ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XVII, nº 3 (2008), p. 461-493.

<http://afst.cedram.org/item?id=AFST_2008_6_17_3_461_0>

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Annales de la Faculté des Sciences de Toulouse Vol. XVII, n° 3, 2008
pp. 461–493

Existence, uniqueness and stability for spatially inhomogeneous Becker-Döring equations with diffusion and convection terms[(*)]

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Abstract. — We consider the spatially inhomogeneous Bekker-Döring infinite-dimensional kinetic system describing the evolution of coagulating and fragmenting particles under the influence of convection and diffusion. The simultaneous consideration of opposite coagulating and fragmenting processes causes many additional difficulties in the investigation of spatially inhomogeneous problems, where the space variable changes differently for distinct particle sizes. To overcome these difficulties, we use a modified maximum principle and establishes the local-in-time existence and uniqueness of continuous solutions for unbounded kinetic coefficients that allow their linear growth. The global-in-time existence, uniqueness, and stability theorems for classical solutions are also obtained for bounded kinetic coefficients, and these are based on a new trick, which enables to obtain new a priori estimates for classical solutions regardless of the above mentioned non-uniform change of the spatial variable in the distribution function. We also show that the solutions are stable with respect to small perturbations in l^1 of both initial data and kinetic coefficients. Our methods allow to treat zero diffusion coefficients limit for some sizes of the particles and, moreover, can be employed to prove the vanishing diffusion limit that the solution of the system with diffusion approaches to the solution of the system with the transport terms only. We establish the uniform stability theorems in L^1 for purely coagulating or purely fragmenting kinetic systems. This new stability result is based on the explicit construction of robust Lyapunov functionals and their decay estimates in time.

Résumé. — We consider the spatially inhomogeneous Bekker-Döring infinite-dimensional kinetic system describing the evolution of coagulating and fragmenting particles under the influence of convection and diffusion. The simultaneous consideration of opposite coagulating and fragmenting processes causes many additional difficulties in the investigation of

(*) Reçu le 18 octobre 2007, accepté le 25 septembre 2008
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spatially inhomogeneous problems, where the space variable changes differently for distinct particle sizes. To overcome these difficulties, we use a modified maximum principle and establishes the local-in-time existence and uniqueness of continuous solutions for unbounded kinetic coefficients that allow their linear growth. The global-in-time existence, uniqueness, and stability theorems for classical solutions are also obtained for bounded kinetic coefficients, and these are based on a new trick, which enables to obtain new a priori estimates for classical solutions regardless of the above mentioned non-uniform change of the spatial variable in the distribution function. We also show that the solutions are stable with respect to small perturbations in l^1 of both initial data and kinetic coefficients. Our methods allow to treat zero diffusion coefficients limit for some sizes of the particles and, moreover, can be employed to prove the vanishing diffusion limit that the solution of the system with diffusion approaches to the solution of the system with the transport terms only. We establish the uniform stability theorems in L^1 for purely coagulating or purely fragmenting kinetic systems. This new stability result is based on the explicit construction of robust Lyapunov functionals and their decay estimates in time.

1. Introduction

One of the basic mechanisms in the evolution of droplets, clusters and polymer chains is the mechanism of merging and splitting. Consider the situation where the cluster growth is caused by joining an active monomer to its neighboring clusters, and its decrease is also due to the splitting to a monomer and the remaining cluster. In the kinetic picture, the spatial-temporal evolution of clusters is described by the spatially inhomogeneous Becker-Döring system : For $i \geqslant 1, N \in \{1,2,3\}$,

$$\partial_t c_i + \mathrm{div}_z(v_i c_i) = Q_i(c), \qquad z \in \mathbf{R}^N,\ t > 0. \tag{1.1}$$

Here Q_i is the collision operator of the following form :

$$Q_i(c) = \begin{cases} -k_1 c_1^2 - \displaystyle\sum_{j=1}^{\infty} k_j c_j c_1 + f_2 c_2 + \sum_{j=2}^{\infty} f_j c_j, & i = 1, \\ k_{i-1} c_{i-1} c_1 - k_i c_i c_1 + f_{i+1} c_{i+1} - f_i c_i, & i \geqslant 2, \end{cases}$$

subject to initial condition :

$$c_i(z,0) = c_i^{(0)}(z) \geqslant 0,\ z \in \mathbf{R}^N. \tag{1.2}$$

Here $c_i = c_i(z,t)$ are the nonnegative distribution functions of size (volume) proportional to $i \geqslant 1$ (i-mers) at the spatial point $z \in \mathbf{R}^N$ and the time $t > 0$. The velocities of spatial transport $v_i = v_i(z,t) \in \mathbf{R}^N$ are known and usually determined by the viscosity properties of the medium where the colliding particles move around. Nonnegative kinetic coefficients k_i and f_i reflect the intensity of i-mer and monomer merging and $i+1$-mer splitting respectively, and are assumed to be known from the nature of the process under consideration.

We next discuss some mathematical literatures regarding the system (1.1). The spatially homogeneous problem to (1.1)-(1.2) was intensively studied in [2, 3, 16, 17]. The mathematical theory for more general spatially homogeneous coagulation-fragmentation equations is presented in [7] and the papers cited therein. In contrast, the spatially inhomogeneous transport problem for the Becker-Döring system with the convection term $\mathrm{div}(v_i c_i)$ has not been analyzed yet. The spatially inhomogeneous transport equation (1.1) was studied by Slemrod [18, 19] and some relevant arguments can be also found in [20]. Below, we briefly discuss the existence theory for the system (1.1).

In [11] the existence theorem was studied by Galkin for pure coagulation equation with the convection term. However, this result cannot be applied to the system (1.1) even without fragmentation terms because of the condition $k_1 = 0$ in [11]. Moreover, Galkin's existence theorem was proved in a class of generalized mild solutions. One of our motivation for the current study is to justify their continuity properties. In [5, 7] the existence of a continuous solution to (1.1) was also established provided that initial data are sufficiently small.

In [8] the second author established the global existence of continuous solutions to the continuous version of the coagulation-fragmentation transport equation. However, this result cannot be applied to the system (1.1), because the analysis there was performed for the continuous coagulation-fragmentation equation only, and hence cannot be transformed to the discrete coagulation-fragmentation version like (1.1).[1]

In the last few years, a number of interesting works regarding the system (1.1) supplemented with diffusion term has been appeared (e.g., [4, 6, 14, 21]). This problem mathematically amounts to adding the term $\mathrm{div}_z(d_i \nabla_z c_i)$ to the right-hand side of the system (1.1) :

$$\partial_t c_i + \mathrm{div}_z(v_i c_i) = Q_i(c) + \mathrm{div}_z(d_i \nabla_z c_i), \qquad z \in \mathbf{R}^N, \quad t > 0. \tag{1.3}$$

For the simplicity of presentation, it is assumed that diffusion coefficients d_i are independent of (z, t). This regularizing diffusion term simplifies the

(1) The most of the results obtained for continuous versions of the coagulation–fragmentation models can be easily transformed to the discrete case and vice versa.

analysis, and this is why most of the results in the papers cited are obtained for strictly positive diffusion coefficients d_i and further simplifying assumptions (e.g. decay conditions for the kinetic coefficients) in cases when the zero diffusion $d = 0$ is admitted. Unfortunately, these results cannot be considered for the most interesting case of vanishing diffusion when $d_i \to 0$ (see [4, 6, 14, 21] for the system (1.3) with no convection term). One of our results is to justify the vanishing diffusion limit ($d_i \to 0$). Let us point out that the number of monomers that constitute particles involved in the evolution process is independent in time. Hence the system (1.1)-(1.2) satisfies formally the mass conservation law

$$\int_{\mathbf{R}^N} \sum_{i=1}^{\infty} ic_i(z,t)dz = \int_{\mathbf{R}^N} \sum_{i=1}^{\infty} ic_i^{(0)}(z)dz = \text{const.} \tag{1.4}$$

This fact follows from the property of the collision operator $Q_i(c)$:

$$\int_{\mathbf{R}^N} \sum_{i=1}^{\infty} iQ_i(c)(z,t)dz = 0, \quad t \geqslant 0, \tag{1.5}$$

provided that uniformly with respect to $i \geqslant 1$ $\lim_{|z|\to\infty}(v_i c_i)(z,t) = 0$, and c is such that the summation and integration operations in (1.5) are valid and can be mutually replaced.

The purpose of this paper is to develop the well-posedness theory to the initial value transport problem (1.1)-(1.2), and to establish the vanishing diffusion limit if the right-hand side of system (1.1) is supplied by the diffusion summand. The rest of this paper is organized as follows.

In Section 2 we rewrite the problem (1.1)-(1.2) in the integral form, and define the solution to (1.1)-(1.2) as a continuous function satisfying the corresponding integral equation. We state the results for solutions of approximate truncated systems when the kinetic coefficients k_i, f_i are equal to zero as $i > n$. We also discuss some versions of a maximum principle, and prove a lemma regarding the maximum properties of the collision operator Q.

In Section 3 we consider initial value problem (1.1)-(1.2) with unbounded kinetic coefficients with the sub-linear growth in i and prove local existence, uniqueness, and stability theorem (Theorem 3.2) in a certain time interval depending on initial data and the ratio of coagulation and fragmentation kinetic coefficients.

In Section 4, we present the proof of the global existence and uniqueness theorem (Theorem 4.1) at any fixed time interval $[0, T]$, $T < \infty$, for bounded kinetic coefficients. The stability result is obtained with a stability constant depending on the time interval T. It is worth pointing out that this well-posedness result to the spatially inhomogeneous Becker-Döring equations with the convection term is essentially more complicated than that for spatially homogeneous case with $v_i \equiv 0$. The reasons for this are as follows. First, for spatially homogeneous case the stronger mass conservation law is valid (compare with (1.4))

$$\sum_{i=1}^{\infty} ic_i(t) = \sum_{i=1}^{\infty} ic_i^{(0)} = \text{const.} \tag{1.6}$$

Thanks to this equality, the upper bound estimate for Q_1 with sub-linear kinetic coefficients can be easily obtained as is done in [2] for the spatially homogeneous problem. However, in our case $v_i \not\equiv$ const we do not have an estimate like (1.6), and Theorem 3.2 with local result for unbounded sub-linear coefficients reflects this problem (for spatially homogeneous case the global existence theorem for unbounded kinetic coefficients with sub-linear growth holds). Secondly, in the spatially inhomogeneous case the solution c_i depends on the additional spatial variable. Problem (1.1)-(1.2) written in integral form (2.1) contains functions c_j taken at the spatial point z_i, where the indices i and j have no direct relation. Therefore the detailed analysis given in [2], where the variables are independent of the indices, cannot be applied. It is worth also noting that for spatially inhomogeneous problems with diffusion taken into account the variables are also independent of the indices. Hence, in this sense the problems with diffusion are more "homogeneous" then transport problems with hyperbolic differential operator, which we are concerned with.

In Section 5 we consider the initial value problems for equation (1.3) with the explicit diffusion phenomenon taken into account and prove corresponding existence, uniqueness and stability theorems in the one-dimensional case (Theorem 5.3). It is worth noting that our mixed system may contain both parabolic equations with diffusion term and hyperbolic ones. This case is reasonable if we recall that for large-sized particles, the influence of diffusion can be neglected.

