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G.K Immink

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Exact asymptotics of nonlinear difference equations with levels 1 and $1^{+(*)}$

G.K. Immink[(1)]

Abstract. — We study a class of nonlinear difference equations admitting a 1-Gevrey formal power series solution which, in general, is not 1- (or Borel-) summable. Using right inverses of an associated difference operator on Banach spaces of so-called *quasi-functions*, we prove that this formal solution can be lifted to an analytic solution in a suitable domain of the complex plane and show that this analytic solution is an *accelero-sum* of the formal power series.

Résumé. — On étudie une classe d'équations aux différences finies, non-linéaires, possédants une solution formelle en forme de série 1-Gevrey qui, en général, n'est pas Borel-sommable. En utilisant des inverses à droite d'un opérateur aux différences associé, définies sur des espaces Banach de *quasi-fonctions*, on démontre qu'à la solution formelle peut être associée, de façon unique, une solution analytique sur un domaine approprié, qui est une *accéléro-somme* de la solution formelle.

1. Introduction

This paper is concerned with the summability of formal power series solutions of certain nonlinear difference equations, i.e. with the existence of analytic solutions, represented asymptotically by the formal solution in some unbounded domain and characterized, in some way, by their asymptotic properties. We begin by discussing two very simple examples of linear difference equations, which may be regarded as building blocks for the class of equations considered below.

Example 1.1. — The equation

$$y(z+1) - ay(z) = \frac{b}{z},\ a, b \in \mathbf{C}^*, a \neq 1 \tag{1.1}$$

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(1) Faculty of Economics, University of Groningen, P.O. Box 800, 9700 AV Groningen, g.k.immink@rug.nl

is a difference equation of *level* 1. It has a formal power series solution $\hat{f} = \sum_{h=1}^{\infty} a_h z^{-h}$ with the property that $|a_h| \leqslant A^h h^h$ for all $h \geqslant 1$, where A is a positive constant, i.e. $\hat{f}$ is 1-Gevrey. Its formal Borel transform

$$\phi(t) = \hat{\mathcal{B}}\hat{f}(t) = \sum_{h=1}^{\infty} \frac{a_h}{(h-1)!} t^{h-1}$$

has a positive radius of convergence. ϕ is the germ of a meromorphic function, to be denoted by the same symbol :

$$\phi(t) = \frac{b}{e^{-t} - a}.$$

The directions $\alpha_l = \arg(-\text{Log } a + 2l\pi i)$, where Log denotes the principal value of the logarithm and $l \in \mathbb{Z}$, are so-called *singular directions* and the directions $-\pi/2 - \alpha_l$, $l \in \mathbb{Z}$, are *Stokes directions* of (1.1). Let $|a| < 1$. The Laplace integrals

$$\int_0^{e^{i\alpha}\infty} \phi(t) e^{-tz} dt = \int_0^{e^{i\alpha}\infty} \frac{b}{e^{-t} - a} e^{-tz} dt, \quad \text{Re } z e^{i\alpha} > 0$$

with $\alpha \in (\alpha_{l-1}, \alpha_l)$, can be 'glued together' to yield an analytic solution y_l of (1.1) with asymptotic expansion $\hat{f}$ as $z \to \infty$ in the (maximal) sector :

$$S_l := \cup_{\alpha \in (\alpha_{l-1}, \alpha_l)} \{z : \text{Re } z e^{i\alpha} > 0\},$$

uniformly on closed subsectors. This solution y_l is called the 1-sum or Borel-sum of $\hat{f}$ in a direction $\alpha \in (\alpha_{l-1}, \alpha_l)$, or on (α_{l-1}, α_l). It is uniquely determined by its asymptotic properties in S_l. The difference of two 'neighbouring' 1-sums is an exponential function of order 1 :

$$y_{l+1}(z) - y_l(z) = \frac{2\pi i b}{a} e^{(\text{Log } a - 2l\pi i)z}$$

The asymptotic behaviour of y_l changes when we cross the Stokes ray with direction $-\pi/2 - \alpha_l$, or the anti-Stokes ray with direction $\pi/2 - \alpha_{l-1}$.

Example 1.2. —

$$y(z+1) - \frac{a}{z} y(z) = \frac{b}{z}, \ a, b \in \mathbf{C}^* \tag{1.2}$$

This is a difference equation of *level* 1^+ (cf. [2, 6, 7], the nonlinear case is discussed in [8]). It has a formal power series solution $\hat{f} = \sum_{h=1}^{\infty} a_h z^{-h}$ with the property that $|a_h| \leqslant A^h (\frac{h}{\log h})^h$ for all $h \geqslant 2$, which we will call

1^+-Gevrey. Its formal Borel transform has infinite radius of convergence and defines an entire function with supra-exponential growth :

$$\phi(t) = \hat{\mathcal{B}}\hat{f}(t) = be^{-a}\exp(t + ae^t)$$

The so-called *critical variable* for the level 1^+ is $z\log z$, or any variable $\psi(z)$ equivalent to it, in the sense that $\lim_{z\to\infty}\psi(z)(z\log z)^{-1} = 1$. The formal Borel transform of $\hat{f}$ with respect to the critical variable $\psi_\theta(z) := z\log(ze^{i\theta})$:

$$\hat{\mathcal{B}}_{1^+,\theta}(\hat{f})(t) = \frac{1}{2\pi i}\sum_{h=1}^{\infty} a_h \int_U z^{-h}e^{t\psi_\theta(z)}d\psi_\theta(z)$$

where $\theta \in \mathbb{R}$ and U is a U-shaped contour, consisting of the half line from $-\infty - i\delta$ to $\delta - i\delta$, the segment from $\delta - i\delta$ to $\delta + i\delta$ and the half line from $\delta + i\delta$ to $-\infty + i\delta$, $\delta > 0$, converges for small positive t and defines a *quasi-analytic* function ϕ_θ on the positive real axis, provided $\theta \neq -\mathrm{Arg}\ a + 2l\pi$ for any $l \in \mathbf{Z}$. (By Arg a we denote the value of arg a in $(-\pi, \pi]$.) The directions $\theta_l = -\mathrm{Arg}\ a + 2l\pi$, $l \in \mathbb{Z}$, are called *pseudo-Stokes directions.* The Laplace integrals

$$\int_0^\infty \phi_\theta(t)e^{-t\psi_\theta(z)}dt$$

with $\theta \in (\theta_{l-1}, \theta_l)$, can be glued together to yield an analytic solution y_l of (2), represented asymptotically by $\hat{f}$ as $z \to \infty$ in

$$D_l = \cup_{\theta\in(\theta_{l-1},\theta_l)}\{z : \mathrm{Re}\ \psi_\theta(z) \geqslant c\},$$

uniformly on subdomains of the type $\cup_{\theta\in I}\{z : \mathrm{Re}\ \psi_\theta(z) \geqslant c'\}$, where I is a closed subinterval of (θ_{l-1}, θ_l) and c' some sufficiently large positive number. The functions y_l satisfy certain generalized Gevrey conditions and can be viewed as '1^+-sum' of $\hat{f}$ on D_l, as they are characterized by their asymptotic properties in D_l. Accordingly, the difference of two neighbouring 1^+-sums is an exponential function 'of order 1^+' :

$$y_{l+1}(z) - y_l(z) = \frac{2\pi i b e^{-a}}{a}e^{(\mathrm{Log}\ a - 2l\pi i)z}\Gamma(z)^{-1}$$

$$= e^{-z\log(ze^{i\theta_l})(1+o(1))} \text{ as } z \to \infty \text{ in } D_l \cap D_{l+1}$$

uniformly on subdomains of the type $\cap_{\theta\in[\theta_l-\delta,\theta_l+\delta]}\{z : \mathrm{Re}\ \psi_\theta(z) \geqslant c_\delta\}$, where δ and $c_\delta > 0$.

Combining the equations in examples 1 and 2, we obtain the simplest (and, admittedly, somewhat trivial) example of a difference equation with *levels* 1 *and* 1^+ :

$$\begin{aligned} &y_1(z+1) - a_1 y_1(z) = \tfrac{b_1}{z},\ a_1, b_1 \in \mathbf{C}^*, a_1 \neq 1 \\ &y_2(z+1) - \tfrac{a_2}{z} y_2(z) = \tfrac{b_2}{z},\ a_2, b_2 \in \mathbf{C}^* \end{aligned} \tag{1.3}$$

Its formal solution $\hat{f} \in \mathbb{C}^2[[z^{-1}]]$ is 1-Gevrey, but not 1-summable in any direction $\alpha \in (-\pi/2, \pi/2)$. Nor is it multi-summable in any of these directions. Its formal Borel transform $\phi = \hat{\mathcal{B}}\hat{f}$ has a positive radius of convergence. ϕ can be continued analytically in any direction $\alpha \neq \arg(-\mathrm{Log}\, a_1 + 2l\pi i)$, where $l \in \mathbb{Z}$, but has supra-exponential growth. If 0 is not a singular direction of level 1, it can be *accelerated* to level 1^+ by means of a *weak acceleration operator*, which is an extension of a Laplace transformation (i.e. an inverse (ordinary) Borel transformation), followed by a Borel transformation with respect to the variable $\psi_\theta(z)$. The accelerate ϕ_θ defines a quasi-analytic function on the positive real axis, provided θ is not a pseudo-Stokes direction of level $1^+ : \theta \neq \theta_l := -\mathrm{Arg}\, a_2 + 2l\pi$, for all $l \in \mathbb{Z}$. The Laplace transforms in the variable $\psi_\theta(z)$, of the functions ϕ_θ, with $\theta \in (\theta_{l-1}, \theta_l)$, can be glued together to yield an *accelero-sum* of the formal solution and a solution of (1.3). This accelero-sum is a particular case of the $(1, 1^+)$-sum introduced in [1], where it was proved that formal solutions of linear systems of difference equations with levels 1 and 1^+ are $(1, 1^+)$-summable on suitable domains, provided 0 is not a singular direction of level 1. It is the purpose of the present paper to extend this result to nonlinear systems of difference equations and lift the restrictive condition on the singular directions of level 1. The equations we consider can be represented in the form

$$\varphi(z, y(z), y(z+1)) = 0 \tag{1.4}$$

where φ is a $\mathbb{C}^n$-valued function, analytic in a neighbourhood of (∞, y_0, y_0), $y_0 \in \mathbb{C}^n$, or in a more general type of domain (cf. §3 for the exact conditions). We *assume* that (1.4) possesses a formal power series solution $\hat{f} = \sum_{h=0}^{\infty} a_h z^{-h/p}$, with $a_0 = y_0$ and $p \in \mathbb{N}$, and that the (formal) difference operator obtained by linearization about the formal solution has no levels < 1 (cf. §2.1 for more details).

In general, the accelero-sums of the formal solution are not characterized by their asymptotic expansion and the corresponding Gevrey type error bounds, as the domain in which the asymptotic expansion is valid is usually not large enough. Therefore, instead of considering individual solutions, we work with so-called quasi-functions (introduced by Ramis in [12]). In our case, these will be pairs of solutions, defined on overlapping domains and differing by an exponentially small function on the intersection of these domains. Quasi-functions can in many ways be treated like ordinary functions. In particular, they may satisfy Gevrey type conditions and admit an asymptotic expansion, which is necessarily the same for each individual function. If the union of the individual domains is sufficiently large, they can, under some additional conditions, be characterized by the asymptotic expansion. In this paper we prove that the equations considered here have unique quasi-function solutions, which turn out to be accelero-sums of the formal solution.

In the case that 0 is a singular direction of level 1, the singularities of $\phi := \hat{\mathcal{B}}\hat{f}$ on the positive real axis present a problem, due to the fact that the acceleration operator we would like to apply involves integration of ϕ along this axis. It turns out that these singularities can, in a certain sense, be 'circumvented' or 'regularized' by replacing the ordinary formal Borel transformation by a Borel transformation with respect to a variable $r_\theta(z)$ defined by

$$r_\theta(z) = \frac{\psi_\theta(z)}{\log z} = z + i\theta \frac{z}{\log z},$$

where θ is a suitable real number, different from 0. Note that $r_\theta(z)$ is equivalent to the variable z, as $\lim_{z\to\infty} r_\theta(z)z^{-1} = 1$.

The domains considered in this paper are 'right' domains, invariant under $z \mapsto z+1$. Analogous results can be derived for 'left' domains, invariant under $z \mapsto z-1$. As there exists a simple relation between results for the two types of domains (cf. [5]), we restrict ourselves to domains of the first type.

The paper is organized as follows. In §2.1 and §2.2 we introduce some basic notions and summarize the main properties of the curves $C_\theta(z)$ (level curves of Re ψ_θ) and the domains $D_I(z)$ and $\widetilde{D}_I(z)$, which play a major role in the theory, comparable to that of sectors of aperture $\leqslant \pi$ and $\geqslant \pi$, respectively, in problems of level 1 (the generic case). In §2.3 we define classes of analytic functions admitting asymptotic expansions with prescribed error bounds and recall some of their properties. §3 contains the main existence results : Theorems 3.1 and 3.2. The first theorem is concerned with existence and uniqueness of Gevrey type solutions of (1.4), while the second one deals with *quasi-function* solutions. Both theorems are based on the existence of right inverses of the difference operator Δ^c, on suitable Banach spaces of analytic functions, or quasi-functions, respectively. The existence of these right inverses is proved in §4.1 for ordinary functions and in §4.2 for quasi-functions. In §5 (Theorem 5.9) it is shown that the quasi-functions in Theorem 3.2 consist of two accelero-sums of the formal solution.

2. Preliminaries

2.1. Formal theory

By τ we denote the 'shift operator', defined by

$$\tau y(z) = y(z+1).$$

We use the same symbol for the automorphism of $End(n, \mathcal{P})$, where

$$\mathcal{P} = \cup_{p\in\mathbb{N}}\mathbb{C}[[z^{-1/p}]][z^{1/p}],$$

defined by

$$\tau(z^{1/p}) = z^{1/p}\sum_{h=0}^{\infty}\binom{1/p}{h}z^{-h},\ p\in\mathbb{N}.$$

Let $\hat{A}$ and $\hat{B} \in Gl(n, \mathcal{P})$, and let $\hat{\Delta} := \hat{A} + \hat{B}\tau$. By a transformation

$$\hat{\Delta} \mapsto (\tau\hat{F})^{-1}\hat{B}^{-1}\hat{\Delta}\hat{F} \tag{2.1}$$

with $\hat{F} \in Gl(n; \mathcal{P})$, $\hat{\Delta}$ can be reduced to a *canonical form* (cf. [11, 5])

$$\Delta^c = \oplus_{j=1}^m \Delta_j^c \tag{2.2}$$

where

$$\Delta_j^c := \tau - y_j^c(z+1)y_j^c(z)^{-1} \tag{2.3}$$

y_j^c is an $n_j \times n_j$- matrix function of the form

$$y_j^c(z) = z^{d_j z}e^{\mu_j z + q_j(z)}z^{C_j} \tag{2.4}$$

where $d_j \in \mathbb{Q}$, $\mu_j \in \mathbb{C}$, $q_j(z)$ is a polynomial in $z^{1/p}$ for some $p \in \mathbb{N}$, of degree $< p$ and without constant term if $q_j \not\equiv 0$, and C_j is a Jordan block of order $n_j : C_j = N_j + \gamma_j I_{n_j}$, with eigenvalue γ_j, $j = 1, ..., m$. The number μ_j is determined up to a multiple of $2\pi i$ and will be chosen such that

$$0 \leqslant \text{Im}\ \mu_j < 2\pi.$$

Furthermore, we *assume* that $q_j \equiv 0$ for all $j \in \{1, ..., m\}$ such that $d_j = \mu_j = 0$. In that case, the equation has no levels less than 1. If there is a $j \in \{1, ..., m\}$ such that $d_j \neq 0$, $\hat{\Delta}$ is said to possess a level 1^+.

Definition 2.1 (Gevrey conditions). — *By $\hat{\mathcal{O}}_1$ we denote the ring of* 1-*Gevrey formal power series of the form $\sum_{h=0}^{\infty} a_h z^{-h/p}$ where $p \in \mathbb{N}$. These are characterized by the property that there exists a positive number A such that, for all $h \geqslant 1$,*

$$|a_h| \leqslant A^h h^{\frac{h}{p}}.$$

DEFINITION 2.2 (Stokes directions). — *Let $\hat{\Delta} = \hat{A} + \hat{B}\tau$, where $\hat{A}$ and $\hat{B} \in Gl(n, \mathcal{P})$. Suppose that $\hat{\Delta}$ has a canonical form Δ^c, with blocks Δ_j^c of the form (2.3). The directions $\pi/2 - \arg(\mu_j + 2l\pi i)$, where $j \in \{1, ..., m\}$ such that $d_j = 0$, $l \in \mathbb{Z}$, $l \neq 0$ if $\mu_j = 0$, are the Stokes directions of $\hat{\Delta}$ of level* 1. *By $\Theta^{\pm}(\hat{\Delta})$ we denote the following set :*

$$\Theta^{\pm}(\hat{\Delta}) = \{\theta \in \mathbb{R} : d_j\theta = \textit{ Im } \mu_j \bmod 2\pi \textit{ for some } j \in \{1, ..., m\}$$

$$\textit{such that } \pm d_j > 0\}.$$

We call the elements of $\Theta(\hat{\Delta}) := \Theta^+(\hat{\Delta}) \cup \Theta^-(\hat{\Delta})$ the pseudo-Stokes directions of $\hat{\Delta}$, of level 1^+.

Note that $\Theta(\hat{\Delta}) = \Theta(\Delta^c) = \cup_{j=1}^m \Theta(\Delta_j^c)$.

Remark 2.3. — In 'right' domains, invariant under $z \mapsto z+1$, the pseudo-Stokes directions belonging to $\Theta^+(\hat{\Delta})$ may be disregarded, whereas in domains invariant under $z \mapsto z-1$ the same is true of the pseudo-Stokes directions belonging to $\Theta^-(\hat{\Delta})$.

2.2. Domains

By S_+ we denote the sector

$$S_+ := \{z : |\arg z| < \pi\}$$

of the Riemann surface of the logarithm. Let $\theta \in \mathbb{R}$, $z \in S_+$ and

$$\psi_\theta(z) := z(\log z + i\theta)$$

We consider two types of domains : $D_I(z)$ and $\widetilde{D}_I(z)$, which play a crucial role, similar to sectors of aperture $\leqslant \pi$ and $\geqslant \pi$, respectively, in problems of level 1. Each domain contains a sector of the form $\{z \in S_+ : |\arg z| < \pi/2 - \delta, |z| > R\}$ for every $\delta \in (0, \pi/2)$ and some sufficiently large R, and is bounded by curves with limiting directions $\pm\pi/2$.

