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On f-rings that are not formally real

James J. Madden[(1)]

To Mel Henriksen for his 80th Birthday

Abstract. — Henriksen and Isbell showed in 1962 that some commutative rings admit total orderings that violate equational laws (in the language of lattice-ordered rings) that are satisfied by all totally-ordered fields. In this paper, we review the work of Henriksen and Isbell on this topic, construct and classify some examples that illustrate this phenomenon using the valuation theory of Hion (in the process, answering a question posed in [E]) and, finally, prove that a base for the equational theory of totally-ordered fields consists of the f-ring identities of the form $0 = 0 \vee (f_1 \wedge \cdots \wedge f_n)$, $n = 1, 2, \ldots$, where $\{ f_1, \ldots, f_n \} \subseteq Z[X_1, X_2, \ldots]$ is not a subset of any positive cone.

Résumé. — Henriksen et Isbell ont montré en 1962 que certains anneaux commutatifs admettent des ordres totaux qui ne vérifient pas les lois equationnelles (dans le language des anneaux réticulés) vérifiées par tous les corps totalement ordonnés. Dans cet article, nous revisitons le travail de Henriksen et Isbell sur ce sujet. En suite nous construisons et classifions quelques exemples qui testifient à ce phenomène utilisant la théorie des valuations de Hion (ce que nous permet, en particulier, de répondre á la question posée dans [E]). Finalement, nous montrons qu'une base pour la théorie equationnelle des corps totalement ordonnés consiste des identités dans les f-anneaux de la forme $0 = 0 \vee (f_1 \wedge \cdots \wedge f_n)$, $n = 1, 2, \ldots$, où $\{ f_1, \ldots, f_n \} \subseteq Z[X_1, X_2, \ldots]$ n'est contenu dans aucun cône positif.

0. Introduction

In the early 20th century, ordered fields appeared in the work of Hilbert (*Grundlagen der Geometrie*), in the work of Hahn (representations of ordered fields) and in the work of Artin and Schreier (on Hilbert's 17th Problem). Since the middle of the 20th century, the study of ordered rings has

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been strongly tied with real algebraic geometry. Reduced rings (*i.e.*, rings without nilpotents) have been the most important algebraic objects in this context. Standard (but by no means trivial) abstract geometric methods allow one to generalize the theory of ordered fields systematically to build up a theory of orderings for reduced rings; see, *e.g.*, [SM].

As stressed by Brumfiel [B], rings with nilpotents arise naturally in geometric settings. As yet, they have not been examined deeply, and understanding the orderings of such rings presents a set of problems quite different from those involved in understanding ordered fields. Relevant work was done in the middle of the last century, when ordered structures were studied abstractly for their own sake. Some landmarks include: a) Birkhoff and Pierce's 1956 work [BP], which defined and studied lattice-ordered rings from the perspective of universal algebra, b) Hion's 1957 work on generalized valuations [Hi], and c) the 1962 work of Henriksen and Isbell [HI] on so-called "formally real" f-rings. Henriksen and Isbell greatly deepened the connections to universal algebra, showing that ordered fields obey equational laws that are not implied by the f-ring identities. Isbell developed the theory further in the very original paper [I], showing that any equational base for the equational theory of totally ordered fields requires infinitely many variables.

Here is a summary of the contents of this paper. Sections 1 through 4 trace ideas relevant to our central theme through the three sources just listed. At the end of section 4, we give an example of a totally-ordered finite-dimensional algebra over $\mathbb{R}$ that violates an order-theoretic law that is satisfied by all totally-ordered fields. Most of the rest of this paper is concerned with understanding how and why this example works and finding more general settings that make sense of a larger class of similar examples. In sections 5 and 6, we examine the example and its generalizations from the perspective of semigroup rings and we exhibit a twisted semigroup ring that shows that the Hion valuation may fail to detect violations of the equations of totally-ordered fields. In the last part of the paper, we prove a new theorem that exhibits a base for the equational theory of totally ordered fields in the language of lattice-ordered rings consisting of equations of a particularly simple form (similar to the equation in the example in section 4).

One area of current research where a better understanding of ordered rings with nilpotents is needed occurs in relation to the Pierce-Birkhoff Conjecture (PBC). This challenging problem was first stated in [BP]; a more recent source is [M]. The approach that seems most promising (to me) requires one to consider the set of all the orderings of a polynomial ring A that agree up to a certain order of vanishing with a given order. To understand the structure of this set, it would be useful to have an explicit statement of the conditions that allow an order on A/I to lift to A, where I is a given ideal that is convex for some given order; I, of course, need not be prime. I hope to develop the connections to the PBC in a future paper.

