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Non-solvable base change for Hilbert modular representations and zeta functions of twisted quaternionic Shimura varieties

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Abstract. — In this paper we prove some non-solvable base change for Hilbert modular representations, and we use this result to show the meromorphic continuation to the entire complex plane of the zeta functions of some twisted quaternionic Shimura varieties. The zeta functions of the twisted quaternionic Shimura varieties are computed at all places.

Résumé. — Dans cet article, nous montrons un changement de base non-résoluble pour certaines représentations modulaires de Hilbert et nous utilisons ce résultat pour établir le prolongement méromorphe à tout le plan complexe des fonctions zêta de certaines variétés de Shimura quaternioniques tordues. Les fonctions zêta des variétés de Shimura quaternioniques tordues sont calculées à toutes les places.

1. Introduction

In the first part of this article we prove the following non-solvable base change for Hilbert modular representations:

Theorem 1.1. — *Let F be a totally real number field, and let π be a cuspidal automorphic representation of weight $k \geqslant 2$ of $GL(2)/F$. Let F' be a finite solvable extension of a totally real number field containing F. Then there exists a number field F'' containing F' which is a solvable extension of a totally real field, such that F'' is Galois over $\mathbb{Q}$, and such that the representation π admits a base change to $GL(2)/F''$. If F' is a totally real number field, then F'' can be chosen to be a totally real number field.*

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To show this theorem, we use some results from Taylor's papers [HSBT] and [T2]. We recall that from Langlands [L] and Arthur-Clozel [AC], we know that if π is an automorphic representation of $\mathrm{GL}(n)/F$, where F is a number field, and F' is a solvable extension of F, then π admits a base change to $\mathrm{GL}(n)/F'$.

In the second part of this article, we compute the zeta function of some "twisted" quaternionic Shimura varieties in terms of automorphic representations, and as an application of Theorem 1.1 we prove that their the zeta function could be meromorphically continued to the entire complex plane and satisfies a functional equation. In [BL], Brylinski-Labesse computed the zeta function of quaternionic Shimura varieties associated to a totally indefinite quaternion algebra D over a totally real field F i.e. all the infinite places of F are unramified in D. In his book [R], Reimann generalized the result in [BL] and computed the semisimple zeta function of quaternionic Shimura varieties associated to indefinite quaternion algebras D. Then in [B], Blasius generalized the result in [R] and obtained the expression of the zeta function of quaternionic Shimura varieties at all places.

More exactly, in the second part of this article, we consider F a totally real field, $O := O_F$ the ring of integers of F and D an indefinite quaternion algebra over F. Let G be the algebraic group over F defined by the multiplicative group $D^\times$ of D and let $\bar{G} := \mathrm{Res}_{F/\mathbb{Q}}(G)$. We fix a prime ideal $\wp$ of O_F, such that $G(F_\wp)$ is isomorphic to $\mathrm{GL}_2(F_\wp)$. Let $S_{\bar{G},\mathbf{K}} = S_{\mathbf{K}}$ be the canonical model of the quaternionic Shimura variety associated to an open compact subgroup $\mathbf{K} := K_\wp \times H$ of $\bar{G}(\mathbb{A}_f)$, where $K_\wp$ is the set of elements of $\mathrm{GL}_2(O_\wp)$ that are congruent to 1 modulo $\wp$, H is an open compact subgroup of the restricted product of $(D \otimes_F F_\mathfrak{p})^\times$ where $\mathfrak{p}$ runs over all the finite places of F, $\mathfrak{p} \neq \wp$ and $\mathbb{A}_f$ is the finite part of the ring of adeles $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$. Then $S_\mathbf{K}$ is a quasi-projective variety defined over a totally real number field E called the canonical field of definition.

The variety $S_\mathbf{K}$ has a natural action of $\mathrm{GL}_2(O/\wp)$ (see §3.2). For H sufficiently small this action is free. We fix such a small group H. If K is a number field, we denote $\Gamma_K := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$. Consider a continuous Galois representation $\varphi : \Gamma_E \to \mathrm{GL}_2(O/\wp)$ and let $S'_\mathbf{K}$ be the variety defined over E obtained from $S_\mathbf{K}$ via twisting by φ composed with the natural action of $\mathrm{GL}_2(O/\wp)$ on $S_\mathbf{K}$ (see §3.2 for details).

From Corollary 11.8 of [R] and Theorem 3 of [B] (see also Propositions 3.3 and 3.4 below), we know that the zeta function $L(s, S_{\mathbf{K}})$ of $S_{\mathbf{K}}$ is given by the formula (see §3.1 for notations):

$$L(s, S_{\mathbf{K}}) = \prod_{\pi} L(s - d'/2, \pi, r)^{m(\pi_\infty)m(\pi_f^{\mathbf{K}})}.$$

Here the product is taken over automorphic cohomological representations π of $\bar{G}(\mathbb{A}_{\mathbb{Q}})$ of weight 2, d' is the dimension of $S_{\mathbf{K}}$, $m(\pi_f^{\mathbf{K}})$ is the dimension of $\pi_f^{\mathbf{K}}$ ($\pi_f^{\mathbf{K}}$ denotes the subspace of $\mathbf{K}$-invariants of π_f), r is a well specified representation of the L-group ${}^L\bar{G}|_{\Gamma_E}$ associated to $\bar{G}$ (see §3.1 for the definition of r) and $m(\pi_\infty)$ will be defined in §3.1.

In this article we obtain the following result (see §3.1 for notations and also Proposition 3.5):

Theorem 1.2. — *The zeta function $L(s, S'_{\mathbf{K}})$ of $S'_{\mathbf{K}}$ is given by the formula:*

$$L(s, S'_{\mathbf{K}}) = \prod_{\pi} L(s - d'/2, \pi, r \otimes (\pi_f^{\mathbf{K}} \circ \varphi))^{m(\pi_\infty)},$$

where the product is taken over automorphic cohomological representations π of $\bar{G}(\mathbb{A}_{\mathbb{Q}})$ of weight 2, such that $\pi_f^{\mathbf{K}} \neq 0$.

