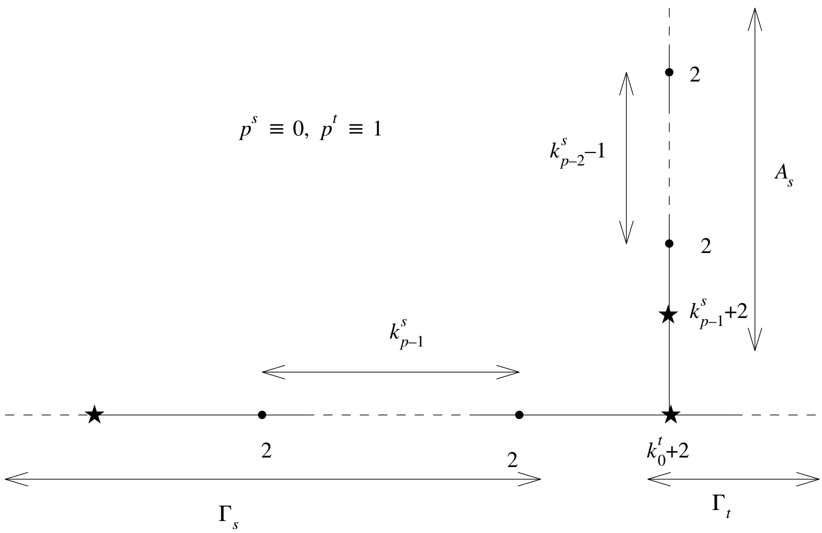
where $q_i = k_i - 1$ (resp. $q_i = k_i$) if the sequence which follows s_{k_i} is singular (resp. regular). Therefore the contribution of this term is

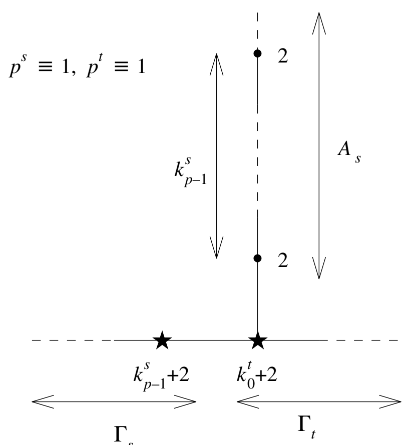
$$\det B = \prod_{i \notin A} k_i \prod_{i \in A} (k_i + 1) = \prod_{i=0}^{N-1} k_i + \sum_{i \in A} \prod_{j \neq i} k_j \mod (k_0, \ldots, k_{N-1})^{N-2}.$$

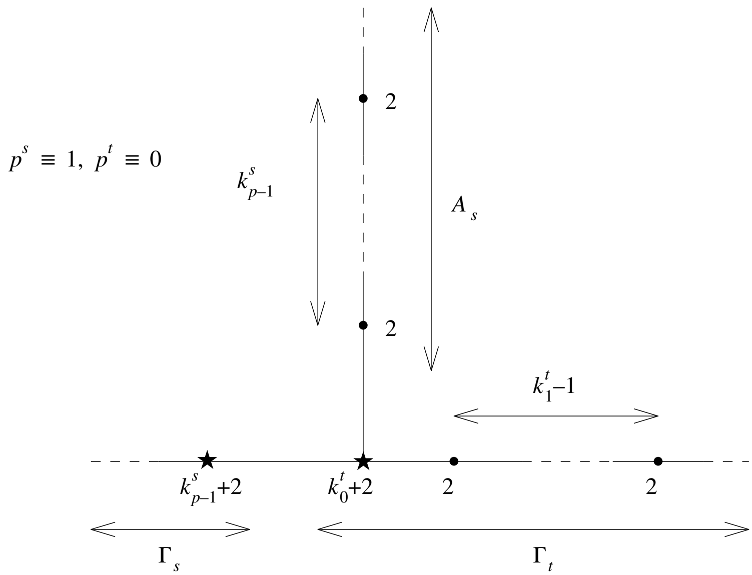
It remains to compute det B_i: By lemma 4.4.5, det B_i is a polynomial of degree at most N and we have to determine when this degree is precisely N. On that purpose, suppose that the index i corresponds to a vertex between two chains of vertices of weight 2, that is to say we have a subgraph