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1. Basic facts

In this section, we summarize the basic information about ordered rings that we use later. All rings in this paper are commutative and have a multiplicative identity. The zero ring $\{0\}$ is not excluded from consideration, but occasionally it may require special treatment. We leave it to the reader to supply this when needed.

Definition. — *A partially ordered ring – or "poring" for short – is a ring A equipped with a partial order $\leqslant$ that satisfies the following conditions:*

- *for all $a, b, c \in A$, if $a \leqslant b$ then $a + c \leqslant b + c$;*
- *for all $a, b, c \in A$, if $a \leqslant b$ and $0 \leqslant c$ then $ac \leqslant bc$;*
- *for all $a \in A$, $0 \leqslant a^2$.*

If the order is total, we call A a *toring*. A *poring morphism* is an order-preserving ring homomorphism between porings. An ideal I in a toring A is said to be *convex* if for all $x, y \in A$,

$$(0 \leqslant x \leqslant y \ \& \ y \in I) \Rightarrow x \in I.$$

Fact 1. — *The kernel of any poring morphism is convex. If I is convex, then*

$$x + I \leqslant y + I :\Leftrightarrow x \leqslant y$$

unambiguously defines a poring order on A/I.

Fact 2. — *Any toring that is not a domain contains nilpotent elements (for if $0 \leqslant x \leqslant y$ and $xy = 0$, then $x^2 = 0$).*

We recall some facts about positive cones. Let A be a ring. We define a *positive cone* to be a subset $P \subseteq A$ such that

$$P + P \subseteq P\ ,\quad PP \subseteq P\ ,\quad A^2 \subseteq P \quad \& \quad P \cap -P = \{0\}.$$

If $P \cup -P = A$, we say that P is *total.*

Fact 3. — *If $\leqslant$ is a partial ring order, then $P_{\leqslant} := \{\, a \in A \mid 0 \leqslant a\}$ is a positive cone. Conversely, if P is a positive cone, the relation $\leqslant_P$ defined by: $x \leqslant_P y \Leftrightarrow y - x \in P$ is a partial ring order. These operations provide a bijection between the positive cones and the partial ring orders. A ring order is total if and only if its associated cone is total. A homomorphism $\phi : (A, \leqslant_P) \to (B, \leqslant_Q)$ between ordered rings is order-preserving if and only if $\phi(P) \subseteq Q$.*

Fact 4. — *If P is a positive cone in A, $x \in A$ and $(xP) \cap -P = \{0\}$, then $P + xP$ is a positive cone. In particular, if A is a field and $-x \notin P$, then $P + xP$ is a positive cone.*

Fact 5. — *Assume A is a domain and k its field of fractions. If P is a positive cone in A, then $Pk^2 = \{\, p/q \mid p, q \in P\ ,\ q \neq 0\}$ is a positive cone in k and $Pk^2 \cap A$ is a positive cone on A (which may properly contain P).*

From Facts 4 and 5, it follows that any positive cone in a domain that is not total has a proper extension. Moreover, given a non-total positive cone P in a field k and an element not in P, a proper extension that excludes that element exists. Thus:

Fact 6. — *Any positive cone in a domain is contained in a total cone. In a field, every positive cone is the intersection of the total cones that contain it.*

All these facts are well-known and easy to prove directly from the definitions.

2. The work of Hion (1957)

Hion's paper [Hi] defines and studies a natural "generalized valuation" on any toring. We say "generalized" because the Hion valuation takes values in a tomonoid (*i.e.*, a commutative monoid with total order satisfying $x \leqslant y \Rightarrow x + z \leqslant y + z$) rather than in a totally-ordered group. In this section, we shall summarize the parts of Hion's work that are relevant to our present purposes. Note that the "Hion algebra" construction that is defined at the end of this section plays an import ant role in [HI], and is sketched (without attribution) in the middle of page 548. If a is an element of a totally ordered ring, we define $|a| := a$ if $a \geqslant 0$ and $|a| := -a$ if $a < 0$.

Definition. — *Let $(A, \leqslant)$ be a toring and let $a, b \in A$. We say that a and b are archimedean equivalent if there are natural numbers m and n with $|b| \leqslant m|a|$ and $|a| \leqslant n|b|$.*

Archimedean equivalence is an equivalence relation on A. The equivalence class of a is denoted $h(a)$. It is called the *archimedean class of* a. Let $\mathcal{H}(A)$ denote the set of all archimedean classes of A. In $\mathcal{H}(A)$, the following rules define a monoid operation and an order. (We leave to the reader the straightforward verification that these definitions are independent of the representatives chosen.)