If $d' = 1$ or 2 and the field $L := \bar{\mathbb{Q}}^{\mathrm{Ker}(\varphi)}$ is a solvable extension of a totally real field, then the zeta function $L(s, S'_{\mathbf{K}})$ can be meromorphically continued to the entire complex plane and satisfies a functional equation.

The first part of Theorem 1.2 is proved in §3.3 by taking the injective limit of the representations of $\Gamma_E \times \mathbb{H}_K$ on the étale cohomology of Shimura varieties S_K and using some linear algebra. Here K is an open compact subgroup of $\bar{G}(\mathbb{A}_f)$ and $\mathbb{H}_K$ is the Hecke algebra of level K.

The second part of Theorem 1.2 regarding the meromorphic continuation of the zeta function $L(s, S'_{\mathbf{K}})$ is proved in §4. We show using Theorem 1.1 (see Theorem 4.3) that if $d' = 1$ or 2 and π is a representation as in the product of Theorem 1.2 and ω is an Artin representation of Γ_E such that the field $K := \bar{\mathbb{Q}}^{\mathrm{Ker}(\omega)}$ is a solvable extension of a totally real field, then the L-function $L(s, \pi, r \otimes \omega)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation. We prove also the meromorphic continuation and functional equation of $L(s, S'_{\mathbf{K}})$ when $d' \geqslant 3$ and the field $L := \bar{\mathbb{Q}}^{\mathrm{Ker}(\varphi)}$ is a solvable extension of a totally real field, if we assume that some other Langlands L-functions can be meromorphically continued to the entire complex plane and satisfy a functional equation (see Lemma 4.1).

We remark that when $D = \mathrm{M}_2(F)$ the Shimura variety is not compact and in this case we use the l-adic intersection cohomologies of the Baily-Borel compactification of the Shimura variety.

In this article, if π is an automorphic representation of $\bar{G}(\mathbb{A}_{\mathbb{Q}})$, we denote the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ ($\mathbb{A}_F$ is the ring of adeles of F), obtained from π by Jacquet-Langlands correspondence (usually denoted $JL(\pi)$) by the same symbol π.

2. The proof of Theorem 1.1

Let π be a cuspidal automorphic representation of weight $k \geqslant 2$ of $\mathrm{GL}(2)/F$, where F is a totally real field. Then from [T1], we know that there exists a λ-adic representation (for λ a prime of the ring of coefficients $\mathbf{O}$ of π, such that $\lambda | l$ for some rational prime l)

$$\rho_\pi := \rho_{\pi,\lambda} : \Gamma_F \to \mathrm{GL}_2(\mathbf{O}_\lambda) \hookrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_l),$$

that is unramified outside the primes dividing $\mathbf{n}l$, where $\mathbf{n}$ is the level of π. We denote by $\bar{\rho}_\pi = \bar{\rho}_{\pi,\lambda}$ the reduction of $\rho_\pi = \rho_{\pi,\lambda} : \Gamma_F \to \mathrm{GL}_2(\mathbf{O}_\lambda)$ mod λ. An l-adic representation $\rho : \Gamma_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ is called automorphic (or modular) if $\rho \cong \rho_\pi$ for some π as above. Proving Theorem 1.1 is equivalent to proving the potential automorphy of the corresponding l-adic representations.

For $F = \mathbb{Q}$ and $k = 2$, Theorem 1.1 is Theorem 3.7 of [V1]. The proof in [V1] uses the positivity of the density of the set of ordinary primes that is known for cuspidal automorphic representations of $\mathrm{GL}(2)/\mathbb{Q}$. This fact is not known for cuspidal automorphic representations of $\mathrm{GL}(2)/F$ for general totally real field F. To prove Theorem 1.1 for general totally real field F, one uses some results from [HSBT] and [T2].

We say that the automorphic representation π of $\mathrm{GL}(2)/F$, where F is a totally real field, is of CM-*type* if there exists some Galois character $\eta : I_F/F^\times \to \bar{\mathbb{Q}}_l^\times$, where I_F denotes the idele group of F, with $\eta \neq 1$ such that $\pi \cong \pi \otimes \eta$. It is known (see Theorem 7.11 of [G]) that if π is of CM-type, then π admits a base change to $\mathrm{GL}(2)/L$ for any finite extension L/F.

So it is sufficient to prove Theorem 1.1 when the representation π are non-CM. We assume this fact from now on.

We assume first that the field F' from Theorem 1.1 is totally real and prove Theorem 1.1 in this case.

We know the following result (see Theorem 3.1 of [HSBT] (unitary case) and Theorem 3.3 of [T2] for details):

Theorem 2.1. — *Suppose that F is a totally real number field and let $k \geqslant 2$ be an integer. Suppose that $l > \max\{3, k\}$ is a rational prime which is unramified in F. Suppose also that*

$$\rho : \Gamma_F \to GL_2(\bar{\mathbb{Q}}_l)$$

is a continuous odd representation (i.e. $\det \rho(c) = -1$ for each complex conjugation c) which is unramified at all but finitely many primes and satisfies the following proprieties:

1. The image of $\bar{\rho}$ contains $SL_2(\mathbb{F}_l)$;

2. If $w|l$ is a prime of F, then the representation $\rho|_{D_w}$, where D_w is the decomposition group at w, is crystalline with Hodge-Tate weights 0 and $k-1$.

Then there exists a totally real finite extension M/F which is Galois over $\mathbf{Q}$, such that each $\rho|_{\Gamma_M}$ is automorphic.

We want to apply Theorem 2.1 to the l-adic representation $\rho_\pi|_{\Gamma_{F'}}$ (see Theorem 1.1 and the beginning of §2 for notations). From the properties of ρ_π we have that $\det \rho_\pi|_{\Gamma_{F'}}(c) = -1$ for each complex conjugation c.

Since the representation π is non-CM, we know the following result (see Proposition 3.8 of [D]):

Proposition 2.2. — *For l sufficiently large, the image of the residual representation $\bar{\rho}_\pi$ contains $SL_2(\mathbb{F}_l)$.*

Hence we can choose a sufficiently large l such that the images of all representations $\bar{\rho}_\pi$ contain $\mathrm{SL}_2(\mathbb{F}_l)$. Since F' is totally real, the image of $\bar{\rho}_\pi|_{\Gamma_{F'}}$ contains $\mathrm{SL}_2(\mathbb{F}_l)$ (see Proposition 3.5 of [V1]). Thus the condition 1 of Theorem 2.1 is satisfied.