DEFINITION 2.4. — *Let $z \in S_+$ such that $\operatorname{Re}\psi_\theta(z) > 1/e + |\theta|$. By $C_\theta(z)$ we denote the level curve of Re ψ_θ through z :*

$$C_\theta(z) = \{\zeta \in S_+ : \textit{ Re } \psi_\theta(\zeta) = \textit{Re } \psi_\theta(z)\}$$

In particular, if $R > 0$, such that $R\log R > 1/e + |\theta|$

$$C_\theta(R) = \{z \in S_+ : \textit{Re } (z\log z + i\theta z) = R\log R\}$$

We define $C_\theta^+(z)$ *and* $C_\theta^-(z)$ *by*

$$C_\theta^\pm(z) = \{\zeta \in C_\theta(z) : \pm \ Im\ (\zeta - z) \geqslant 0\}.$$

For all $R > 0$ *such that* $R \log R > 1/e + |\theta|$, $C_\theta^+(R)$ *is contained in the sector* $\{z \in S_+ : |\arg z| < \pi/2, |z| > 1\}$ *iff* $\theta \geqslant -\pi/2$ *and* $C_\theta^-(R)$ *is contained in this sector iff* $\theta \leqslant \pi/2$ *(cf. Figure 1).*

By $D_\theta(z)$ *we denote the domain*

$$D_\theta(z) := \{\zeta \in S_+ : Re\ \psi_\theta(\zeta) \geqslant Re\ \psi_\theta(z)\}.$$

Let I *be a finite interval of* $\mathbb{R}$, $\theta_1 = \inf I$ *and* $\theta_2 = \sup I$. *Let* $z \in S_+$ *and suppose that* $Re\ \psi_\theta(z) > 1/e + |\theta|$ *for all* $\theta \in I$. *By* $D_I(z)$ *we denote the domain*

$$D_I(z) = \cap_{\theta \in I} D_\theta(z) = D_{\theta_1}(z) \cap D_{\theta_2}(z)$$

and by $\widetilde{D}_I(z)$

$$\widetilde{D}_I(z) = \cup_{\theta \in I} D_\theta(z).$$

By R_I *we denote the positive number such that*

$$R_I \log R_I = 1/e + \sup\{|\theta| : \theta \in I\}.$$

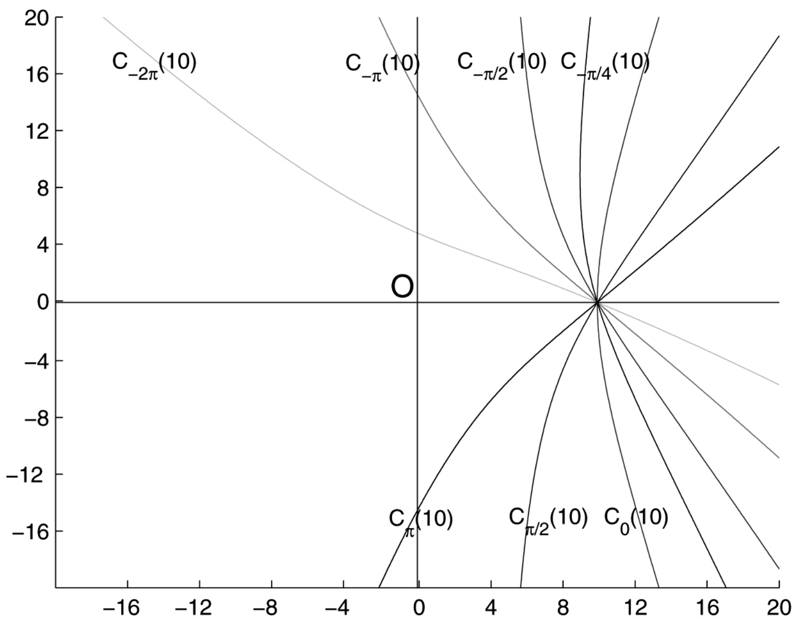

Figure 1. — Examples of $C_\theta(R)$ for various values of θ and $R = 10$

Remark 2.5. — $D_I(z)$ is a closed domain, bounded by $C^+_{\theta_2}(z)$ and $C^-_{\theta_1}(z)$, whereas $\widetilde{D}_I(z)$ is bounded by $C^+_{\theta_1}(z)$ and $C^-_{\theta_2}(z)$ (cf. Figure 2). $\widetilde{D}_I(z)$ is open when I is open and closed when I is closed. Note that $I \subset I'$ implies $\widetilde{D}_I(z) \subset \widetilde{D}_{I'}(z)$, but $D_{I'}(z) \subset D_I(z)$.
The condition $\text{Re}\ \psi_\theta(z) > 1/e + |\theta|$ ensures that $|\zeta| > 1$ for all $\zeta \in C_\theta(z)$ (cf. [9]).
Let I be a finite interval and $z \in S_+$ such that $\text{Re}\ \psi_\theta(z) > 1/e + |\theta|$ for all $\theta \in I$. Then there exist positive numbers R and R' such that $R \log R = \inf_{\theta \in I} \text{Re}\ \psi_\theta(z)$ and $R' \log R' = \sup_{\theta \in I} \text{Re}\ \psi_\theta(z)$, and we have : $D_I(R') \subset D_I(z) \subset D_I(R)$ and $\widetilde{D}_I(R') \subset \widetilde{D}_I(z) \subset \widetilde{D}_I(R)$. Conversely, let $\phi \in (-\pi/2, \pi/2)$ be a fixed number and $R \log R > 1/e + |\theta|$ for all $\theta \in I$. Since $\text{Re}\ \psi_\theta(R'' e^{i\phi}) = R'' \log R'' \cos\phi(1 + o(1))$ as $R'' \to \infty$, uniformly on I, there exists a positive number R'' such that $\inf_{\theta \in I} \text{Re}\ \psi_\theta(R'' e^{i\phi}) \geqslant R \log R$. This implies that $\widetilde{D}_I(R'' e^{i\phi}) \subset \widetilde{D}_I(R)$.

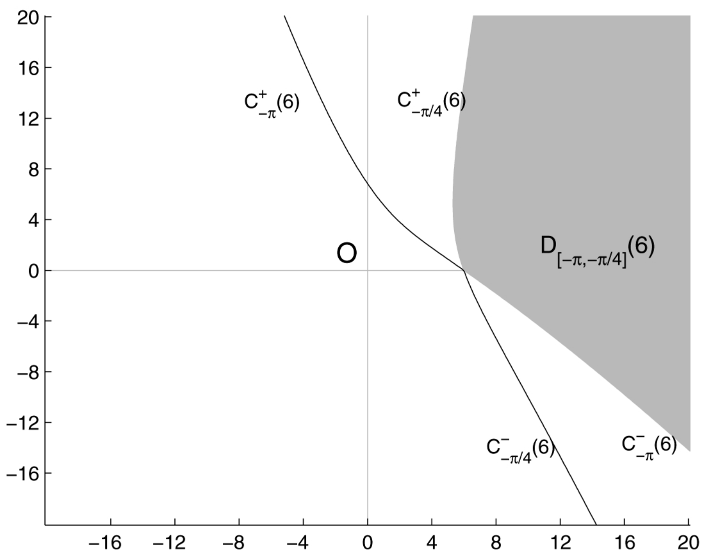

Figure 2. — $\widetilde{D}_{[-\pi, -\frac{\pi}{4}]}(6)$ is the large domain, bounded by $C^-_{-\frac{\pi}{4}}(6)$ and $C^+_{-\pi}(6)$

For a detailed discussion of the curves $C_\theta(z)$ the reader is referred to [9]. Here we give a brief survey of those properties that will be needed here. For all $z \in S_+$ such that $\text{Re}\ \psi_\theta(z) > 1/e + |\theta|$, $C_\theta(z)$ admits a parameter representation of the form :

$$C_\theta(z) = \{\zeta(x) = \rho(x) + ix : x \in \mathbb{R}\}.$$

where $x = \text{Im}\ \zeta(x)$. It is easily seen that

$$\rho(x) = O(\frac{|x|}{\log|x|}) \text{ as } |x| \to \infty$$

Putting $\arg\zeta(x) = \phi(x)$, $|\zeta(x)| = r(x)$ and Re $\psi_\theta(z) = c(z)$ we have

$$\rho'(x) = \frac{\phi(x) + \theta}{1 + \log r(x)} \tag{2.5}$$

$$\phi'(x) = \frac{\rho(x) + c(z)}{r(x)^2(1 + \log r(x))} \tag{2.6}$$

$$\inf_{x\in\mathbb{R}} \phi(x) \to -\frac{\pi}{2} \text{ and } \sup_{x\in\mathbb{R}} \phi(x) \to \frac{\pi}{2} \text{ as } c(z) \to \infty \tag{2.7}$$

$$r'(x) = \sin\phi(x) + \frac{\phi(x) + \theta}{1 + \log r(x)} \cos\phi(x) \tag{2.8}$$

$$1 < \min_{x\in\mathbb{R}} r(x) = \frac{c(z)}{\log c(z)}(1 + o(1)) \text{ as } c(z) \to \infty \tag{2.9}$$

(cf. [9]). (2.6) shows that $\phi'(x) > 0$ whenever $\rho(x) > 0$. From (2.5), (2.7) and (2.9) we infer that $\rho' < 0$ if $\theta < -\pi/2$ and $\rho' > 0$ if $\theta > \pi/2$, provided $c(z)$ is sufficiently large.

Lemma 2.6. — *Let $I = [\theta_1, \theta_2]$, $z_0 \in S_+$ such that Re $\psi_\theta(z_0) > 1/e + |\theta|$ for all $\theta \in I$, and $D = D_I(z_0)$ or $D = \widetilde{D}_I(z_0)$.*
(i) $C_\theta^-(z) \subset D$ for every $z \in D$ and every $\theta \leqslant \theta_1$, and $C_\theta^+(z) \subset D$ for every $z \in D$ and every $\theta \geqslant \theta_2$.
(ii) Let $\alpha \in (-\pi/2, \pi/2)$. If $|z_0|$ is sufficiently large, the half line from z to ∞ with direction α is contained in D for all $z \in D$.
(iii) $C_\theta(z) \subset \widetilde{D}_I(z_0)$ for all $z \in D_I(z_0)$ and all $\theta \in I$.

Proof. — (i) We give the proof for $C_\theta^-(z)$, the proof for $C_\theta^+(z)$ is analogous. Let $z \in D$, $\theta \leqslant \theta_1$ and $\zeta \in C_\theta^-(z)$. For $i \in \{1, 2\}$ we have

$$\text{Re } \psi_{\theta_i}(\zeta) - \text{Re } \psi_{\theta_i}(z) = \text{Re } \psi_\theta(\zeta) - \text{Re } \psi_\theta(z) + (\theta - \theta_i)\ \text{Im } (\zeta - z)$$

and the right-hand side is nonnegative, as Re $\psi_\theta(\zeta)$ = Re $\psi_\theta(z)$, $\theta \leqslant \theta_1 < \theta_2$ and $\text{Im}\ \zeta \leqslant \text{Im}\ z$. If $z \in D_I(z_0)$, then it follows that Re $\psi_{\theta_i}(\zeta) \geqslant$ Re $\psi_{\theta_i}(z) \geqslant$ Re $\psi_{\theta_i}(z_0)$ for both $i = 1$ and $i = 2$. If, on the other hand, $z \in \widetilde{D}_I(z_0)$, then Re $\psi_{\theta_i}(\zeta) \geqslant$ Re $\psi_{\theta_i}(z) \geqslant$ Re $\psi_{\theta_i}(z_0)$ for either $i = 1$ or $i = 2$. In both cases this implies that $\zeta \in D$.
(ii) follows easily from (2.5).

(iii) Let $z \in D_I(z_0)$, $\theta \in I$ and $\zeta \in C_\theta(z)$. Then Re $\psi_\theta(\zeta) =$ Re $\psi_\theta(z) \geqslant$ Re $\psi_\theta(z_0)$ and, consequently, $\zeta \in \widetilde{D}_I(z_0)$. □

In §4.1 we will use the following technical lemma (cf. [9, Lemma 3.10], [10, Lemma 2.5]).

LEMMA 2.7. — *(i) Let $s > 0$, $d_j \in \mathbb{R}$. There exists a positive number K_s such that, for all $z \in S_+$ with the property that $|z+x| \geqslant 1$ for all $x \geqslant 0$ and for all $\sigma \in [0,s]$,*

$$|y_j^c(z) y_j^c(z+\sigma)^{-1}| \leqslant K_s |z|^{-d_j \sigma}$$

(ii) For each $\delta > 0$ there exists a positive number K'_δ, such that, for all z, $\zeta \in S_+$, such that $d(\zeta, z+\mathbb{Z}) \geqslant \delta$,

$$|e^{\pm 2\pi i(\zeta - z)} - 1|^{-1} \leqslant K'_\delta$$

2.3. Asymptotic expansions with Gevrey-type error bounds

In this section we define classes of functions admitting asymptotic expansions with particular types of error bounds and discuss some of their properties. The sets $\mathcal{A}_1(I)$ defined below consist of functions that are 1-Gevrey, uniformly on closed subsectors of $\{z \in S_+ : |\arg z| < \pi/2\}$, and satisfy additional conditions on $D_{I'}(R)$ for any open interval I' containing $\overline{I}$, which can be expressed in terms of a convenient variable $r_\theta(z)$, equivalent to z.

DEFINITION 2.8. — *For all $z \in S_+ : z \neq 1$ and $\theta \in \mathbb{R}$ we define*

$$r_\theta(z) = \frac{\psi_\theta(z)}{\log z} \text{ and } \rho_\theta(z) = \mathrm{Re}\, r_\theta(z)$$

Remark 2.9. — Obviously, $r_0(z) = z$. $r_\theta(z)$ is equivalent to z in the sense that

$$\frac{r_\theta(z)}{z} = 1 + \frac{i\theta}{\log z} = 1 + o(1) \text{ as } z \to \infty$$

In [10] it is shown that, for any $\theta' \in \mathbb{R}$ such that $\theta' \neq \theta + \pi/2$,

$$\rho_\theta(z) = \frac{(\theta - \theta' + \frac{1}{2}\pi)|z|}{\log|z|}\left(1 + O\left(\frac{1}{\log|z|}\right)\right) \text{ as } z \to \infty \text{ on } C^-_{\theta'}(R) \tag{2.10}$$

and, for any $\theta' \neq \theta - \pi/2$,

$$\rho_\theta(z) = \frac{(\theta' - \theta + \frac{1}{2}\pi)|z|}{\log|z|}\left(1 + O\left(\frac{1}{\log|z|}\right)\right) \text{ as } z \to \infty \text{ on } C^+_{\theta'}(R) \tag{2.11}$$

Hence it can be deduced that $\rho_\theta(z) \to \infty$ on $D_I(R)$, where $I = [\theta_1, \theta_2]$, if $\theta \in (\theta_1 - \pi/2, \theta_2 + \pi/2)$, whereas $\rho_\theta(z) \to \infty$ on $\widetilde{D}_I(R)$ if $\theta_2 - \theta_1 < \pi$ and $\theta \in (\theta_2 - \pi/2, \theta_1 + \pi/2)$.

DEFINITION 2.10 (Generalized Gevrey classes). — *Let I be a finite interval of $\mathbb{R}$ and let $\overline{I} = [\theta_1, \theta_2]$. By $|I|$ we denote the length of I :*

$$|I| = \sup I - \inf I = \theta_2 - \theta_1$$

By $\mathcal{A}(I)$ or $\mathcal{A}_0(I)$ we denote the set of continuous functions $f : S_+ \to \mathbb{C}$, admitting an asymptotic expansion $\hat{f} = \sum_{h=0}^{\infty} a_h z^{-h/p}$, with $p \in \mathbb{N}$, such that, for any open interval I' containing $\overline{I}$ and some sufficiently large $R > R_I$ (depending on I'), f is holomorphic in int $D_{I'}(R)$ and for all $N \in \mathbb{N}$, there exists a positive constant $M_N(I')$ such that

$$|R_N(f; z)| := |f(z) - \sum_{h=0}^{N-1} a_h z^{-h/p}| < M_N(I')|z|^{-N/p}$$

uniformly on $D_{I'}(R)$. By $\mathcal{A}_1(I)$ we denote the set of $f \in \mathcal{A}(I)$ with the property that, for any open interval $I' = (\theta_1', \theta_2')$ containing $\overline{I}$ and some (or any, cf. Remark 2.11 below) $\theta \in (\theta_1' - \frac{1}{2}\pi, \theta_2' + \frac{1}{2}\pi)$, there exist positive constants A' and $R > R_I$ (depending on I'), such that, for all $N \in \mathbb{N}$,

$$|R_N(f; z)| < A'^N N^{N/p} \rho_\theta(z)^{-N/p}$$

uniformly on $D_{I'}(R)$. We write $\mathcal{A}(\theta)$ instead of $\mathcal{A}([\theta, \theta])$. By $\mathcal{A}_{0,0}(I)$ we denote the set of $f \in \mathcal{A}(I)$ such that $\hat{f} = 0$, and $\mathcal{A}_{1,0}(I) := \mathcal{A}_{0,0}(I) \cap \mathcal{A}_1(I)$.