- $h(a) + h(b) := h(ab)$;
- $h(a) \leqslant h(b)$: if $|b| \leqslant |a|$.

We call $\mathcal{H}(A)$ the *Hion tomonoid of* A. Note that h reverses order. This conforms to the notational customs of valuation theory. Also note that $\infty := h(0)$ is the largest element of $\mathcal{H}(A)$, and it is absorbing: $\infty + h(a) = \infty$ for all a. (A monoid can have at most one absorbing element, for if x and y are both absorbing, $x = x + y = y$.)

In addition to the properties already mentioned, the reader may easily verify that $\mathcal{H}(A)$ satisfies

- $h(a + b) \geqslant \min\{h(a), h(b)\}$, with equality whenever $h(a) \neq h(b)$.

Thus, h possesses all the properties of a valuation, except that the target is only a monoid, not a group with an absorbing element adjoined, as in a typical valuation.

Next, we identify a property that characterizes the tomonoids that arise as $\mathcal{H}(A)$.

Definition. — *Let H be a tomonoid. We call H a Hion tomonoid if it has a largest element ∞, this element is absorbing and H satisfies the following weak cancellation law: for all $x, y, z \in H$:*

$$x + z = y + z \neq \infty \quad \Rightarrow \quad x = y.$$

Theorem (Hion). — *1) If A is a toring, then $\mathcal{H}(A)$ is a Hion tomonoid. 2) Moreover, for any Hion tomonoid H, there is a toring A such that $\mathcal{H}(A)$ is isomorphic to H.*

Proof of 1). — Suppose $a, b, c \in A$. We prove the contrapositive. Suppose $h(a) \neq h(b)$. Without loss of generality, $h(a) < h(b)$. Then $n|b| \leqslant |a|$ for all $n \in N$, and therefore $n|bc| \leqslant |ac|$ for all $n \in N$. Suppose $h(bc) = h(ac)$. Pick $m \in N$ such that $|ac| \leqslant m|bc|$. Then we get $n|bc| \leqslant m|bc|$ for all $n \in \mathbb{N}$, so $|bc| = 0$, so $h(b) + h(c) = \infty$.

Proof of 2). — If R is a ring and S is a monoid, the monoid ring $R[S]$ is the set of all finite formal sums $r_1X^{s_1} + \cdots + r_nX^{s_n}$, where X is an indeterminate, $r_i \in R$ and $s_i \in S$. Multiplication is defined by the rule $X^sX^t = X^{s+t}$ and distributivity. If S is a tomonoid, then we say $g \in R[S] \setminus \{0\}$ is in *normal form* when it is written with exponents of ascending order:

$$g = r_1X^{s_1} + \cdots + r_nX^{s_n},$$

with $r_i \neq 0$ and $s_1 < s_2 < \cdots < s_n$. Now suppose H is a Hion tomonoid and R is an archimedean toring with no zero-divisors. Let $R[H]^*$ denote the quotient of $R[H]$ obtained by identifying X^∞ with 0, ordered in such a way that an element $r_1X^{h_1} + \cdots$ in normal form is positive iff $r_1 > 0_R$. (The elements of this ring can be written unambiguously as R-linear combinations of elements X^h, where $h \in H \setminus \{\infty\}$.) The Hion condition suffices to show that products of nonnegative elements are nonnegative, as the reader may check. The map $h(r_1X^{h_1} + \cdots) \mapsto h_1 : \mathcal{H}(R[H]^*) \to H$ shows that $R[H]^*$ fulfills the condition required by 2) for A. $\square$

Definition. — *The tomonoid algebra* $R[H]^* := R[H]/(X^\infty)$ *introduced in the proof will be called* the Hion algebra of H over R.

3. Work of Birkhoff and Pierce (1956)

Birkhoff and Pierce initiated the study of f-rings in [BP]. An f-ring is a member of the equational class of rings-with-binary-operation $\vee$, whose laws are the laws of commutative rings together with the following laws for $\vee$:

$$\begin{aligned}
x \vee (y \vee z) &= (x \vee y) \vee z, \\
x \vee y &= y \vee x, \\
x \vee x &= x, \\
(x \vee y) + z &= (x + z) \vee (y + z), \\
(0 \vee z)(x \vee y) &= \big((0 \vee z)\, x\big) \vee \big((0 \vee z)\, y\big).
\end{aligned}$$

In [BP], f-rings that fail to have identity were also considered. The defining identities for such f-rings differ in non-obvious ways from those given above, but these subtleties are not important in the present context.