In Section 6 we prove the convergence of the vanishing diffusion solution to the purely convection solution (Theorem 6.1). The existence of such limit solution is proved in sections 4 and 5. The methods employed in sections 3-6 use a priori boundedness estimates based on the maximum principle [7, 8] and an anzatz that allows us to control uniformly the "tails" of infinite series involved in the right-hand side of (1.1).

Finally Section 7 is devoted to the study of uniform L^1 stability estimates, which are valid at the whole time half-axis $t \geqslant 0$ (Theorems 7.3

and 7.6). The constants in these estimates are time-independent, this is the essential difference from the stability results of Section 4. Unfortunately, the results hold for small initial data only. The method is based on the careful treatment of the convection velocities and explicit construction of a nonlinear functional, which is equivalent to the L^1-distance between two solutions. Similar reasonings can be found in Liu-Yang [15] and Ha-Tzavaras [12, 13].

In summary we have proved the unique solvability and stability results for the initial value problems to (1.1), (1.3) along with passage to the zero diffusion limit.

2. Auxiliary properties and statements

Let us fix arbitrary $T > 0$. Let us denote by l^1_λ, $\lambda > 0$, the space of sequences with bounded series $\sum_{i=1}^\infty \lambda^i |c_i|$. Let $\tilde{l}^1 = \bigcup\limits_{\lambda>1} l^1_\lambda$. Notation $c \in l^1_\lambda$ means $\sum_{i=1}^\infty \lambda^i \sup\limits_{z,t} |c_i(z,t)| < \infty$.

Definition 2.1. — *Let $c = (c_i)$ be a solution of the initial value problem (1.1)-(1.2), if and only if $c_i = c_i(z,t)$ is a continuous function in $(z,t) \in \mathbf{R}^N \times [0,T]$ satisfying the following integral equation arising from (1.1)-(1.2) :*

$$c_i(z,t) = c_i^{(0)}(z_i(0)) + \int_0^t Q_i(c)(z_i(s),s)ds, \ i \geqslant 1. \tag{2.1}$$

Here $z_i(s)$ is the value of characteristic curve $dz/dt = v_i(z,t)$ passing through (z,t), $0 \leqslant s \leqslant t$. If v_i are independent of z and t, then $z_i(s) = z - v_i \cdot (t-s)$.

Lemma 2.2. — *Let kinetic coefficients k_i, f_i be non-negative and differ from zero at a finite set of indices i only, $1 \leqslant i \leqslant n$. Let initial distribution $c_i^{(0)}$ be a continuous non-negative function, its integral (1.4) be bounded, $c_i^{(0)} = 0$, $i \geqslant n+1$, and*

$$\sup_{z\in\mathbf{R}^N,\ 1\leqslant i\leqslant\infty} c_i^{(0)}(z) < \infty. \tag{2.2}$$

Let for any $i \geqslant 1$ $v_i \in C^{1,0}_{z,t}(\mathbf{R}^N \times [0,T]) \bigcap L^\infty(\mathbf{R}^N \times [0,T])$ and

$$\inf_{z\in\mathbf{R}^N, 0\leqslant t\leqslant T, i\geqslant 1} \mathrm{div}_z v_i(z,t) \geqslant 0. \tag{2.3}$$

Let the characteristic equation

$$\frac{dz}{dt} = v_i(z,t) \tag{2.4}$$

be uniquely continuously solvable for all $i \geqslant 1$, $z \in \mathbf{R}^N$, $t \in [0,T]$. *Then there exists a unique continuous non-negative solution of the initial value problem (1.1), (1.2)* $c_i \in C(\mathbf{R}^N \times [0,T]) \bigcap C([0,T]; L^1_+(\mathbf{R}^N))$, $1 \leqslant i \leqslant n$. *Also, the mass conservation law (1.4) holds. If (2.15) holds, then we also have* $c_i \in C([0,T], L^\infty(\mathbf{R}^N))$, $i \geqslant 1$.

If initial distribution is summable with weight ϕ_i *uniformly in* $z \in \Omega$ *then this property is conserved for all* $t > 0$ *:*

$$\sum_{i=1}^{\infty} \phi_i \sup_{z\in\mathbf{R}^N} c_i(z,t) < \infty,\ \ 0 \leqslant t \leqslant T. \tag{2.5}$$

If additionally the initial function $c_i^{(0)}$ *is continuously differentiable, then the solution has continuous derivatives in both arguments.*

The lemma's proof is similar to that of [5, 7, 8], it is based on the boundedness of sums in the right-hand side of (1.1) due to the truncation of kinetic coefficients. Using the contraction mapping theorem the local solvability in time is proved first. The non-negativity of the solution can be proved using Lemma 2 from [8]. Then the use of simple boundedness estimates allows us to extend the solution to all times $0 \leqslant t \leqslant T$. The continuous differentiability of the solution along characteristics $z_i(s)$ follows from the continuity of the integrand in (2.1) due to the finite number of summands in the truncated integrand. The summability of the solution with weight ϕ, follows directly from the zero right-hand side of (1.1) at $i > n+1$.

Remark 2.3. — The inequality (2.3) is not actually restrictive because for any $b > 0$ it may be easily replaced by

$$\inf_{z\in\mathbf{R}^N, 0\leqslant t\leqslant T, i\geqslant 1} \mathrm{div}_z v_i(z,t) \geqslant -b.$$

In this case the right-hand sides of all the estimates presented below should be multiplied by e^{bt}, which is bounded at $0 \leqslant t \leqslant T$. For example, the inequality (2.7) below will be replaced by e^{bt} in its right-hand side. So, to clarify the key issues of the paper we have just simplified the estimates without the loss of generality.

We next recall the existence of fundamental solution to the following linearized version of (1.3) due to Aronson [1] and Friedman [9] :

$$\partial_t c_i + \mathrm{div}_z(v_i c_i) = d_i \Delta_z c_i, \qquad i \geqslant 1. \tag{2.6}$$

Below, we denote by C to be a generic positive constant independent of t.

Lemma 2.4. — [1, 9] *Let the conditions of Lemma 2.2 hold and initial data $c^{(0)}$ and coefficient functions $d_i, v_i, \partial_z v_i$ satisfy the following conditions.*

1. *$c^{(0)}$ is continuous and satisfy the growth condition*

$$c^{(0)}(z) \leqslant Ce^{h|z|^2}, \qquad \text{for some } h > 0,$$

2. *Coefficients d_i, v_i and $\partial_z v_i$ are uniformly continuous, bounded in $\mathbf{R}^N \times [0,T]$, and satisfy the uniform Hölder condition with exponent γ, $0 < \gamma \leqslant 1$, with respect to $z \in \mathbf{R}^N$, that is, there exist positive constants C and δ_i such that for any $z', z \in \mathbf{R}^N$ and $i \geqslant 1$,*

$$d_i > 0, \quad |v_i(z',t) - v_i(z,t)| \leqslant C|z'-z|^\gamma,$$
$$|\partial_z v_i(z',t) - \partial_z v_i(z,t)| \leqslant C|z'-z|^\gamma.$$

Then there exists a unique fundamental solution $G_i(z,\xi,t) \equiv G_i(z,t;\xi,0) > 0$ to (2.6) satisfying the following estimate : For $i \geqslant 1$,

$$\int_{\mathbf{R}^N} G_i(z,\xi,t)d\xi \leqslant C, \qquad \text{for some constant} \quad C > 0, \tag{2.7}$$

and there exists a unique continuous non-negative solution of the initial value problem (1.3), (1.2) $c_i \in C^{2,1}_{z,t}(\mathbf{R}^N \times [0,T]) \bigcap C([0,T]; L^1_+(\mathbf{R}^N))$, and this solution can be expressed in the form

$$c_i(z,t) = \int_{\mathbf{R}^N} G_i(z,\xi,t)c_i^{(0)}(\xi)d\xi + \int_0^t\!\!\int_{\mathbf{R}^N} G_i(z,\xi,t-s)Q_i(c)(\xi,s)d\xi ds. \tag{2.8}$$

Also, relations (1.4) and (2.5) hold. If additionally (2.15) holds, then $c_i \in C([0,T]; L^\infty(\mathbf{R}^N))$.

Let us consider the following version of a maximum principle. Let Ω be a compact subset in $\mathbf{R}^N$. We consider the "tube" $T_\Omega(t) \subset \mathbf{R}^N \times [0,t]$, which is formed by all points at $0 \leqslant s \leqslant t \leqslant T$ lying on the characteristics

$$\frac{dz}{dt} = v_i(z,t), \ 1 \leqslant i \leqslant I, \tag{2.9}$$

that finish in Ω at time t :

$$T_\Omega(t) = \{(z_i(s), s) : \ 0 \leqslant s \leqslant t, \ dz_i/ds = v_i(z_i, s), \ z_i(t) \in \Omega, \ 1 \leqslant i \leqslant I\}.$$

Certainly, $T_\Omega(0) = \Omega$. We say that transport velocities v_i satisfy the tube conditions if for any compact Ω and $I < \infty$ the tube $T_\Omega(t)$ is compact and no other characteristic curve, which starts at $t = 0$ outside the tube, can intersect the tube. By other words, if a characteristic curve intersects the tube then it cannot leave the tube in the future and, hence, belong to the tube. Obviously, such tube properties hold if, for example, v_i are independent of z and t for any $i \geqslant 1$.

Lemma 2.5 (Maximum principle). — *Let Ω be a compact simply connected subset of $\mathbf{R}^N$. Let v_i satisfy the tube conditions. Let characteristic equations (2.9) be uniquely solvable for any initial conditions. Let $c_i(z,t)$ be continuous in the tube $T_\Omega(T)$ along with its characteristic derivative dc_i/dt, $1 \leqslant i \leqslant I$. For any fixed $t \in (0,T]$ let (i_0, z_0, t), where $(z_0, t) \in T_\Omega(t)$ and $1 \leqslant i_0 \leqslant I$, be a maximum point, that is,*

$$c_{i_0}(z_0, t) = \max_{1 \leqslant i \leqslant I, \ (z,t) \in T_\Omega(t)} c_i(z,t). \tag{2.10}$$

Suppose that there exists a positive constant $m > 0$ such that whenever $v_{i_0}(z_0, t) \geqslant m$, then

$$\frac{dc_{i_0}(z_0,t)}{dt} \leqslant 0. \tag{2.11}$$

Here d/dt means the derivative along the characteristic curve $z_{i_0}(t)$. Then either the maximum of c does not exceed m, or this maximum is attained at the initial time moment :

$$\max_{1 \leqslant i \leqslant I, \ (z,t) \in T_\Omega(T)} c_i(z,t) = max\Big\{m, \max_{1 \leqslant i \leqslant I, \ (z,0) \in T_\Omega(T)} c_i^{(0)}(z)\Big\}. \tag{2.12}$$

Proof. — To simplify the reasoning, let us consider first the strengthened inequality (2.11) :

$$\frac{dc_{i_0}(z_0,t)}{dt} < 0. \tag{2.13}$$

Then we instantly obtain that there exists a point $t_1 \in (0,t)$ such that $c_{i_0}(z_{i_0}(t_1), t_1) > c_{i_0}(z_{i_0}(t), t)$. However, by the construction, this point belongs to the tube. Hence, this contradicts to the maximum value at (i_0, z_0, t).