We know the following result (see for example Corollary 2.10 of [D]):

Proposition 2.3. — *Assume that π is a cuspidal automorphic representation of weight $k \geqslant 2$ of $GL(2)/F$, where F is a totally real field. Then* *if l is a sufficiently large rational prime, we have that for each prime v of F dividing l, the l-adic representation $\rho_{\pi,\lambda}|_{D_v}$, where $\lambda|l$ is a prime of the field of coefficients of π and D_v is the decomposition group at v, is crystalline with Hodge-Tate weights 0 and $k-1$.*

Hence for l sufficiently large the condition 2 of Theorem 2.1 is also satisfied and Theorem 1.1 is proved if F' is a totally real number field.

Now we prove Theorem 1.1 for F' a solvable extension of a totally real field F_0 containing F. From Theorem 1.1 applied to the totally real extension F_0 of F, we deduce that there exists a Galois totally real extension F'' of F_0 such that the representation π admits a base change to $\mathrm{GL}(2)/F''$. Then $F'F''$ is a solvable extension of F''. Since F'' is Galois over $\mathbb{Q}$, we deduce that the Galois closure F''' of $F'F''$ over $\mathbb{Q}$ is a solvable extension of F''. Hence from Langlands base change for solvable extensions we get that the representation π admits base changes to $\mathrm{GL}(2)/F'''$ and thus Theorem 1.1 is proved.

3. Quaternionic Shimura varieties

Consider a totally real number field F of degree d over $\mathbb{Q}$ and let D be a quaternion algebra over F. Let $\mathbb{A}_f$ be the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$. We denote by J_F the set of infinite places of F and we identify J_F as a $\Gamma_{\mathbb{Q}}$-set with $\Gamma_F \setminus \Gamma_{\mathbb{Q}}$. Let J'_F be the subset of places of J_F where D is ramified. Let d' :=the cardinal of $J_F - J'_F$. We assume that $d' > 0$, i.e. D is indefinite over F.

Let G be the algebraic group over F defined by the multiplicative group $D^\times$ of D. Consider the algebraic group $\bar{G} := \mathrm{Res}_{F/\mathbb{Q}}(G)$ over $\mathbb{Q}$ defined by the propriety: $\bar{G}(A) = G(A \otimes_{\mathbb{Q}} F)$ for all $\mathbb{Q}$-algebras A. The L-group associated to $\bar{G}$ is defined by the semidirect product:

$$^L\bar{G} :=^L \bar{G}^0 \rtimes \Gamma_{\mathbb{Q}},$$

where $^L\bar{G}^0$ is the product of d copies of $\mathrm{GL}_2(\mathbb{C})$ indexed by elements $\sigma \in \Gamma_F \setminus \Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ acts on $^L\bar{G}^0$ by permuting the factors in the natural way. It is easy to see that $\bar{G}(\mathbb{R})$ is isomorphic to $\mathrm{GL}_2(\mathbb{R})^{d'} \times \mathbf{H}^{\times(d-d')}$, where $\mathbf{H}$ is the algebra of quaternions over $\mathbb{R}$.

For $v \in J_F - J'_F$, we fix an isomorphism of $G(F_v)$ with $\mathrm{GL}_2(\mathbb{R})$. We have $\bar{G}(\mathbb{R}) = \prod_{v \in J_F} G(F_v)$. Let $J := (J_v) \in \bar{G}(\mathbb{R})$, where

$$J_v := \begin{cases} 1 & \text{for } v \in J'_F; \\ 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} & \text{for } v \in J_F - J'_F. \end{cases}$$

Let K_∞ be the centralizer of J in $\bar{G}(\mathbb{R})$. Put

$$X := \bar{G}(\mathbb{R})/K_\infty.$$

It is well known that X is complex analytically isomorphic to $(\mathfrak{H}_\pm)^{d'}$, where $\mathfrak{H}_\pm = \mathbb{C} - \mathbb{R}$. For each open compact subgroup $K \subset \bar{G}(\mathbb{A}_f)$ put

$$S_K(\mathbb{C}) := \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/K.$$

For K sufficiently small, $S_K(\mathbb{C})$ is a complex manifold which is the set of complex points of a quasi-projective variety. In general $S_K(\mathbb{C})$ is not connected and is a finite disjoint union of quotients $\Gamma\backslash{\mathfrak{H}_\pm}^{d'}$, where $\Gamma \subset \bar{G}(\mathbb{Q})$ is a congruence subgroup. The subfield E of $\bar{\mathbb{Q}}$ having the propriety that Γ_E is the stabilizer of the subset $J'_F \subset \Gamma_F \setminus \Gamma_{\mathbb{Q}}$, for the natural right action of $\Gamma_{\mathbb{Q}}$ on $\Gamma_F \setminus \Gamma_{\mathbb{Q}}$, is called the canonical field of definition. It is known (see [DE]) that $S_K(\mathbb{C})$ has a canonical model over E that is denoted by S_K. Then S_K is called a quaternionic Shimura variety. The dimension of S_K is equal to d'.

3.1. Zeta function of quaternionic Shimura varieties

In this section we introduce some notations and we shall expose the computation of the zeta function for quaternionic Shimura varieties following closely [RT].

Let π be an automorphic representation of $\bar{G}(\mathbb{A}_{\mathbb{Q}}) = G(\mathbb{A}_F)$. Then $\pi = \otimes\pi_v$, where the restricted tensor product is taken over all places v of F and π_v is a representation of $G(F_v)$, where F_v is the completion of F at v. An irreducible representation π_v of $\mathrm{GL}_2(F_v)$ is called unramified if π_v contains a nonzero vector that is fixed under $\mathrm{GL}_2(O_v)$, where O_v the completion of the ring of integers O_F at v. For almost all v, the representation π_v is unramified. We define $L(s,\pi) := \prod_v L(s,\pi_v)$, where if π_v is unramified

$$L(s,\pi_v) := \det(1 - Nv^{-s} g(\pi_v))^{-1},$$

and

$$g(\pi_v) = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$$

denotes the Langlands class of π_v.