Any toring is an f-ring with respect to the operation $x \vee y := \max\{x, y\}$, but in general an f-ring need not be totally-ordered. Besides the torings, the most extensively studied examples of f-rings are the rings $C(X)$ of all continuous real-valued functions on a topological space X. In fact, the "f" in "f-ring" stands for "function". Note that every f-ring is a poring, with order given by the relation $x \leqslant y :\Leftrightarrow x \vee y = y$.

An ℓ-ring is a ring endowed with a binary operation $\vee$ that satisfies all but the last of the f-ring identities, above. In [BP], Birkhoff and Pierce showed that every f-ring—but not every ℓ-ring—is both a subring *and* sub-$\vee$-lattice of a product of totally ordered rings. In particular, a lattice-ring identity that is violated by an f-ring is violated by a toring. This provides a way to use the tools of universal algebra to treat questions about torings by rephrasing them as questions about f-rings, and vice versa.

4. Work of Henriksen and Isbell (1962)

Henriksen and Isbell undertook a deep study of the equational theory of f-rings in [HI]. In their work, they did not assume the rings they dealt with to have identity. Actually, a substantial amount of [HI] is devoted to the question of when an identity can be adjoined, but we do not consider this part of their paper here. The results of [HI] concerning the equational laws of ordered fields are no less interesting when a multiplicative identity is assumed as part of the definition of a ring. The arguments from [HI] that we refer to below remain valid under this assumption—with occasional, obvious, minor changes.

A pivotal technical result, Theorem 3.3 of [HI], states that using the defining equations for f-rings, any f-ring word w can be rewritten as a supremum of finitely many infima of finitely many polynomials with integer coefficients:

$$w = \bigvee_{i=1}^{k} \bigwedge_{j=1}^{\ell_i} f_{i,j}\ , \quad f_{i,j} \in \mathbb{Z}[X_1, \dots, X_n].$$

This implies the following

LEMMA ([HI], 3.6). — *Any f-ring identity is equivalent to a conjunction of identities of the form*

$$f_1 \wedge \cdots \wedge f_s \leqslant 0\ , \quad f_i \in \mathbb{Z}[X_1, \dots, X_n].$$

To see this, observe first that $w \leqslant 0$ is equivalent to $w \vee 0 = 0$, so the use of the symbol $\leqslant$ is permissible in stating an equational law. Next, note that $w = 0$ is equivalent to $w \leqslant 0 \;\&\; -w \leqslant 0$. By [HI] 3.3, both w and $-w$ may be written as suprema of infima. Finally, $\bigvee_{i=1}^{k} w_i \leqslant 0$ is equivalent to the conjunction of the equations $w_i \leqslant 0$. □

Theorem 1 [HI]. — *All totally-ordered fields satisfy the same lattice-ring identities, and not all of these identities are implied by the f-ring identities.*

The reader is referred to [HI], Theorem 3.8, for the proof of the first assertion. Later in this section, we will present an example that proves the second. Note that all totally-ordered integral domains satisfy all the lat tice-ring identities of totally-ordered fields (by Fact 5 in §1), but they may satisfy more. For example, in $\mathbb{Z}$ we have $x^2 \vee x = x^2$. See [HI], page 550 for additional examples.

Henriksen and Isbell called an f-ring that satisfies all lattice-ring identities satisfied by a totally-ordered field "formally real". By Theorem 1 and Birkhoff's characterization of equational classes, the formally real f-rings are the f-rings that are f-ring homomorphic images of sub-f-rings of products of copies of $\mathbb{Q}$. In universal algebra, this class is often denoted $\mathbf{HSP}_f(\mathbb{Q})$, the subscript f showing that we are referring to algebras with the operations of f-rings.

It also follows from Theorem 1 by standard facts of universal algebra that the free formally real f-ring $F(E)$ on any set E of generators is the sub-f-ring of the f-ring of all $\mathbb{Q}$-valued functions on $\mathbb{Q}^E$ that is generated by the projections x_e, $e \in E$. This assertion is Theorem 4.4 of [HI]. The functions in $F(E)$ are in fact the $f : \mathbb{Q}^E \to \mathbb{Q}$ such that f is a supremum of finitely many infima of finitely many of polynomials with integer coefficients:

$$f(x) = \bigwedge_{i=1}^{k} \bigvee_{j=1}^{\ell_i} f_{i,j}(x)\,, \quad f_{i,j} \in \mathbb{Z}[\, x_e \mid e \in E \,] \subseteq \mathbb{Q}^{\mathbb{Q}^E}.$$

(Note that statement is modified from the statement in [HI] to accommodate our assumption that rings have a multiplicative identity element.)

Henriksen and Isbell also proved the following characterization of formally real torings.