Weakening strict inequality (2.13) to (2.11) is based on the consideration of the sequence

$$c_i(z,t) + \frac{1}{l}(T - t), \ l \geqslant 1,$$

and the limit pass as $l \to \infty$. Lemma 2.5 has been proved. □

This proof is similar to that for classical parabolic equation with the replacement of the classical parabolic cylinder by the tube $T_\Omega(t)$. The closest variants of this maximum principle and its proof can be found for pure coagulation equation in [10], and for the coagulation–fragmentation equation in [7, 8]. It is worth noting that the introduction of the constant m as a parameter in the maximum principle is essential because allows us to treat the fragmentation case.

The following assertion follows from Lemma 2.5.

Corollary 2.6. — *Let the conditions of Lemma 2.5 hold with the following change : the basic inequality (2.11) holds in the interior of a compact simply connected domain only. Then there appears an additional option for the maximum value : it can be also attained at the boundary of this domain.*

Lemma 2.7. — *Let the conditions of Corollary 2.6 hold for any parabolic cylinder $\Omega \times [0,T]$. Let function $c_i(z,t)$, $1 \leqslant i \leqslant I$, be non-negative, $z \in \mathbf{R}^1$ and also*

$$\int_{\mathbf{R}^1} c_i(z,t)dz < \infty,\ \ t \geqslant 0,\ \ 1 \leqslant i \leqslant I. \tag{2.14}$$

Then the maximum value in $\mathbf{R}^1 \times [0,T]$ is either less than m or is attained at $t = 0$.

Proof. — Let the maximum in a compact cylinder $\Omega \times [0,T]$ be attained at its boundary $\partial\Omega \times [0,T]$. Let us enlarge the cylinder "bottom" Ω to cover the whole line $\mathbf{R}^1$. During this continuous enlargement the maximum value remains to be achieved on the parabolic boundary. However, such a value can correspond to different indices i and time variable t. If the assertion of the lemma is not valid then the non-negative function

$$\tilde{c}(z,t) = \max_{1 \leqslant i \leqslant I} c_i(z,t)$$

is monotonically increasing. So, its integral over $z \in \mathbf{R}^1$ is infinite. However, in view of (2.14),

$$\int_{\mathbf{R}^1} \tilde{c}(z,t)dz \leqslant \int_{\mathbf{R}^1} \sum_{i=1}^{I} c_i(z,t)dz < \infty.$$

This contradiction proves Lemma 2.7. $\square$

It is worth pointing out that Lemma 2.7 cannot be proved in the multidimensional case because the finite nature of an integral over $\mathbf{R}^2$ does not contradict to the unboundedness of a non-negative function.

LEMMA 2.8 (Maximum property of the collision operator). — *Let there exist a constant $m > 0$ such that*

$$k_i m \geqslant f_i, \; i \geqslant 3, \quad k_2 m \geqslant 2 f_2 \tag{2.15}$$

and $c_i(z,t) \geqslant 0$ at any $i \geqslant 1$. Let (z_0, t_0) be a maximum point of c_1 and let $c_1(z_0, t_0)$ exceed m :

$$c_1(z_0, t_0) = \sup_{z \in T_\Omega(T), I=1, 0 \leqslant t \leqslant T} c_1(z,t) \geqslant m. \tag{2.16}$$

Then

$$Q_1(c)(z_0, t_0) \leqslant 0. \tag{2.17}$$

Let, besides (2.15),

$$k_{i-1} + f_{i+1} \leqslant k_i + f_i, \; i \geqslant 2, \tag{2.18}$$

and $f_i = 0, \; i > n$. Then

$$Q_{i_0}(c)(z_0, t_0) \leqslant 0 \tag{2.19}$$

where (i_0, z_0, t_0) is the maximum point of c in $T_\Omega(t_0)$, $1 \leqslant i \leqslant n$:

$$c_{i_0}(z_0, t_0) = \sup_{1 \leqslant i \leqslant n, (z,t) \in T_\Omega(t_0)} c_i(z,t).$$

Remark 2.9. — Let us note that (2.18) corresponds to (2.15) for linear coefficients $k_i = f_i = i$. As a particular case, (2.18) holds if the coagulation coefficients increase while fragmentation ones decrease : $k_{i-1} \leqslant k_i, \; f_{i+1} \leqslant f_i, \; i \geqslant 2$.

Proof. — We rewrite Q_1 in the following form :

$$Q_1(c)(z,t) = -2k_1 c_1^2(z,t) - (k_2 c_1(z,t) - 2f_2) c_2(z,t) - \sum_{i=3}^{\infty} \Big(k_i c_1(z,t) - f_i \Big) c_i(z,t).$$

Then assertion (2.17) follows from (2.15), (2.16) instantly. To obtain estimate (2.19) we write

$$\begin{aligned} Q_{i_0}(c)(z_0, t_0) &= (k_{i_0 - 1} c_{i_0 - 1} - k_{i_0} c_{i_0}) + (f_{i_0+1} c_{i_0+1} - f_{i_0} c_{i_0}) \\ &\leqslant (k_{i_0-1} - k_{i_0} + f_{i_0+1} - f_{i_0}) c_{i_0}(z_0, t_0). \end{aligned}$$

Using (2.18), we arrive at (2.19). Lemma 2.8 has been proved. □

3. Local existence with unbounded kinetic coefficients

The function $c_1(z,t)$ plays a special role in the collision operator Q. Let us estimate its upper value. The following lemma easily follows from Lemmas 2.5 and 2.8.

LEMMA 3.1. — *Let the conditions of Lemma 2.2 and (2.15) hold. Let $c_i(z,t) \geqslant 0$ be a continuous solution to initial value problem (1.1), (1.2) at any $i \geqslant 1$. Then*

$$c_1(z,t) \leqslant \max\Big\{m, \sup_{z\in\mathbf{R}^N} c_1^{(0)}(z)\Big\} = M_0,\ 0 \leqslant t \leqslant T. \tag{3.1}$$

If, in addition, (2.18) holds then

$$c_i(z,t) \leqslant \sup_{2\leqslant i\leqslant n, z\in\mathbf{R}^N} c_i^{(0)}(z) \leqslant C_0 < \infty,\ 0 \leqslant t \leqslant T. \tag{3.2}$$

Let $c_i^{(n)}$, $n \geqslant 1$, be the solution of truncated initial value problem (1.1), (1.2). To proceed further, we introduce the auxiliary function $g_i^{(n)}(t) = \sup_z c_i^{(n)}(z,t)$ and, using (3.1), we obtain from (2.1)

$$g_i^{(n)}(t) \leqslant g_i^{(0)} + \int_0^t \Big[M_0 k_{i-1} g_{i-1}^{(n)}(s) + f_{i+1} g_{i+1}^{(n)}(s)\Big]\, ds, \quad i \geqslant 2,$$

Here $g_i^{(0)} = \sup_z c_i^{(0)}(z)$. Let us verify that

$$g_i^{(n)}(t) < h_i(t),\ i,n \geqslant 1, \tag{3.3}$$

where

$$\frac{dh_i(t)}{dt} = M_0 k_{i-1} h_{i-1}(t) + f_{i+1} h_{i+1}(t),\ h_i^{(0)} = g_i^{(0)} + \frac{1}{i!}. \tag{3.4}$$

Note, if (3.3) is not valid then in view of $g_i^{(n)}(t) \equiv 0$, $i \geqslant n+2$, there exists the "first" (both in t and i) point (i_0,t_0) where $h_{i_0}(t_0) = g_{i_0}^{(n)}(t_0)$ and $h_{i_0\pm 1}(s) \geqslant g_{i_0\pm 1}^{(n)}(s),\ 0 \leqslant s < t_0$. Then, taking into account (3.4), we obtain

$$g_{i_0}^{(n)}(t_0) < h_{i_0}^{(0)} + \int_0^t [M_0 k_{i-1} h_{i_0-1}(s) + f_{i_0+1} h_{i_0+1}(s)]\, ds = h_{i_0}(t),$$

and thus we arrive at the contradiction $g_{i_0}(t_0) \neq h_{i_0}(t_0)$. This proves (3.3). We need to demonstrate the existence of such functions h_i, $1 \leqslant i < \infty$, and to find an upper estimate for the majorant functions h_i. Let us introduce the generating function $H(\lambda, t) = \sum_{i=1}^{\infty} \lambda^i h_i(t)$ and impose the following sub-linear conditions on the kinetic coefficients

$$k_i \leqslant K \cdot i, \ f_i \leqslant F \cdot i, \ i \geqslant 1. \tag{3.5}$$

We assume that there exists $\Lambda > 1$ such that

$$H_0(\lambda) = \sum_{i=1}^{\infty} \lambda^i h_i^{(0)} < \infty, \ 0 < \lambda \leqslant \Lambda, \ \Lambda \leqslant \infty.$$

We multiply (3.4) by λ^i and sum the equations over $2 \leqslant i < \infty$. Then for $0 < \lambda < \Lambda$ we obtain

$$H_t'(\lambda, t) = \lambda h_1'(t) + \left(M_0 K \lambda^2 + F\right) H_\lambda'(\lambda, t), \ H\Big|_{t=0} = H_0(\lambda). \tag{3.6}$$

This problem satisfies the conditions of the Cauchy-Kovalevskaya theorem and possesses a unique analytic local in time solution. Hence, the local existence of functions h_i, $1 \leqslant i < \infty$ holds. To extend the function $H(\lambda, t)$ for larger time interval we point out that any solution of the corresponding characteristic equation

$$\frac{d\lambda}{dt} = -(M_0 K \lambda^2 + F),$$

starting from λ_0 at $t = 0$, intersects the line $\lambda = 1$ not later than

$$t_1 = \frac{1}{\sqrt{M_0 K F}} \left(\frac{\pi}{2} - \arctan \sqrt{M_0 K / F}\right) \tag{3.7}$$

So, this way allows us to obtain the uniform (in n) upper estimate for the solutions at the finite time interval $0 \leqslant t < t_1$ only. Unfortunately, we cannot say for sure if this fact reflects the real absence of the global in time solution or not. We have picked up $\lambda = 1$ as the lowest value of λ since the desired solution must at least be summable with weight i (to ensure, in view of (3.5), the boundedness of series in the right-hand side of (1.1)). We obtain from (3.4), (3.6) the upper estimate

$$\sup_{0 \leqslant t \leqslant t_1, \ 0 \leqslant \lambda \leqslant \lambda(t)} H(\lambda, t) \leqslant \sup_{1 \leqslant \lambda \leqslant \Lambda} H_0(\lambda) + \Lambda \cdot (M_0 + 1) = C_2.$$

Hence, at $t \in [0, t_1)$

$$c_i(z, t) \leqslant \frac{C_2}{\lambda^i}, \ z \in \mathbf{R}^N, \ t \in [0, t_1), \ \lambda > 1, \ i \geqslant 1. \tag{3.8}$$

We are in position now to formulate the following theorem.