For a continuous representation $r : {}^L\bar{G}^0 \rtimes \Gamma_E \to \mathrm{GL}_n(\mathbb{C})$ and an automorphic representation $\pi = \bigotimes \pi_v$ of $\bar{G}(\mathbb{A}_{\mathbb{Q}}) = G(\mathbb{A}_F)$, where π_v denotes a representation of $G(F_v)$, one can define the L-function $L(s,\pi,r) := \prod_v L(s,\pi_v,r)$, where if π_v is unramified

$$L(s,\pi_v,r) := \det(1 - Nv^{-s} r(g(\pi_v)))^{-1}.$$

If $\omega : \Gamma_E \to \mathrm{GL}_m(\mathbb{C})$ is an Artin representation, then we denote by the same symbol the representation of ${}^L\bar{G}^0 \rtimes \Gamma_E$ that extends ω and restricts to the trivial representation on ${}^L\bar{G}^0$. Then one can define as above the L-function $L(s, \pi, r \otimes \omega)$.

Consider ${}^L\bar{T}^0$ to be the subgroup of ${}^L\bar{G}^0$ of elements (t_σ) such that for all σ, t_σ is diagonal and let ν be the character of ${}^L\bar{T}^0$ defined by

$$\nu((t_\sigma)) := \prod \nu_\sigma(t_\sigma),$$

$$\nu_\sigma \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} := \begin{cases} a & \text{for } \sigma \in J_F - J'_F; \\ 1 & \text{for } \sigma \in J'_F. \end{cases}$$

Then Γ_E stabilizes the character ν.

We denote by r the finite dimensional representation of ${}^L\bar{G}^0$ whose highest weight with respect to the standard Borel subgroup is ν. Since Γ_E stabilizes ν, the representation r could be uniquely extended to ${}^L\bar{G}^0 \rtimes \Gamma_E$ such that Γ_E acts as the identity on the ν-weight space. Then, the dimension of r is $2^{d'}$. From now on in this paper we fix this representation r.

Let K be an open compact subgroup of $\bar{G}(\mathbb{A}_f)$. If l is a prime number, let $\mathbb{H}_K$ be the Hecke algebra generated by the bi-K-invariant $\bar{\mathbb{Q}}_l$-valued compactly supported functions on $\bar{G}(\mathbb{A}_f)$ under the convolution. If $\pi = \pi_f \otimes \pi_\infty$ is an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$, we denote by π_f^K the space of K-invariants in π_f. The Hecke algebra $\mathbb{H}_K$ acts on π_f^K.

We have an action of the Hecke algebra $\mathbb{H}_K$ and an action of the Galois group Γ_E on the étale cohomology $H^i_{et}(S_K, \bar{\mathbb{Q}}_l)$ and these two actions commute. We say that the representation π is *cohomological* if $H^*(\mathfrak{g}, K_\infty, \pi_\infty) \neq 0$, where $\mathfrak{g}$ is the Lie algebra of K_∞ (the cohomology is taken with respect to $(\mathfrak{g}, K_\infty)$-module associated to π_∞). Then we know (see for example Proposition 1.8 of [RT]):

Proposition 3.1. — *The representation of $\Gamma_E \times \mathbb{H}_K$ on the étale cohomology $H^i_{et}(S_K, \bar{\mathbb{Q}}_l)$ is isomorphic to*

$$\oplus_\pi \sigma^i(\pi) \otimes \pi_f^K,$$

where $\sigma^i(\pi)$ is a representation of the Galois group Γ_E. The above sum is over weight 2 irreducible cohomological automorphic representations π of $\bar{G}(\mathbb{A}_\mathbb{Q})$, such that $\pi_f^K \neq 0$ and the $\mathbb{H}_K$-representations π_f^K are irreducible and mutually inequivalent.

The automorphic representations π which appear in Proposition 3.1 are one-dimensional or cuspidal and infinite-dimensional and we know the following result (see Propositions 1.5 and 1.8 of [RT]):

Proposition 3.2. — *i) If π is infinite-dimensional, then*

$$\dim \sigma^i(\pi) = \begin{cases} 2^{d'} & \text{for} \quad i = d', \\ 0 & \text{for} \quad i \neq d'. \end{cases}$$

ii) If π is one-dimensional, then

$$\dim \sigma^i(\pi) = \begin{cases} \binom{d'}{i'} & \text{for} \quad i = 2i', \\ 0 & \text{for} \quad i \text{ odd}. \end{cases}$$

Fix an isomorphism $i_l : \bar{\mathbb{Q}}_l \to \mathbb{C}$, and define the L-function

$$L(s, S_K) := \prod_\pi \prod_v \prod_i \det(1 - Nv^{-s} i_l(\sigma^i(\pi)(\phi_v)) | H^i_{et}(S_K, \bar{\mathbb{Q}}_l)^{I_v})^{-1^{i+1}},$$

where ϕ_v is a geometric Frobenius element at a finite place v of E and I_v is the inertia group at v (in order to define the local factors at the places of E dividing l one has to use actually the l'-adic cohomology for some $l' \neq l$ and Theorem 3 of [B] which gives us the expression of the local factors of the zeta functions of quaternionic Shimura varieties).

For π cohomological, we define

$$m(\pi_\infty) = \begin{cases} (-1)^{d'} & \text{if } \pi_\infty \text{ is infinite-dimensional;} \\ 1 & \text{if } \pi_\infty \text{ is one-dimensional.} \end{cases}$$

We know (see Corollary 1.10 of [RT]):

Proposition 3.3. — *There exists a finite set S of primes of E, such that for all primes v of E not in S, and for π cohomological of weight 2:*

$$\prod_i det(1 - Nv^{-s} i_l(\sigma^i(\pi)(\phi_v)))^{(-1)^{i+1}} = det(1 - Nv^{-s+(d'/2)} r_v(g(\pi_v)))^{-m(\pi_\infty)},$$

where r_v denotes the restriction of r to ${}^L\bar{G}^0 \rtimes G_v$, and G_v is a decomposition group at v.

