ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XIX, nº S1 (2010), p. 101-110.

<http://afst.cedram.org/item?id=AFST_2010_6_19_S1_101_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIX, n° Spécial, 2010
pp. 101–110

Classes of Commutative Clean Rings

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Abstract. — Let A be a commutative ring with identity and I an ideal of A. A is said to be *I-clean* if for every element $a \in A$ there is an idempotent $e = e^2 \in A$ such that $a - e$ is a unit and ae belongs to I. A filter of ideals, say $\mathcal{F}$, of A is *Noetherian* if for each $I \in \mathcal{F}$ there is a finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I$. We characterize I-clean rings for the ideals 0, $n(A)$, $J(A)$, and A, in terms of the frame of multiplicative Noetherian filters of ideals of A, as well as in terms of more classical ring properties.

Résumé. — Soit A une anneau commutatif unitaire et I and idéal de A. L'anneau A est dit I-propre si pour chaque élément $a \in A$ il existe un idempotent $e = e^2 \in A$ tel que $a - e$ est une unité et que $ae \in I$. Un filtre $\mathcal{F}$ d'idéaux de A est *noetherien* si pour tout $I \in \mathcal{F}$, il existe un idéal finiment engendré $J \in \mathcal{F}$ tel que $J \subseteq I$. Nous caractérisons les anneaux I-propres pour les idéaux 0 $n(A)$, $J(A)$ et A en termes du filtre multiplicatif noetherien des idéaux de A ainsi que en termes de propriétés plus classiques de théorie des anneaux.

1. Introduction

Let A be a commutative ring with identity. We say A is a *clean ring* if every element is the sum of a unit and an idempotent. In this article, we follow the ideas used in the three papers [2], [3], and [6] where the authors used lattice and frame theory to characterize clean rings. For the rest of

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this section we introduce the main tools that will be used to study clean rings. In the subsequent section we characterize certain well known classes of commutative clean rings. Throughout we assume that A is a commutative ring with identity. $J(A)$ and $n(A)$ denote the Jacobson radical and the nilradical of A, respectively.

A frame is a complete distributive lattice $(L, \wedge, \vee, 0, 1)$ which satisfies the strengthened distributive law

$$a \wedge \bigvee_{s \in S} s = \bigvee \{a \wedge s | s \in S\}$$

for all $a \in L$ and all $S \subseteq L$. This equality is known as the *frame law*. We denote the bottom and top elements of a frame by 0 and 1, respectively (and assume that $0 \neq 1$).

A frame is necessarily a pseudocomplemented lattice (in the sense of Birkhoff [4]). In particular, for $a \in L$ the pseudo-complement of a is given by

$$a^{\perp} = \bigvee \{t \in L : t \wedge a = 0\}.$$

An element of the form $a^{\perp}$ is called a *pseudocomplement of* L. When $a \vee a^{\perp} = 1$, we say that a is a *complemented element of* L and that a and $a^{\perp}$ form a complementary pair. It is not true that if $a^{\perp}$ is complemented, then a is complemented. We point out that frames are also known as complete Brouwerian lattices or complete Heyting algebras. They are also very often called locales.

Definition 1.1. — We now recall some basic notions regarding frames. Throughout, L denotes a frame.

(i) Let $c \in L$. We call c *compact* if whenever $c \leqslant \bigvee_{i \in I} a_i$, then there is a finite subset of I, say $\{i_1, \cdots, i_n\}$, such that $c \leqslant a_{i_1} \vee \cdots \vee a_{i_n}$. If the top element of L is compact, then we call L *compact*. Whenever every element of L is the supremum of compact elements, L is called an *algebraic frame*. We denote the set of compact elements of L by $\mathfrak{k}(L)$. This collection is closed under finite joins. When $\mathfrak{k}(L)$ is closed under nonempty finite meets, then we say that L has the *finite intersection property*, or that L satisfies the FIP. A compact algebraic frame that satisfies the FIP is known as a *coherent frame*.

(ii) L is said to be *zero-dimensional* when every element is a supremum of complemented elements. It is straightforward to check that for an algebraic frame being zero-dimensional is equivalent to having the property that every compact element is complemented.

(iii) Suppose L is an algebraic frame. L is called a *projectable frame* if for every $c \in \mathfrak{k}(L)$ the element $c^{\perp}$ is a complemented element of L. L is called *feebly projectable* if for any disjoint $a, b \in \mathfrak{k}(L)$ there exists a $c \in \mathfrak{k}(L)$ such that $c^{\perp}$ is complemented, $a \leqslant c^{\perp}$, and $b \leqslant c^{\perp\perp}$. Every zero-dimensional frame is projectable and every projectable frame is feebly projectable.