Let $y_0 \in \mathbb{C}^n$. By $\mathcal{A}(I; y_0)$ we denote the set of functions $\varphi : S_+ \times \mathbb{C}^{2n} \to \mathbb{C}$ with the following properties :
(i) There exists a neighbourhood U of y_0, such that φ is holomorphic on $D_{I'}(R) \times U \times U$ for any open interval I' containing $\overline{I}$ and some $R > R_I$.
(ii) There exist $p \in \mathbb{N}$ and (holomorphic) functions $\varphi_h : U \times U \to \mathbb{C}$ with the property that, for any open interval I' containing $\overline{I}$ and some $R > R_I$, and for all $N \in \mathbb{N}$, there exists a positive constant $M_N(I')$ such that

$$|R_N(\varphi; z, y_1, y_2)| := |\varphi(z, y_1, y_2) - \sum_{h=0}^{N-1} \varphi_h(y_1, y_2) z^{-h/p}| < M_N(I')|z|^{-N/p},$$

uniformly on $D_{I'}(R) \times U \times U$. By $\mathcal{A}_1(I; y_0)$ we denote the set of functions $\varphi \in \mathcal{A}(I; y_0)$ with the property that, for any open interval $I' = (\theta_1', \theta_2')$

containing $\overline{I}$ *and some* $\theta \in (\theta_1' - \frac{1}{2}\pi, \theta_2' + \frac{1}{2}\pi)$*, there exist positive constants* A' *and* $R > R_I$ *(depending on* I'*), such that, for all* $N \in \mathbb{N}$,

$$|R_N(\varphi; z, y_1, y_2)| < A'^N N^{N/p} \rho_\theta(z)^{-N/p},$$

uniformly on $D_{I'}(R) \times U \times U$.

Let I *be a finite, open interval. We define*

$$\tilde{\mathcal{A}}(I) = \tilde{\mathcal{A}}_0(I) := \cap_{\theta' \in I} \mathcal{A}(\theta')$$

and

$$\tilde{\mathcal{A}}_1(I) := \cap_{\theta' \in I} \mathcal{A}_1(\theta')$$

or, equivalently, $\tilde{\mathcal{A}}_1(I)$ *is the set of* $f \in \tilde{\mathcal{A}}(I)$*, with the property that, for any closed interval* $I' = [\theta_1', \theta_2'] \subset I$ *of length* $|I'| < \pi$ *and some* $\theta \in (\theta_2' - \frac{1}{2}\pi, \theta_1' + \frac{1}{2}\pi)$*, there exist positive constants* A' *and* $R > R_I$ *(depending on* I'*), such that, for all* $N \in \mathbb{N}$,

$$|R_N(f; z)| < A'^N N^{N/p} \rho_\theta(z)^{-N/p}$$

uniformly on $\widetilde{D}_{I'}(R)$*. By* $\tilde{\mathcal{A}}_{1,0}(I)$ *we denote the set of* $f \in \tilde{\mathcal{A}}_1(I)$ *such that* $\hat{f} = 0$.

Let I *be a finite, open interval and* $y_0 \in \mathbb{C}^n$*. We define*

$$\tilde{\mathcal{A}}(I; y_0) := \cap_{\theta' \in I} \mathcal{A}(\theta'; y_0)$$

and

$$\tilde{\mathcal{A}}_1(I; y_0) := \cap_{\theta' \in I} \mathcal{A}_1(\theta'; y_0)$$

or, equivalently, the set of functions $\varphi \in \tilde{\mathcal{A}}(I; y_0)$ *with the property that, for any closed interval* $I' = [\theta_1', \theta_2'] \subset I$ *of length* $|I'| < \pi$ *and some* $\theta \in (\theta_2' - \frac{1}{2}\pi, \theta_1' + \frac{1}{2}\pi)$*, there exist positive constants* A' *and* $R > R_I$ *(depending on* I'*), such that, for all* $N \in \mathbb{N}$,

$$|R_N(\varphi; z, y_1, y_2)| < A'^N N^{N/p} \rho_\theta(z)^{-N/p},$$

uniformly on $\widetilde{D}_{I'}(R) \times U \times U$.

Remark 2.11. — It can be shown that $\mathcal{A}_1(I)$, $\mathcal{A}_1(I; y_0)$, $\tilde{\mathcal{A}}_1(I)$ etc. are independent of the choice of θ (cf. [10]).

The elements of $\mathcal{A}_1(I)$ need not be Gevrey of order 1, uniformly on $D_{I'}(R)$ for any open interval I' containing $\overline{I}$, but they are Gevrey of order 1, uniformly on closed subsectors of $\{z \in S_+ : |\arg z| < \pi/2,\ |z| \geqslant R\}$.

Definition 2.12. — *Let I be a finite interval of $\mathbb{R}$. By $\mathcal{A}_{1^+}(I)$ we denote the set of $f \in \mathcal{A}(I)$, with the property that, for any open interval I' containing $\overline{I}$ and some $\theta \in \overline{I}$, there exist a positive constant A' and a positive number $R > R_I$ (depending on I'), such that, for all $N \geqslant 2$,*

$$|R_N(f;z)| < (\frac{A'N}{\log N})^{N/p} d_\theta(z)^{-N/p} \tag{2.12}$$

where $d_\theta(z) = \min\{|\zeta| : \ Re\ \psi_\theta(\zeta) = \ Re\ \psi_\theta(z)\}$, uniformly on $D_{I'}(R)$. By $\mathcal{A}_{1^+,0}(I)$ we denote the set of $f \in \mathcal{A}_{1^+}(I)$ such that $\hat{f} = 0$.

We shall need the following **Borel-Ritt type theorem** (cf. [10]), which generalizes a well-known result (the case $\theta = 0$).

Theorem 2.13. — *Let $\hat{f} = \sum_{h=0}^{\infty} a_h z^{-h/p} \in \hat{\mathcal{O}}_1$, $\theta \in \mathbb{R}$ and $R > 1$. There exists a function f, holomorphic on $\rho_\theta(z) \geqslant R$, with the property that, for all $N \in \mathbb{N}$,*

$$|R_N(f;z)| < A'^N N^{N/p} \rho_\theta(z)^{-N/p}$$

Elements of $\mathcal{A}_{1,0}(I)$ and $\mathcal{A}_{1^+,0}(I)$ are characterized by their rate of decrease at ∞ in appropriate domains. For the proofs of the following lemmas we refer the reader to [10].

Lemma 2.14. — *(i) Let I be a finite interval of $\mathbb{R}$ and let $\overline{I} = [\theta_1, \theta_2]$. $f \in \mathcal{A}_{1,0}(I)$ iff for each open interval I' containing $\overline{I}$ there exist positive constants R and a such that*

$$\sup_{z \in D_{I'}(R)} |f(z) e^{a\rho_\theta(z)}| < \infty$$

where $\theta \in [\theta_1 - \frac{\pi}{2}, \theta_2 + \frac{\pi}{2}]$.
(ii) Let I be a finite interval of $\mathbb{R}$ and let $\overline{I} = [\theta_1, \theta_2]$. $f \in \mathcal{A}_{1^+,0}(I)$ iff for each open interval I' containing $\overline{I}$ there exist positive constants R and a such that

$$\sup_{z \in D_{I'}(R)} |f(z) e^{a\psi_\theta(z)}| < \infty$$

where $\theta \in [\theta_1, \theta_2]$.

Lemma 2.15. — *1. Let I be a finite interval of $\mathbb{R}$, $\overline{I} = [\theta_1, \theta_2]$. and R a sufficiently large number. Let either $D(R) = D_I(R)$ and $\theta \in (\theta_1 - \frac{1}{2}\pi, \theta_2 + \frac{1}{2}\pi)$, or $|I| < \pi$, $D(R) = \widetilde{D}_I(R)$ and $\theta \in (\theta_2 - \frac{1}{2}\pi, \theta_1 + \frac{1}{2}\pi)$. Let*

$f : D(R) \to \mathbb{C}$ a continuous function, holomorphic in int $D(R)$. Then the following statements are equivalent.
(i). There exist positive numbers a and C, such that, for all $z \in D(R)$,

$$|f(z)| \leqslant Ce^{-a\rho_\theta(z)}$$

(ii). There exist positive numbers δ and C, such that, for all $z \in D(R)$,

$$|f(z)| \leqslant Ce^{-\delta \frac{|z|}{\log |z|}}$$

2. If f is an analytic function on a domain $\widetilde{D}_I(R)$, where I is a finite interval of $\mathbb{R}$ of length $|I| > \pi$ and $R > R_I$, with the property that

$$|f(z)| \leqslant Ce^{-c \frac{|z|}{\log |z|}}$$

for all z in this domain, where C and c are positive constants, then $f \equiv 0$.

Corollary 2.16. — *If I is an open interval of $\mathbb{R}$ of length $|I| > \pi$, then any $f \in \tilde{\mathcal{A}}_1(I)$ is uniquely determined by its asymptotic expansion.*

Lemma 2.17. — *1. Let $\theta_1 < \theta_2$, $I = [\theta_1, \theta_2]$, $R > R_I$ and $f : D_I(R) \to \mathbb{C}$ a continuous function, holomorphic in int $D_I(R)$. Let $\theta \in (\theta_1, \theta_2)$. Then the following statements are equivalent.*
(i). There exist positive numbers t and C, such that, for all $z \in D_I(R)$,

$$|f(z)| \leqslant Ce^{-tRe\ \psi_\theta(z)}$$

(ii). There exist positive numbers δ and C, such that, for all $z \in D_I(R)$,

$$|f(z)| \leqslant Ce^{-\delta |z|}$$

2. (cf. [7]). Let $\theta \in \mathbb{R}$. If there exist positive numbers δ and C, such that $|f(z)| \leqslant Ce^{-\delta|z|}$, uniformly on $D_\theta(R)$, then $f \equiv 0$.

3. Existence theorems

The first theorem concerns the existence of ordinary solutions, characterized by their asymptotic expansion $\hat{f}$. The class of equations to which it applies is larger than that mentioned in the introduction. Instead of assuming φ analytic at (∞, y_0, y_0), we assume that it satisfies certain Gevrey conditions : $\varphi \in \mathcal{A}_1(I; y_0)^n$ (cf. Definition 2.10).

THEOREM 3.1. — *Let I be a finite interval of $\mathbb{R}$, $y_0 \in \mathbb{C}^n$ and $\varphi \in \mathcal{A}_1(I;y_0)^n$, admitting an asymptotic expansion $\hat{\varphi}$. Suppose that equation (1.4) possesses a formal solution $\hat{f} \in \hat{\mathcal{O}}_1^n$ with constant term y_0, and that the following conditions are satisfied :*

I. The formal matrix functions $\hat{A}$ and $\hat{B}$ defined by

$$\hat{A} := \hat{\varphi}_1'(\hat{f}, \tau\hat{f}),\ \hat{B} := \hat{\varphi}_2'(\hat{f}, \tau\hat{f})^1$$

belong to $Gl(n, \mathcal{P})$.

II. The difference operator $\hat{\Delta} := \hat{A} + \hat{B}\tau$ has a canonical form (2.2), where Δ_j^c is of the form (2.3), with $q_j \equiv 0$ for all $j \in \{1, ..., m\}$ such that $d_j = \mu_j = 0$ (i.e. $\hat{\Delta}$ has no levels different from 1 and 1^+).

III. I contains no pseudo-Stokes directions of $\hat{\Delta}$ of level 1^+ (cf. Definition 2.2 and Remark 2.3; more precisely : $I \cap \Theta^-(\hat{\Delta}) = \emptyset$).

IV. (i) : I is closed and, if $-\pi/2$ is a Stokes direction of $\hat{\Delta}$ of level 1, then $I \cap [-\frac{\pi}{2}, \frac{\pi}{2}] = \emptyset$, or (ii) : I is an open interval : $I = (\theta_1, \theta_2)$, $\varphi \in \tilde{\mathcal{A}}_1(I;y_0)^n$ and, if $-\pi/2$ is a Stokes direction of $\hat{\Delta}$ of level 1, then either $\theta_1 < -\pi/2$ and $\theta_2 < \pi/2$, or $\theta_1 > -\pi/2$ and $\theta_2 > \pi/2$.

Then the equation (1.4) has a unique *solution $y \in \mathcal{A}_1(I)^n$ or $y \in \tilde{\mathcal{A}}_1(I)^n$, respectively, with asymptotic expansion $\hat{f}$.*

In the case that $\hat{f} = 0$, the statements remain valid if the condition $\varphi \in \mathcal{A}_1(I;y_0)^n$ or $\varphi \in \tilde{\mathcal{A}}_1(I;y_0)^n$ is replaced by : $\varphi \in \mathcal{A}(I;0)^n$ and $\varphi_0 \in \mathcal{A}_{1,0}(I)^n$, or $\varphi \in \tilde{\mathcal{A}}(I;0)^n$ and $\varphi_0 \in \tilde{\mathcal{A}}_{1,0}(I)^n$, respectively, where φ_0 is defined by $\varphi_0(z) = \varphi(z,0,0)$.

Proof. — In the cases that condition IV (i) is satisfied, or that condition IV (ii) is satisfied and $|I| \leqslant \pi$, the statements of Theorem 3.1 can be deduced from proposition 4.4 below, with the aid of Theorem 2.13, by means of a classical argument (cf. [10]). Now suppose that condition IV (ii) is satisfied and $|I| = \theta_2 - \theta_1 > \pi$. Then we have either $\theta_1 + \pi < \pi/2$ and $\theta_2 - \pi < -\pi/2$, or $\theta_1 + \pi > \pi/2$ and $\theta_2 - \pi > -\pi/2$. In both cases, (1.4) has unique solutions $y_1 \in \tilde{\mathcal{A}}_1(\theta_1, \theta_1 + \pi)^n$ and $y_2 \in \tilde{\mathcal{A}}_1(\theta_2 - \pi, \theta_2)^n$. Moreover, it has a unique solution $y \in \tilde{\mathcal{A}}_1(\theta_2 - \pi, \theta_1 + \pi)^n$ if $\theta_2 - \theta_1 < 2\pi$, or $y \in \mathcal{A}_1([\theta_1 + \pi, \theta_2 - \pi])^n$ if $\theta_2 - \theta_1 \geqslant 2\pi$. The uniqueness of these solutions implies that y_1 and y_2 are analytic continuations of y. □

Our main result is an existence and uniqueness theorem for quasi-function solutions (f_1, f_2), where f_1 and f_2 are represented asymptotically by $\hat{f}$ in

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(1) By $\hat{\varphi}_i'(y_1, y_2)$ we denote the Jacobian matrix of $\hat{\varphi}$ with respect to y_i

overlapping domains and differ by an exponentially small function on the intersection. The most interesting case is that where $|I| > \pi$: then the solutions f_1 and f_2 are accelero-sums of $\hat{f}$ (cf. Theorem 5.9).

THEOREM 3.2. — *Let $y_0 \in \mathbb{C}^n$, let I_1, I_2 be open intervals of $\mathbb{R}$: $I_i = (\theta_i^-, \theta_i^+)$ such that $\theta_1^+ \leqslant \theta_2^-$. Let $I_{12} = [\theta_1^+, \theta_2^-]$ and $I = (\theta_1^-, \theta_2^+)$. Assume that*

I. $\varphi \in \tilde{\mathcal{A}}_1(I; y_0)^n$.

II.(1.4) has a formal solution $\hat{f} \in \hat{\mathcal{O}}_1^n$ and conditions I. and II. of Theorem 3.1 are satisfied.

III. $I_1 \cup I_2$ contains no pseudo-Stokes directions of $\hat{\Delta}$ of level 1^+ (more precisely : $(I_1 \cup I_2) \cap \Theta^-(\hat{\Delta}) = \emptyset$).

IV. If $-\pi/2$ is a Stokes direction of $\hat{\Delta}$ of level 1, then either one of the following conditions holds.
a) $\theta_1^- < -\pi/2$ and $\theta_2^+ < \pi/2$,
b) $\theta_1^- > -\pi/2$ and $\theta_2^+ > \pi/2$.

Then the equation (1.4) has unique solutions $f_i \in \tilde{\mathcal{A}}_1(I_i)^n$, $i = 1, 2$, represented asymptotically by $\hat{f}$, with the property that $f_2 - f_1 \in \mathcal{A}_{1^+,0}(I_{12})^n$. Moreover, if I_{12} contains no pseudo-Stokes directions of $\hat{\Delta}$ of level 1^+, then $f_2 = f_1$.

Proof .— To begin with, assume that $|I| \leqslant \pi$. According to Theorem 2.13 there exists a function $f \in \widetilde{\mathcal{A}}_1(I)^n$, with asymptotic expansion $\hat{f}$. The substitution

$$(z, y_1, y_2) \mapsto (z, y_1 + f(z), y_2 + f(z + 1))$$

changes φ to a function $\tilde{\varphi} \in \tilde{\mathcal{A}}_1(I; 0)^n$, such that the function φ_0 defined by $\varphi_0(z) = \tilde{\varphi}(z, 0, 0)$ belongs to $\tilde{\mathcal{A}}_{1,0}(I)^n$, and the corresponding difference equation has a formal power series solution with vanishing coefficients. By another simple transformation it can be reduced to an equation in the 'prepared form' (3.1) below, and thus the theorem can be deduced from Theorem 3.3 below, except for the last statement, which follows immediately from Theorem 3.1.