Theorem 2 [HI]. — *A toring A is formally real (i.e., satisfies all the lattice-ring identities that are true in $\mathbb{Q}$) if and only if for any ring homo-*

morphism $\phi : \mathbb{Z}[\, x_e \mid e \in E\,] \to A$ there is a total cone $T \subseteq \mathbb{Z}[\, x_e \mid e \in E\,]$ whose image under ϕ is contained in the positive cone of A.

This is Theorem 4.7 of [HI]. The "if" direction is easy: given A, there is a surjective ϕ, and endowing the domain of ϕ with an order as specified exhibits A as a homomorphic image of a totally ordered domain. For the "only if" part, we will sketch a proof that is slightly different form the proof given in [HI], though similar in its basic idea. Suppose A is a formally real toring. There is an f-ring surjection $\phi : F(E) \to A$ from some free formally real f-ring onto A. The kernel is an ℓ-prime ℓ-group ideal, hence is contained in a *minimal ℓ-prime ℓ-group ideal*, J, say. But any such ℓ-group ideal is also a ring ideal, as shown in standard references on ordered algebra, *e.g.*, [BKW]. The proof is completed by showing that any minimal prime of $F(E)$ meets the subring $\mathbb{Z}[\, x_e \mid e \in E\,] \subseteq F(E)$ at $\{0\}$ only. This can be seen from the fact that for any minimal prime, there is a prime filter of closed semialgebraic sets (in $\mathbb{Q}^E$) with nonempty interior such that f is in the p rime if and only if f vanishes on an element of the filter; see [DM]. Clearly, 0 is the only element of $\mathbb{Z}[\, x_e \mid e \in E\,]$ that satisfies this condition. Thus, the quotient $F(E)/J$ is in fact a total ordering of the polynomial ring $\mathbb{Z}[\, x_e \mid e \in E\,] \subseteq F(E)$.

In [HI], the authors presented a 9-generator toring that is *not* formally real and that has the additional property that all 8-generator sub-torings are formally real. This shows that at least 9 variables are required to axiomatize the equational theory of formally real f-rings. Their example was a Hion algebra over a tomonoid with 80 elements. In 1972 Isbell [I] showed by generalizing this example that the equational theory of formally real f-rings does not have a base with a finite number of variables. Indeed, for each $n \geqslant 6$, he produced a non-formally-real totally-ordered algebra over $\mathbb{R}$ with n generators in which every subalgebra generated by fewer than n elements is formally real. In a letter to the author dated February 24, 1997, Isbell suggested an example of a non-formally-real toring on 4 generators, admitted not knowing whether an example on 3 generators was possible and speculated that no example on 2 generators would exist.

Example. — Here is a ring-lattice identity true in all totally-ordered fields, but violated in a toring with three generators :

$$0 \;=\; 0 \vee \big(x \wedge y \wedge z \wedge (x^3 - yz) \wedge (y^2 - xz) \wedge (z^2 - x^2y)\big). \qquad (1)$$

This is true in every totally-ordered field, for in a totally-ordered field it is impossible for x, y, z as well as all the binomials to be positive all at once. Indeed, if the variables and the first two binomials are positive, then $x^3 > yz$ and $y^2 > xz$, so $x^3y^2 > xyz^2$. Multiplying by $x^{-1}y^{-1} > 0$, we get $x^2y > z^2$. This makes the last binomial negative. Yet identity (1) is *not* an f-ring identity. Here is a toring in which it fails. Let S be the monoid whose elements are the integers $0, 9, 12, 16, 18, 21, 24, 25, 27, 28, 30, 32$ together with an absorbing element ∞. The monoid operation is standard integer addition, unless $a + b > 32$, in which case we take $a + b = \infty$. Now, order S so that

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32 < 30 < \infty.$$

Note that 32 is not in its "usual" place. It is easy—if tedious—to check that with this order S is a Hion tomonoid. Hence we may form the Hion algebra $\mathbb{R}[S]^*$. If we set $x = X^9$, $y = X^{12}$ and $z = X^{16}$, we get a counterexample to the identity. All the strict inequalities $x^3 > yz$, $y^2 > xz$, and $z^2 > x^2y$ hold, and the right hand side of (1) is equal to $z^2 - x^2y = X^{32} - X^{30} \neq 0$. (Recall that the order in $\mathbb{R}[A]^*$ reverses the order in S.)

This example was found in 1997. (I discussed it with Isbell.) It appears to be one of the simplest examples possible. All the examples that appear in [HI] and [I] are similar to this one in that they are all Hion algebras over Hion tomonoids with peculiar orders. The pathology lies in the monoid.