Example 1.2. — There are several ways of generating frames from rings. For example, it is known that the collection of ideals of a commutative ring forms a complete lattice which is algebraic. It is a frame precisely when the ring is an arithmetical ring, i.e., its lattice of ideals is distributive. Restricting to the collection of radical ideals (or semiprime ideals) of a ring A, one obtains a coherent frame. (This frame is the main tool in [2, 3].)

Another source of frames arises from studying filters of ideals. Let A be a ring and let $\mathcal{L}(A)$ denote the complete lattice of ideals of A ordered by inclusion. A subset $\mathcal{F} \subseteq \mathcal{L}(A)$ is a *filter of ideals* if the following conditions hold for $I, J \in \mathcal{L}(A)$:

(i) $\emptyset \neq \mathcal{F}$;

(ii) if $I \in \mathcal{F}$ and $I \subseteq J$, then $J \in \mathcal{F}$;

(iii) if $I, J \in \mathcal{F}$, then $I \cap J \in \mathcal{F}$.

The filter $\mathcal{F}$ is called a *multiplicative filter* if

(iv) $IJ \in \mathcal{F}$ whenever $I, J \in \mathcal{F}$.

Definition 1.3. — A filter of ideals $\mathcal{F}$ is said to be *Noetherian* if every ideal belonging to $\mathcal{F}$ contains a finitely generated ideal, which also belongs to $\mathcal{F}$.

We denote the collection of all multiplicative filters of ideals of A by $\mathfrak{M}_A$ partially ordered by inclusion. In [6] it is shown that $\mathfrak{M}_A$ is a coherent frame. Let $\mathfrak{C}_A$ be the subcollection of multiplicative Noetherian filters. We now prove that $\mathfrak{C}_A$ is a coherent frame as well, though we do point out that the technique is slightly different. We will need to use the following notation.

Given any ideal I of A there is a smallest multiplicative filter containing I called the *multiplicative filter generated by* I. Namely,

$$\mathcal{F}_I = \{J \in \mathcal{L}(A) : I^n \subseteq J, \text{ for some } n \in \mathbb{N}\}.$$

THEOREM 1.4. — *$\mathfrak{C}_A$ is a coherent frame and the compact elements are precisely the filters of the form $\mathcal{F}_I$ for some finitely generated ideal $I \in (A)$.*

Proof. —For an arbitrary collection $\{\mathcal{F}_i\}$ of elements of $\mathfrak{C}_A$, it is easily seen that

$$\bigvee \mathcal{F}_i = \{I \in \mathfrak{L}(A) : I_{i_1}...I_{i_n} \subseteq I \text{ for some finitely generated } I_{i_k} \in \mathcal{F}_{i_k}, n \in \mathbb{N}\}.$$

From this we deduce that $\mathfrak{C}_A$ is a complete lattice. Note that the intersection of two Noetherian filters is Noetherian. For if $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{C}_A$ and $I \in \mathcal{F}_1 \cap \mathcal{F}_2$, then there are finitely generated ideals J_1, J_2 in $\mathcal{F}_1, \mathcal{F}_2$, respectively, such that $J_1, J_2 \subseteq I$, hence $J_1 + J_2 \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $J_1 + J_2 \subseteq I$. Therefore, $\mathcal{F}_1 \wedge \mathcal{F}_2 = \mathcal{F}_1 \cap \mathcal{F}_2$.

Next we prove that $\mathfrak{C}_A$ is distributive. We have to show that $\mathcal{F} \wedge (\mathcal{F}_1 \vee \mathcal{F}_2) = (\mathcal{F} \wedge \mathcal{F}_1) \vee (\mathcal{F} \wedge \mathcal{F}_2)$ for arbitrary elements of $\mathfrak{C}_A$. Let $I \in \mathcal{F} \wedge (\mathcal{F}_1 \vee \mathcal{F}_2)$ and $\mathcal{F}$, $\mathcal{F}_1$, and $\mathcal{F}_2 \in \mathfrak{C}_A$, so I must be in $\mathcal{F}$ and in $\mathcal{F}_1 \vee \mathcal{F}_2$. Therefore, there exist finitely generated ideals J, J_1, J_2 in $\mathcal{F}$, $\mathcal{F}_1, \mathcal{F}_2$, respectively, such that $J \subseteq I$, $J_1 J_2 \subseteq I$. Clearly then $(J + J_1)(J + J_2) \subseteq I$. But this implies that $I \in (\mathcal{F} \wedge \mathcal{F}_1) \vee (\mathcal{F} \wedge \mathcal{F}_2)$. Since the other inclusion is trivial we conclude that $\mathfrak{C}_A$ is distributive.