Next, suppose that $|I| > \pi$. If condition IV a) holds, we define $I_2' := (\theta_2^-, \theta_2')$, where $\theta_2' = \theta_2^+$ if $|I_2| < \pi$ and $\theta_2' \in (\theta_2^-, \theta_2^- + \pi)$ if $|I_2| \geqslant \pi$, and we define $\tilde{I}_1 := (\tilde{\theta}_1^-, \tilde{\theta}_1^+)$, where $\tilde{\theta}_1^- = \max\{\theta_1^-, \theta_2' - \pi\} (< \theta_2^-)$ and $\tilde{\theta}_1^+ \in (\tilde{\theta}_1^-, \theta_2^-]$ such that $\tilde{I}_1 \cap \Theta^-(\hat{\Delta}) = \emptyset$. As both θ_1^- and $\theta_2' - \pi$ are less than $-\pi/2$, so is $\tilde{\theta}_1^-$. Moreover, $\theta_2' - \tilde{\theta}_1^- \leqslant \pi$, and thus (1.4) has solutions $\tilde{f}_1 \in \tilde{\mathcal{A}}_1(\tilde{I}_1)^n$ and $f_2' \in \tilde{\mathcal{A}}_1(I_2')^n$, represented asymptotically by $\hat{f}$, with the property that

$f_2' - \tilde{f}_1 \in \mathcal{A}_{1^+,0}([\tilde{\theta}_1^+, \theta_2^-])^n$. Without loss of generality we may assume that $\tilde{\theta}_1^+ \geqslant \theta_1^+$, hence $f_2' - \tilde{f}_1 \in \mathcal{A}_{1^+,0}([\theta_1^+, \theta_2^-])^n$. Note that either $I_2' = I_2$, in which case we put $f_2' = f_2$, or $\theta_2^- \leqslant \theta_2^+ - \pi < -\pi/2$. In the latter case, according to Theorem 3.1, (1.4) has unique solutions $f_2 \in \tilde{\mathcal{A}}_1(I_2)^n$ and $f_2' \in \tilde{\mathcal{A}}_1(I_2')^n$ with asymptotic expansion $\hat{f}$. The uniqueness implies that f_2 is the analytic continuation of f_2'. Again by Theorem 3.1, (1.4) has unique solutions $f_1 \in \tilde{\mathcal{A}}_1(I_1)^n$ and $\tilde{f}_1 \in \tilde{\mathcal{A}}_1(\tilde{I}_1)^n$ with asymptotic expansion $\hat{f}$. In the case that $\theta_2' - \pi < \theta_1^+$ this implies that f_1 is the analytic continuation of $\tilde{f}_1$ and the result follows. If $\tilde{\theta}_1^- = \theta_2' - \pi \geqslant \theta_1^+$, we proceed with $\tilde{I}_1$ in the role of I_2 and define $\tilde{I}_2 := (\tilde{\theta}_2^-, \tilde{\theta}_2^+)$, where $\tilde{\theta}_2^- = \max\{\theta_1^-, \tilde{\theta}_1^+ - \pi\}$ and $\tilde{\theta}_2^+ \in (\tilde{\theta}_2^-, \tilde{\theta}_1^-]$ such that $\tilde{\theta}_2^+ \geqslant \theta_1^+$ and $\tilde{I}_2 \cap \Theta^-(\hat{\Delta}) = \emptyset$. In this manner, we obtain a finite number of open subintervals $\tilde{I}_j = (\tilde{\theta}_j^-, \tilde{\theta}_j^+)$ of I, $j = 1, ..., N$, such that $\theta_1^+ \leqslant \tilde{\theta}_j^+ \leqslant \tilde{\theta}_{j-1}^- < \theta_2^-$ and $\tilde{\theta}_{j-1}^+ - \tilde{\theta}_j^- \leqslant \pi$ for $j = 2, ..., N$ and $\tilde{\theta}_N^- < \theta_1^+$, and unique solutions $\tilde{f}_j \in \tilde{\mathcal{A}}_1(\tilde{I}_j)^n$ with asymptotic expansion $\hat{f}$, such that $\tilde{f}_j - \tilde{f}_{j-1} \in \mathcal{A}_{1^+,0}([\tilde{\theta}_j^+, \tilde{\theta}_{j-1}^-])^n \subset \mathcal{A}_{1^+,0}([\theta_1^+, \theta_2^-])^n$. The uniqueness of f_1 and $\tilde{f}_N$ implies that these two solutions coincide. Consequently, $f_1 - f_2 = \sum_{j=2}^N \tilde{f}_j - \tilde{f}_{j-1} + \tilde{f}_1 - f_2 \in \mathcal{A}_{1^+,0}([\theta_1^+, \theta_2^-])^n$. The remaining cases can be proved similarly. □

THEOREM 3.3. — *Let I_1, I_2 be open intervals of $\mathbb{R}$: $I_i = (\theta_i^-, \theta_i^+)$, such that $\theta_1^+ \leqslant \theta_2^-$ and $\theta_2^+ - \theta_1^- \leqslant \pi$. Let $I_{12} = [\theta_1^+, \theta_2^-]$, $I = (\theta_1^-, \theta_2^+)$ and $\varphi \in \tilde{\mathcal{A}}(I; 0)^n$. Assume that*

I. φ can be written in the form

$$\varphi(z, y_1, y_2) = \varphi_0(z) + A(z)y_1 + B(z)y_2 + \psi(z, y_1, y_2)$$

where $\varphi_0 \in \tilde{\mathcal{A}}_{1,0}(I)^n$, A and $B \in$ End $(n; \tilde{\mathcal{A}}_{0,0}(I))$ and for any closed subinterval I' of I, there exists a positive number $R > R_I$, such that $\psi_2'(z, 0, 0) = \psi_3'(z, 0, 0) = 0$ for all $z \in \widetilde{D}_{I'}(R)$.

II. $I_1 \cup I_2$ contains no pseudo-Stokes directions of $\hat{\Delta}$ of level 1^+ (more precisely : $(I_1 \cup I_2) \cap \Theta^-(\hat{\Delta}) = \emptyset$).

III. If $-\pi/2$ is a Stokes direction of $\hat{\Delta}$ of level 1, then either $\theta_1^- < -\pi/2$ or $\theta_2^+ > \pi/2$. Then the equation

$$\Delta^c y(z) = \varphi(z, y(z), y(z+1)) \tag{3.1}$$

where Δ^c is of the form (2.3), with $q_j \equiv 0$ for all $j \in \{1, ..., m\}$ such that $d_j = \mu_j = 0$, has unique solutions $f_i \in \tilde{\mathcal{A}}_{1,0}(I_i)^n$, $i = 1, 2$, with the property that $f_2 - f_1 \in \mathcal{A}_{1^+,0}(I_{12})^n$.

An outline of the proof of this theorem can be found at the end of §4.2.

4. Right inverses of Δ_j^c

4.1. Right inverses of Δ_j^c on Banach spaces of analytic functions

Definition 4.1. — *Let D be a closed domain of S_+. Let $\theta \in \mathbb{R}$ and $a > 0$. By $b^1_{\theta,a}(D)$ we denote the Banach space of continuous functions $f : D \to \mathbb{C}$ that are holomorphic in int D and have the property that*

$$\|f\|^\theta_{a,D} := \sup_{z\in D} |e^{ar_\theta(z)} f(z)| < \infty$$

By $b^{1^+}_{\theta,a}(D)$ we denote the Banach space of continuous functions $f : D \to \mathbb{C}$ that are holomorphic in int D and have the property that

$$|\!|\!| f |\!|\!|^\theta_{a,D} := \sup_{z\in D} |e^{a\psi_\theta(z)} f(z)| < \infty$$

We will consider the Banach spaces $b^1_{\theta,a}(D)^n$ and $b^{1^+}_{\theta,a}(D)^n$ equipped with the norms $\|(f_1, ..., f_n)\|^\theta_{a,D} := \max_{i\in\{1,...,n\}} \|f_i\|^\theta_{a,D}$ and $|\!|\!|(f_1, ..., f_n)|\!|\!|^\theta_{a,D} := \max_{i\in\{1,...,n\}} |\!|\!|f_i|\!|\!|^\theta_{a,D}$, respectively.

Definition 4.2. — *Let I be a finite interval of $\mathbb{R}$. We define*

$$J_1(I) = \begin{cases} \{j \in \{1, ..., m\} : d_j = 0, \mu_j \neq 0 : \arg \mu_j = \pi\} & \text{if } I \cap (-\frac{\pi}{2}, \frac{\pi}{2}) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

$$\widetilde{J}_1(I) = \begin{cases} J_1(I) & \text{if } I \subset (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \emptyset & \text{otherwise} \end{cases}$$

$$J_{1^+}(I) := \{j \in \{1, ..., m\} : \Theta^-(\Delta_j^c) \cap I \neq \emptyset\}$$

Remark 4.3. — $J_1(I)$ or $\widetilde{J}_1(I) \neq \emptyset$ implies that $D_I(R_I)$ or $\widetilde{D}_I(R_I)$ is contained in the right half plane and $-\pi/2$ is a Stokes direction of level 1. $J_1(I) \cup J_{1^+}(I)$ is the set of indices $j \in \{1, ..., m\}$ such that, for some $l \in \mathbb{Z}$ (equal to 0 in the case that $d_j = 0$, $\mu_j \neq 0$ and $\arg \mu_j = \pi$), $y_j^c(z)e^{2l\pi iz} \sim 0$ as $z \to \infty$, uniformly on $D_I(R_I)$. Obviously, $J_1(I) = J_1(\overline{I})$.

$j \in \widetilde{J}_1(I)$ implies that $y_j^c(z) \sim 0$ as $z \to \infty$ on $\widetilde{D}_I(R_I)$, uniformly on $\widetilde{D}_{I'}(R_I)$, for any closed subinterval $I' \subset$ *int* I and uniformly on $\widetilde{D}_I(R_I)$ if $\overline{I} \subset (-\pi/2, \pi/2)$.

In what follows we shall associate with certain closed intervals I of $\mathbb{R}$ two types of closed domains $D(R)$ with the property that

$$d(R) := \min\{|z| : z \in D(R)\} = R(1 + o(1)) \text{ as } R \to \infty \tag{4.1}$$

These are defined as follows. Let $I = [\theta_1, \theta_2]$, $R > R_I$ and assume that $\{\theta_1, \theta_2\} \cap \{-\pi/2, \pi/2\} = \emptyset$. If $J_1(I) = \emptyset$, $D(R) = D_I(R)$ or $\widetilde{D}_I(R)$ (this is the case when $-\pi/2$ is not a Stokes direction of level 1, or, when either $\theta_2 < -\pi/2$, or $\theta_1 > \pi/2$). Next, suppose that $\tilde{J}_1(I) = \emptyset$, but $J_1(I) \neq \emptyset$, and $|I| < \pi$. This implies that either $\theta_1 < -\pi/2 < \theta_2 < \pi/2$ or $-\pi/2 < \theta_1 < \pi/2 < \theta_2$. In the first case we choose $\phi \in (-\pi/2, -\theta_2)$, in the second case we choose $\phi \in (-\theta_1, \pi/2)$, and in both cases we define : $D(R) := \widetilde{D}_I(Re^{i\phi})$. From (2.5) and (2.7) it can be seen that, on the boundary of $D(R)$, Re z decreases monotonely to $-\infty$ (and thus $e^{\mu_j z}$ increases monotonely to ∞ for all $j \in J_1(I)$) as Im $z \to \infty$ in the first and as $\operatorname{Im} z \to -\infty$ in the second case, provided R is sufficiently large. (Cf. Figure 3.)

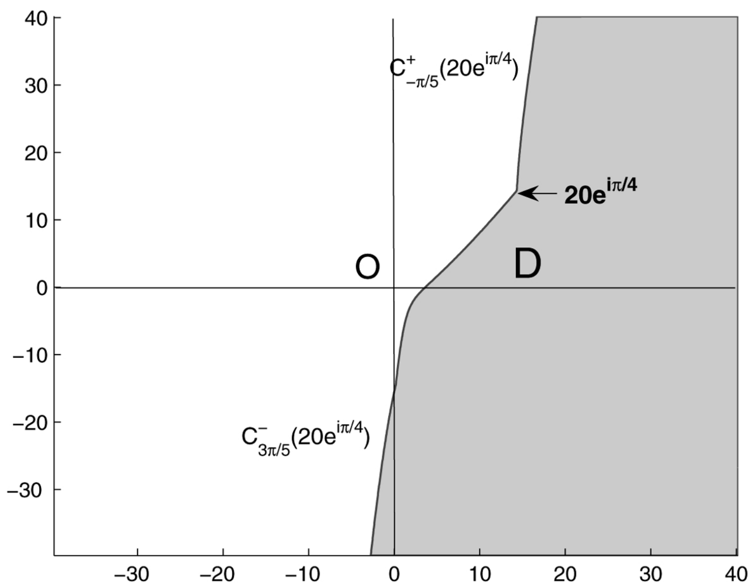

Figure 3. — In this picture, $R = 20$, $\phi = \frac{\pi}{4}$ and $D = \widetilde{D}_{[-\frac{\pi}{5}, \frac{3\pi}{5}]}(Re^{i\phi})$

Proposition 4.4. — *Let $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 < \theta_2$ and $I = [\theta_1, \theta_2]$. Suppose that $J_{1^+}(I) = \tilde{J}_1(I) = \{\theta_1, \theta_2\} \cap \{-\pi/2, \pi/2\} = \emptyset$. We consider the following two cases :*
a) $J_1(I) = \emptyset$, $D(R) = D_I(R)$ and $\theta \in (\theta_1 - \pi/2, \theta_2 + \pi/2)$, or
b) $|I| < \pi$ and $D(R) = \widetilde{D}_I(Re^{i\phi})$, where $\phi = 0$ if $J_1(I) = \emptyset$ and, in the case that $J_1(I) \neq \emptyset$, $\phi \in (-\pi/2, -\theta_2)$ if $\theta_1 < -\pi/2 < \theta_2 < \pi/2$, $\phi \in (-\theta_1, \pi/2)$ if $-\pi/2 < \theta_1 < \pi/2 < \theta_2$, and $\theta \in (\theta_2 - \pi/2, \theta_1 + \pi/2)$.

There exist positive numbers a_0 and R_0 such that, for each $j \in \{1, ..., m\}$, Δ_j^c is a bijective mapping from $\cup_{(a,R)} b_{\theta,a}^1(D(R))^{n_j}$ onto itself, where the union is taken over all $a \in (0, a_0)$ and $R > R_0$. Its inverse Λ_j^c has the following properties :
(i) There exists a real number ν_j such that, for all $a \in (0, a_0)$ and $R > R_0$,

$$\Lambda_j^c(b_{\theta,a}^1(D(R))^{n_j}) \subset z^{\nu_j} b_{\theta,a}^1(D(R))^{n_j}$$

(ii) There exists a positive constant C_j', independent of R, such that

$$\|z^{-\nu_j}\Lambda_j^c f\|_{a,D(R)}^\theta \leqslant C_j' \|f\|_{a,D(R)}^\theta \tag{4.2}$$

for all $f \in b_{\theta,a}^1(D(R))^{n_j}$, provided $a \in (0, a_0)$ and $R > R_0$.

Proof. — Let $j \in \{1, ..., m\}$, $\theta \in \mathbb{R}$, $R > R_I$, $a > 0$ and $f \in b_{\theta,a}^1(D(R))^{n_j}$. We define $\Lambda_j^c f$ by

$$\Lambda_j^c f(z) = y_j^c(z) \int_{C_j(z')} \frac{e^{-2l_j\pi i(\zeta - z)} y_j^c(\zeta+1)^{-1} f(\zeta)}{e^{2\pi i(\zeta - z)} - 1} d\zeta - y_j^c(z) y_j^c(z+1)^{-1} f(z)$$

where l_j is a suitable integer and $C_j(z')$ is a path in $D(R)$ with limiting directions $\alpha^- \in [-\pi/2, 0)$ and $\alpha^+ \in (0, \pi/2]$, intersecting the line $\operatorname{Im} \zeta = \operatorname{Im} z$ exactly once, at a point z' on the segment $(z, z+1)$, described in the direction of increasing imaginary part. We shall briefly discuss each of the different cases that may occur and derive estimates of the form (4.2). For more details we refer the reader to [5, 9], where similar proofs can be found. Without loss of generality, we may assume that $\|f\|_{a,D(R)}^\theta = 1$. We begin by noting that, due to lemma 2.7(i), we have, for all $z \in D(R)$,

$$|y_j^c(z) y_j^c(z+1)^{-1}| \leqslant K_1 |z|^{-d_j} \tag{4.3}$$

and for all $\zeta \in D(R)$, choosing $z' = z + 1/2$,

$$|y_j^c(z) y_j^c(\zeta+1)^{-1}| \leqslant K_1 K_{1/2} |z|^{-d_j/2} |\zeta|^{-d_j} |y_j^c(z') y_j^c(\zeta)^{-1}| \tag{4.4}$$

provided R is sufficiently large.

Case 1. $d_j = 0$ and $\mu_j = 0$. We can take $C_j(z')$ to consist of a half line from $\infty e^{i\alpha^-}$ to z' and a half line from z' to $\infty e^{i\alpha^+}$, where $\alpha^- \in (-\pi/2, 0)$ and $\alpha^+ \in (0, \pi/2)$. With the aid of residue calculus we find

$$\Lambda_j^c f(z) = -y_j^c(z) \sum_{h=0}^{\infty} y_j^c(z+h+1)^{-1} f(z+h) \tag{4.5}$$

(Or, alternatively, $\Lambda_j^c f$ could be defined by the above expression.) We have

$$\frac{d}{dz}\log y_j^c(z) = \frac{C_j}{z}$$

and

$$r'_\theta(z) = 1 + i\theta\left(\frac{1}{\log z} - \frac{1}{(\log z)^2}\right) \tag{4.6}$$

Hence we deduce that, for all $z \in D(R)$ and all $h \in \mathbb{N}$,

$$e^{a\rho_\theta(z) - a\rho_\theta(z+h)}|y_j^c(z)y_j^c(z+h+1)^{-1}| \leqslant Ke^{-ah/2}$$

where K is a positive number, independent of R, provided R is sufficiently large. It follows that

$$e^{a\rho_\theta(z)}|\Lambda_j^c f(z)| \leqslant K\sum_{h=0}^{\infty} e^{-ah/2}$$

Obviously, the right-hand side is independent of R. Thus $\Lambda_j^c f \in b_{\theta,a}^1(D(R))^{n_j}$ ($\nu_j = 0$) and $\|\Lambda_j^c f\|_{a,D(R)}^\theta \leqslant C'_j$, where the constant $C'_j = K(1 - e^{-a/2})^{-1}$ is independent of R, provided R is sufficiently large.

Case 2. $d_j > 0$. We can choose a path $C_j(z')$, similar to that in case 1. Here again, the use of residue calculus yields the representation (4.5), and, with the aid of (4.3), we obtain an estimate of the form

$$e^{a\rho_\theta(z)}|\Lambda_j^c f(z)| \leqslant \sum_{h=0}^{\infty} K_1^h e^{-ah/2}|z(z+1)...(z+h)|^{-d_j} \leqslant C'_j|z|^{-d_j}$$

where C'_j is a positive constant, independent of R. This implies that $\Lambda_j^c f \in z^{-d_j}b_{\theta,a}^1(D(R))^{n_j}$ ($\nu_j = -d_j$) and $\|z^{d_j}\Lambda_j^c f\|_{a,D(R)}^\theta \leqslant C'_j$, where C'_j is a positive constant, independent of R, provided R is sufficiently large.