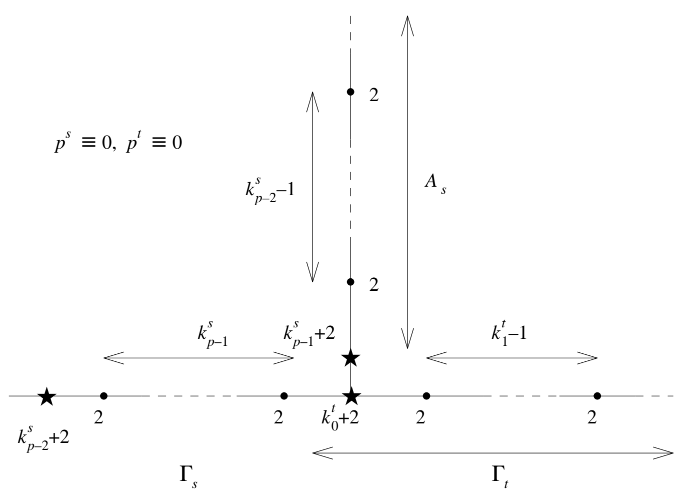


By lemma 4.4.1 the determinant of this connected component is $k + k'$, hence of degree 1 and det B_i will be of degree at most $N - 1$. Therefore we are only interested in vertices which are the root of a branch or linked to a root. By theorem 1.1.9, for $\rho(S) \geqslant 1$ and $t = s + 1 \mod \rho(S)$ there are four possible situations:


$p^s \equiv 0,\ p^t \equiv 1$
2
$k^s_{p-2}-1$
A_s
2
$k^s_{p-1}+2$
k^s_{p-1}
2
2
k^t_0+2
Γ_s
Γ_t