Since a finite product of finitely generated ideals is again finitely generated, it follows that for any finitely generated ideal I of A, $\mathcal{F}_I \in \mathfrak{C}_A$. Moreover, $\mathcal{F}_I$ is a compact element of $\mathfrak{C}_A$, for if $\mathcal{F}_I \leqslant \bigvee \mathcal{F}_i$, then $I_{i_1}...I_{i_n} \subseteq I$ for some finite collection of finitely generated ideals $I_{i_k} \in \mathcal{F}_{i_k}$, hence $\mathcal{F}_I \leqslant \mathcal{F}_{i_1} \vee \cdots \vee \mathcal{F}_{i_n}$. Furthermore, for any $\mathcal{F} \in \mathfrak{C}_A$,

$$\mathcal{F} = \bigvee \{\mathcal{F}_I : I \text{ is finitely generated and } I \in \mathcal{F}\}$$

and thus $\mathfrak{C}_A$ is an algebraic distributive lattice, whence a frame. To see that $\mathfrak{C}_A$ has the FIP observe that for any finitely generated ideals $I, J \in (A)$, $\mathcal{F}_I \wedge \mathcal{F}_J = \mathcal{F}_{I+J}$. Finally, if $\mathcal{F}$ is also compact, then $\mathcal{F} = \mathcal{F}_{I_1} \vee ... \vee \mathcal{F}_{I_n} = \mathcal{F}_{I_1 ... I_n}$ for appropriate finitely generated ideals $I_1, \cdots, I_n \in \mathcal{F}$. This concludes the proof. □

Remark. — We note that $\mathfrak{C}_A$ is a subframe of $\mathfrak{M}_A$, specifically $\mathfrak{C}_A$ is a frame which is a sublattice of $\mathfrak{M}_A$ whose arbitrary union in $\mathfrak{C}_A$ agrees with $\mathfrak{M}_A$.

Lemma 1.5 *The complemented elements of* $\mathfrak{C}_A$ *are precisely the elements of the form* $\mathcal{F}_{Ae}$ *for some idempotent* $e \in A$.

Proof. —Let $\mathcal{F} \vee \mathcal{G} = 1$ and $\mathcal{F} \wedge \mathcal{G} = 0$, then there exist finitely generated ideals $I \in \mathcal{F}$ and $J \in \mathcal{G}$ such that $IJ = 0$. In addition for all $K \in \mathcal{F}$, $K + J = A$, so $I = I(K + J) = IK \subseteq K$ therefore, $\mathcal{F} \subseteq \mathcal{F}_I$, but then $\mathcal{F} = \mathcal{F}_I$. Now it is easy to show that $IJ = 0, I + J = A$ implies that $I = Ae$ for some idempotent e. □

Lemma 1.6. — *Let* $I \in \mathcal{A}(A)$ *be a finitely generated ideal. Then* $\mathcal{F}_I^{\perp} = \{K \in \mathfrak{L}(A)$: *there is a finitely generated ideal* $J \subseteq K$ *such that* $I + J = A\}$.

Proof. — It suffices to note that if J_1, J_2 are finitely generated ideals such that $I + J_1 = A$ and $I + J_2 = A$, then $J_1 J_2$ is finitely generated and $I + J_1 J_2 = A$, hence the above defined set is a multiplicative Noetherian filter. □

2. Clean Rings

Let I be an ideal of A. A is said to be an *I-clean* ring if for every $a \in A$ there is an idempotent e such that $a - e$ is a unit and $ae \in I$. Note that an A-clean ring is precisely a clean ring.

More generally, A is said to be a *weakly I-clean* ring if for every $a \in A$ there is an idempotent e such that $ae \in I$ and at least one of $a + e$ and $a - e$ is a unit. We shall state this by saying that $a \pm e$ is a unit. It turns out that in most cases considered here weakly I-clean is equivalent to I-clean. See [1] for more information on weakly clean rings.

The purpose of this section is to characterize such rings for the ideals 0, $n(A)$, $J(A)$, and A, in terms of the frame of multiplicative Noetherian filters of ideals of A, and in terms of more classical ring properties. In particular, we give a common structure to clean, zero dimensional, and von Neumann regular rings, and add to the results in [5] pp. 10, 11.