3.2. Twisted quaternionic Shimura varieties

Let $\wp$ be a prime ideal of O_F such that $G(F_\wp)$ is isomorphic to $\mathrm{GL}_2(F_\wp)$. Consider $\mathbf{K} := K_\wp \times H$, where $K_\wp$ is the set of elements of $\mathrm{GL}_2(O_\wp)$ that are congruent to 1 modulo $\wp$ and H is some open compact subgroup of the restricted product of $(D \otimes_F F_\mathfrak{p})^\times$, where $\mathfrak{p}$ runs over all the finite places of F, with $\mathfrak{p} \neq \wp$. Then it is well known (see for example [C], Corollary 1.4.1.3) that for H sufficiently small, the group $\mathrm{GL}_2(O/\wp)$ acts freely on $S_\mathbf{K}$. We fix such a small H. Then the action of $\mathrm{GL}_2(O/\wp)$ on

$$S_\mathbf{K}(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/\mathbf{K}$$

can be described in the following way : we have that $\mathrm{GL}_2(O_\wp) \hookrightarrow \bar{G}(\mathbb{A}_\mathbb{Q})$ by $\alpha \mapsto (1, ..., \alpha, 1, ..., 1)$, α at the $\wp$ component. Using the isomorphism $\mathrm{GL}_2(O/\wp) \cong \mathrm{GL}_2(O_\wp)/K_\wp$, the action of an element $g \in \mathrm{GL}_2(O_\wp)$ is given by the right multiplication at the $\wp$-component.

We fix a continuous representation

$$\varphi : \Gamma_E \to \mathrm{GL}_2(O/\wp).$$

Let L be the finite Galois extension of E defined by $L := (\overline{\mathbb{Q}})^{\mathrm{Ker}(\varphi)}$.

Let

$$S' = S_\mathbf{K} \times_{\mathrm{Spec}(E)} \mathrm{Spec}(L).$$

The group $\mathrm{GL}_2(O/\wp)$ acts on $S_\mathbf{K}$. Since $\varphi : \mathrm{Gal}(L/E) \hookrightarrow \mathrm{GL}_2(O/\wp)$, the group $\mathrm{Gal}(L/E)$ acts on $S_\mathbf{K}$. We denote this action of $\mathrm{Gal}(L/E)$ on $S_\mathbf{K}$ by φ'. The Galois group $\mathrm{Gal}(L/E)$ has a natural action on $\mathrm{Spec}(L)$ and we can descend via the quotient process S' to $S'_\mathbf{K}/\mathrm{Spec}(E)$ using the diagonal action

$$\mathrm{Gal}(L/E) \ni \sigma \to \varphi'(\sigma) \otimes \sigma$$

on S'. Thus, we obtain a quasi-projective variety $S'_\mathbf{K}/\mathrm{Spec}(E)$. This is the twisted quaternionic Shimura variety that we mentioned in the title.

3.3. Computation of the zeta function of twisted quaternionic Shimura varieties

We consider the injective limit:

$$V^i := \varinjlim_K H^i_{et}(S_K, \bar{\mathbb{Q}}_l) \cong \varinjlim_K \oplus_\pi U^i(\pi) \otimes_{\bar{\mathbb{Q}}_l} \pi_f^K,$$

where $U^i(\pi)$ is the $\bar{\mathbb{Q}}_l$-space that corresponds to $\sigma^i(\pi)$ (see Proposition 3.1 for notations).

Then the π-component $V^i(\pi)$ of V^i is isomorphic to $\sigma^i(\pi)\otimes\pi_f$ as $\Gamma_E\times\mathbb{H}$-module. Taking the $\mathbf{K}$-fixed vectors, we deduce that $V^i(\pi)^{\mathbf{K}}$ is isomorphic to $\sigma^i(\pi)\otimes\pi_f^{\mathbf{K}}$ as $\Gamma_E\times \mathrm{GL}_2(O/\wp O)$-module. Since the varieties $S_{\mathbf{K}}$ and $S'_{\mathbf{K}}$ become isomorphic over $\bar{\mathbb{Q}}$, we have the isomorphism $H^i_{et}(S_{\mathbf{K}},\bar{\mathbb{Q}}_l)\cong H^i_{et}(S'_{\mathbf{K}},\bar{\mathbb{Q}}_l)$. The actions of Γ_E on these cohomologies that give the expression of the zeta functions of these varieties are different. If we consider the component $V^{i'}(\pi)$ that corresponds to π of $H^i_{et}(S'_{\mathbf{K}},\bar{\mathbb{Q}}_l)$ (see the decomposition of Proposition 3.1), we get that $V^{i'}(\pi)$ is isomorphic to $\sigma^i(\pi)\otimes(\pi_f^{\mathbf{K}}\circ\varphi)$ as Γ_E-module.

We consider a local field L of characteristic 0 and residue characteristic p. Let $W_L\subset\Gamma_L$ be the Weil group. The $Weil-Deligne$ group WD_L of L is defined as the semidirect product of W_L with $\mathbb{C}$ by the relation

$$\sigma z\sigma^{-1}=|\sigma|z$$

for all $\sigma\in W_L$ and $z\in\mathbb{C}$, where $|\ |$ is the norm map: $|\ |:W_L\to q^{\mathbb{Z}}\subset\mathbb{Q}^\times$, where q denotes the cardinality of the residue field of L, and $|\ |=1$ on the inertia group $I_L\subset W_L$ and $|\Phi|=q$, where $\Phi\in W_L$ is an arithmetic Frobenius.

Fix a prime number l different from p. For a vector space V of finite dimension over $\mathbb{Q}_l$, let $\rho:\Gamma_L\to\mathrm{GL}(V)$ be a continuous l-adic representation. We denote also by ρ its restriction to W_L. To the l-adic representation ρ, one can associate (see for example [B] for details) a pair (ρ^*,N) called *Frobenius semisimple parameter* of ρ, where $N\in\mathrm{End}(V)$ is a nilpotent endomorphism and ρ^* is a representation of WD_L having the propriety that $\rho^*|_{W_L}$ is semisimple and for all $\sigma\in W_L$, $\rho^*(\sigma)$ is semisimple.