Case 3. If $d_j = 0$, $\mu_j \neq 0$ and $|\arg \mu_j| < \pi$ (in view of the definition of μ_j this implies $0 \leqslant \arg \mu_j < \pi$), we take $l_j = -1$ and $C_j(z')$ to consist of two half lines $C_j^-(z')$ and $C_j^+(z')$ from z' to ∞ with directions

$$\alpha_j^- \in (-\pi/2, \min\{0, \pi/2 - \arg(\mu_j + a)\})$$

and

$$\alpha_j^+ \in (\max\{0, 3\pi/2 - \arg(\mu_j + a - 2\pi i)\}, \pi/2),$$

respectively. Here we choose $\arg(\mu_j + a) \in [0, \pi)$ and $\arg(\mu_j + a - 2\pi i) \in (\pi, 2\pi)$ (note that Im $(\mu_j + a - 2\pi i) < 0$). According to lemma 2.6(ii),

$C_j^-(z')$ and $C_j^+(z')$ are contained in $D(R)$ if R is sufficiently large. Putting $\zeta = z' + e^{i\alpha_j^+}x$ if $\zeta \in C_j^+(z')$ and using (4.6) and (4.1), we have

$$a(\rho_\theta(z') - \rho_\theta(\zeta)) + \text{Re } (\mu_j - 2\pi i)(z' - \zeta)$$

$$= -(\text{Re } ((\mu_j + a - 2\pi i)e^{i\alpha_j^+}) + O(\frac{1}{\log R}))x$$

Similarly, putting $\zeta = z' + e^{i\alpha_j^-}x$ if $\zeta \in C_j^-(z')$, we have

$$a(\rho_\theta(z') - \rho_\theta(\zeta)) + \text{Re } \mu_j(z' - \zeta) = -(\text{Re } ((\mu_j + a)e^{i\alpha_j^-}) + O(\frac{1}{\log R}))x$$

Using lemma 2.7(ii) and the fact that both $\cos(\arg(\mu_j + a - 2\pi i) + \alpha_j^+) > 0$ and $\cos(\arg(\mu_j + a) + \alpha_j^-) > 0$, one easily obtains an estimate of the form

$$e^{a\rho_\theta(z)}|y_j^c(z')| \int_{C_j(z')} |\frac{e^{2\pi i(\zeta - z)}y_j^c(\zeta)^{-1}f(\zeta)}{e^{2\pi i(\zeta - z)} - 1}| \, |d\zeta| \leqslant K \int_0^\infty e^{-\delta x}dx$$

where K and δ are positive numbers, independent of R, provided R is sufficiently large. With the aid of (4.3) and (4.4) we conclude that $\Lambda_j^c f \in b_{\theta,a}^1(D(R))^{n_j}$ $(\nu_j = 0)$ and $\|\Lambda_j^c f\|_{a,D(R)}^\theta \leqslant C_j'$, where C_j' is a positive constant, independent of R.

Case 4. If $d_j < 0$, we choose $C_j(z')$ to consist of $C_{\theta_1}^-(z')$ and $C_{\theta_2}^+(z')$. According to lemma 2.6(i), these paths lie in $D(R)$ for all $z \in D(R)$. Let $\theta' \in \mathbb{R}$. Putting $\zeta = \zeta(x)$ if $\zeta \in C_{\theta'}(z)$ and using the notation of §2.1, the identities (2.5), (2.8) and (4.6), we have

$$\zeta'(x) = \rho'(x) + i = \frac{\phi(x) + \theta'}{1 + \log r(x)} + i, \tag{4.7}$$

$$\frac{d}{dx}\log r(x) = O(\frac{1}{r(x)}), \tag{4.8}$$

$$\frac{d}{dx}\rho_\theta(\zeta(x)) = \text{Re } \{[1 + i\theta(\frac{1}{\log\zeta(x)} - \frac{1}{(\log\zeta(x))^2})]\zeta'(x)\} =$$

$$= \frac{\phi(x) + \theta' - \theta}{\log r(x)} + O(\frac{1}{(\log r(x))^2}) \tag{4.9}$$

and

$$\frac{d}{dx}\log|y_j^c(\zeta(x))e^{2l_j\pi i\zeta(x)}| = \text{Re }\{[d_j(\log\zeta(x)+1)+(\mu_j+2l_j\pi i)]\zeta'(x)\}+$$

$$+O(r(x)^{-1/p}) = d_j\theta' - \text{Im }\mu_j - 2l_j\pi + O(\frac{1}{\log r(x)}) \qquad (4.10)$$

Due to the assumption that $J_{1+}(I)=\emptyset$, there exists a unique integer n with the property that

$$\frac{\text{Im }\mu_j + 2(n+1)\pi}{d_j} < \theta_1 \text{ and } \frac{\text{Im }\mu_j + 2n\pi}{d_j} > \theta_2$$

Now, choose l_j to be this integer. Then we have

$$d_j\theta_2 - \text{Im }\mu_j - 2l_j\pi > 0 \text{ and } d_j\theta_1 - \text{Im }\mu_j - 2(l_j+1)\pi < 0 \qquad (4.11)$$

Using (4.3), (4.4), (4.10), (4.11) and lemma 2.7(ii), we obtain an estimate of the form

$$e^{a\rho_\theta(z)}|z^{d_j}y_j^c(z)|\int_{C_j(z')}|\frac{e^{-2l_j\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f(\zeta)}{e^{2\pi i(\zeta-z)}-1}||d\zeta| \leqslant K\int_0^\infty e^{-\delta x}dx$$

where K and δ are positive numbers, independent of R, provided R is sufficiently large. With the aid of (4.4) we conclude that $\Lambda_j^c f \in z^{-\frac{3}{2}d_j}b^1_{\theta,a}(D(R))^{n_j}$ ($\nu_j = -\frac{3}{2}d_j$) and $\|z^{\frac{3}{2}d_j}\Lambda_j^c f\|^\theta_{a,D(R)} \leqslant C'_j$, where C'_j is a positive constant, independent of R. (Note that a similar result holds for any $\nu_j > -d_j$, as we can choose z' arbitrarily close to z.)

Case 5. Now suppose $d_j=0$, $\mu_j\neq 0$, $\arg\mu_j=\pi$. The assumption $\tilde{J}_1(I)=\emptyset$ implies that $\theta_1 < -\pi/2$, or $\theta_2 > \pi/2$. We shall discuss the case that $\theta_1 < -\pi/2$ and take $l_j=0$.

First, suppose that $J_1(I)=\emptyset$, so $\theta_2 < -\pi/2$ as well. Let $D(R)=D_I(R)$ and $\theta\in(\theta_1-\pi/2,\theta_2+\pi/2)$ (case a) of proposition 4.4), or $|I|<\pi$, $D(R)=\widetilde{D}_I(R)$ and $\theta\in(\theta_2-\pi/2,\theta_1+\pi/2)$ (case b)). In both cases, $\theta<0$. We choose $C_j(z')$ to consist of a half line $C_j^-(z')$ from z' to ∞ with direction $\alpha_j^-\in(-\pi/2,\min\{0,\pi/2-\arg(\mu_j+a+2\pi i)\})$ and $C^+_{\theta_2}(z')$. The integral over $C_j^-(z')$ is similar to that in case 3 above. Using lemma 2.7(ii) and (4.4), we find that, for all $\zeta\in C^+_{\theta_2}(z')$,

$$e^{a\rho_\theta(z)}|\frac{y_j^c(z)y_j^c(\zeta+1)^{-1}f(\zeta)}{e^{2\pi i(\zeta-z)}-1}| \leqslant Ke^{a(\rho_\theta(z)-\rho_\theta(\zeta))}|y_j^c(z')y_j^c(\zeta)^{-1}| \qquad (4.12)$$

where K is a positive constant, independent of R, provided R is sufficiently large. We put $\zeta=\zeta(x)$ for all $\zeta\in C_{\theta_2}(z')$ and use the notation of §2.1. In view of (2.7), there exists a positive number δ such that $\phi(x)+\theta_2<-\delta$ for all $x\in\mathbb{R}$, provided R is sufficiently large. From (2.5) we deduce that ρ is monotone decreasing on $\mathbb{R}$. With the aid of (4.7) and (4.9) we find

$$\frac{d}{dx}\rho_\theta(\zeta(x)) = (\frac{\phi(x)+\theta_2-\theta}{\phi(x)+\theta_2}+O(\frac{1}{\log R}))\rho'(x),$$

$$\frac{d}{dx}\log|y_j^c(\zeta(x))| = \mu_j\rho'(x)+O(r(x)^{-1/p}) = (\mu_j+O(\frac{1}{\log R}))\rho'(x),$$

$$|d\zeta(x)| = O(\log|\zeta(x)|)|d\rho(x)|$$

and

$$\frac{d}{dx}\log\log|\zeta(x)| = O(\frac{1}{R})\rho'(x)$$

After a change of variable, setting $\rho(z')=\rho_0$ and noting that, for all $x\in\mathbb{R}$, $\frac{\theta}{\phi(x)+\theta_2}>0$, we obtain

$$\int_{C^+_{\theta_2}(z')} e^{a(\rho_\theta(z')-\rho_\theta(\zeta))}(\log|z'|)^{-1}|y_j^c(z')y_j^c(\zeta)^{-1}||d\zeta|$$

$$\leqslant \int_{-\infty}^{\rho_0} e^{(\mu_j+a+O(\frac{1}{\log R}))(\rho_0-\rho)}d\rho$$

provided $a<|\mu_j|$ and R is sufficiently large. With the aid of (4.12) we conclude that $\Lambda_j^c f\in z^{\nu_j}b^1_{\theta,a}(D(R))^{n_j}$ for $a<|\mu_j|$ and any positive number ν_j, and $\|z^{-\nu_j}\Lambda_j^c f\|^\theta_{a,D(R)}\leqslant C_j'$, where C_j' is a positive constant, independent of R, provided R is sufficiently large.

Next, we consider the case that $|I|<\pi$, $D(R)=\widetilde{D}_I(R)$ and $J_1(I)\neq\emptyset$, so $-\pi/2<\theta_2<\theta_1+\pi<\pi/2$, and $\theta\in(\theta_2-\pi/2,\theta_1+\pi/2)$. In this case, $D(R)=\widetilde{D}_I(Re^{i\phi})$, with $\phi\in(-\pi/2,-\theta_2)$. If $z'\in D_{\theta_1}(Re^{i\phi})$, we can proceed as above. If $z'\notin D_{\theta_1}(Re^{i\phi})$, we have to choose a slightly more complicated path of integration, consisting of a half line $C_j^-(z')$ as above, the arc of $C^+_{\theta(z)}(z')$ connecting z' and $Re^{i\phi}$, where $\theta(z)\in[\theta_1,\theta_2]$ is defined by $\mathrm{Re}\ \psi_{\theta(z)}(z')=\mathrm{Re}\ \psi_{\theta(z)}(Re^{i\phi})$, and $C^+_{\theta_1}(Re^{i\phi})$. The integrals over the path from z' to $Re^{i\phi}$ and $C^+_{\theta_1}(Re^{i\phi})$ can be estimated similarly to the integral over $C^+_{\theta_2}(z')$ in the previous case, due to the fact that, on each of these paths, $\mathrm{Re}\ \zeta$ decreases sufficiently fast as $\mathrm{Im}\ \zeta$ increases (cf. the proof of proposition 4.7). The case $\theta_2>\pi/2$ is similar.

Moreover, a careful analysis reveals that R_0 can be taken independent of a, provided $a<a_0<\inf\{|\mu_j| : j\in\{1,...,m\}, d_j=0, \arg\mu_j=\pi\}$. This completes the proof of (i) and (ii).

The proof of the fact that $\Lambda_j^c \Delta_j^c f = \Delta_j^c \Lambda_j^c f = f$ for all $f \in b^1_{\theta,a}(D(R))^{n_j}$ is straightforward and similar to, for example, the proof of the corresponding part of proposition 3.6 in [9]. It involves deformation of contours and application of Cauchy's theorem. □

4.2. Right inverses of Δ_j^c on Banach spaces of quasi-functions

In this section we consider Banach spaces of so-called **quasi-functions** : pairs of functions, defined on overlapping domains $D_1(R)$ and $D_2(R)$, and differing by an exponentially small function on the intersection of the domains. The domains $D_1(R)$ and $D_2(R)$ are defined as follows. Let I_1 and I_2 be finite, closed intervals of $\mathbb{R}$: $I_i = [\theta_i^-, \theta_i^+]$, such that $\theta_1^+ < \theta_2^-$, $\theta_2^+ - \theta_1^- < \pi$. Let R be a sufficiently large positive number. If $\theta_1^- < -\pi/2 < \theta_2^+ < \pi/2$ we choose $\phi_{12} \in (-\pi/2, -\theta_2^+)$, if $-\pi/2 < \theta_1^- < \pi/2 < \theta_2^+$ we choose $\phi_{12} \in (-\theta_1^-, \pi/2)$, otherwise we take $\phi_{12} = 0$. In all cases we define $z_{12} = Re^{i\phi_{12}}$ and $D_i(R) := \widetilde{D}_{I_i}(z_{12})$ for $i = 1, 2$. Furthermore, we put $D_1 \cap D_2 = D_{12}$ and $D_1 \cup D_2 = D$. Note that D is defined similarly to $D(R)$ in proposition 4.4 b), with $I = [\theta_1^-, \theta_2^+]$. (cf. Figure 4.)

DEFINITION 4.5. — *Let I_1, I_2 be finite, closed intervals of $\mathbb{R}$: $I_i = [\theta_i^-, \theta_i^+]$, such that $\theta_1^+ < \theta_2^-$ and $\theta_2^+ < \theta_1^- + \pi$. Let R be a sufficiently large positive number and, for $i = 1, 2$, let $D_i := D_i(R)$ denote the domain defined above and $D_{12} = D_1 \cap D_2$. Let $\theta \in (\theta_2^+ - \pi/2, \theta_1^- + \pi/2)$, $\theta' \in (\theta_1^+, \theta_2^-)$, a and $b > 0$. By $B^{a,\theta}_{b,\theta'}(R)$ we denote the Banach space of quasi-functions $F = (f_1, f_2) \in b^1_{\theta,a}(D_1) \times b^1_{\theta,a}(D_2)$ with the property that $f_1 - f_2 \in b^{1^+}_{\theta',b}(D_{12})$, equipped with the norm*

$$\|F\| := \max\{\|f_1\|^\theta_{a,D_1}, \|f_2\|^\theta_{a,D_2}, K(R)\|\!|f_1 - f_2\|\!|^{\theta'}_{b,D_{12}}\}$$

where $K(R) = \sup_{z\in D_{12}} |e^{ar_\theta(z) - b\psi_{\theta'}(z)}|$.
By τF and $\Delta_j^c F$ we denote the quasi-functions

$$\tau F := (\tau f_1, \tau f_2), \ \ \Delta_j^c F := (\Delta_j^c f_1, \Delta_j^c f_2)$$

and by $z^\nu F$ the quasi-function (g_1, g_2), where $g_i(z) = z^\nu f_i(z)$ for all $z \in D_i$, $i = 1, 2$.

Remark 4.6. — Obviously, $K(R)$ also depends on a, b, θ and θ'. For all $z \in D_{12}$ we have

$$\mathrm{Re}\ (\psi_{\theta'}(z) - \psi_{\theta'}(z_{12})) \geqslant \max\{(\theta' - \theta_1^+)\mathrm{Im}\ (z_{12} - z), (\theta_2^- - \theta')\mathrm{Im}\ (z - z_{12})\}$$

With the aid of this inequality it is easily seen that, for every $\theta' \in (\theta_1^+, \theta_2^-)$ and every $\theta \in \mathbb{R}$, $K(R) = |e^{ar_\theta(z_{12}) - b\psi_{\theta'}(z_{12})}|$, provided R is sufficiently large.

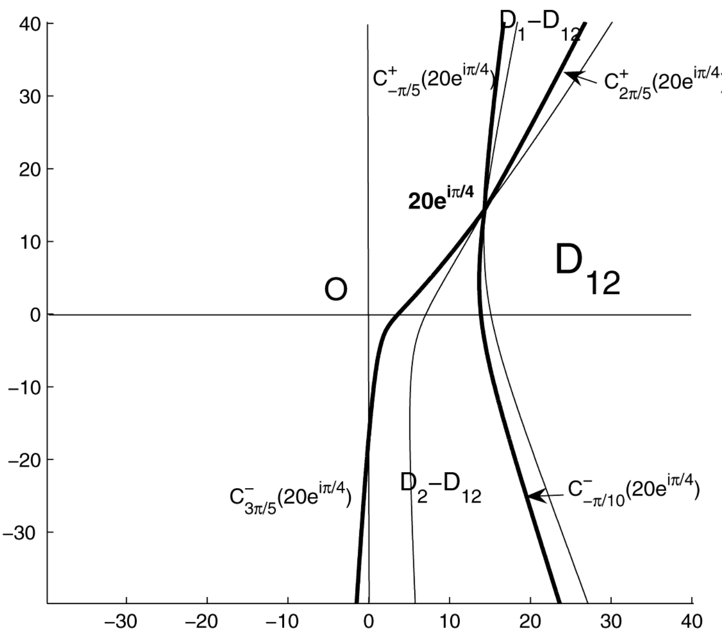

Figure 4. — In this picture, $I_1 = [-\frac{\pi}{5}, -\frac{\pi}{10}]$, $I_2 = [\frac{2\pi}{5}, \frac{3\pi}{5}]$, $R = 20$ and $\phi = \frac{\pi}{4}$

PROPOSITION 4.7. — *Let I_1, I_2 be finite, closed intervals of $\mathbb{R}$: $I_i = [\theta_i^-, \theta_i^+]$, $i \in \{1, 2\}$, such that $\theta_1^+ < \theta_2^-$, $\theta_2^+ < \theta_1^- + \pi$ and, for $i = 1, 2$, $J_{1^+}(I_i) = \{\theta_i^-, \theta_i^+\} \cap \{-\pi/2, \pi/2\} = \emptyset$. Moreover, if there is a $j \in \{1, ..., m\}$ such that $d_j = 0$, $\mu_j \neq 0$ and $\arg \mu_j = \pi$, we assume that either $\theta_1^- < -\pi/2$ or $\theta_2^+ > \pi/2$.*

Let $\theta \in (\theta_2^+ - \pi/2, \theta_1^- + \pi/2)$ and $\theta' \in (\theta_1^+, \theta_2^-)$. There exist positive numbers a_0, b_0 and R_0 such that, for each $j \in \{1, ..., m\}$, Δ_j^c is a bijective mapping from $\cup_{(a,b,R)} B_{b,\theta'}^{a,\theta}(R)^{n_j}$ onto itself, where the union is over all $a \in (0, a_0)$, $b \in (0, b_0)$ and $R > R_0$. Its inverse Λ_j^q has the following properties :
(i) There exists a real number ν_j such that, for all $a \in (0, a_0)$, $b \in (0, b_0)$ and $R > R_0$,

$$\Lambda_j^q(B_{b,\theta'}^{a,\theta}(R)^{n_j}) \subset z^{\nu_j} B_{b,\theta'}^{a,\theta}(R)^{n_j}$$

(ii) There exists a positive constant C_j', independent of R, such that

$$\|z^{-\nu_j} \Lambda_j^q F\| \leqslant C_j' \|F\|$$

for all $F \in B_{b,\theta'}^{a,\theta}(R)^{n_j}$, provided $a \in (0, a_0)$, $b \in (0, b_0)$ and $R > R_0$.