5. Peculiarities detectable by the Hion valuation

All examples of non-formally real torings up to the present have been Hion algebras over Hion tomonoids with deviant properties. This raises several questions which we examine in the present section and the next: What properties of a Hion tomonoid H are necessary and sufficient for $R[H]^*$ to be formally real? What special properties do the Hion tomonoids of formally real torings possess? If a toring fails to be formally real, must this be manifest in its Hion tomonoid?

DEFINITION. — *Suppose S is a monoid and $K \subseteq S$. K is called a monoid ideal if $k \in K$ & $s \in S \;\Rightarrow\; k + s \in K$. If S is a tomonoid and K is an ideal, we say that K is convex if $x \geqslant y \in K \;\Rightarrow\; x \in K$.*

LEMMA. — *Let S be a tomonoid with convex ideal K. On the set $(S \setminus K) \cup \{\infty\}$, there is a tomonoid operation $+$ defined by*

$$a + b := \begin{cases} a +_S b, & \text{if } a +_S b \notin K; \\ \infty & \text{if } a = \infty \text{ or } b = \infty \text{ or } a +_S b \in K. \end{cases}$$

This tomonoid is denoted S/K and it is called a truncation *of S at K. If $K = \{\, x \mid x \geqslant a \,\}$, we write S/a as an alternative notation for S/K.*

Note that K may be empty, in which case the construction adjoins to S a new element that is absorbing. If S already contains an absorbing element a and K is empty, then in S/K, a is no longer absorbing since $a+\infty=\infty$.

Definition. — *A formally real tomonoid is a tomonoid that is isomorphic (as a tomonoid) to a truncation of a subtomonoid of a totally ordered abelian group.*

Observe that every formally real tomonoid is Hion. The following propositions explain the connection to formally real torings.

Proposition. — *If S is a formally real tomonoid and R is a totally-ordered domain, then $R[S]^*$ is a formally real toring.*

Proof. — If S were a counterexample, $R[S]^*$ would contain finitely many elements violating a tofield identity. These elements would be contained in a subalgebra of $R[S]^*$ of the form $R[S']^*$ with $S' \subseteq S$ finitely generated, providing a finitely generated counterexample. Thus, it suffices to prove the assertion when S is finitely generated. In this case, $S = T/K$ is a truncation of a subtomonoid T of a totally ordered copy of $\mathbb{Z}^n$. Now, $R[T]$ is a totally-ordered domain since it is a subring of the ring of Laurent polynomials $R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Also, $R[T]$ has a natural toring order induced by the order on T. Finally there is a natural, order-preserving surjection $R[T] \to R[S]^*$. $\square$

Proposition. — *If A is a formally real toring, then $\mathcal{H}(A)$ is formally real.*

Remark. — The converse is false. An example will be given in the next section.

Proof. — There is a toring surjection $\phi : D \to A$, where D is a totally ordered domain. Let F be the ordered field of fractions of D. Then $\mathcal{H}(D) \subseteq \mathcal{H}(F)$, and the latter (with ∞ removed) is a totally ordered group. Now it suffices to show that if $h(\phi(x)) = h(\phi(y))$, then either $h(x) = h(y)$ or $h(\phi(y)) = \infty$. Suppose $x, y \in D^+$, $h(x) < h(y)$ and $\phi(y) \neq 0$. Then $x > ny$ for all $n \in \mathbb{Z}$, and $x > z$ for all $z \in \ker \phi$. It follows that $h(\phi(x)) < h(\phi(y))$. $\square$

Putting the previous two propositions together, we get the following,

Corollary. — *Let S be a Hion tomonoid. Then S is formally real as a tomonoid if and only if $\mathbb{R}[S]^*$ is formally real as a toring.* $\square$

There is an important class of tomonoids that is closely related to the formally real tomonoids but is properly larger. A tomonoid is *formally integral* if its order lifts to one – hence to any – free monoid of which it is an image. This property is explored at length in [E]. A formally real tomonoid is formally integral, but not every formally integral tomonoid is Hion and even if it is, it need not be formally real.

Here is an example of a formally integral Hion tomonoid that is not formally real. Let U be the quotient of $\langle 9, 12, 16\rangle/33$ obtained by identifying 30 and 32. Using a, b and c to denote 9, 12 and 16, respectively, we have:

$$U = \{\, 0 < a < b < c < 2a < a+b < 2b < a+c < 3a < b+c < 2a+b = 2c < \infty \,\}.$$

U is a formally integral Hion tomonoid but it is not formally real, since $2a+b < 2c$ in any totally ordered group in which $2b < a+c$ and $3a < b+c$.