We know the following result which is a generalization of Proposition 3.3 above (see Theorem 3 of [B]):

Proposition 3.4. — *Let l be a prime number and π be a cuspidal automorphic representation as in Proposition 3.1. Then for each finite place v of E whose residue characteristic p is different from l, the isomorphism class of the Frobenius semisimple parameter $(\rho^*_{K,v},N_v)$ of the Weil-Deligne group WD_{E_v} of E_v defined by the restriction of $\sigma^{d'}(\pi)$ to a decomposition group D_v at v coincides with the class*

$$((r\otimes|\ |^{d'/2})\circ\sigma(\pi_v)),N_{K,v})$$

obtained by the restriction of $(r\otimes|\ |^{d'/2})\circ\sigma$ *to the decomposition group* D_v*, where* $\sigma:\Gamma_E\hookrightarrow^L\bar{G}^o\rtimes\Gamma_E$ *is the inclusion and* $\sigma(\pi_v):WD_{E_v}\rightarrow^L\bar{G}^0\rtimes\Gamma_E$ *is the standard homomorphism.*

Proof. — The idea of the proof of the Proposition 3.4 (for details see [B]) is that the representations $\sigma^{d'}(\pi)$ and $((r\otimes|\ |^{d'/2})\circ\sigma)(\phi_v)$ satisfy (see §5.2 of [B]) the *Weight Monodromy Conjecture* (i.e. the eigenvalues of $\sigma^{d'}(\pi)(\phi_v)$ and $((r\otimes|\ |^{d'/2})\circ\sigma)(\phi_v)$ are *Nv-Weil numbers of weight* d', which means that the eigenvalues are algebraic integers α having the propriety that for each automorphism σ of $\bar{\mathbb{Q}}$, we have $|\sigma(\alpha)|=|Nv|^{d'/2})$ and thus for each v their corresponding nilpotent data $N_{K,v}$ and N_v are uniquely determined (for details see §1.12 of [B]) by the semisimple representations $\rho^*_{K,v}$ and $(r\otimes|\ |^{d'/2})\circ\sigma(\pi_v)$. Hence it is sufficient to prove that

$$\rho^*_{K,v}=(r\otimes|\ |^{d'/2})\circ\sigma(\pi_v).$$

But we know that for almost all v, (i) $N_{K,v}=0$ and $N_v=0$, (ii) $\rho^*_{K,v}$ and $(r\otimes|\ |^{d'/2})\circ\sigma(\pi_v)$ are unramified and from the computation of the unramified zeta function (see [R] and [BL], or Proposition 3.3 above) shows that this formula is true. From Cebotarev theorem, we see that the semisimplification $\sigma^{d'}(\pi)^{ss}$ is isomorphic to $(r\otimes|\ |^{d'/2})\circ\sigma$ and thus their restrictions to the decomposition group D_v for $v\nmid l$ are isomorphic and thus they give rise to isomorphic parameters $\rho^*_{K,v}$ and $(r\otimes|\ |^{d'/2})\circ\sigma(\pi_v)$ and we conclude Proposition 3.4. □

We prove now the following result which implies the first part of Theorem 1.2:

Proposition 3.5. — *Let* l *be a prime number and* π *be a cuspidal automorphic representation as in Proposition 3.1. Then for each finite place* v *of* E *whose residue characteristic is different from* l*, the isomorphism class of the Frobenius semisimple parameter* $(\rho'^*_{\mathbf{K},v},N'_v)$ *of the Weil-Deligne group* WD_{E_v} *of* E_v *defined by the restriction to the decomposition group* D_v *at* v *of* $\sigma^{d'}(\pi)\otimes(\pi_f^{\mathbf{K}}\circ\varphi)$ *coincides with the class of*

$$(((r\otimes|\ |^{d'/2})\otimes(\pi_f^{\mathbf{K}}\circ\varphi))\circ\sigma(\pi_v),N'_{K,v})$$

obtained by the restriction to the decomposition group D_v *of* $((r\otimes|\ |^{d'/2})\otimes(\pi_f^{\mathbf{K}}\circ\varphi))\circ\sigma$*.*

Proof. — From the proof of Proposition 3.4, we know that $\sigma^{d'}(\pi)$ and $(r\otimes|\ |^{d'/2})\circ\sigma$ satisfy the *Weight Monodromy Conjecture*, and thus from Brauer's induction theorem, we get that the representations $\sigma^{d'}(\pi)\otimes$

$(\pi_f^{\mathbf{K}} \circ \varphi)$ and $((r \otimes |\ |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma$ satisfy also the *Weight Monodromy Conjecture* and thus for each v their corresponding nilpotent data N'_v and $N'_{\mathbf{K},v}$ are uniquely determined by the semisimple representations $\rho'^{*}_{\mathbf{K},v}$ and $((r \otimes |\ |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma(\pi_v)$. Thus it is sufficient to show that

$$\rho'^{*}_{\mathbf{K},v} = ((r \otimes |\ |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma(\pi_v).$$

But again we know that for almost all v, (i) $N'_v = 0$ and $N'_{\mathbf{K},v} = 0$, (ii) $\rho'^{*}_{\mathbf{K},v}$ and $((r \otimes |\ |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma(\pi_v)$ are unramified and Proposition 3.3 combined with Brauer's induction theorem shows that this formula is true. From Cebotarev theorem, we see that the semisimplification $(\sigma^{d'}(\pi) \otimes (\pi_f^{\mathbf{K}} \circ \varphi))^{ss}$ is isomorphic to $((r \otimes |\ |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma$ and thus their restrictions to the decomposition group D_v for $v \nmid l$ are isomorphic and thus they give rise to isomorphic parameters $\rho'^{*}_{\mathbf{K},v}$ and $((r \otimes |\ |^{d'/2}) \otimes (\pi_f^{\mathbf{K}} \circ \varphi)) \circ \sigma(\pi_v)$ and we conclude Proposition 3.5. □

4. Meromorphic continuation

As an application to Theorem 1.1, in this section we prove the second part of Theorem 1.2 regarding the meromorphic continuation of zeta functions of the twisted quaternionic Shimura varieties defined in §3.