Proof. — Let $j \in \{1, ..., m\}$, a and $b > 0$, $R > R_I$, where $I = [\theta_1^-, \theta_2^+]$, let $F = (f_1, f_2) \in B^{a,\theta}_{b,\theta'}(R)^{n_j}$ and $i \in \{1, 2\}$. Again, without loss of generality, we may assume that $\|F\| = 1$. If $j \notin \tilde{J}_1(I_i)$ (cf. definition 4.2) we define $g_i := (\Lambda_j^q F)_i = \Lambda_j^c f_i$, where Λ_j^c denotes the mapping mentioned in proposition 4.4. According to proposition 4.4, $\Lambda_j^c f_i \in z^{\nu_j} b^1_{\theta,a}(D_i)^{n_j}$ and

$$\|z^{-\nu_j}\Lambda_j^c f_i\|^\theta_{a,D_i} \leqslant C_j^i \|f_i\|^\theta_{a,D_i} \leqslant C_j^i$$

where C_j^i is a positive constant, independent of R, provided R is sufficiently large.

Now suppose that $j \in \widetilde{J}_1(I_i)$. This implies that $d_j = 0$, $\mu_j \neq 0$ and $\arg\mu_j = \pi$, and $I_i \subset (-\pi/2, \pi/2)$. We shall discuss the case that $i = 2$, i.e. $j \notin \tilde{J}_1(I_1)$, but $j \in \tilde{J}_1(I_2)$. The case that $j \notin \tilde{J}_1(I_2)$, but $j \in \tilde{J}_1(I_1)$ is similar. For all $z \in D_2$ we define

$$g_2(z) := (\Lambda_j^q F)_2(z) := y_j^c(z) \int_{C_j^+(z_*)} \frac{y_j^c(\zeta+1)^{-1} f_1(\zeta)}{e^{2\pi i(\zeta - z)} - 1} d\zeta +$$

$$-y_j^c(z) \int_{C_j^-(z')} \frac{y_j^c(\zeta+1)^{-1} f_2(\zeta)}{e^{2\pi i(\zeta - z)} - 1} d\zeta + y_j^c(z) \int_{C_j(z', z_*)} \frac{y_j^c(\zeta+1)^{-1} f_2(\zeta)}{e^{2\pi i(\zeta - z)} - 1} d\zeta$$

$$+y_j^c(z) \int_{C_j^{12}(z_*)} \frac{y_j^c(\zeta+1)^{-1} (f_2 - f_1)(\zeta)}{e^{2\pi i(\zeta - z)} - 1} d\zeta - y_j^c(z) y_j^c(z+1)^{-1} f_2(z)$$

Here z_* is some point in $D_{\theta_1^-}(z_{12}) \cap D_2$, $C_j^+(z_*) = C^+_{\theta_1^-}(z_*)$, $C_j(z', z_*)$ is a suitable path from z' to z_*, $C_j^-(z')$ is a half line from z' to ∞ in a direction $\alpha_j^- \in (-\pi/2, \pi/2 - \arg(\mu_j + a + 2\pi i))$ and $C_j^{12}(z_*)$ is a half line from z_* to ∞ in a direction $\alpha_j^+ \in (0, \pi/2)$. If $a < |\mu_j|$, we can choose $\arg(\mu_j + a + 2\pi i) \in (\pi/2, \pi)$, so that $\alpha_j^- \in (-\pi/2, 0)$. (Note that, essentially, the path $C_j^-(z') \cup C_j(z', z_*) \cup C_j^+(z_*)$ is the same as the path $C_j(z')$ in the last part of the proof of case 5 of proposition 4.4, if we replace θ_1 and θ_2 in that proof by θ_1^- and θ_2^+, respectively. The fact that the function in the integrand changes from f_2 to f_1 at some point on this path, necessitates a "corrective term" involving the difference $f_2 - f_1$.)
We deduce from lemma 2.6 (iii) that $C_j^+(z_*) \subset D_1$, $C_j^{12}(z_*) \subset D_{12}$, and $C_j^-(z') \subset D_2$ for all $z \in D_2$ if R is sufficiently large. It is easily seen that, within certain limits, the above definition is independent of the choice of z_*. If Re $\psi_{\theta_1^-}(z') \geqslant$ Re $\psi_{\theta_1^-}(z_{12})$ (i.e. $z' \in D_2 \cap D_{\theta_1^-}(z_{12}) \subset D_{12}$), we take $z_* = z'$, otherwise we choose it to be the intersection of ∂D_1 and ∂D_2, i.e. z_{12}. In the latter case we have Re $\psi_{\theta_2^+}(z) \geqslant$ Re $\psi_{\theta_2^+}(z_{12})$ and

$\text{Re } \psi_{\theta_1^-}(z) \leqslant \text{Re } \psi_{\theta_1^-}(z_{12})$, or, equivalently,

$$\theta_1^- \text{Im } (z_{12} - z) \leqslant \text{Re } (z_{12} \log z_{12} - z \log z) \leqslant \theta_2^+ \text{Im } (z_{12} - z)$$

for all $z \in D_2 - D_{\theta_1^-}(z_{12})$. Thus, if $z' \in D_2 - D_{\theta_1^-}(z_{12})$, there is a $\theta(z) \in [\theta_1^-, \theta_2^+]$, such that $\text{Re } \psi_{\theta(z)}(z') = \text{Re } \psi_{\theta(z)}(z_*)$, (more precisely, $\theta(z)\text{Im } (z_{12} - z) = \text{Re } (z_{12} \log z_{12} - z' \log z'))$ and we take $C_j(z', z_*)$ to be the arc of $C_{\theta(z)}(z_{12})$ between z' and z_*.
The integral over $C_j^-(z')$ can be estimated in a way similar to the corresponding integrals in cases 3 and 5 in the proof of proposition 4.4 above. The integral over $C_j^+(z_*)$ is similar to the integral over $C_{\theta_2}^+(z')$ in case 5 of that proof (with z_* instead of z' and θ_1^- instead of θ_2), and we have the following estimate

$$(\log |z_*|)^{-1} e^{a\rho_\theta(z_*)} |y_j^c(z_*) \int_{C_j^+(z_*)} \frac{y_j^c(\zeta+1)^{-1} f_1(\zeta)}{e^{2\pi i(\zeta - z)} - 1} d\zeta| \leqslant C_j' \tag{4.13}$$

where C_j' is a positive number, independent of R. If $z' \in D_{\theta_1^-}(z_{12})$ we are done. If $z' \in D_2 - D_{\theta_1^-}(z_{12})$ we have $z_* = z_{12}$, $\text{Im } z \leqslant \text{Im } z_{12}$ and $\arg z' \leqslant \phi_{12} < -\theta_2^+$. Putting $\zeta = \zeta(x)$ if $\zeta \in C_{\theta(z)}(z')$ and using the notation of §2.1 and (2.5) and (2.8), we have

$$\frac{d}{dx}[a\rho_\theta(\zeta(x)) + \log |y_j^c(\zeta(x))| - \log\log |\zeta(x)|]$$

$$= \frac{a(\phi(x) + \theta(z) - \theta) + \mu_j(\phi(x) + \theta(z))}{\log r(x)} + O(\frac{1}{(\log r(x))^2})$$

As $\phi(x) \leqslant \phi_{12} < -\theta_2^+$ for all $x \leqslant \text{Im } z_{12}$ and $\theta(z) \leqslant \theta_2^+$, the left-hand side is positive for all $x \leqslant \text{Im } z_{12}$, provided a is a sufficiently small positive number and R is sufficiently large. Consequently,

$$(\log |z'|)^{-1} \log |z_{12}| e^{a(\rho_\theta(z') - \rho_\theta(z_{12}))} |y_j^c(z') y_j^c(z_{12})^{-1}| \leqslant 1 \tag{4.14}$$

for all $z' \in D_2 - D_{\theta_1^-}(z_{12})$, provided a is sufficiently small and R is sufficiently large. With (4.3), (4.4) and (4.13) this implies that, for all $z \in D_2$,

$$(\log |z|)^{-1} e^{a\rho_\theta(z)} |y_j^c(z) \int_{C_j^+(z_*)} \frac{y_j^c(\zeta+1)^{-1} f_1(\zeta)}{e^{2\pi i(\zeta - z)} - 1} d\zeta| \leqslant C_j''$$

where C_j'' is a positive number, independent of R. The integral over $C_j(z', z_{12})$ can be estimated in a similar manner, due to the fact that, for all $\zeta \in$

$C_j(z', z_{12})$, $\arg\zeta + \theta(z) \leqslant \phi_{12} + \theta_2^+ < 0$. Due to the supra-exponential decrease of $f_2 - f_1$ on $C_j^{12}(z_*)$, one easily verifies that

$$|e^{b\psi_{\theta'}(z_*)} y_j^c(z_*) \int_{C_j^{12}(z_*)} \frac{y_j^c(\zeta+1)^{-1}(f_2 - f_1)(\zeta)}{e^{2\pi i(\zeta - z)} - 1} d\zeta| \leqslant C_j''' \|f_2 - f_1\|_{b,D_{12}}^{\theta'}$$

where C_j''' is a positive number, independent of R. Since $K(R)\|f_2 - f_1\|_{b,D_{12}}^{\theta'} \leqslant \|F\| = 1$, it suffices to prove that $K(R)^{-1} e^{ar_\theta(z) - b\psi_{\theta'}(z_*)} y_j^c(z) y_j^c(z_*)^{-1}$ is uniformly bounded on D_2, by a constant independent of R. This is obviously true if $z' \in D_{\theta_1^-}(z_{12})$, in which case $z_* = z'$. For $z' \in D_2 - D_{\theta_1^-}(z_{12})$ it follows easily from (4.4) and (4.14). Combining the above estimates, we find that $g_2 \in z^{\nu_j} b_{\theta,a}^1(D_2)^{n_j}$ for any positive number ν_j and

$$\|z^{-\nu_j} g_2\|_{a,D_2}^{\theta} \leqslant C_j^{(2)}$$

where $C_j^{(2)}$ is a positive constant, independent of R.

It remains to be proved that, for $\theta' \in (\theta_1^+, \theta_2^-)$, sufficiently small b and sufficiently large R, $g_1 - g_2 \in z^{\nu_j} b_{\theta',b}^{1^+}(D_{12})$ and

$$\|z^{-\nu_j}(g_1 - g_2)\|_{b,D_{12}}^{\theta'} \leqslant C\|f_2 - f_1\|_{b,D_{12}}^{\theta'}$$

where C is a positive constant, independent of R. By means of residue calculus we find, if $d_j \geqslant 0$,

$$g_1(z) - g_2(z) = y_j^c(z) \sum_{h=0}^{\infty} y_j^c(z+h+1)^{-1}(f_1 - f_2)(z+h)$$

Furthermore, we have

$$\psi_{\theta'}'(z) = \log z + 1 + i\theta'$$

Hence it follows that, for all $h \geqslant 0$,

$$\text{Re } (\psi_{\theta'}(z) - \psi_{\theta'}(z+h)) \leqslant -h \log d(R)$$

where $d(R) = \min\{|z| : z \in D_{12}\}$. With the aid of lemma 2.7(i) and (4.1) we find

$$\|z^{d_j}(g_1 - g_2)\|_{b,D_{12}}^{\theta'} \leqslant \sum_{h=0}^{\infty} K_1^h d(R)^{-(d_j+b)h} \|f_2 - f_1\|_{b,D_{12}}^{\theta'} < 2\|f_2 - f_1\|_{b,D_{12}}^{\theta'}$$

if R is sufficiently large. Thus, we can conclude that, if $d_j \geqslant 0$ and R is sufficiently large, there exists a $\nu_j \in \mathbb{R}$ such that $\Lambda_j^q F \in z^{\nu_j} B_{b,\theta'}^{a,\theta}(R)^{n_j}$ and

$$\|z^{-\nu_j}\Lambda_j^q F\| \leqslant C_j \tag{4.15}$$

where C_j is a positive constant, independent of R.

Now consider the case that $d_j < 0$. Then we have for $i \in \{1,2\}$ (cf. Proposition 4.4, case 4)

$$g_i(z) = y_j^c(z)\int_{C_j(z')} \frac{e^{-2l_i\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f_i(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta - y_j^c(z)y_j^c(z+1)^{-1}f(z)$$

where $C_j(z')$ is the path consisting of $C^-_{\theta_i^-}(z')$ and $C^+_{\theta_i^+}(z')$, l_1 and l_2 are the unique integers with the property that, for $i \in \{1,2\}$,

$$d_j\theta_i^+ - \operatorname{Im}\mu_j - 2l_i\pi > 0 \text{ and } d_j\theta_i^- - \operatorname{Im}\,\mu_j - 2(l_i+1)\pi < 0 \tag{4.16}$$

By lemma 2.6(i) we have, for all $z \in D_{12}$, $C^+_{\theta_2^+}(z') \subset D_1$ (as $\theta_2^+ > \theta_1^+$) and $C^-_{\theta_1^-}(z') \subset D_2$ (as $\theta_1^- < \theta_2^-$). From (4.16) we deduce that

$$\int_{C^+_{\theta_1^+}(z')} \frac{e^{-2l_2\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f_1(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta = \int_{C^+_{\theta_2^+}(z')} \frac{e^{-2l_2\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f_1(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta$$

and

$$\int_{C^-_{\theta_2^-}(z')} \frac{e^{-2l_1\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f_2(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta = \int_{C^-_{\theta_1^-}(z')} \frac{e^{-2l_1\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f_2(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta$$

Putting $f_1 - f_2 = f_{12}$ we obtain

$$y_j^c(z)^{-1}(g_1(z)-g_2(z)) = \int_{C^+_{\theta_2^+}(z')} \frac{e^{-2l_2\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}(f_{12})(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta + \int_{C^-_{\theta_1^-}(z')} \frac{e^{-2l_1\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}(f_{12})(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta$$

$$+\int_{C^+_{\theta_1^+}(z')} \frac{(e^{-2l_1\pi i(\zeta-z)} - e^{-2l_2\pi i(\zeta-z)})y_j^c(\zeta+1)^{-1}f_1(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta \tag{4.17}$$

$$+\int_{C^-_{\theta_2^-}(z')} \frac{(e^{-2l_1\pi i(\zeta-z)} - e^{-2l_2\pi i(\zeta-z)})y_j^c(\zeta+1)^{-1}f_2(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta$$

For all $\zeta \in C^-_{\theta_1^-}(z')$ we have, due to (4.16),

$$\log|(z'/\zeta)^{d_j}y_j^c(z')y_j^c(\zeta)^{-1}e^{2(l_1+1)\pi i(z-\zeta)}| + b\text{Re }(\psi_{\theta'}(z') - \psi_{\theta'}(\zeta))$$

$$= \{d_j\theta_1^- - \text{Im }\mu_j - 2(l_1+1)\pi + b(\theta_1^- - \theta') + O(\frac{1}{\log R})\}\text{Im }(z-\zeta) < -\delta\text{Im }(z-\zeta)$$

and, for all $\zeta \in C^+_{\theta_2^+}(z')$,

$$\log|(z'/\zeta)^{d_j}y_j^c(z')y_j^c(\zeta)^{-1}e^{2l_2\pi i(z-\zeta)}| + b\text{Re }(\psi_{\theta'}(z') - \psi_{\theta'}(\zeta)) =$$

$$= \{d_j\theta_2^+ - \text{Im }\mu_j - 2l_2\pi + b(\theta_2^+ - \theta') + O(\frac{1}{\log R})\}\text{Im }(z-\zeta)$$

$$< \delta\text{Im }(z-\zeta)$$

where δ is a positive number, independent of R, provided R is sufficiently large. Hence we deduce, with the aid of (4.4), an estimate of the form

$$|z^{3d_j/2}y_j^c(z)\{\int_{C^+_{\theta_2^+}(z')} \frac{e^{-2l_2\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f_{12}(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta +$$

$$\int_{C^-_{\theta_1^-}(z')} \frac{e^{-2l_1\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f_{12}(\zeta)}{e^{2\pi i(\zeta-z)}-1}d\zeta\}|$$

$$\leqslant C_j'\|f_{12}\|_{b,D_{12}}^{\theta'}|e^{-b\psi_{\theta'}(z)}|, \quad z \in D_{12}$$

where C_j' is a positive number, independent of R.