Recall that a ring A is *von Neumann regular* if for every $a \in A$ there is an $x \in A$ such that $a = a^2 x$. A ring is *zero-dimensional* if every prime ideal is maximal.

Definition 2.1. — Recall that *Spec*(A) denotes the collection of all prime ideals of A endowed with the hull-kernel or Zariski topology. If I is an ideal of A, then $U(I)$ is the set of all those prime ideals of A that do not contain I. All open sets are determined this way. The complement of $U(I)$ is denoted $V(I)$. For the principal ideal Aa we denote $U(a) = U(Aa)$ and $V(a) = V(Aa)$. $Max(A)$ is the subspace of *Spec*(A) consisting of the maximal ideals of A. Its open and closed sets are $U_M(I) = U(I) \cap Max(A)$ and $V_M(I) = V(I) \cap Max(A)$, respectively.

Theorem 2.2. — *The following are equivalent:*

(1) A is 0-clean.

(2) A is weakly 0-clean.

(3) A is a von Neumann regular ring.

(4) For finitely generated ideals I and J with $\mathcal{F}_I = \mathcal{F}_J$, then $I = J$.

(5) For finitely generated ideals I and J with $V(I) = V(J)$ in Spec(A), then $I = J$.

Proof. — (1) implies (2). Clear.

(2) implies (3). Let $a \in A$. Then $a = u \pm e$ and $ae = 0$ where u is a unit and e an idempotent. Therefore, $a^2 = au$ and so $a^2u^{-1} = a$ whence A is a von Neumann regular ring.

(3) implies (4). It is well known that in a von Neumann regular ring, each finitely generated ideal I is a principal ideal generated by an idempotent, so $I^n = I$ for any natural number n. From this it follows that I is smallest in $\mathcal{F}_I$, hence the assertion is clear.

(4) implies (5). We argue by contradiction. If $I \neq J$, then $\mathcal{F}_I \neq \mathcal{F}_J$, thus say $I^n \not\subseteq J$ for all natural n. Let $a_1, \ldots, a_m$ be a set of generators of I. We need only show that $\{a_i^n\}_n \cap J = \emptyset$ for some a_i and apply Zorn's Lemma. But this must be the case, otherwise there would be an n with $a_i^n \in J$ for all a_i, hence $I^{nm} \subseteq J$, which is a contradiction. We conclude that $V(I) \neq V(J)$.

(5) implies (1). Let $a \in A$, then $V(a) = V(a^2)$ implies $Aa = Aa^2$, so $a = xa^2$ for some x. Clearly xa is an idempotent and $a(1 - xa) = 0$. Moreover, for any maximal ideal M, $a \in M$ iff $1 - xa \notin M$, so $a - (1 - xa)$ is a unit therefore, A is 0-clean. $\square$

Theorem 2.3. — *The following are equivalent:*

(1) A is $n(A)$-clean.

(2) A is weakly $n(A)$-clean.

(3) For every $a \in A$ there is an idempotent e such that $V(a) = U(e)$ in Spec(A).

(4) *For every finitely generated ideal I of A there is an idempotent e such that $V(I) = U(e)$ in Spec(A).*

(5) *For every finitely generated ideal I of A there is an idempotent e such that $I^n = Ae$ for some natural number n.*

(6) *$\mathfrak{C}_A$ is a zero-dimensional frame.*

(7) *For every $a \in A$ there exists a natural number n and an idempotent e such that $Aa^n = Ae$.*

(8) *A is zero dimensional.*

Proof. — (1) implies (2). Clear.

(2) implies (3). Given $a \in A$, let $a = u \pm e$ with $ae \in n(A)$, u a unit and e an idempotent. Let P be a prime ideal, then $ae \in P$. Since P is prime at least one of a and e must be in P, and since u is a unit, at most one of a and e may be in P. This translates into $V(a) = U(e)$.

(3) implies (4). Let $a_1,\ldots,a_n$ be a set of generators for I, and e_i, the corresponding idempotents. Then $V(I) = \cap V(a_i) = \cap U(e_i) = U(e_1 \cdots e_n)$ where $e_1 \cdots e_n$ is an idempotent.