In this section, under some conditions, we continue meromorphically the zeta function $L(s, S'_{\mathbf{K}})$ to the entire complex plane and show that it satisfies also a functional equation.

Let $\omega = \pi_f^{\mathbf{K}} \circ \varphi$. We define $L := \bar{\mathbb{Q}}^{\mathrm{Ker}(\varphi)}$ and $K := \bar{\mathbb{Q}}^{\mathrm{Ker}(\omega)}$. From now on, in this paper, we assume that L is a solvable extension of a totally real field. Since $K \subseteq L$, the field K is also a solvable extension of a totally real field.

4.1. Definition of the representation $\rho(\pi)$

One can find a representation $\rho(\pi)$ of Γ_E ([BR] §7.2) such that

$$L(s, \rho(\pi)) = L(s - d'/2, \pi, r).$$

We describe now the representation $\rho(\pi)$. Let G be a group and K and H be two subgroups of G. We consider a representation

$$\tau : H \to \mathrm{GL}(W)$$

and a double coset $H\sigma K$ such that $d(\sigma) = |H \setminus H\sigma K| < \infty$. We define a representation $\tau_{H\sigma K}$ of K on the $d(\sigma)$-fold tensor product $W^{\otimes d(\sigma)}$. Consider the representatives $\{\sigma_1, \ldots, \sigma_{d(\sigma)}\}$ such that $H\sigma K = \cup H\sigma_j$. If $\gamma \in K$, then there exists $\xi_j \in H$ and an index $\gamma(j)$ such that

$$\sigma_j \gamma = \xi_j \sigma_{\gamma(j)}.$$

We define the representation:

$$\tau_{H\sigma K}(\gamma)(\omega_1 \otimes \cdots \otimes \omega_{d(\sigma)}) = \tau(\xi_1)\omega_{\gamma^{-1}(1)} \otimes \cdots \otimes \tau(\xi_{d(\sigma)})\omega_{\gamma^{-1}(d(\sigma))}.$$

One can prove easily that the equivalence class of $\tau_{H\sigma K}$ is independent of the choice of the representatives $\sigma_1, \ldots, \sigma_{d(\sigma)}$.

Let $J_F - J'_F = \{\delta_1, \ldots, \delta_{d'}\}$, and $S := \bigcup \Gamma_F \delta_i$. We write S as a disjoint union of double cosets

$$S = \cup_{j=1}^k \Gamma_F \sigma_j \Gamma_E$$

and we denote by ρ_j the representation of Γ_E defined by $\rho_{\pi,\lambda}$, and the double coset $\Gamma_F \sigma_j \Gamma_E$. Then our representation $\rho(\pi)$ is isomorphic to $\rho_1 \otimes \cdots \otimes \rho_k$. Thus

$$L(s - d'/2, \pi, r) = L(s, \rho(\pi)) = L(s, \rho_1 \otimes \cdots \otimes \rho_k)$$

and we obtain also that

$$L(s - d'/2, \pi, r \otimes \omega) = L(s, \rho(\pi) \otimes \omega) = L(s, \rho_1 \otimes \cdots \otimes \rho_k \otimes \omega).$$

4.2. Base change and Brauer's theorem

LEMMA 4.1. — *Let ϕ be an l-adic representation of Γ_E. Suppose that there exists a Galois extension F' of $\mathbb{Q}$, which contains the field $K := \bar{\mathbb{Q}}^{ker(\omega)}$, and that the L-function $L(s, \phi|_{\Gamma_{F''}} \otimes \chi)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation for any subfield F'' of F' containing E such that F' is a solvable extension of F'' and any finite order character χ of $\Gamma_{F''}$. Then $L(s, \phi \otimes \omega)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation.*

Proof. — From Brauer's theorem (see Theorems 16 and 19 of [SE]), we know that there exist some subfields $F_i \subset F'$ such that $\mathrm{Gal}(F'/F_i)$ are solvable, some characters $\chi_i : \mathrm{Gal}(F'/F_i) \to \bar{\mathbb{Q}}^\times$ and some integers m_i, such that the representation

$$\omega : \mathrm{Gal}(F'/E) \to \mathrm{Gal}(K/E) \to \mathrm{GL}_N(\bar{\mathbb{Q}}_l),$$

can be written as $\omega = \sum_{i=1}^{i=k} m_i \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_E} \chi_i$ (a virtual sum). Then

$$L(s, \phi \otimes \omega) = \prod_{i=1}^{i=k} L(s, \phi \otimes \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_E} \chi_i)^{m_i}$$

$$= \prod_{i=1}^{i=k} L(s, \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_E} (\phi|_{\Gamma_{F_i}} \otimes \chi_i))^{m_i} = \prod_{i=1}^{i=k} L(s, \phi|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}.$$

We know from our assumption that the L-functions $L(s, \phi|_{\Gamma_{F_i}} \otimes \chi_i)$ have a meromorphic continuation to the entire complex plane and verify a functional equation. Thus $L(s, \phi \otimes \omega)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation. □

4.3. Meromorphic continuation of the zeta functions for curves and surfaces

In this section we prove the second part of Theorem 1.2, that is a consequence of Theorem 4.3 below.

It is known (Theorem M of [RA1]) that:

Proposition 4.2. — *If π_1 and π_2 are two cuspidal automorphic representations of $GL(2)/L$, where L is a number field, then $\pi_1 \otimes \pi_2$ is an automorphic (isobaric) representation of $GL(4)/L$.*

We prove now the following result:

Theorem 4.3. — *If $K := \bar{\mathbb{Q}}^{\mathrm{Ker}(\omega)}$ is a solvable extension of a totally real field and $d' = 1$ or $d' = 2$, then the function $L(s - d'/2, \pi, r \otimes \omega) = L(s, \rho(\pi) \otimes \omega)$ can be meromorphically continued to the entire complex plane and satisfies a functional equation.*

Proof. — It is sufficient to show that there exists a Galois extension F' of $\mathbb{Q}$, which contains F and K, such that $\rho(\pi)|_{\Gamma_{F'}}$ satisfies the conditions of Lemma 4.1.