THEOREM 3.2. — *Let the conditions of Lemma 2.2 for $c^{(0)}$, v hold and*

$$\sum_{i=1}^{\infty} \lambda^i \sup_{z \in \mathbf{R}^N} c_i^{(0)}(z) < \infty, \ 1 < \lambda \leqslant \Lambda, \ \Lambda \leqslant \infty. \tag{3.9}$$

Let (2.15), (3.5) hold, v_i, $1 \leqslant i < \infty$, satisfies the tube conditions and be such that $\forall\ \varepsilon > 0\ \exists\ \delta > 0\ \forall z', z \in \mathbf{R}^N$ if $|z' - z| < \delta$ then

$$|z_i'(t) - z_i(t)| < \varepsilon, \ i \geqslant 1, \ 0 \leqslant t \leqslant T. \tag{3.10}$$

Then there exists a unique local in time continuous solution of the initial value problem (1.1), (1.2)

$$c \in \tilde{l}^1, \ c_i \in C([0, t_1], L_+^1(\mathbf{R}^N) \bigcap L^\infty(\mathbf{R}^N)) \bigcap C(\mathbf{R}^N \times [0, t_1]), \ i \geqslant 1.$$

The solution is continuously differentiable along the characteristics and satisfies the mass conservation law (1.4). It is stable with respect to the perturbations of initial data and kinetic coefficients in l_λ^1, $\lambda > 1$. The time interval of the existence depends on the initial data, it can be estimated by (3.7).

Remark 3.3. — Condition (3.10) holds, e.g., for v_i independent of z and t, in this case $\delta = \varepsilon$.

Proof of Theorem 3.2. — As we have already pointed out, in view of Lemma 2.2 we construct the sequence of solutions $\{c^{(n)}\}$ of problem (1.1), (1.2) with kinetic coefficients equal to zero at $i > n$. The elements of this sequence satisfy (3.8) uniformly because this estimate is independent of n. Hence, the functions $\{c^{(n)}\}$ are not only bounded but also uniformly summable. Let us consider the continuity modulus $|c_i^{(n)}(z', t') - c_i^{(n)}(z, t)|$ on the compact set $1 \leqslant i \leqslant I, \ |z| \leqslant Z, \ 0 \leqslant t \leqslant T$. Using (2.1) we obtain that continuity modulae $|c_i^{(n)}(z', t) - c_i^{(n)}(z, t)|$ and $|c_i^{(n)}(z, t') - c_i^{(n)}(z, t)|$ satisfy the Gronwall inequality with respect to continuity modulae of initial function $c^{(0)}$. Then the uniform smallness of "tails" of infinite sums is guaranteed by estimate (3.8) :

$$\sum_{i=k}^{\infty} c_i^{(n)}(z, t) \leqslant C_2 \sum_{i=k}^{\infty} \frac{1}{\lambda^i} \to 0, \ k \to \infty. \tag{3.11}$$

Hence, the sequence $\{c^{(n)}\}$ is equicontinuous. Using the Arzela theorem we obtain its limit point on a certain compact set. Then we twin the compacta, repeat the reasonings, and so on. Using the standard diagonal process we choose a subsequence that converge as $n \to \infty$ to a continuous function $c_i(z, t)$ for all $i \geqslant 1$, $z \in \mathbf{R}^N$, $0 \leqslant t \leqslant T$. We should now demonstrate that this limit function really satisfies (2.1), that is, that the limit pass under the infinite sum is possible. However, this fact follows from (3.8) and the estimate, which is similar to (3.11) :

$$\sum_{i=k}^{\infty}(k_i+f_i)c_i^{(n)}(z,t)\leqslant 2C_2\sum_{i=k}^{\infty}\frac{i}{\lambda^i}\to 0,\ k\to\infty\ (\lambda>1).$$

This proves existence. Since the solution $c_i(z,t)$ is continuous for any $i\geqslant 1$ and (3.8) holds then the integrand in (2.1) is continuous. Hence, the solution c_i is, in addition, continuously differentiable along the characteristics for any $i\geqslant 1$. The mass conservation law (1.4) follows from the correctness of all operations leading to (1.4). Such a correctness is due to (3.8). To prove uniqueness and stability let us consider two solutions of (1.1), (1.2) with different initial data. We denote the modulus of their difference by $u_i(z,t)$. Then we obtain from (2.1), (3.5), (3.8)

$$\begin{aligned} u_i(z,t) &\leqslant u_i^{(0)}(z_i(0)) \\ &+C_2\int_0^t\Big(Kiu_{i-1}+2Kiu_1+(K+F)iu_i+Fiu_{i+1}\Big)(z_i(s),s)ds,\ i\geqslant 2, \end{aligned} \tag{3.12}$$

$$u_1(z,t)\leqslant u_1^{(0)}+\int_0^t\left(KM_0u_1+KM_0\sum_{i=1}^{\infty}iu_i+KC_2u_1+Fu_2+F\sum_{i=2}^{\infty}iu_i\right)ds. \tag{3.13}$$

Perturbations of the kinetic coefficients can be added in (3.12), (3.13) in the form $C\cdot(|k_i-\tilde{k}_i|+|f_i-\tilde{f}_i|)\in l^1_\lambda$. We sum the solutions with weight λ^i and use the above argument regarding the upper function like $H(\lambda,t)$. This function has small initial data and, hence, it is small on any fixed time interval. This proves stability in the space l^1_λ, $\lambda>1$. The uniqueness result follows from the stability. This proves Theorem 3.2. □

Corollary 3.4. — *If additional condition (2.18) holds then the uniqueness result is valid for the whole time interval* $[0,T]$.

Proof. — Due to (3.2) inequalities (3.13) and (3.13) can be summed for any time $t\in[0,T]$ with weight $\lambda<1$ and zero values of $u_i^{(0)}$. □

4. Global existence for bounded kinetic coefficients

Let us assume from now on that there exists a constant M such that

$$k_i\leqslant M,\quad f_i\leqslant M,\quad i\geqslant 1. \tag{4.1}$$

Also, let the maximum principle conditions (2.15) hold. Then (3.1) is valid. Again, we multiply (2.1) by λ^i, $i \geqslant 1$, $1 < \lambda \leqslant \Lambda < \infty$ but arrange the functions using another substitution $g_i^{(n)}(t) = \sup_{z \in \mathbf{R}^N} \lambda^i c_i^{(n)}(z,t)$. Then we obtain the following expressions :

$$g_1^{(n)}(t) \leqslant g_1^{(0)} + \int_0^t \left[f_2 g_2^{(n)}(s) + \sum_2^\infty f_i g_i^{(n)}(s) \right] ds,$$

$$g_i^{(n)}(t) \leqslant g_i^{(0)} + \int_0^t \left[\lambda M_0 k_{i-1} g_{i-1}^{(n)}(s) + \lambda^{-1} f_{i+1} g_{i+1}^{(n)}(s) \right] ds, \quad i \geqslant 2.$$

Here $g_i^{(0)} = \lambda^i \sup_{z \in \mathbf{R}^N} c_i^{(0)}(z)$. Summation of these inequalities yields

$$G(t) \leqslant G_0 + M_0 \lambda \int_0^t \left(\sum_{i=1}^\infty k_i g_i(s) \right) ds + (1 + \lambda^{-1}) \int_0^t \left(\sum_{i=2}^\infty f_i g_i(s) \right) ds$$
$$\leqslant G_0 + M M_0 \Lambda \int_0^t G(s) ds + 2M \int_0^t G(s) ds,$$

where $G(t) = \sum_{i=1}^\infty g_i(t)$. Hence, we obtain the Gronwall inequality that results in $G(t) \leqslant C_2 = \text{const}$, $0 \leqslant t \leqslant T$. Consequently, we obtain apriori estimate (3.8), which holds at the whole interval $t \in [0,T]$. Hence, we arrive at the following theorem.

Theorem 4.1. — *Let kinetic coefficients k_i, f_i be non-negative and conditions (2.15), (4.1) hold. Let initial function $c_i^{(0)}$ be a continuous non-negative function, its integral (1.4) be bounded and (2.2), (3.9) hold. Let transport velocities v_i satisfy (2.3), (2.4), (3.10) and the tube conditions. Then there exists a unique continuous non-negative solution "in the whole" of initial value problem (1.1), (1.2) $c \in l^1_\lambda$, $\lambda > 1$, $c_i \in C([0,T], L^1_+(\mathbf{R}^N) \bigcap L^\infty(\mathbf{R}^N))$, $i \geqslant 1$. This solution is stable with respect to perturbations of initial data and kinetic coefficients in l^1, it satisfies the mass conservation law (1.4) and is continuously differentiable along characteristics.*

The proof is like that of Theorem 3.2.

5. The problem with the diffusion taken into account

Starting from this section we consider one-dimensional initial value problem (1.3), (1.2) where the diffusion term is taken into account. Usually diffusion makes a considerable impact to small particles. Hence, $d_i \to 0$, $i \to \infty$, or we just set $d_i = 0$ for sufficiently large values of i. Consequently, the system may contain both hyperbolic (1.1) and parabolic (1.3) equations. To simplify the reasonings (without loss of generality), we assume $d_1 > 0$ and $d_i = 0$, $i \geqslant 2$. However, all the reasonings are valid even if the system has infinite number of both hyperbolic and parabolic equations. So, for definiteness we have

$$\partial_t c_i(z,t) + \partial_z(v_i c_i(z,t)) = Q_i(c), \;\; i \geqslant 2, \tag{5.1}$$
$$\partial_t c_1(z,t) + \partial_z(v_1 c_1(z,t)) = Q_1(c)(z,t) + d_1 \partial^2_{zz} c_1(z,t). \tag{5.2}$$

DEFINITION 5.1. — *Under solution of initial value problem to (5.1), (5.2) we understand continuous functions $c_i(z,t)$ that satisfy the following integral equations :*

$$c_1(z,t) = \int_{-\infty}^{+\infty} G_1(z,\xi,t) c_1^{(0)}(\xi) d\xi + \int_0^t \int_{-\infty}^{+\infty} G_1(z,\xi,t-s) Q_1(c)(\xi,s) d\xi ds, \tag{5.3}$$

$$c_i(z,t) = c_i^{(0)}(z_i(0)) + \int_0^t Q_i(c)(z_i(s),s) ds, \;\; i \geqslant 2. \tag{5.4}$$

The function $G_1(z,\xi,t)$ is the fundamental solution for equation (5.2).

From Lemmas 2.2, 2.4 we easily obtain the following lemma for mixed hyperbolic-parabolic system (5.1), (5.2).

LEMMA 5.2. — *Let $z \in \mathbf{R}^1$, the conditions of Lemma 2.4 and (2.15) be fulfilled. Then there exists a unique continuous non-negative solution of the initial value problem to (5.1), (5.2) $c_i \in C(\mathbf{R}^1 \times [0,T]) \bigcap C([0,T], L^\infty_+ \bigcap L^1(\mathbf{R}))$, $1 \leqslant i \leqslant n$, $c_1 \in C^{2,1}_{z,t}(\mathbf{R} \times [0,T])$, and the mass conservation law (1.4) holds.*

For the problem with diffusion the maximum principle can be applied in the form of Corollary 2.6. Hence, we have to control the boundary values $\partial\Omega$. However, in view of Lemma 2.7 we are able to control the values in the one-dimensional case only. To apply the one-dimensional version of the maximum principle (Lemma 2.7) we note that at a maximum point inside a parabolic cylinder $\Omega \times [0,T]$ the time derivative of c_1 is non-positive provided that conditions (2.15) hold. Hence, from Lemma 2.7 we arrive at (3.1).