For all $z \in D_{12}$, such that $\text{Im } z \neq \text{Im } z_{12}$, there exists a real number $\theta(z) : \theta(z) \leqslant \theta_1^+$ if $\text{Im } z < \text{Im } z_{12}$, $\theta(z) \geqslant \theta_2^-$ if $\text{Im } z > \text{Im } z_{12}$, with the property that

$$\text{Re }\psi_{\theta(z)}(z') = \text{Re }\psi_{\theta(z)}(z_{12})$$

By deformation of contours, one easily verifies that the sum of the last two integrals in (4.17) is equal to

$$\sum_{h=l_2+1}^{l_1}\int_{C_j(z',z_{12})} y_j^c(\zeta+1)^{-1}f_{12}(\zeta)e^{-2h\pi i(\zeta-z)}d\zeta$$

$$+\sum_{h=l_2+1}^{l_1}[\int_{C^+_{\theta_1^+}(z_{12})} e^{-2h\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f_1(\zeta)d\zeta$$

$$+\int_{C^-_{\theta_2^-}(z_{12})} e^{-2h\pi i(\zeta-z)}y_j^c(\zeta+1)^{-1}f_2(\zeta)d\zeta]$$

where $C_j(z',z_{12})$ is the (directed) segment from z' to z_{12} if Im $z=$ Im z_{12}, and the arc of $C_{\theta(z)}(z')$ connecting z' and z_{12} otherwise. We shall discuss the case that Im $z<$ Im z_{12}. In that case we have, for all $\zeta\in C_j(z',z_{12})$ and all integers $h\leqslant l_1$,

$$\log|(z'/\zeta)^{d_j}y_j^c(z')y_j^c(\zeta)^{-1}e^{2h\pi i(z-\zeta)}|+b\mathrm{Re}\ (\psi_{\theta'}(z')-\psi_{\theta'}(\zeta))=$$

$$\{(d_j+b+O(\frac{1}{\log R}))\theta(z)-\mathrm{Im}\ \mu_j-2h\pi-b\theta'+O(\frac{1}{\log R})\}\mathrm{Im}\ (z-\zeta)$$

As $\theta(z)\leqslant\theta_1^+$ while $\theta'>\theta_1^+$, the right-hand side is less than

$$\{d_j\theta_1^+-\mathrm{Im}\ \mu_j-2l_1\pi+b(\theta_1^+-\theta')+O(\frac{1}{\log R})\}\mathrm{Im}\ (z-\zeta)$$

provided $b<-d_j$ and R is sufficiently large. With (4.16) it follows that, for all $\zeta\in C_j(z',z_{12})$ and all integers $h\leqslant l_1$,

$$\log|(z'/\zeta)^{d_j}y_j^c(z')y_j^c(\zeta)^{-1}e^{2h\pi i(z-\zeta)}|+b\mathrm{Re}\ (\psi_{\theta'}(z')-\psi_{\theta'}(\zeta))<\delta\mathrm{Im}\ (z-\zeta) \tag{4.18}$$

where δ is a positive number, independent of R, provided b is sufficiently small and R is sufficiently large. Hence we deduce, with the aid of (4.4), an estimate of the form

$$|z^{3d_j/2}e^{b\psi_{\theta'}(z)}y_j^c(z)\sum_{h=l_2+1}^{l_1}\int_{C_j(z',z_{12})}y_j^c(\zeta+1)^{-1}f_{12}(\zeta)e^{-2h\pi i(\zeta-z)}d\zeta|$$

$$\leqslant C_j''\|f_{12}\|^{\theta'}_{b,D_{12}}$$

where C_j'' is a positive constant, independent of R, and we have used the fact that Re $\psi_{\theta'}(z)\leqslant$ Re $\psi_{\theta'}(z')$.

For all $\zeta\in C^+_{\theta_1^+}(z_{12})$ we have, due to (4.16),

$$\log|(z_{12}/\zeta)^{d_j}y_j^c(z_{12})y_j^c(\zeta)^{-1}e^{2h\pi i(z_{12}-\zeta)}|+a(\rho_\theta(z_{12})-\rho_\theta(\zeta))$$

$$=\{d_j\theta_1^+-\mathrm{Im}\ \mu_j-2h\pi+O(\frac{1}{\log R})\}\mathrm{Im}\ (z_{12}-\zeta)<\delta\mathrm{Im}\ (z_{12}-\zeta) \tag{4.19}$$

if $h \leqslant l_1$. From (4.18) we infer that, for all $h \leqslant l_1$,

$$\log|(z'/z_{12})^{d_j} y_j^c(z') y_j^c(z_{12})^{-1} e^{2h\pi i(z-z_{12})}| + b\mathrm{Re}\,(\psi_{\theta'}(z') - \psi_{\theta'}(z_{12}) \leqslant 0$$

provided b is sufficiently small and R is sufficiently large. Using (4.4) and (4.19), we find

$$|z^{3d_j/2} e^{b\psi_{\theta'}(z)} y_j^c(z) \sum_{h=l_2+1}^{l_1} \int_{C^+_{\theta_1^+}(z_{12})} e^{-2h\pi i(\zeta-z)} y_j^c(\zeta+1)^{-1} f_1(\zeta) d\zeta| \leqslant$$

$$K_1 |e^{b\psi_{\theta'}(z_{12})} z_{12}^{d_j} y_j^c(z_{12})| \sum_{h=l_2+1}^{l_1} \int_{C^+_{\theta_1^+}(z_{12})} |\zeta^{-d_j} y_j^c(\zeta)^{-1} e^{2h\pi i(z_{12}-\zeta)} f_1(\zeta) d\zeta|$$

$$< K_1 |e^{b\psi_{\theta'}(z_{12}) - ar_\theta(z_{12})}| \sum_{h=l_2+1}^{l_1} \int_{C^+_{\theta_1^+}(z_{12})} e^{\delta \mathrm{Im}\,(z_{12}-\zeta)} |d\zeta| \|f_1\|^{\theta}_{a,D_1}$$

With the aid of remark 4.6 it follows that

$$K(R)|z^{3d_j/2} e^{b\psi_{\theta'}(z)} y_j^c(z) \sum_{h=l_2+1}^{l_1} \int_{C^+_{\theta_1^+}(z_{12})} e^{-2h\pi i(\zeta-z)} y_j^c(\zeta+1)^{-1} f_1(\zeta) d\zeta| \leqslant \tilde{C}_j$$

where $\tilde{C}_j$ is a positive constant, independent of R. In a similar manner one proves that

$$K(R)|z^{3d_j/2} e^{b\psi_{\theta'}(z)} y_j^c(z) \sum_{h=l_2+1}^{l_1} \int_{C^-_{\theta_2^-}(z_{12})} e^{-2h\pi i(\zeta-z)} y_j^c(\zeta+1)^{-1} f_2(\zeta) d\zeta| \leqslant \tilde{C}'_j$$

where $\tilde{C}'_j$ is a positive constant, independent of R. Combining the above estimates, we conclude that, if $d_j < 0$, $\Lambda_j^q F \in z^{-3d_j/2} B^{a,\theta}_{b,\theta'}(R)^{n_j}$ and

$$\|z^{3d_j/2} \Lambda_j^q F\| \leqslant C'_j \tag{4.20}$$

where C'_j is a positive constant, independent of R. This completes the proof of (i) and (ii). For the proof of the fact that Λ_j^q inverts Δ_j^c we refer the reader to [9]. □

We shall need the following simple lemma, which is a straightforward generalization of a lemma proved by Wasow (cf. [13, lemma 14.3]).

Lemma 4.8. — *Let $\psi : D \times U \to \mathbb{C}$ be a holomorphic function, where D is a domain of $\mathbb{C}$ and U a convex subset of $\mathbb{C}^n$ containing O. Suppose that* *the Jacobian matrix of $\psi(z,u)$ with respect to u vanishes at $u=0$ for all $z\in D$ and the Hessian matrix of $\psi(z,u)$ with respect to u is bounded on $D\times U$. Then there exists a positive constant K such that, for all $z\in D$ and all u_1, u_2, v_1 and $v_2\in U$,*

$$|\psi(z,u_1)-\psi(z,u_2)-\psi(z,v_1)+\psi(z,v_2)| \leqslant$$

$$K\left[\max\{|u_2-u_1|,|v_2-v_1|\}\max\{|u_1-v_1|,|u_2-v_2|\}\right.$$
$$\left.+\max\{|u_1|,|u_2|,|v_1|,|v_2|\}|u_2-u_1-v_2+v_1|\right]$$

Proof. — For all $z\in D$ and all u_1, u_2, v_1 and $v_2\in U$,

$$|\psi(z,u_1)-\psi(z,u_2)-\psi(z,v_1)+\psi(z,v_2)| =$$

$$|\int_0^1 \frac{d}{dt}\psi(z,u_2+t(u_1-u_2))-\psi(z,v_2+t(v_1-v_2))dt| =$$

$$|\int_0^1\int_0^1 \frac{\partial^2}{\partial s\partial t}\psi(z,v_2+t(v_1-v_2)+s(u_2-v_2+t(u_1-u_2-v_1+v_2)))dsdt|$$

Due to the properties of ψ, there exist positive numbers K_1 and K_2 such that, for all $z\in D$, all u_1, u_2, v_1 and $v_2\in U$ and $(s,t)\in(0,1)\times(0,1)$,

$$|\frac{\partial^2}{\partial s\partial t}\psi(z,v_2+t(v_1-v_2)+s(u_2-v_2+t(u_1-u_2-v_1+v_2)))|$$

$$\leqslant K_1|v_1-v_2+s(u_1-u_2-v_1+v_2)||u_2-v_2+t(u_1-u_2-v_1+v_2)|+$$

$$+K_2|v_2+t(v_1-v_2)+s(u_2-v_2+t(u_1-u_2-v_1+v_2))||u_1-u_2-v_1+v_2|$$

Hence the result follows. □

Let $U\subset\mathbb{C}^n$, $D_i\subset S_+$, $i=1,2$, and ψ a $\mathbb{C}^n$-valued function on $(D_1\cup D_2)\times U$. For $i=1,2$, let $f_i: D_i\to U$ be a given function and let $g_i: D_i\to\mathbb{C}^n$ be defined by

$$g_i(z)=\psi(z,f_i(z)) \text{ for all } z\in D_i$$

Putting $(f_1,f_2)=F$, we define

$$\psi^q(F) := G := (g_1,g_2)$$

In particular, if U is a neighbourhood of O, $D_i=D_i(R)$ and F is a given element of $B^{a,\theta}_{b,\theta'}(R)^n$, then $\psi^q(F)$ is well-defined if $\|F\|$ is sufficiently small, or R is sufficiently large. From lemma 4.8 we derive the following corollary.

Corollary 4.9. — *Let I_1, I_2 be finite, closed intervals of $\mathbb{R}$: $I_i = [\theta_i^-, \theta_i^+]$, $i \in \{1,2\}$, such that $\theta_1^+ < \theta_2^-$ and $\theta_2^+ < \theta_1^- + \pi$, and let $I = [\theta_1^-, \theta_2^+]$. For $i = 1,2$, let $D_i := D_i(R)$ denote a domain of the type defined at the beginning of section 4.2, let $D_{12} = D_1 \cap D_2$ and $D = D_1 \cup D_2$. Let $\psi \in \widetilde{\mathcal{A}}(I', 0)^n$, where I' is some finite open interval containing I, and suppose that $\psi(z,0,0) = 0$ and $\psi_2'(z,0,0) = \psi_3'(z,0,0) = 0$ for all $z \in D$. Let $\theta \in (\theta_2^+ - \pi/2, \theta_1^- + \pi/2)$, $\theta' \in (\theta_1^+, \theta_2^-)$, a and $b > 0$ and $\nu \in \mathbb{R}$. For any positive number M, let $B_M(R)$ denote the ball*

$$B_M(R) = \{Y \in B_{b,\theta'}^{a,\theta}(R)^n : \|Y\| \leqslant M\}$$

Then $\psi^q(Y, \tau Y) \in B_{b,\theta'}^{a,\theta}(R)^n$ for all $Y \in z^\nu B_M(R)$, provided R is sufficiently large. Moreover, there exists a positive constant $K_\nu'(R)$, such that $K_\nu'(R) \to 0$ as $R \to \infty$ and, for all Y_1, $Y_2 \in z^\nu B_M(R)$,

$$\|\psi^q(Y_1, \tau Y_1) - \psi^q(Y_2, \tau Y_2)\| \leqslant M K_\nu'(R) \|\tilde{Y_1} - \tilde{Y_2}\|$$

where $\tilde{Y_i} = z^{-\nu} Y_i$, $i = 1,2$.

Proof. — Let $Y = (y_1, y_2) = z^\nu(\tilde{y_1}, \tilde{y_2}) \in z^\nu B_M(R)$. Thus, $\tilde{y_i} \in b_{\theta,a}^1(D_i)^n$ for $i = 1,2$, $\tilde{y_1} - \tilde{y_2} \in b_{\theta',b}^{1^+}(D_{12})^n$ and $\|(\tilde{y_1}, \tilde{y_2})\| \leqslant M$. It is easily seen that $\rho_\theta(z+1) - \rho_\theta(z) > 0$ and Re $(\psi_{\theta'}(z+1) - \psi_{\theta'}(z)) > 0$ for sufficiently large $|z|$. Hence, $\tau\tilde{y_i} \in b_{\theta,a}^1(D_i)^n$ for $i = 1,2$ and $\tau(\tilde{y_1} - \tilde{y_2}) \in b_{\theta',b}^{1^+}(D_{12})^n$, $\|\tau\tilde{y_i}\|_{a,D_i}^\theta \leqslant \|\tilde{y_i}\|_{a,D_i}^\theta \leqslant M$ and $\|\!|\tau(\tilde{y_1} - \tilde{y_2})\|\!|_{b,D_{12}}^{\theta'} \leqslant \|\!|\tilde{y_1} - \tilde{y_2}\|\!|_{b,D_{12}}^{\theta'}$, if R is sufficiently large. ψ is analytic on $D \times U \times U$, where $U \subset \mathbb{C}^n$ is a (convex) neighbourhood of 0. For all $z \in D_i$ we have $|y_i(z)| \leqslant M|z|^\nu e^{-a\rho_\theta(z)}$, $i = 1,2$. Hence we infer, with the aid of lemma 2.15, that both $y_i(z)$ and $y_i(z+1) \in U$ if R is sufficiently large. Applying lemma 4.8, with $u_1 = (y_i(z), y_i(z+1))$, $i \in \{1,2\}$, and $u_2 = v_1 = v_2 = 0$, we have, for all $z \in D_i$,

$$|\psi(z, y_i(z), y_i(z+1))| \leqslant 2K|(y_i(z), y_i(z+1))|^2 \leqslant 2M^2 K_\nu |z|^{2\nu} e^{-2a\rho_\theta(z)}$$

where K and K_ν are positive constants, independent of R, if R is sufficiently large. As $|z|^{2\nu} e^{-a\rho_\theta(z)}$ is bounded on D, it follows that $\psi(z, y_i, \tau y_i) \in b_{\theta,a}^1(D_i)^n$, $i = 1,2$. Again applying lemma 4.8, now with $u_i = (y_i(z), y_i(z+1))$ for $i = 1,2$, and $v_1 = v_2 = 0$, we find that, for all $z \in D_{12}$,

$$|\psi(z, y_1(z), y_1(z+1)) - \psi(z, y_2(z), y_2(z+1))|$$

$$\leqslant 2K|(y_1(z), y_1(z+1)) - (y_2(z), y_2(z+1))| \max_{i=1,2} |(y_i(z), y_i(z+1))|$$

$$\leqslant 2MK_\nu |z^{2\nu} e^{-b\psi_{\theta'}(z) - ar_\theta(z)}| \|\!|\tilde{y_1} - \tilde{y_2}\|\!|_{b,D_{12}}^{\theta'}$$

This shows that $\psi(z,y_1,\tau y_1)-\psi(z,y_2,\tau y_2)\in b^{1^+}_{\theta',b}(D_{12})^n$ and thus $\psi^q(Y,\tau Y)$ $\in B^{a,\theta}_{b,\theta'}(R)^n$. In a similar manner one deduces from lemma 4.8 that, for $i\in\{1,2\}$ and $y_1,y_2\in z^\nu b^1_{\theta,a}(D_i)^n$ such that $\|\tilde{y_j}\|^\theta_{a,D_i}\leqslant M$, $j\in\{1,2\}$,

$$\|\psi(z,y_1,\tau y_1)-\psi(z,y_2,\tau y_2)\|^\theta_{a,D_i}\leqslant 2MK_\nu\sup_{z\in D_i}|z|^{2\nu}e^{-a\rho_\theta(z)}\|\tilde{y_1}-\tilde{y_2}\|^\theta_{a,D_i}$$

Now let $Y_1=(u_1,u_2)=z^\nu\tilde{Y_1}=z^\nu(\tilde{u_1},\tilde{u_2})\in z^\nu B_M(R)$ and $Y_2=(v_1,v_2)=z^\nu\tilde{Y_2}=z^\nu(\tilde{v_1},\tilde{v_2})\in z^\nu B_M(R)$. Let $\psi^q(Y_1,\tau Y_1)-\psi^q(Y_2,\tau Y_2)=G=(g_1,g_2)$. Then we have for $i=1,2$: $\|\tilde{u_i}\|^\theta_{a,D_i}\leqslant\|\tilde{Y_1}\|\leqslant M$, $\|\tilde{v_i}\|^\theta_{a,D_i}\leqslant\|\tilde{Y_2}\|\leqslant M$, $\|\tilde{u_i}-\tilde{v_i}\|^\theta_{a,D_i}\leqslant\|\tilde{Y_1}-\tilde{Y_2}\|$, hence

$$\begin{aligned}\|g_i\|^\theta_{a,D_i} &= \|\psi(z,u_i,\tau u_i)-\psi(z,v_i,\tau v_i)\|^\theta_{a,D_i}\\ &\leqslant 2MK_\nu\sup_{z\in D_i}|z|^{2\nu}e^{-a\rho_\theta(z)}\|\tilde{Y_1}-\tilde{Y_2}\| \qquad (4.21)\end{aligned}$$

and, applying once more lemma 4.8,

$$|\!|\!|g_1-g_2|\!|\!|^{\theta'}_{b,D_{12}}=|\!|\!|\psi(z,u_1,\tau u_1)-\psi(z,u_2,\tau u_2)-\psi(z,v_1,\tau v_1)+\psi(z,v_2,\tau v_2)|\!|\!|^{\theta'}_{b,D_{12}}$$

$$\leqslant K_\nu\sup_{z\in D}|z|^{2\nu}e^{-a\rho_\theta(z)}\Big[\max_{i=1,2}\|\tilde{u_i}-\tilde{v_i}\|^\theta_{a,D_i}\max\{|\!|\!|\tilde{u_1}-\tilde{u_2}|\!|\!|^{\theta'}_{b,D_{12}},|\!|\!|\tilde{v_1}-\tilde{v_2}|\!|\!|^{\theta'}_{b,D_{12}}\}$$

$$+\max\{\|\tilde{u_1}\|^\theta_{a,D_1},\|\tilde{u_2}\|^\theta_{a,D_2},\|\tilde{v_1}\|^\theta_{a,D_1},\|\tilde{v_2}\|^\theta_{a,D_2}\}|\!|\!|\tilde{u_1}-\tilde{v_1}-\tilde{u_2}+\tilde{v_2}|\!|\!|^{\theta'}_{b,D_{12}}\Big]$$