We have two cases:

a) $d' = 1$. We assume for simplicity that $J_F - J'_F = \{1\}$, where 1 is the trivial embedding of F in $\bar{\mathbb{Q}}$. In this case $E = F$ and $\rho(\pi) \cong \rho_{\pi,\lambda}$. From Theorem 1.1, we deduce that one can find a field F' which is Galois over $\mathbb{Q}$, and contains F and K, such that $\rho_{\pi,\lambda}|_{\Gamma_{F'}}$ is modular. From Langlands base change for GL(2) for solvable extensions ([L]), we obtain that $\rho(\pi)|_{\Gamma_{F'}}$ satisfies the conditions of Lemma 4.1.

b) $d' = 2$. We assume for simplicity that $J_F - J'_F = \{1, c\}$, where 1 is the trivial embedding of F in $\bar{\mathbb{Q}}$. We denote by the same symbol c the extension of c to $\bar{\mathbb{Q}}$. Then,

$$S = \Gamma_F \cup \Gamma_F c.$$

The stabilizer of S is Γ_E. It is easy to see that the stabilizer of S is equal to $(\Gamma_F c \cap c^{-1}\Gamma_F) \cup (\Gamma_F \cap c^{-1}\Gamma_F c)$. Thus we get

$$\Gamma_E = (\Gamma_F c \cap c^{-1}\Gamma_F) \cup (\Gamma_F \cap c^{-1}\Gamma_F c).$$

We distinguish two cases:

i) $\Gamma_F c \cap c^{-1}\Gamma_F = \emptyset$. Then, $\Gamma_E = \Gamma_F \cap c^{-1}\Gamma_F c$. Thus,

$$F \subset E \subset F^{gal}$$

where F^{gal} is the Galois closure of F. We have

$$S = \Gamma_F \cup \Gamma_F c = \Gamma_F 1 \Gamma_E \cup \Gamma_F c \Gamma_E.$$

If $\gamma \in \Gamma_E$, then

$$\tau_{\Gamma_F 1 \Gamma_E}(\gamma)(\omega_1) = \rho_{\pi,\lambda}(\gamma)(\omega_1)$$

and

$$\tau_{\Gamma_F c \Gamma_E}(\gamma)(\omega_1) = \rho_{\pi,\lambda}(c\gamma c^{-1})(\omega_1).$$

Thus

$$\rho(\pi) \cong \rho_{\pi,\lambda}|_{\Gamma_E} \otimes \rho_{\pi,\lambda}|^c_{\Gamma_E},$$

where

$$\rho_{\pi,\lambda}|^c_{\Gamma_E}(\gamma) = \rho_{\pi,\lambda}(c\gamma c^{-1}).$$

The representation π is one-dimensional or cuspidal and infinite-dimensional. If π is one-dimensional, then $\pi(g) = \rho_\pi(N(g))|N(g)|^{1/2}$, where N is the reduced norm map and $|\ \ |$ denotes the ideles norm and ρ_π is a Hecke character.

From Theorem 1.1, we deduce that one can find a solvable extension of a totally real field F' which is Galois over $\mathbb{Q}$, and contains F and K, such that $\rho_{\pi,\lambda}|_{\Gamma_{F'}}$ is modular. Thus

$$\rho(\pi)|_{\Gamma_{F'}} \cong \rho_{\pi,\lambda}|_{\Gamma_{F'}} \otimes \rho_{\pi,\lambda}|^c_{\Gamma_{F'}}$$

is a tensor product of two automorphic representations and from Langlands base change for GL(2) for solvable extensions ([L]) and Proposition 4.2, we obtain that $\rho(\pi)|_{\Gamma_{F'}}$ satisfies the conditions of Lemma 4.1.

ii) $\Gamma_F c \cap c^{-1}\Gamma_F \neq \emptyset$. Let $\Gamma_{E_1} := \Gamma_F \cap c^{-1}\Gamma_F c$. Thus

$$F \subset E_1 \subset F^{gal}.$$

Since it is obvious now that $\Gamma_{E_1} \subset \Gamma_E$, $[\Gamma_E : \Gamma_{E_1}] = 2$ and $\Gamma_E \nsubseteq \Gamma_F$, we get $[E_1 : E] = 2$ and $F \nsubseteq E$. If $F_1 := E \cap F$, then $[F : F_1] = 2$ and we can easily see that c when restricted to F_1 is the trivial embedding. Hence c is the nontrivial automorphism of F over F_1 and we get that $\Gamma_{E_1} = \Gamma_F \cap c^{-1}\Gamma_F c = \Gamma_F$, which means that $E_1 = F$ and $E = F_1$ and therefore we have $[F : E] = 2$ and c is the nontrivial automorphism of F over E.

We have

$$S = \Gamma_F \cup \Gamma_F c = \Gamma_F 1 \Gamma_E.$$

If $\gamma \in \Gamma_F$, then

$$\tau_{\Gamma_F 1 \Gamma_E}(\gamma)(\omega_1 \otimes \omega_2) = \rho_{\pi,\lambda}(\gamma)\omega_1 \otimes \rho_{\pi,\lambda}(c\gamma c^{-1})\omega_2.$$

If $\gamma \in \Gamma_E - \Gamma_F$, then

$$\tau_{\Gamma_F 1 \Gamma_E}(\gamma)(\omega_1 \otimes \omega_2) = \rho_{\pi,\lambda}(\gamma c^{-1})\omega_2 \otimes \rho_{\pi,\lambda}(c\gamma)\omega_1.$$

Thus $\rho(\pi)$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_F}^{\Gamma_E}(\rho_{\pi,\lambda} \otimes \rho^c_{\pi,\lambda})$$

that satisfies

$$\rho(\pi)|_{\Gamma_F} \cong \rho_{\pi,\lambda} \otimes \rho^c_{\pi,\lambda},$$

and from Theorem [D] of [RA2] we know that $\rho(\pi)$ is automorphic.

As in i) above, we deduce that for some solvable extension of a totally real field F' which is Galois over $\mathbb{Q}$, and contains F and K, the representation $\rho(\pi)|_{\Gamma_{F'}}$ satisfies the conditions of Lemma 4.1. $\square$

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