6. A non-formally real toring whose Hion tomonoid is formally real

The example presented in this section answers a question that was posed in [E], showing that a toring may fail to be formally real even if its Hion tomonoid is formally real. The example is a twisted monoid algebra. Such objects have been studied under the name "binomial algebras" by Sturmfels and others; see [S]. Years ago, Anderson and Ohm pointed out that the ways of twisting a monoid algebra are classified by the second cohomology of the monoid with coefficients in the group of units of the ring of scalars; see [AO]; this theme will be taken up in a separate paper.

Example. — With $a, b, c \in \mathbb{R}$, let $A := A_{a,b,c} = \mathbb{R}[X, Y, Z]/J$, where

$$J = \langle\, X^3 - aYZ,\ Y^2 - bXZ,\ Z^2 - cX^2Y,\ X^2Z,\ X^4,\ X^3Y \,\rangle.$$

Let x, y and z stand for the residues of X, Y and Z in A. Assigning the degrees 3, 4 and 5 to the variables X, Y and Z, respectively, A is graded by the tomonoid

$$H := \{\, 0, 3, 4, 5, 6, 7, 8, 9, 10, \infty \,\}.$$

We have

$$A = A_0 \oplus A_3 \oplus A_4 \oplus \cdots \oplus A_{10},$$

and $\dim_{\mathbb{R}} A_i = 1$ for $i = 0, 3, 4, \ldots, 10$. Assuming that $0 < a, b, c$, we can totally order A as a ring by requiring that x, y, z be positive (thus determining the order of each graded piece) and extending to A lexicographically. That is, we put

$$0 \leqslant \lambda_0 + \lambda_3 x + \lambda_4 y + \lambda_5 z + \lambda_6 x^2 + \lambda_7 xy + \lambda_8 y^2 + \lambda_9 x^3 + \lambda_{10} x^2 y$$

if all the coefficients vanish or the first non-zero coefficient is positive. Note that the Hion tomonoid of A is H, and H is obviously formally real. (Indeed, $H \cong \langle 3,4,5\rangle/11$.)

Assertion. — *With the order described above, $A_{a,b,c}$ is formally real if and only if $abc = 1$.*

Proof. — If $abc = 1$, then A is isomorphic to the tomonoid algebra $\mathbb{R}[H]^*$, which is a quotient of $\mathbb{R}[t^3, t^4, t^5]$, ordered so that $0 < t \ll 1$. (To see this, let $X = a^{2/5}b^{1/5}\overline{X}$, $Y = a^{1/5}b^{3/5}\overline{Y}$ and $Z = \overline{Z}$; then $J = \left\langle \overline{X}^3 - \overline{YZ},\ \overline{Y}^2 - \overline{XZ},\ \overline{Z}^2 - \overline{X}^2\overline{Y},\ \overline{X}^2\overline{Z},\ \overline{X}^4,\ \overline{X}^3\overline{Y} \right\rangle$.) For the converse, suppose that $abc \neq 1$. Let $\phi : \mathbb{R}[X,Y,Z] \to A$ be defined by $\phi(f) := f + J$. It suffices to show that there is no total order $\leqslant$ on $\mathbb{R}[X,Y,Z]$ such that $f \leqslant g \Rightarrow \phi(f) \leqslant \phi(g)$. We treat the case when $abc < 1$, the case $abc > 1$ admitting an analogous treatment. Pick $\delta > 1$ so that $abc\delta^2 < 1$. If $\leqslant_0$ is a total order on $\mathbb{R}[X,Y,Z]$ preserved by ϕ, then

$$0 <_0 X,\ 0 <_0 Y,\ 0 <_0 Z,$$

$$X^3 <_0 a\delta YZ \text{ and } Y^2 <_0 b\delta XZ,$$

and hence

$$X^2Y <_0 ab\delta^2 Z^2.$$

Thus, $\phi(X^2Y) \leqslant_A ab\delta^2\phi(Z^2) = abc\delta^2\phi(X^2Y) < \phi(X^2Y)$. But this is obviously impossible, so no order on $\mathbb{R}[X,Y,Z]$ preserved by ϕ exists. □

7. The lattice-ring equational theory of totally ordered fields

Can we give the equations defining the class of formally real f-rings explicitly? As mentioned at the beginning of §4, every f-ring identity is equivalent to a conjunction of identities—recall that $x \leqslant y$ abbreviates $x \vee y = y$—of the form

$$f_1 \wedge \cdots \wedge f_s \leqslant 0\ , \quad f_i \in \mathbb{Z}[X_1,\ldots,X_n].$$

Thus, there is an equational base for $\mathbf{HSP}(\mathbb{Q})$ consisting of elements of this form. Can we say more about the equations that are in it?