$p^s \equiv 1,\ p^t \equiv 1$
2
k^s_{p-1}
A_s
2
$k^s_{p-1}+2$
k^t_0+2
Γ_s
Γ_t

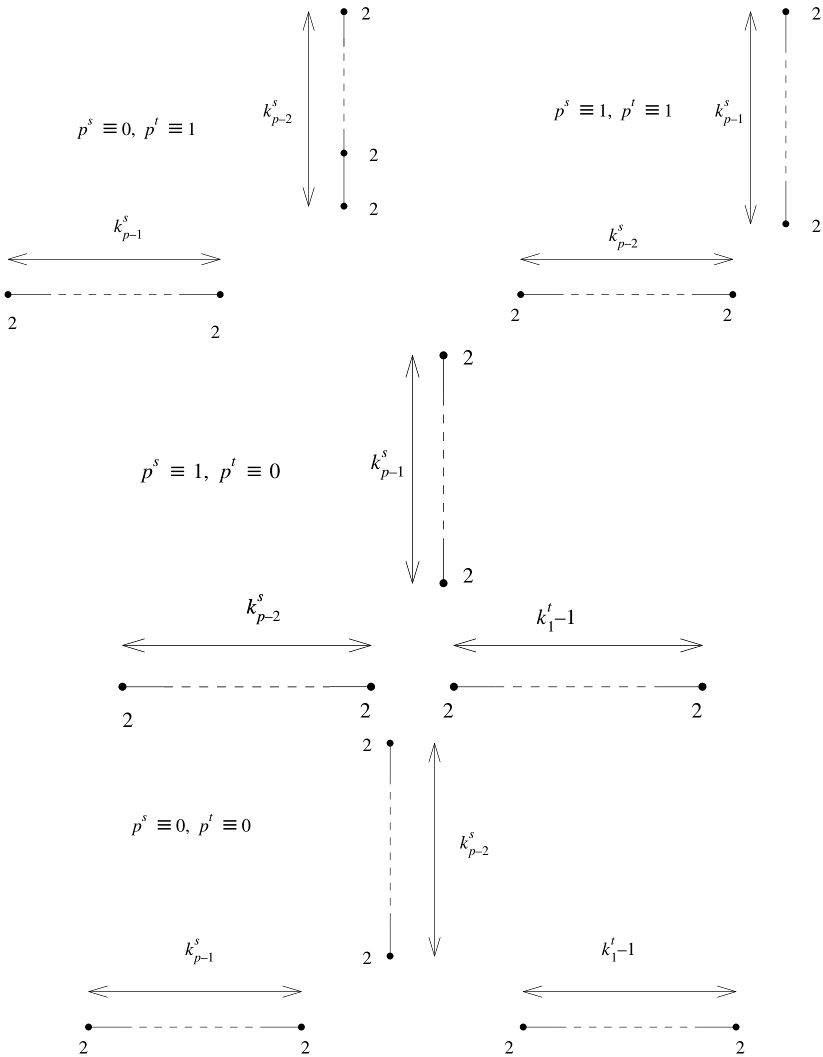




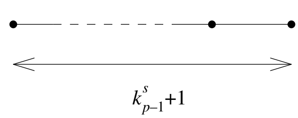
We see that the two only involved vertices are thoose of weight $k^s_{p-1}+2$ and k^t_0+2.

- If s_{k_i} is followed by a regular sequence, i.e. $s_{k_i} = s_{k^s_{p-1}}$ or ($s_{k_i} = s_{k^t_0}$ and $p^t = 1$):

  In the first case the graph of B_i contains one of the subgraphs

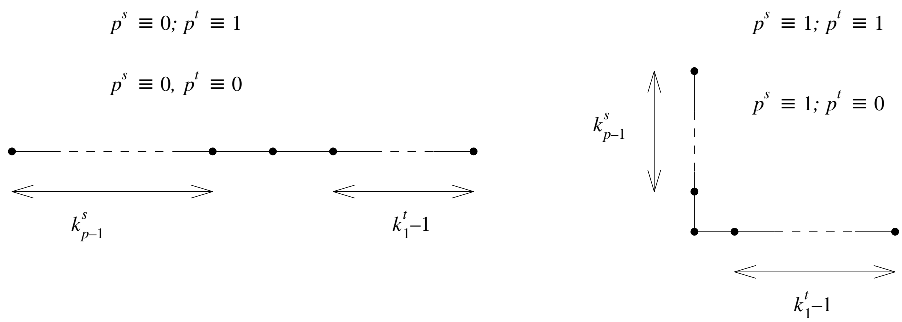


If $p^t = 1$ and p^s is any integer, we have the following connected component



in all these cases deg $B_i = N$ with contribution $\prod_{i=0}^{N-1} k_i$.

- If s_{k_i} is followed by a singular sequence, i.e. $s_{k_i} = s_{k_0^t}$ and (if $p^t \equiv 1$ then $p^t \geqslant 3$): the dual graph contains the following connected components



In all these cases, deg det $B_i = N - 1$.

Finally, we have

$$\begin{aligned} P_\sigma(k_0,\dots,k_{N-1})^2 &= \det M(\sigma) \\ &= \prod_{i=0}^{N-1} k_i \left(\prod_{i=0}^{N-1} k_i + \sum_{i\in A} \prod_{j\neq i} k_j \right) + \sum_{i\in A} \prod_{j\neq i} k_j \prod_{i=0}^{N-1} k_i \\ &= \left(\prod_{i=0}^{N-1} k_i \right)^2 + 2 \sum_{i\in A} \prod_{j\neq i} k_j \prod_{i=0}^{N-1} k_i \\ &\qquad \mod \quad (k_0,\dots,k_{N-1})^{2N-2} \end{aligned}$$

and $P_\sigma = P_A$ by proposition 3.3.4, 3) as wanted. □

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