Since there is no diffusion for $i \geqslant 2$ then the above substitution $g_i(t) = \lambda^i \sup\limits_{z \in \mathbf{R}^1} c_i(z,t)$, $\lambda > 1$, yields

$$g_i(t) \leqslant g_i^{(0)} + \int_0^t \left[\lambda M_0 k_{i-1} g_{i-1}(s) + \lambda^{-1} f_{i+1} g_{i+1}(s)\right] ds, \quad i \geqslant 2.$$

Hence,

$$G(t) \leqslant G_0 + M_0 + M M_0 \lambda \int_0^t G(s) ds + M \int_0^t G(s) ds,\ G(t) = \sum_{i=1}^{\infty} g_i(t).$$

The Gronwall inequality results in (3.8). We are now in position to prove the following theorem.

Theorem 5.3. — *Let the conditions of Lemma 5.2 hold without the truncation condition. Let initial function $c^{(0)} \in l_\lambda^1$ for a certain $\lambda > 1$. Let v satisfies the tube conditions and (3.10) hold. Then there exists a unique continuous non-negative solution "in the whole" of the initial value problem to (5.1), (5.2)*

$$c \in l_\lambda^1,\ c_i \in C(\mathbf{R} \times [0,T]) \bigcap C([0,T], L_+^\infty \bigcap L^1(\mathbf{R})),\ i \geqslant 1.$$

The solution is stable with respect to perturbations of $c^{(0)}$ and kinetic coefficients in l^1, and satisfies the mass conservation law (1.4).

Proof. — In view of Lemma 5.2 we construct the sequence of solutions $\{c^{(n)}\}$ of problem to (5.1) with truncated kinetic coefficients equalled to zero at $i > n$. The elements of this sequence satisfy (3.1), (3.8) uniformly.

Let us consider the continuity modulus $\omega_i^{(n)}(t) = |c_i^{(n)}(z',t) - c_i^{(n)}(z,t)|$ on a compact set $1 \leqslant i \leqslant n,\ \Omega = \{z : \ |z| \leqslant Z\},\ 0 \leqslant t \leqslant T$. We obtain

$$\omega_1^{(n)}(t) \leqslant \int_0^t \int_{\mathbf{R}^1} \left\{|G_1(z',\xi,t-s) - G_1(z,\xi,t-s)||Q_1(c^{(n)})(\xi,s)|\right\} d\xi ds, \tag{5.5}$$

$$\omega_i^{(n)}(t) \leqslant \int_0^t |Q_i(c^{(n)})(z_i'(s),s) - Q_i(c^{(n)})(z_i(s),s)| ds. \tag{5.6}$$

Since in view of (3.1), (3.8), (4.1)

$$|Q_1(c^{(n)})(z,t)| \leqslant 2MC_2(M_0+1) = \text{const},\ z \in \mathbf{R}^1,\ 0 \leqslant t \leqslant T, \tag{5.7}$$

then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|z' - z| < \delta$ then independently of $n \geqslant 1$

$$\omega_1^{(n)}(t) \leqslant \int_0^t \int_{\mathbf{R}^1} |G_1(z', \xi, t-s) - G_1(z, \xi, t-s)||Q_1(c^{(n)})(\xi, s)|d\xi ds \leqslant \varepsilon,$$
$$-Z \leqslant z, z' \leqslant Z,\ 0 \leqslant t \leqslant T. \tag{5.8}$$

Let us estimate the continuity modulus in (5.6) omitting for the simplicity index "n" :

$$\begin{aligned}|Q_i(c)(z_i'(s), s) - Q_i(c)(z_i(s), s)| &\leqslant k_{i-1}|c_1(z_i'(s), s) - c_1(z_i(s), s)|c_{i-1}(z_i'(s), s)\\ &+k_{i-1}c_1(z_i(s), s)|c_{i-1}(z_i'(s), s) - c_{i-1}(z_i(s), s)|\\ &+k_i|c_1(z_i'(s), s) - c_1(z_i(s), s)|c_i(z_i'(s), s)\\ &+k_i c_1(z_i(s), s)|c_i(z_i'(s), s) - c_i(z_i(s), s)|\\ &+f_{i+1}|c_{i+1}(z_i'(s), s) - c_{i+1}(z_i(s), s)|\\ &+f_i|c_i(z_i'(s), s) - c_i(z_i(s), s)|.\end{aligned} \tag{5.9}$$

Due to (3.10) there exists $\delta > 0$ such that $|z_i'(s) - z_i(s)| \leqslant \delta_1$, $0 \leqslant s \leqslant t$. Our aim is to take supremum over $1 \leqslant i \leqslant I$ in (5.9). However, its right-hand side contains the term $|c_{i+1}(z_i'(s), s) - c_{i+1}(z_i(s), s)|$, which is outside $1 \leqslant i \leqslant I$. To control this term we assume that the value of I is sufficiently large and in view of (3.8) it can be estimated as

$$|c_{i+1}(z_i'(s), s) - c_{i+1}(z_i(s), s)| \leqslant \frac{2C_2}{\lambda^I},\ i \geqslant I.$$

We introduce

$$\omega(t) = \sup_{1\leqslant i\leqslant I, z', z\in\Omega, |z'-z|<\delta} \omega_i(t)$$

and, in view of (4.1), (5.6), (5.8), (5.9), we obtain for $\omega(t)$ the Gronwall inequality

$$\omega(t) \leqslant \varepsilon + \int_0^t \omega(s) ds.$$

Hence, there exists $\delta > 0$ such that independently of truncation parameter n

$$\sup_{1\leqslant i\leqslant I, z', z\in\Omega, |z'-z|<\delta} \omega_i(t) < \varepsilon,\ 0 \leqslant t \leqslant T. \tag{5.10}$$

Similar reasonings hold for the continuity modulus $\theta_i(z) = |c_i(z, t') - c_i(z, t)|$:

$$\sup_{1\leqslant i\leqslant I, |t'-t|<\delta} \theta_i(z) < C|t' - t|,\ z \in \Omega,\ 0 \leqslant t, t' \leqslant T. \tag{5.11}$$

Using (3.8), (5.10), (5.11) and Arzela theorem, we conclude that the sequence of approximate solutions $c^{(n)}$ is compact in $C(\Omega \times [0, T])$. By the standard diagonal process we pick up from $\{c^{(n)}\}$ the subsequence, which converges to a continuous function $c_i(z,t) \in C(\mathbf{R} \times [0,T])$. Obviously, this limit function satisfies (3.8). To show that this limit function is the solution to (5.3), (5.4), we replace $c^{(n)}$ by $c^{(n)} - c + c$. Then we obtain

$$[c_1^{(n)} - c_1]\ (z,t) + c_1(z,t) = \int\limits_{-\infty}^{+\infty} G_1(z,\xi,t) c_1^{(0)}(\xi) d\xi$$

$$+ \int\limits_0^t \int\limits_{-\infty}^{+\infty} G_1(z,\xi,t-s) Q_1([c^{(n)} - c] + c)(\xi,s) d\xi ds, \quad (5.12)$$

$$[c_i^{(n)} - c_i](z,t) + c_i(z,t) = c_i^{(0)}(z_i(0)) + \int\limits_0^t Q_i([c^{(n)} - c] + c)(z_i(s),s) ds, \ i \geqslant 2. \quad (5.13)$$

We estimate the right-hand side of (5.12), (5.13) :

$$Q_1([c^{(n)} - c] + c) = -k_1[c_1^{(n)} - c_1](c_i^{(n)} + c_1) - k_1 c_1^2$$
$$- \sum_{i=1}^n k_i [c_1^{(n)} - c_1] c_i^{(n)} - \sum_{i=1}^n k_i c_1 [c_i^{(n)} - c_i] - \sum_{i=1}^n k_i c_1 c_i$$
$$+ \sum_{i=2}^n f_i [c_i^{(n)} - c_i] + \sum_{i=2}^n f_i c_i = Q_1(c) + r_n(z,t). \quad (5.14)$$

Since on each compacta $c = \lim_{n\to\infty} c^{(n)}$ then the contents of brackets goes to zero as $n \to \infty$. However, the sums in (5.14) become infinite and to demonstrate their uniform in n smallness we have to estimate the series

$$\sum_{i=1}^{\infty} k_i c_i^{(n)} [c_1^{(n)} - c_1] \leqslant M M_0 \sum_{i=1}^{A} [c_1^{(n)} - c_1] + 2 M M_0 C_2 \sum_{i=A+1}^{\infty} \lambda^{-i} \to 0, \ A \to \infty,$$
$$n \to \infty.$$

Here we used (3.1), (3.8), (4.1). We first fix sufficiently large A to deal with small series "tail" and then tend n to the infinity. Similar reasonings are true for other summands in (5.14). Consequently, for any $\varepsilon > 0$ there exists n_0 such that for any $n > n_0$ the residual $r_n(z,t) < \varepsilon$, $z \in \Omega$, $t \in [0,T]$. We choose sufficiently large Z to be sure that in view of (5.7)

$$\int_0^t \int_{\mathbf{R}^1 \setminus \Omega} G_1(z,\xi,t-s) Q_1(c)(\xi,s) d\xi ds < \varepsilon, \ \Omega = \{z : \ |z| \leqslant Z\}.$$

Finally, the substitution of these observations in (5.12) yields

$$c_1(z,t) = \int_{-\infty}^{+\infty} G_1(z,\xi,t)c_1^{(0)}(\xi)d\xi + \int_0^t \int_{-\infty}^{+\infty} G_1(z,\xi,t-s)Q_1(c)(\xi,s)d\xi ds.$$

So, the function c_1 satisfies (5.3). Easier reasonings hold for equality (5.13). Consequently, the function c satisfies (5.3), (5.4). The uniqueness and stability properties are proved like in Theorem 3.2 with (2.7) taken into account. This proves Theorem 5.3.

6. Vanishing diffusion limit

Let kinetic coefficients be uniformly bounded and (4.1) hold. Then in view of Theorems 4.1 and 5.3 there exist continuous solutions c and $\overline{c}$ of equations (5.3), (5.4) and (2.1), respectively. Our aim now is to demonstrate that $c \to \overline{c}$ as the diffusion coefficient d_1 in (5.2) goes to zero.