(4) implies (5). We wish to prove that for a finitely generated ideal I, $I^n = A(1-e)$ for some natural number n, and e as in the hypothesis. This would prove our assertion since $1-e$ is an idempotent. Let $a \in I$ and assume that $a^n \notin A(1-e)$ for all natural numbers n. Then $\{a^n\}$ forms a multiplicative closed set disjoint from $A(1-e)$. By a Zorn's Lemma argument, there is a prime ideal P, $a \notin P$ and $A(1-e) \subseteq P$. Thus $e \notin P$ so, by hypothesis, $I \subseteq P$. But this is a contradiction. Therefore, $a^n \in A(1-e)$ for some n. Since I is finitely generated, it follows that $I^n \subseteq A(1-e)$ for some n. Say now that $1-e \notin I^n$, then $1-e \notin I$. Since $\{1-e\}$ is a multiplicative closed set disjoint from I, by the same argument, there is a prime ideal P with $I \subseteq P$ and $1-e \notin P$, so $e \in P$. But this contradicts the hypothesis, so $A(1-e) \subseteq I^n$. This concludes the assertion.

(5) implies (6). All compact elements of $\mathfrak{C}_A$ are of the form $\mathcal{F}_I$, for some finitely generated ideal I. Since $\mathfrak{C}_A$ is algebraic, it suffices to prove that these are complemented. But by hypothesis, $I^n = Ae$, and it is clear from the definition of principal filter that $\mathcal{F}_{I^n} = \mathcal{F}_I$, so $\mathcal{F}_I = \mathcal{F}_{Ae}$. It now follows from Lemma 1.5 that $\mathcal{F}_{Ae}$ is complemented, therefore $\mathfrak{C}_A$ is zero-dimensional.

(6) implies (1). By Lemma 1.5 and the hypothesis, given any finitely generated ideal I, there is an idempotent e such that $\mathcal{F}_I = \mathcal{F}_{Ae}$. In particular

if we let $Aa = I$ for some arbitrary $a \in A$, then for any prime ideal P we have that $a \in P$ iff $P \in \mathcal{F}_{Aa}$ iff $P \in \mathcal{F}_{Ae}$ iff $e \in P$ iff $1 - e \notin P$. Therefore, $a-(1-e) \notin P$ and $a(1-e) \in P$ for all prime ideals P in A, hence $a-(1-e)$ is a unit and $a(1-e) \in n(A)$. In other words A is $n(A)$-clean.

Since (7) and (8) are equivalent ([8], Lemma 5.6), and (5) clearly implies (7), it only remains to show (7) implies (3). Given $a \in A$ and the hypothesis, then for any prime P, $P \in V(a)$ iff $P \in V(a^n)$ iff $P \in V(e)$ iff $P \in U(1-e)$. Therefore, $V(a) = U(1-e)$. $\square$

Theorem 2.4. — *The following are equivalent:*

(1) A is $J(A)$-clean.

(2) A is weakly $J(A)$-clean.

(3) For every $a \in A$ there is an idempotent $e \in A$ such that $V_M(a) = U_M(e)$.

(4) For every finitely generated ideal I of A there is an idempotent $e \in A$ such that $V_M(I) = U_M(e)$.

(5) For every finitely generated ideal I of A there is an idempotent $e \in A$ such that $I + J = A$ iff $Ae \subseteq J$, for any finitely generated ideal J.

(6) $\mathfrak{C}_A$ is a projectable frame.

(7) $A/J(A)$ is von Neumann regular and idempotents lift mod $J(A)$.

Proof. — (1) implies (2). Clear.

(2) implies (3). Given $a \in A$, let $a = u \pm e$ with $ae \in J(A)$, u a unit and e an idempotent. Let M be a maximal ideal, then $ae \in M$. Since M is prime at least one of a and e must be in M, and since u is a unit, at most one of a and e may be in M. This translates into $V(a) = U(e)$.

(3) implies (4). Let $a_1, \ldots, a_n$ be a set of generators for I, and e_i, the corresponding idempotents. Then $V_M(I) = \cap V_M(a_i) = \cap U_M(e_i) = U_M(e_1 \cdots e_n)$ where $e_1 \cdots e_n$ is, of course, an idempotent.

(4) implies (5). Let I, J be finitely generated ideals of A and $V_M(I) = U_M(e)$. If $I + J \neq A$, then there is a maximal ideal M with $I + J \subseteq M$. Now $I \subseteq M$ implies that $e \notin M$, so $e \notin J$. Conversely, assume that $e \notin J$. Note that in this case $J + A(1-e) \neq A$, for otherwise there are $a \in A$, $b \in J$, with $b + a(1-e) = 1$. Multiplying by e gives $e = eb \in J$, a contradiction.