Lemma. — *Let F be a finite set of non-zero elements of $\mathbb{Z}[X_1,\ldots,X_n]$, and let QF be the subsemiring of $\mathbb{Z}[X_1,\ldots,X_n]$ generated by F together with all squares. Then the following are equivalent:*

1) $QF \cap -QF \neq \{0\}$.

2) F *is not contained in any positive cone in* $\mathbb{Z}[X_1, \ldots, X_n]$.

3) F *is not contained in any total cone in* $\mathbb{Z}[X_1, \ldots, X_n]$.

4) The identity $\bigwedge F \leqslant 0$ *is valid in any totally ordered field.*

Proof. — QF is a positive cone if and only if $QF \cap -QF = \{0\}$. If P is a positive cone and P contains F, then P contains QF. Thus, we have the equivalence of 1) and 2). The equivalence of 2) and 3) is immediate from the fact that every positive cone is contained in a total cone. Now suppose 3); we shall prove 4). Let k be a totally ordered field and suppose $\bigwedge F \leqslant 0$ is violated at $a = (a_1, \ldots, a_n) \in k^n$. Then $f(a) > 0$ for all $f \in F$. By [HI], 4.7 (quoted as Theorem 2 of §4, above) there is a total cone $T \subseteq \mathbb{Z}[X_1, \ldots, X_n]$ that contains F, contradicting our supposition. Finally, suppose that 3) is false; we show that 4) is false. In this case, the field of fractions of $\mathbb{Z}[X_1, \ldots, X_n]$ admits a total order in which $f(X) > 0$ for all $f \in F$, exhibiting a violation of $\bigwedge F \leqslant 0$. □

Remark. — This has the following amusing consequence. Suppose that we have finitely many *non-zero* elements $g_i, f_{i,j} \in \mathbb{Z}[X_1, \ldots, X_n]$ such that

$$0 = \sum_i g_i^2 \prod_j f_{i,j}. \qquad (*)$$

Let $F := \{f_{i,j}\}$. Suppose k is a totally ordered field and let $a \in k^n$. If for one or more of the i, $g_i(a) \neq 0$, then it is immediate from $(*)$ that $\bigwedge F(a) \leqslant 0$. The lemma allows us to say more. Evidently by $(*)$, F satisfies the conditions of the lemma, hence *even if* $g_i(a) = 0$ *for all* i it is still true that $\bigwedge F(a) \leqslant 0$. This can be explained topologically. We seek a contradiction from the assumption that $f_{i,j}(a) > 0$ for all i and j. If this is so, then $f_{i,j}(x) > 0$ for all x in some semialgebraic neighborhood of a. But the zerosets of the g_i are nowhere dense, so near a there is a point where none of the g_i vanish and yet all of the $f_{i,j}$ are strictly positive. But $(*)$ makes this impossible.

THEOREM. — *Let* $\mathcal{E}$ *be the set of* f*-ring identities of the form* $\bigwedge F \leqslant 0$*, where* F *is a finite subset of* $\mathbb{Z}[X_1, X_2 \ldots]$ *that is not contained in any positive cone. Then* $\mathcal{E}$ *is an equational base (i.e., a set of equational axioms) for the formally real* f*-rings.*

Proof. — The lemma above shows that every identity in $\mathcal{E}$ is satisfied by all totally ordered fields. Therefore, to prove the theorem we need only to show that a toring B that satisfies $\mathcal{E}$ is formally real. For this, we use

Theorem 2 of section 4. Let $\mathbb{Z}[X] := \mathbb{Z}[X_e \mid e \in E] \xrightarrow{\beta} B$ be a surjective ring homomorphism. We shall find a total cone T on $\mathbb{Z}[X]$ so that $\beta(T) \subseteq Q$, where Q is the positive cone of B. This will show that B is an order-homomorphic image of a totally ordered domain. To this end, we first establish that every finite subset

$$F \subseteq P_\beta := \beta^{-1}\{\, b \in B \mid b > 0\}$$

is contained in a positive cone. If not, then $\bigwedge F \leqslant 0$ is an equation in $\mathcal{E}$ violated in B, contrary to assumption. But if every finite subset of P_β is contained in a positive cone, then P_β itself is, and hence it's contained in a total cone, T, say. Finally, $\beta(T) \subseteq B_{\geqslant 0}$, for if $\beta(t) < 0$, then $\beta(-t) \in P_\beta$, so $-t \in T$, hence $t \notin T$. $\square$

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