THEOREM 6.1. — *Let the conditions of Theorems 4.1 and 5.3 hold. Then*

$$\sum_{i=1}^{\infty} \sup_{z\in\mathbf{R}, 0\leqslant t\leqslant T} |c_i(z,t) - \overline{c}_i(z,t)| \to 0, \ d_1 \to 0, \tag{6.1}$$

provided that the initial data are close in l^1 *:*

$$\sum_{i=1}^{\infty} \sup_{z\in\mathbf{R}} |c_i^{(0)}(z) - \overline{c}_i^{(0)}(z)| \to 0, \ d_1 \to 0. \tag{6.2}$$

Proof. — We estimate the difference $c - \overline{c}$. From (2.1), (5.3), (5.4) we obtain

$$\begin{aligned}(c_1 - \overline{c}_1)(z,t) &= \int_{\mathbf{R}} G_1(z,\xi,t)[c_1^{(0)} - \overline{c}_1^{(0)}](\xi)d\xi \\ &+ \left[\int_{\mathbf{R}} G_1(z,\xi,t)\overline{c}_1^{(0)}(\xi)d\xi - \overline{c}_1^{(0)}(z_1(0))\right] \\ &+ \int_0^t \int_{\mathbf{R}} G_1(z,\xi,t-s)[Q_1(c) - Q_1(\overline{c})](\xi,s)d\xi ds \\ &+ \left[\int_0^t \int_{\mathbf{R}} G_1(z,\xi,t-s)Q_1(\overline{c})(\xi,s)d\xi ds - \int_0^t Q_1(\overline{c})(z_1(s),s)ds\right]\end{aligned} \tag{6.3}$$

Similar expression holds for the difference $c_i - \overline{c}_i$ (however, without function G_1) :

$$(c_i - \overline{c}_i)(z,t) = [c_i^{(0)} - \overline{c}_i^{(0)}](z_i(0)) + \int_0^t [Q_i(c) - Q_i(\overline{c})](z_i(s),s)ds. \quad (6.4)$$

The most complicated term is written in the second line of (6.3) :

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}} G_1(z,\xi,t-s)|Q_1(c) - Q_1(\overline{c})|(\xi,s)d\xi ds \\
&\qquad \leqslant M(1+M_0) \int_0^t \int_{\mathbf{R}} G_1(z,\xi,t-s) \sum_{i=1}^{\infty} |c_i - \overline{c}_i| d\xi ds \\
&\qquad \leqslant M(1+M_0) \int_0^t \sum_{i=1}^{\infty} \sup_{z\in\mathbf{R}} |c_i - \overline{c}_i|(z,s)ds. \qquad (6.5)
\end{aligned}$$

Other summands in brackets go to zero as $d_1 \to 0$. Taking relation (6.5) into account, we take supremum over $z \in \mathbf{R}$ and summarize (6.3), (6.4). Then for any given $\varepsilon > 0$ we obtain

$$\sum_{i=1}^{\infty} \sup_z |c_i - \overline{c}_i|(z,t) \leqslant \varepsilon + 2M(1+M_0) \int_0^t \sum_{i=1}^{\infty} \sup_{z\in\mathbf{R}} |c_i - \overline{c}_i|(z,s)ds \quad (6.6)$$

provided that $d < \delta(\varepsilon)$. The use of the Gronwall inequality finishes the proof of Theorem 6.1.

7. L^1 stability of one-dimensional systems

In this section, we study the L^1 stability of one-dimensional pure coagulation system and pure fragmentation system using the nonlinear functional approach in [12, 13, 15]. We assume that $z \in \mathbf{R}^1$ and no diffusion happens. We denote the weighted L^1 norm by $||\cdot||$

$$||c(\cdot,t)|| \equiv \int_{\mathbf{R}^1} \sum_{i=1}^{\infty} ic_i(z,t)dz.$$

The nonlinear functional approach for the L^1 stability is based on the construction of a Lyapunov functional $\mathcal{H}[t]$ which has the following two key properties : For two nonnegative mass conserving continuous solutions $c(z,t)$ and $\overline{c}(z,t)$, $\mathcal{H}[t]$ satisfies

1. $\frac{1}{C_0}||c(\cdot,t)-\bar{c}(\cdot,t)|| \leqslant \mathcal{H}[t] \leqslant C_0||c(\cdot,t)-\bar{c}(\cdot,t)||$, for some constant $C_0>0$,

2. $\mathcal{H}[t]$ is non-increasing in time $t : \mathcal{H}[t] \leqslant \mathcal{H}[0], \quad t>0$.

In the following, we briefly explain the component functionals of $\mathcal{H}[t]$. Let us assume that $c(z,t)$ and $\bar{c}(z,t)$ are non-negative mass preserving continuous solutions with initial data $c^{(0)}$ and $\bar{c}^{(0)}$. The functional $\mathcal{L}[t]$ is defined to be the weighted L^1 distance $||c(\cdot,t)-\bar{c}(\cdot,t)||$,

$$\mathcal{L}[t] \equiv \int_{\mathbf{R}^1} \sum_{i=1}^{\infty} i|c_i-\bar{c}_i|(z,t)dz \quad (\leqslant ||c^{(0)}||+||\bar{c}^{(0)}||<\infty).$$

Then by the direct calculation (Lemma 7.2), the functional $\mathcal{L}[t]$ can be increasing locally in time t due to the coupling of particles with same and different velocities. However, in the estimates $\dfrac{d\mathcal{L}(t)}{dt}$, the errors due to the coupling of particles with same velocities can be shown to be non-positive so that we only need to control the errors due to the coupling of particles with different velocities. For this, we employ the interaction potential $\mathcal{Q}_d[t]$ as in [12, 13, 15] :

$$\begin{aligned}
\mathcal{Q}_d[t] \quad &\equiv \quad \int\int_{\mathbf{R}^2} \mathbf{1}_{z<y}\Big(\sum_{i=1}^{\infty} i|c_i-\bar{c}_i|(z,t)\Big)c_1(y,t)dz\,dy \\
&+ \quad \int\int_{\mathbf{R}^2} \mathbf{1}_{z<y}\Big(\sum_{i=1}^{\infty} i\bar{c}_i(z,t)\Big)|c_1-\bar{c}_1|(y,t)dz\,dy \\
&= \quad \mathcal{Q}_d^1[t]+\mathcal{Q}_d^2[t],
\end{aligned}$$

where $\mathbf{1}_{z<y}$ is the characteristic function of the set $\{(z,y)\in\mathbf{R}^2 : z<y\}$, and $\mathcal{Q}_d^i[t]$, $(i=1,2)$ are bounded by the quantities depending only on the initial data, i.e.,

$$\mathcal{Q}_d^1[t] \leqslant (||c^{(0)}||+||\bar{c}^{(0)}||)||c^{(0)}||<\infty, \quad \mathcal{Q}_d^2[t] \leqslant ||\bar{c}^{(0)}||(||c^{(0)}||+||\bar{c}^{(0)}||)<\infty.$$

Finally, we combine above two functionals by a linear combination, i.e.,

$$\mathcal{H}[t] := \mathcal{L}[t]+\kappa\mathcal{Q}_d[t] = \int_{\mathbf{R}^1}\sum_{i=1}^{\infty}\mathcal{W}_i(z,t)i|c_i-\bar{c}_i|(z,t)dz,$$

where κ is a large constant which will be determined later, and nonlocal weights $\mathcal{W}_i(z,t)$ are defined as follows

$$\mathcal{W}_1(z,t) = 1+\kappa\left(\int_{-\infty}^{z}\sum_{i=1}^{\infty} i\bar{c}_i(y,t)dy+\int_{z}^{\infty}c_1(y,t)dy\right),$$

$$\mathcal{W}_i(z,t) = 1 + \kappa \left(\int_z^\infty c_1(y,t)dy \right), \quad i \geqslant 2.$$

Then it is easy to see that $\mathcal{H}[t]$ is equivalent to the weighted L^1 distance $||\cdot||$. On the other hand, since the possible quadratic increase of $\mathcal{L}[t]$ will be expected to be compensated by the good quadratic decay of $\mathcal{Q}_d[t]$, $\mathcal{H}[t]$ will be non-increasing in time t. In the following two subsections, we consider the pure coagulation system and pure fragmentation system. At present, the above nonlinear functional approach does not work for the coagulation and fragmentation system, we will discuss this issue at the end of this section.

7.1. Pure coagulation system

In this subsection we prove L^1 stability for the system (1.1) in the one-dimensional case when the particles do not split, and the diffusion can be neglected. So, (1.1) takes the form :

$$\begin{aligned} \partial_t c_i(z,t) &+ \partial_z(v_i c_i(z,t)) \\ &= k_{i-1}c_{i-1}(z,t)c_1(z,t) - k_i c_i(z,t)c_1(z,t), \ i \geqslant 2, \qquad (7.1) \\ \partial_t c_1(z,t) &+ \partial_z(v_1 c_1(z,t)) \\ &= -k_1 c_1^2(z,t) - \sum_{i=1}^\infty k_i c_i(z,t)c_1(z,t), \quad t>0, z \in \mathbf{R}^1. \ (7.2) \end{aligned}$$

Let us denote the source term for the equation of c_i by $Q_i(c)$, and notice that $Q_1(c)$ is non-positive and $Q_i(c)$, $(i \geqslant 3)$ consists of only transversal terms. Let (1.4), (2.15) and (4.1) hold. Then in accordance with Theorem 4.1 there exists a unique continuous non-negative solution to the initial value problem for (7.1), (7.2). This solution satisfies (2.1). Let us assume

$$\inf_{i\geqslant 2} |v_i - v_1| \geqslant v_* > 0, \qquad \sup_{1\leqslant i<\infty} k_i \leqslant M, \quad \text{for some constant } M>0, \ (7.3)$$

and initial data are sufficiently small, i.e.,

$$\sum_{i=1}^\infty \int_{\mathbf{R}^1} i c_i^{(0)}(z)dz \ll 1. \qquad (7.4)$$

To estimate the difference $i|c_i - \overline{c}_i|$ of two solutions to (7.1), (7.2), corresponding different initial data $c^{(0)}$ and $\overline{c}^{(0)}$ and satisfying (7.4), we introduce a simplified notation

$$\delta_i(z,t) = \mathrm{Sgn}(c_i(z,t) - \overline{c}_i(z,t)),$$

and obtain

$$\partial_t|c_1-\overline{c}_1|+\partial_z(v_1|c_1-\overline{c}_1|)=-2k_1(c_1+\overline{c}_1)|c_1-\overline{c}_1|$$
$$-\sum_{i=2}^{\infty}k_i\left(\frac{\delta_1}{\delta_i}c_1|c_i-\overline{c}_i|+\overline{c}_i|c_1-\overline{c}_1|\right):=R_1(c,\overline{c}),\tag{7.5}$$
$$\partial_t(2|c_2-\overline{c}_2|)+\partial_z(2v_2|c_2-\overline{c}_2|)=\frac{2\delta_2}{\delta_1}k_1(c_1+\overline{c}_1)|c_1-\overline{c}_1|$$
$$-2k_2c_1|c_2-\overline{c}_2|-2k_2\frac{\delta_2}{\delta_1}\overline{c}_2|c_1-\overline{c}_1|:=R_2(c,\overline{c}),\tag{7.6}$$
$$\partial_t(i|c_i-\overline{c}_i|)+\partial_z(iv_i|c_i-\overline{c}_i|)=k_{i-1}\left(\frac{\delta_i}{\delta_{i-1}}c_1i|c_{i-1}-\overline{c}_{i-1}|\right.$$
$$\left.+\frac{\delta_i}{\delta_1}\overline{c}_{i-1}i|c_1-\overline{c}_1|\right)-k_i\left(ic_1|c_i-\overline{c}_i|+i\frac{\delta_i}{\delta_1}\overline{c}_i|c_1-\overline{c}_1|\right)$$
$$\leqslant 2k_{i-1}\Big((i-1)c_1|c_{i-1}-\overline{c}_{i-1}|+(i-1)\overline{c}_{i-1}|c_1-\overline{c}_1|\Big)$$
$$-k_i\left(ic_1|c_i-\overline{c}_i|+i\frac{\delta_i}{\delta_1}\overline{c}_i|c_1-\overline{c}_1|\right):=R_i(c,\overline{c}),\ i\geqslant 3.\tag{7.7}$$

Here we have used the fact that $2(i-1)\geqslant i$ for $i\geqslant 3$. Unlike the collision terms $Q_i(c)$ in system (7.1) and (7.2), the term $R_i(c,\overline{c})$ does not satisfy the conservation of mass and notice that $R_i(c,\overline{c})$, $(i\geqslant 3)$ consists of only transversal terms. For the simplicity of notation, we introduce two instantaneous interaction productions as follows.

$$\Lambda[c(\cdot,t)]\equiv\int_{\mathbf{R}^1}\left(\sum_{i=2}^{\infty}ic_i(z,t)\right)c_1(z,t)dz,$$
$$\Lambda[c(\cdot,t),\overline{c}(\cdot,t)]\equiv\Lambda^1[c(\cdot,t),\overline{c}(\cdot,t)]+\Lambda^2[c(\cdot,t),\overline{c}(\cdot,t)],$$
$$\Lambda^1[c(\cdot,t),\overline{c}(\cdot,t)]\equiv\int_{\mathbf{R}^1}\left(\sum_{i=2}^{\infty}i|c_i-\overline{c}_i|(z,t)\right)c_1(z,t)dz,$$
$$\Lambda^2[c(\cdot,t),\overline{c}(\cdot,t)]\equiv\int_{\mathbf{R}^1}\left(\sum_{i=2}^{\infty}i\overline{c}_jiz,t)\right)|c_1-\overline{c}_1|(z,t)dz.$$

In the following theorem, we show that the source terms are integrable in space and time which can be useful to the study of large time behavior of solutions.