So there is a maximal ideal M with $J + A(1-e) \subseteq M$. But then, e cannot be in M, which means that $I \subseteq M$, so $I + J \subseteq M$. Thus $I + J \neq A$.

(5) implies (6). Given a finitely generated ideal I of A there is an idempotent e such that $I + J = A$ iff $Ae \subseteq J$, for any finitely generated ideal J. Now, by Lemma 1.6, it follows that $\mathcal{F}_I^{\perp} = \mathcal{F}_{Ae}$. Therefore, by Lemma 1.5 and Theorem 1.4, $\mathfrak{C}_A$ is a projectable frame.

(6) implies (1). Let $a \in A$, then by Lemma 1.5, $\mathcal{F}_{Aa}^{\perp} = \mathcal{F}_{Ae}$ for some idempotent e. So by Lemma 1.6 for all finitely generated ideals J of A, $Aa + J = A$ iff $Ae \subseteq J$. Let M be a maximal ideal. Now $a \notin M$ iff $Aa + M = A$ iff $Ae \in M$ iff $e \in M$. So, for every maximal ideal M, $a - e \notin M$ and $ae \in M$, which means $a - e$ is a unit and $ae \in J(A)$, thus A is $J(A)$-clean.

(1) implies (7). By Theorem 2.2, $A/J(A)$ is von Neumann regular. Now suppose that $a + J(A)$ is an idempotent, so $a(a-1) = a^2 - a \in J(A)$, therefore, for any maximal ideal M, $a \in M$ iff $a - 1 \notin M$. Now $a = u + e$ with $ae \in J(A)$ so $a \in M$ iff $e \notin M$ iff $1 - e \in M$. From these equivalences it is clear that $a - (1-e) \in M$ for all maximal ideals, so $a - (1-e) \in J(A)$. Therefore, $a + J(A)$ lifts to $1 - e$.

(7) implies (1). By hypothesis and Theorem 2.2, for any $a \in A$, $a - e + J(A)$ is a unit in $A/J(A)$ and $ae \in J(A)$, so $a - e$ is a unit in A, thus A is $J(A)$-clean. □

Theorem 2.5. — *The following are equivalent:*

(1) A is clean.

(2) The collection $\{U(e) \subseteq Max(A) : e \text{ is idempotent}\}$ is a basis of clopen sets for $Max(A)$.

(3) $\mathfrak{C}_A$ is feebly projectable.

Proof. — (1) is equivalent to (2). See [7].

(2) implies (3). It is well known that $Max(A)$ is compact but with our hypothesis it is also Hausdorff, and therefore $Max(A)$ is a normal Hausdorff space. Assume now that $\mathcal{F}_I \wedge \mathcal{F}_J = 0$ where I, J are finitely generated ideals of A. Since $I + J = A$, there are $a \in I$, $b \in J$ with $a + b = 1$. Therefore, $V(a)$ and $V(b)$ are disjoint closed sets. Since we can separate disjoint closed sets by a clopen set in our base there is an idempotent e with $V(a) \subseteq U(e)$ and $V(b) \subseteq U(1-e)$. We claim now that $a(1-e) + be$ cannot be in any maximal ideal. For suppose $a(1-e) + be \in M$, then $a(1-e), be \in M$. If

$1 - e \notin M$, then $a \in M$, so $M \in U(e)$. But this is a contradiction since one of e and $1 - e$ must belong to M. We reach a similar contradiction if we assume that $e \notin M$, so we conclude that $a(1 - e) + be$ is a unit. From this it is easy to show that $\mathcal{F}_{Aa(1-e)}$ and $\mathcal{F}_{Abe}$ contain $\mathcal{F}_I$ and $\mathcal{F}_J$, respectively, and are complementary.

(3) implies (2). Let $M \in U(a)$, so that there is a $b \in M$ such that $a + b = 1$, so $\mathcal{F}_{Aa} \wedge \mathcal{F}_{Ab} = 0$. By hypothesis and Lemma 1.5 there is an idempotent e such that $\mathcal{F}_{Aa} \leqslant \mathcal{F}_{Ae}$ and $\mathcal{F}_{Ab} \leqslant \mathcal{F}_{A(1-e)}$, so $Ae \subseteq Aa$ and $A(1 - e) \subseteq Ab$, thus $M \in U(e) \subseteq U(a)$. We conclude that the $U(e)$'s form a base for the Zariski topology on $Max(A)$. □

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