Theorem 7.1. — *Suppose that $c(z,t)$ is a non-negative continuous solution of system (7.1), (7.2). Then*

$$\int_0^{\infty}\int_{\mathbf{R}^1}\sum_{i=1}^{\infty}ic_i(z,t)c_1(z,t)dz\,dt\leqslant\gamma(\mu^2+\mu),$$

where γ is some positive constant independent of time t, and $\mu = ||c^{(0)}||$ is the initial mass.

Proof. — We only prove the theorem for C^1 solutions. The generalization to continuous solutions is straighforward. First, we introduce the interaction potential $\mathcal{Q}(t)$ in order to show that the transversal terms are bounded by $\mathcal{O}(\mu^2)$. Define a quadratic interaction potential $\mathcal{Q}(t)$ by

$$\mathcal{Q}(c)[t] \equiv \int\int_{\mathbf{R}^2} \mathbf{1}_{z<y} \left(\sum_{i=1}^{\infty} ic_i(z,t) \right) c_1(y,t) dz dy \quad (\leqslant ||c^{(0)}||^2 < \infty).$$

Next we estimate the time-derivative of $\mathcal{Q}[t]$. Recall that

$$\partial_t(ic_i(z)) + \partial_z(iv_i c_i(z)) = iQ_i(c)(z), \tag{7.8}$$

$$\partial_t c_1(y) + \partial_y(v_1 c_1(y)) = Q_1(c)(y). \tag{7.9}$$

Using integration by parts, $\mathbf{1}_{z<y}[(7.8)\cdot c_1(y) + (7.9)\cdot c_i(z)]$ becomes

$$\begin{aligned} \partial_t[\mathbf{1}_{z<y} ic_i(z)c_1(y)] &+ \nabla_{(z,y)}[(v_i, v_1)\mathbf{1}_{z<y} ic_i(z)c_1(y)] \\ &+ \delta(z-y)(v_i - v_1) ic_i(z) c_1(y) \\ &= \mathbf{1}_{z<y}[iQ_i(c)(z)c_1(y) + Q_1(c)(y) ic_i(z)]. \end{aligned} \tag{7.10}$$

If we integrate $\sum_{i=1}^{\infty}(7.10)$ over $\mathbf{R}^2$, then we have

$$\begin{aligned} \frac{d\mathcal{Q}[t]}{dt} &= -\int_{\mathbf{R}^1} \sum_{i=1}^{\infty} (v_i - v_1) ic_i(z) c_1(z) dz \\ &+ \int\int_{\mathbf{R}^2} \left(\sum_{i=1}^{\infty} iQ_i(c)(z) \right) c_1(y) dz dy \\ &+ \int\int_{\mathbf{R}^1} Q_1(c)(y) \left(\sum_{i=1}^{\infty} ic_i(z) \right) dy dz \\ &\leqslant -v_* \int_{\mathbf{R}^1} \sum_{i=1}^{\infty} ic_i(z) c_1(z) dz. \end{aligned} \tag{7.11}$$

In the above calculation, we have used that fact that

$$\inf_{i\neq 1} |v_i - v_1| \geqslant v_* > 0, \quad \sum_{i\geqslant 1} iQ_i(c)(z) = 0, \quad Q_1(c)(y) \leqslant 0.$$

If we integrate (7.11) over $[0,T]$, then we have

$$\begin{aligned} \int_0^T \int_{\mathbf{R}^1} \sum_{i=2}^{\infty} ic_i(z,t) c_1(z,t) dz dt &\leqslant \frac{1}{v_*}[\mathcal{Q}(c)(0) - \mathcal{Q}(c)(T)] \\ &\leqslant \frac{\mathcal{Q}(c)(0)}{v_*} = \mathcal{O}(\mu^2). \end{aligned} \tag{7.12}$$

Since the right hand side does not depend on T, by letting $T \to \infty$, the transversal source terms are bounded by $\mathcal{O}(\mu^2)$. Next we estimate the square terms $\int_0^\infty \int_{\mathbf{R}^1} c_1^2(z,t)dzdt$ as follows. From the equation (7.2), we have

$$\begin{aligned} 2k_1 c_1^2(z,t) &= -\sum_{i=2}^{\infty} k_i c_i(z,t) c_1(z,t) - \partial_t c_1(z,t) - \partial_z(v_1 c_1(z,t)) \\ &\leqslant -\partial_t c_1(z,t) - \partial_z(v_1 c_1(z,t)). \end{aligned}$$

If we integrate above equation from $[0,T] \times \mathbf{R}^1$ in (t,z), then we have

$$\int_0^T \int_{\mathbf{R}^1} c_1^2(z,t)dzdt \leqslant \frac{||c_1^{(0)}||}{2k_1} = \mathcal{O}(\mu). \tag{7.13}$$

Hence combining (7.12) and (7.13), we obtain the desired result. This completes the proof of Theorem 7.1. □

In the following lemma, we estimate the time-evolution of the above nonlinear functionals.

LEMMA 7.2. — *Let c and $\bar{c}$ be continuously differentiable solutions of (7.1) and (7.2) corresponding to continuously differentiable initial data $c^{(0)}$ and $\bar{c}^{(0)}$. Then we have the following estimates for the nonlinear functionals.*

$$\frac{d\mathcal{L}[t]}{dt} \leqslant \mathcal{O}(1)M\Lambda(c,\bar{c})(t), \quad \frac{d\mathcal{Q}_d[t]}{dt} \leqslant -\alpha\Lambda(c,\bar{c})(t), \quad \frac{d\mathcal{H}[t]}{dt} \leqslant -\beta\Lambda(c,\bar{c})(t),$$

where α and β are positive constants independent of time t.

Proof. — We estimate the nonlinear functionals separately.
(i) From the equations for the difference $i|c_i - \bar{c}_i|$, we have

$$\begin{aligned} \frac{d\mathcal{L}[t]}{dt} &= \int_{\mathbf{R}^1} \partial_t \left(\sum_{i=1}^{\infty} i|c_i - \bar{c}_i|(z) \right) dz \\ &\leqslant \int_{\mathbf{R}^1} k_1 \left(-2 + 2\frac{\delta_2}{\delta_1} \right) (c_1 + \bar{c}_1)|c_1 - \bar{c}_1|(z)dz + \mathcal{O}(1)M\Lambda(c,\bar{c})(t) \\ &\leqslant \mathcal{O}(1)M\Lambda(c,\bar{c})(t). \end{aligned}$$

(ii) First we consider $\dfrac{d\mathcal{Q}_d^1[t]}{dt}$. Recall that

$$\partial_t(i|c_i - \bar{c}_i|) + \partial_z(iv_i|c_i - \bar{c}_i|) = R_i(c,\bar{c}), \tag{7.14}$$
$$\partial_t c_1 + \partial_y c_1 = Q_1(c). \tag{7.15}$$

Using integration by parts, $\mathbf{1}_{z<y}[(7.14)\cdot c_1(y)+(7.15)\cdot i|c_i-\bar{c}_i|(z)]$ becomes

$$\begin{aligned}&\partial_t[\mathbf{1}_{z<y}i|c_i-\bar{c}_i|(z)c_1(y)]+\nabla_{(z,y)}[(v_i,v_1)\mathbf{1}_{z<y}i|c_i-\bar{c}_i|(z)c_1(y)]\\&\quad+\delta(z-y)(v_i-v_1)i|c_i-\bar{c}_i|(z)c_1(y)\\&\quad=\mathbf{1}_{z<y}[R_i(c,\bar{c})(z)c_1(y)+Q_1(c)(y)i|c_i-\bar{c}_i|(z)].\end{aligned}\tag{7.16}$$

If we integrate $\sum_{i=1}^{\infty}(7.16)$ over $\mathbf{R}^2$, then we have

$$\begin{aligned}\frac{d\mathcal{Q}_d^1[t]}{dt}&\leqslant -v_*\int_{\mathbf{R}^1}\sum_{i=2}^{\infty}i|c_i-\bar{c}_i|(z)c_1(z)dz\\&+\left(\int_{\mathbf{R}^1}\sum_{i=3}^{\infty}R_i(c,\bar{c})z)dz\right)\left(\int_{\mathbf{R}^1}c_1(y)dy\right)\\&+\left(\int_{\mathbf{R}^1}(R_2(c,\bar{c})(z)+R_1(c,\bar{c})(z))dz\right)\left(\int_{\mathbf{R}^1}c_1(y)dy\right)\\&\leqslant\left(-v_*+\mathcal{O}(1)M\int_{\mathbf{R}^1}c_1(y)dy\right)\Lambda^1(c,\bar{c})(t).\end{aligned}$$

In the above calculation, we have used that fact that $\sum_{i=3}^{\infty}R_i(c,\bar{c})$ and $R_2(c,\bar{c})+R_1(c,\bar{c})$ are bounded by transversal terms and $Q_1(c)(y)\leqslant 0$. Since $\int_R c_1(y)dy\ll 1$, we have

$$\frac{d\mathcal{Q}_d^1[t]}{dt}\leqslant -\frac{v_*}{2}\Lambda^1(c,\bar{c})(t).$$

Next, we consider $\mathcal{Q}_d^2[t]$. Good terms from the convection part in the estimates of $\dfrac{d\mathcal{Q}_d^2[t]}{dt}$ are :

$$-\int_{\mathbf{R}^1}\sum_{i=2}^{\infty}(v_i-v_1)i\bar{c}_i(z)|c_1-\bar{c}_1|(z)dz\leqslant -v_*\int_{\mathbf{R}^1}\left(\sum_{i=2}^{\infty}i\bar{c}_i(z)\right)|c_1-\bar{c}_1|(z)dz.$$

In contrast, we have possible bad terms from the source terms :

$$\begin{aligned}I_1&\equiv\int\int_{\mathbf{R}^2}\mathbf{1}_{z<y}\left(\sum_{i=1}^{\infty}iQ_i(\bar{c})(z)\right)|c_1-\bar{c}_1|(y)dzdy,\\I_2&\equiv\int\int_{\mathbf{R}^2}\mathbf{1}_{z<y}\left(\sum_{i=1}^{\infty}i\bar{c}_i(z)\right)R_1(c,\bar{c})(y)dzdy.\end{aligned}$$

By the conservation of mass $\sum_{i=1}^{\infty}iQ_i(\bar{c})(z)=0$, we have

$$I_1=\int_{\mathbf{R}^1}\int_{\mathbf{R}^1}\mathbf{1}_{z<y}\left(\sum_{i=1}^{\infty}iQ_i(\bar{c})(z)\right)|c_1-\bar{c}_1|(y)dzdy=0.$$

On the other hand, we have

$$
\begin{aligned}
I_2 &= \int\int_{\mathbf{R}^2} \mathbf{1}_{z<y} \left(\sum_{i=1}^{\infty} i\bar{c}_i(z) \right) R_1(c,\bar{c})(y)dzdy \\
&= \int\int_{\mathbf{R}^1} \mathbf{1}_{z<y} \left(\sum_{j=1}^{\infty} j\bar{c}_j(z) \right) (-2k_1(c_1+\bar{c}_1)(y)|c_1-\bar{c}_1|(y) \\
&\qquad\qquad + \text{transversal terms})\ dzdy \\
&\leqslant \mathcal{O}(1)M \left(\int_{\mathbf{R}^1} \sum_{i=1}^{\infty} i\bar{c}_i(z)dz \right) \Lambda^2(c,\bar{c})(t) \ll \Lambda^2(c,\bar{c})(t).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\frac{d\mathcal{Q}_d^2[t]}{dt} &\leqslant \left(-v_* + \mathcal{O}(1)M \int_{\mathbf{R}} \sum_{i=1}^{\infty} i\bar{c}_i(z)dz \right) \Lambda^2(c,\bar{c})(t) \\
&\leqslant -\frac{v_*}{2}\Lambda^2(c,\bar{c})(t).
\end{aligned}
$$

Combining the estimates for $\mathcal{Q}_d^1[t]$ and $\mathcal{Q}_d^2[t]$, we obtain

$$
\frac{d\mathcal{Q}_d[t]}{dt} \leqslant -\frac{v_*}{2}\Lambda(c,\bar{c})(t).
$$

(iii) By definition of $\mathcal{H}[t]$, we have

$$
\begin{aligned}
\frac{d\mathcal{H}[t]}{dt} &= \frac{d\mathcal{L}[t]}{dt} + \kappa\frac{d\mathcal{Q}_d[t]}{dt} \\
&= \left(\mathcal{O}(1)M - \frac{\kappa v_*}{2} \right) \Lambda(c,\bar{c})(t).
\end{aligned}
$$

Choose κ large enough so that

$$
\mathcal{O}(1)M - \frac{\kappa v_*}{2} < 0.
$$

Then for such κ, we have

$$
\frac{d\mathcal{H}[t]}{dt} \leqslant -\beta\Lambda(c,\bar{c})(t), \quad \text{for some constant } \beta > 0.
$$

This completes the proof of Lemma 7.2. $\square$

Using the above decay estimate of the nonlinear functional $\mathcal{H}[t]$, we establish the L^1 stability of solutions.

THEOREM 7.3. — *Let c and $\bar{c}$ be continuous solutions corresponding to two initial data $c^{(0)}$ and $\bar{c}^{(0)}$ whose initial masses are sufficiently small. Then we have the L^1 stability.*

$$||c(\cdot,t) - \bar{c}(\cdot,t)|| \leqslant G||c^{(0)}(\cdot) - \bar{c}^{(0)}(\cdot)||,$$

where G is a generic constant independent of time t.

Remark 7.4. — The existence of such continuous solutions is proved in Theorem 4.1.

Proof. — Let c and $\bar{c}$ be solutions corresponding to two initial data $c^{(0)}$ and $\bar{c}^{(0)}$. Let $c^{(n)}$ and $\bar{c}^{(n)}$ be C^1-finite approximations of c and $\bar{c}$ such that for given t,

$$c^{(n)}(z,t) \to c(z,t),\ \bar{c}^{(n)}(z,t) \to \bar{c}(z,t), \quad \text{in } L^1(\mathbf{R}) \text{ as } n \to \infty.$$

We define a nonlinear functional $\mathcal{H}(t)$ for c and $\bar{c}$ as follows.

$$\mathcal{H}(c,\bar{c})[t] \equiv \lim_{n\to\infty} \mathcal{H}(c^{(n)},\bar{c}^{(n)})[t].$$

Then by the estimate of $\mathcal{H}(c^{(n)},\bar{c}^{(n)})[t]$, we have

$$\frac{1}{C_0}||c(\cdot,t) - \bar{c}(\cdot,t)|| \leqslant \mathcal{H}[t] \leqslant C_0||c(\cdot,t) - \bar{c}(\cdot,t)||, \quad \mathcal{H}[t] \leqslant \mathcal{H}[0].$$

Using above two key properties of $\mathcal{H}[t]$, we have

$$||c(\cdot,t) - \bar{c}(\cdot,t)|| \leqslant C_0\mathcal{H}[t] \leqslant C_0\mathcal{H}[0] \leqslant C_0^2||c^{(0)}(\cdot) - \bar{c}^{(0)}(\cdot)||.$$

This completes the proof of Theorem 7.3. $\square$

In the following section, we consider the pure fragmentation system and show that pure fragmentation system is L^1-contractive.

7.2. Pure fragmentation system

In this subsection, we consider the pure fragmentation system which contains the only linear source terms.

$$\partial_t c_1 + \partial_z(v_1 c_1) = f_2 c_2 + \sum_{i=2}^{\infty} f_i c_i,$$
$$\partial_t(ic_i) + \partial_z(v_i i c_i) = i f_{i+1} c_{i+1} - i f_i c_i, \quad i \geqslant 2.$$

As in the previous subsection, we derive equations for the difference $i|c_i - \bar{c}_i|$.

$$\partial_t|c_1 - \bar{c}_1| + \partial_z(v_1|c_1 - \bar{c}_1|) = \frac{\delta_1}{\delta_2} f_2|c_2 - \bar{c}_2| + \sum_{i=2}^{\infty} \frac{\delta_1}{\delta_i} f_i|c_i - \bar{c}_i|,$$

$$\partial_t(i|c_i - \bar{c}_i|) + \partial_z(v_i i|c_i - \bar{c}_i|) = \frac{i\delta_i}{\delta_{i+1}} f_{i+1}|c_{i+1} - \bar{c}_{i+1}| - i f_i|c_i - \bar{c}_i|.$$

Then in the following lemma, we estimates the time evolution of $\mathcal{L}[t]$.

Lemma 7.5. — *Let c and $\bar{c}$ be continuously differentiable solutions corresponding to continuously differentiable initial data $c^{(0)}$ and $\bar{c}^{(0)}$. Then the functional $\mathcal{L}[t]$ is non-increasing in time t.*

$$\frac{d\mathcal{L}[t]}{dt} \leqslant 0.$$

Proof. — Summing all equations for $i|c_i - \bar{c}_i|$ over i, we have

$$\begin{aligned}
&\partial_t \left(\sum_{i=1}^{\infty} i|c_i - \bar{c}_i| \right) + \partial_z \left(\sum_{i=1}^{\infty} i v_i |c_i - \bar{c}_i| \right) \\
&= \left(-2 + 2\frac{\delta_1}{\delta_2} \right) f_2|c_2 - \bar{c}_2| + \sum_{i=3}^{\infty} \left(\frac{\delta_1}{\delta_i} + (i-1)\frac{\delta_{i-1}}{\delta_i} - i \right) f_i|c_i - \bar{c}_i| \leqslant 0.
\end{aligned}$$

Hence, we have the following differential inequality :

$$\partial_t \left(\sum_{i=1}^{\infty} i|c_i - \bar{c}_i| \right) + \partial_z \left(\sum_{i=1}^{\infty} i v_i |c_i - \bar{c}_i| \right) \leqslant 0.$$

Integrating the above inequality over $\mathbf{R}^1$, we obtain

$$\frac{d\mathcal{L}[t]}{dt} = \frac{d}{dt} \int_{\mathbf{R}^1} \sum_{i=1}^{\infty} i|c_i - \bar{c}_i|(z) dz \leqslant 0.$$

This completes the proof of Lemma 7.5. $\square$

Using the same density argument as in Theorem 7.3, we obtain the following L^1 contraction of the solution.

Theorem 7.6. — *Let c and $\bar{c}$ be mass conserving continuous solutions corresponding to two initial data $c^{(0)}$ and $\bar{c}^{(0)}$. Then we have a L^1 contraction*

$$||c(\cdot, t) - \bar{c}(\cdot, t)|| \leqslant ||c^{(0)}(\cdot) - \bar{c}^{(0)}(\cdot)||, \ 0 \leqslant t < \infty.$$

Remark 7.7. — Let us discuss the question why we have considered only pure coagulation and pure fragmentation cases. Unfortunately, the original system (1) which takes into account both processes cannot be analyzed to obtain L^1 stability estimate. In fact by a direct calculation, one can show that

$$\frac{d\mathcal{L}[t]}{dt} \leqslant |\mathcal{O}(1)|\Lambda(c, \bar{c}).$$

The local errors in the increment of $\mathcal{L}(t)$ can be dominated by the errors from the coupling of particles with different velocities as pure coagulation system. However, in the estimates of $\dfrac{d\mathcal{Q}_d[t]}{dt}$, one will have uncontrollable terms due to the linear fragmentation source terms. More precisely, in the time-evolution estimates of $\dfrac{d\mathcal{Q}_d^2[t]}{dt}$, one needs to estimates the following quantity :

$$\begin{aligned} \frac{d\mathcal{Q}_d^2[t]}{dt} &= \int_{\mathbf{R}^1} \mathbf{1}_{z<y} \partial_t \left(\sum_{i=1}^{\infty} i\bar{c}_i(z,t) \right) |c_1 - \bar{c}_1|(y,t) dz dy \\ &+ \int_{\mathbf{R}^1} \mathbf{1}_{z<y} \left(\sum_{i=1}^{\infty} i\bar{c}_i(z,t) \right) \partial_t |c_1 - \bar{c}_1|(y,t) dz dy. \end{aligned} \tag{7.17}$$

Then the first term in the right hand side of (7.17) will be zero because of conservation of mass, and one of the bad terms in second term will be equal to

$$\int\int_{\mathbf{R}^2} \mathbf{1}_{z<y} \left(\sum_{i=1}^{\infty} i\bar{c}_i(z,t) \right) |c_2 - \bar{c}_2|(y,t) dz dy,$$

which will not be controlled by quadratic good decay terms. Because of this, we are not able to show the decay of the interaction potential and, consequently, the global L^1-stability for the combined coagulation-fragmentation kinetic system.

Acknowledgment. — The authors would like to use this opportunity to express deep gratitude to Professor M. Slemrod for his numerous fruitful discussions and advices regarding this paper. P. Dubovski wishes to thank him for kind hospitality, which was greatly appreciated. The work of P. B. Dubovski is supported by NSF grant DMS-0071463 Mod. 01, and the work of S.-Y. Ha is supported by KOSEF R01-2006-000-10002-0.

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