Noting that $K(R)|\!|\!|\tilde{u_1}-\tilde{u_2}|\!|\!|^{\theta'}_{b,D_{12}}\leqslant M$, $K(R)|\!|\!|\tilde{v_1}-\tilde{v_2}|\!|\!|^{\theta'}_{b,D_{12}}\leqslant M$ and $K(R)|\!|\!|\tilde{u_1}-\tilde{v_1}-\tilde{u_2}+\tilde{v_2}|\!|\!|^{\theta'}_{b,D_{12}}\leqslant\|\tilde{Y_1}-\tilde{Y_2}\|$, we obtain the inequality

$$K(R)|\!|\!|g_1-g_2|\!|\!|^{\theta'}_{b,D_{12}}\leqslant 2MK_\nu\sup_{z\in D}|z|^{2\nu}e^{-a\rho_\theta(z)}\|\tilde{Y_1}-\tilde{Y_2}\| \qquad (4.22)$$

From (4.21) and (4.22) the result follows, with $K'_\nu(R)=2K_\nu\sup_{z\in D}|z|^{2\nu}e^{-a\rho_\theta(z)}$. □

Proof of Theorem 3.3. — If there is no $j\in\{1,...,m\}$ such that $d_j=0$, $\mu_j\neq 0$ and $\arg\mu_j=\pi$, i.e. if $-\pi/2$ is not a Stokes direction of level 1, then the statements of the theorem can be deduced from Theorem 3.1. We shall discuss the case that $-\pi/2$ is a Stokes direction of level 1 and $\theta_1^-<-\pi/2$ (the case that $\theta_2^+>\pi/2$ can be proved in a similar manner). For $i=1,2$ let $I'_i=[\hat\theta_i^-,\hat\theta_i^+]$ be a closed subinterval of I_i such that $\{\hat\theta_i^-,\hat\theta_i^+\}\cap\{-\pi/2,\pi/2\}=\emptyset$ and $\hat\theta_1^-<-\pi/2$. Then $I'_1\not\subset(-\pi/2,\pi/2)$, so $\tilde J_1(I'_1)=\emptyset$, but $\tilde J_1(I'_2)$ may be nonempty (viz., if $\hat\theta_2^->-\pi/2$). Obviously, $\theta_1^+\leqslant\theta_2^-$ and $\theta_2^+-\theta_1^-\leqslant\pi$ implies that $\hat\theta_1^+<\hat\theta_2^-$ and $\hat\theta_2^+-\hat\theta_1^-<\pi$. Let $\theta\in(\hat\theta_2^+-\pi/2,\hat\theta_1^-+\pi/2)$ and $\theta'\in(\hat\theta_1^+,\hat\theta_2^-)$. Let D be a domain of the type mentioned in definition 4.5 with I_1

and I_2 replaced by I_1' and I_2'. In view of lemma 2.14(i), the assumption that $\varphi_0 \in \tilde{\mathcal{A}}_{1,0}(I)^n$ implies the existence of a positive number a_1 such that $\varphi_0 \in b^1_{\theta,a}(D)^n$ for all $a \in (0, a_1)$, if R is sufficiently large. Let $a \in (0, \min\{a_0, a_1\})$, $b \in (0, b_0)$, $\nu = \max\{\nu_j : j \in \{1, ..., m\}\}$, where a_0, b_0 and ν_j, $j = 1, ..., m$, are the real numbers mentioned in proposition 4.7, and let $\Lambda^q := \oplus_{j=1}^m \Lambda_j^q$. Let $M > \|z^{-\nu}\Lambda^q(\varphi_0, \varphi_0)\|^{\theta}_{a,D}$. φ is analytic on $D \times U \times U$, where U is a neighbourhood of O. For $Y = (y_1, y_2) \in z^\nu B^{a,\theta}_{b,\theta'}(R)^n$, with the property that $y_i(z) \in U$ for all $z \in D_i$, $i = 1, 2$, we define $\varphi^q(Y, \tau Y) := (g_1, g_2)$, where $g_i(z) = \varphi_0(z) + A(z)y_i(z) + B(z)y_i(z+1) + \psi(z, y_i(z), y_i(z+1))$. By virtue of corollary 4.9, $\varphi^q(Y, \tau Y) \in B^{a,\theta}_{b,\theta'}(R)^n$ for all $Y \in z^\nu B_M(R)$, provided R is sufficiently large (depending on M). Furthermore, for all Y_1, $Y_2 \in z^\nu B_M(R)$,

$$\|\varphi^q(Y_1, \tau Y_1) - \varphi^q(Y_2, \tau Y_2)\| \leqslant \sup_{z \in D}(|z^\nu A(z)| + |(z+1)^\nu B(z)|)\|\tilde{Y}_1 - \tilde{Y}_2\|$$

$$+ M K'_\nu(R)\|\tilde{Y}_1 - \tilde{Y}_2\|$$

where $\tilde{Y}_i = z^{-\nu} Y_i$ for $i = 1, 2$ and R is supposed sufficiently large. Due to the fact that $A(z) \sim 0$, $B(z) \sim 0$ as $z \to \infty$, uniformly on $\widetilde{D}_{I'}(R')$ for any closed subinterval I' of I and some sufficiently large R', and $K'_\nu(R) \to 0$ as $R \to \infty$, and in view of proposition 4.7, this implies that, for sufficiently large R, the mapping $T : B_M(R) \to B^{a,\theta}_{b,\theta'}(R)^n$ defined by

$$T\tilde{Y} = z^{-\nu}\Lambda^q \varphi^q(Y, \tau Y) = z^{-\nu}\Lambda^q(\varphi^q(Y, \tau Y) - \varphi^q(0, 0)) + z^{-\nu}\Lambda^q(\varphi_0, \varphi_0)$$

where $\Lambda^q := \oplus_{j=1}^m \Lambda_j^q$, is a contraction. Consequently, there exists a unique quasi-function $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2) \in B_M(R)$, such that $T\widetilde{F} = \widetilde{F}$. With proposition 4.7 it follows that $z^\nu \widetilde{F}$ is a unique quasi-function solution of (3.1) in $z^\nu B_M(R)$, and thus, both $F_1 := z^\nu \widetilde{F}_1$ and $F_2 := z^\nu \widetilde{F}_2$ are solutions of (3.1). Increasing I_1' and I_2' and using the uniqueness of the quasi-function solutions and lemma 2.14, we conclude that F_i can be analytically continued to an element f_i of $\tilde{\mathcal{A}}_{1,0}(I_i)^n$, $i = 1, 2$, such that $f_2 - f_1 \in \tilde{\mathcal{A}}_{1^+,0}(I_i)^n$.

Remark 4.10 . — Note that the conditions on A and B in Theorem 3.3 can be relaxed. All we need is that $\sup_{z \in \widetilde{D}_{I'}(R')} |z^\nu A(z)|$ and $\sup_{z \in \widetilde{D}_{I'}(R')} |z^\nu B(z)|$ tend to 0 as $R' \to \infty$ for any closed subinterval I' of I. Moreover, both in Proposition 4.4 and Theorem 3.3, the condition $q_j \equiv 0$ for all $j \in \{1, \ldots, m\}$ such that $d_j = \mu_j = 0$ can be lifted.

5. Accelero-sums of $\hat{f}$

In this section we generalize results of [7, §4] and show that the solutions f_1 and f_2, mentioned in Theorem 3.2, can be obtained from the formal solution $\hat{f}$ by means of a summation procedure, known as 'accelero-summation', provided $|I| > \pi$. It is shown that, for suitable values of θ and θ', the formal Borel transform $\phi_{1,\theta}(s)$ of $\hat{f}$ with respect to the variable $r_\theta(z)$ can be continued quasi-analytically to the half line $\arg s = 0$ and has so-called accelerable growth. Its accelerate $\mathcal{A}_{\theta,\theta'}(\phi_{1,\theta})$ to the level 1^+ (with corresponding 'critical variable' $\psi_{\theta'}(z)$, cf. (5.9) and (5.11)), has at most exponential growth of order 1 and, depending on the values of θ and θ', the Laplace transform of $\mathcal{A}_{\theta,\theta'}(\phi_{1,\theta})$, in the variable $\psi_{\theta'}(z)$, coincides with either f_1 or f_2. This Laplace transform is an *accelero-sum* of $\hat{f}$.

Let $\theta \in \mathbb{R}$ and let f be a continuous function on (R, ∞), where $R > 0$. If f has at most subexponential growth as $z \to \infty$, then the function $\tilde{\phi}_{1,\theta}$ defined by

$$\tilde{\phi}_{1,\theta}(s) := \frac{1}{2\pi i}\int_R^\infty f(z)e^{sr_\theta(z)}dr_\theta(z) \tag{5.1}$$

is analytic in the half plane Re $s < 0$. If f is analytic in $\widetilde{D}_I(R)$, where $I = (\theta_1, \theta_2)$ is an open interval of $\mathbb{R}$, and f has subexponential growth as $z \to \infty$ in $\widetilde{D}_I(R)$, then $\tilde{\phi}_{1,\theta}$ can be continued analytically to the sector $0 < \arg s < 2\pi$.

Lemma 5.1. — *Let $\theta \in \mathbb{R}$ and let f be a continuous function on (R, ∞). (i) Suppose that f is analytic in $\widetilde{D}_I(R)$, where $I = (\theta_1, \theta_2)$ and f satisfies a growth condition of the form*

$$\sup_{z\in\widetilde{D}_I(R)} |f(z)|e^{-\epsilon\frac{|z|}{\log|z|}} < \infty \text{ for all } \epsilon > 0 \tag{5.2}$$

If $\theta_1 < \theta - \pi/2$, then $\tilde{\phi}_{1,\theta}$ defined by (5.1), is continuous on the sector $0 \leqslant \arg s < 2\pi$ and quasi-analytic on $\arg s = 0$[2]*. If $\theta_2 > \theta + \pi/2$, then $\tilde{\phi}_{1,\theta}$ is continuous on the sector $0 < \arg s \leqslant 2\pi$ and quasi-analytic on $\arg s = 2\pi$. (ii) If there exist positive numbers C and ω such that*

$$|f(z)| \leqslant Ce^{-\omega z\log z} \text{ for all } z \in (R, \infty) \tag{5.3}$$

then $\tilde{\phi}_{1,\theta}$ is an entire function satisfying a growth condition of the form

$$\tilde{\phi}_{1,\theta}(s) = \exp\{c\exp(\frac{Re\ s}{\omega})\}O(1) \text{ as } Re\ s \to \infty, \tag{5.4}$$

uniformly on closed subsectors of $|\arg s| < \pi/2$, where c is a positive number.

(2) More precisely, the restriction of $\tilde{\phi}_{1,\theta}$ to $[a, b]$ belongs to the Denjoy class ${}^1D[a, b]$ for all $0 < a < b$

Proof. — (i) Suppose that $\theta_1 < \theta - \pi/2$. By deformation of the path of integration in (5.1), it is easily seen that the right-hand side can be continued analytically to the upper half plane and that it can be represented by

$$\tilde{\phi}_{1,\theta}(s) = \frac{1}{2\pi i}\int_{C^+_{\theta_1}(R)} f(z)e^{sr_\theta(z)}dr_\theta(z), \quad 0 < \arg s < \pi \tag{5.5}$$

According to (2.11), there exists $\delta > 0$ such that $\rho_\theta(z) \leqslant -\delta\frac{|z|}{\log|z|}$ for all $z \in C^+_{\theta_1}(R)$ with sufficiently large absolute value. Thus, in view of (5.2), the function defined by the right-hand side of (5.5) is continuous on $0 \leqslant \arg s < \pi$ and C^∞ on $\arg s = 0$, and we have, for all $m \in \mathbb{N}$,

$$\tilde{\phi}^{(m)}_{1,\theta}(s) = \frac{1}{2\pi i}\int_{C^+_{\theta_1}(R)} f(z)(r_\theta(z))^m e^{sr_\theta(z)}dr_\theta(z), \quad \arg s = 0$$

From (5.2) and (2.11) we deduce an inequality of the form

$$|\tilde{\phi}^{(m)}_{1,\theta}(s)| \leqslant e^{c's}C^m_\epsilon \int_{C^+_{\theta_1}(R)} |z|^m e^{(\epsilon-\delta s)\frac{|z|}{\log|z|}}|dz|, \quad \arg s = 0$$

where c' and $C_\epsilon > 0$ and $0 < \epsilon < \delta s$. Application of the Laplace method to the integral on the right-hand side yields an estimate of the form

$$|\tilde{\phi}^{(m)}_{1,\theta}(s)| \leqslant K^m_{a,b}(m\log m)^m, \quad m \geqslant 2, \ s \in [a,b],$$

where $K_{a,b} > 0$, for any interval $[a,b] \subset (0,\infty)$. Hence it follows that the restriction of $\tilde{\phi}_{1,\theta}$ to $[a,b]$ belongs to the Denjoy class ${}^1D[a,b]$ (cf. [4]). The statement for the case that $\theta_2 > \theta + \pi/2$ can be proved similarly, using the representation

$$\tilde{\phi}_{2,\theta}(s) = \frac{1}{2\pi i}\int_{C^-_{\theta_2}(R)} f(z)e^{sr_\theta(z)}dr_\theta(z), \quad \pi < \arg s < 2\pi \tag{5.6}$$

(ii) If f decreases supra-exponentially as $z \to \infty$, the right-hand side of (5.1) obviously defines an entire function. Moreover, if (5.3) is satisfied, we have, for all $z \in (R,\infty)$,

$$|f(z)e^{sr_\theta(z)}| \leqslant Ce^{-\omega z\log z + \mathrm{Re}\ sz - \theta \mathrm{Im}\ \frac{sz}{\log z}}$$

The right-hand side of this inequality attains its maximum at a point $z(s)$ with the property that $\log z(s) = (\frac{\mathrm{Re}\ s}{\omega} - 1)(1 + O(\frac{\mathrm{Im}\ s}{(\mathrm{Re}\ s)^2}))$ as $\frac{\mathrm{Im}\ s}{(\mathrm{Re}\ s)^2} \to 0$, and

$$-\omega z(s)\log z(s) + \mathrm{Re}\ sz(s) - \theta\mathrm{Im}\ \frac{sz(s)}{\log z(s)} = z(s)\{\omega + O(\frac{\mathrm{Im}\ s}{\mathrm{Re}\ s})\}$$

hence (5.4) follows easily. $\square$

Let I_1, I_2 be open intervals of $\mathbb{R}$: $I_i = (\theta_i^-, \theta_i^+)$, $i \in \{1,2\}$, such that $\theta_1^- < \theta_2^+ - \pi$ and let $\theta \in (\theta_1^- + \frac{\pi}{2}, \theta_2^+ - \frac{\pi}{2})$. Suppose we are given functions f_i, analytic on $\widetilde{D}_{I_i}(R)$ and satisfying a growth condition of the form (5.2) on $\widetilde{D}_{I_i}(R)$ instead of $\widetilde{D}_I(R)$, $i = 1,2$, such that

$$|f_2(z) - f_1(z)| \leqslant Ce^{-\omega z \log z} \text{ on } (R, \infty),$$

where C and ω are positive numbers. Let $\tilde{\phi}_{1,\theta}^{[i]}(s) := \int_R^\infty f_i(z) e^{s r_\theta(z)} dr_\theta(z)$, $i = 1,2$. According to lemma 5.1, $\tilde{\phi}_{1,\theta}^{[1]}(s)$ is quasi-analytic on the half line $\arg s = 0$, $\tilde{\phi}_{1,\theta}^{[2]}(s)$ is quasi-analytic on the half line $\arg s = 2\pi$ and $\tilde{\phi}_{1,\theta}^{[2]} - \tilde{\phi}_{1,\theta}^{[1]}$ is an entire function satisfying a growth condition of the form (5.4). Hence it follows that $\tilde{\phi}_{1,\theta}^{[1]}(s)$ is quasi-analytic on the half line $\arg s = 2\pi$ as well and the function $\phi_{1,\theta}$ defined by

$$\phi_{1,\theta}(s) = \tilde{\phi}_{1,\theta}^{[1]}(s) - \tilde{\phi}_{1,\theta}^{[1]}(se^{2\pi i}) = \tilde{\phi}_{1,\theta}^{[2]}(s) - \tilde{\phi}_{1,\theta}^{[2]}(se^{2\pi i}), \ \arg s = 0 \quad (5.7)$$

is quasi-analytic on the half line $\arg s = 0$ (more precisely, for all $0 < a < b$, the restriction of $\phi_{1,\theta}$ to $[a,b]$ belongs to the Denjoy class ${}^1D[a,b]$).

Definition 5.2. — *We call $\phi_{1,\theta}$ defined by (5.7) the Borel transform of the quasi-function $F := (f_1, f_2)$ with respect to the variable $r_\theta(z)$, and denote it by $\mathcal{B}_{1,\theta}(F)$.*

Remark 5.3. — If $f_i \in \mathcal{A}_1(I_i)$ for $i = 1,2$, with common asymptotic expansion $\hat{f} = \sum_{h=0}^\infty a_h z^{-h/p}$, then, for sufficiently small s, the Borel transform $\phi_{1,\theta}(s)$ coincides with the analytic function defined by the (formal) Borel transform of $\hat{f}$ with respect to the variable $r_\theta(z)$, i.e. :
$\sum_{h=1}^\infty \frac{a_h}{2\pi i} \int_U z^{-h/p} e^{s r_\theta(z)} dr_\theta(z)$, where U is a suitable contour in S_+. This is well-known in the case of the ordinary Borel transform, i.e. $\theta = 0$.

Proposition 5.4. — *Let $I_i = [\theta_i^-, \theta_i^+]$, $i \in \{1,2\}$, such that $\theta_1^- < \theta_2^+ - \pi$, let $I = (\theta_1^-, \theta_2^+)$, and $\theta \in (\theta_1^- + \frac{\pi}{2}, \theta_2^+ - \frac{\pi}{2})$. Suppose we are given functions f_j, analytic on $\widetilde{D}_{I_i}(R)$ and satisfying a growth condition of the form (5.2) on $\widetilde{D}_{I_i}(R)$ instead of $\widetilde{D}_I(R)$, $i = 1,2$, such that*

$$|f_2(z) - f_1(z)| \leqslant Ce^{-\omega z \log z} \text{ on } (R, \infty), \quad (5.8)$$

where C and ω are positive numbers. Then the Borel transform $\phi_{1,\theta}$ of (f_1, f_2) satisfies a growth condition of the form (5.4) as $s \to \infty$, $\arg s = 0$. Furthermore, $\phi_{1,\theta}$ admits the following integral representation, for all $s > 0$,

$$\phi_{1,\theta}(s) = \frac{1}{2\pi i}\left[\int_{C^+_{\theta_1^-}(R)} f_1(z)e^{sr_\theta(z)}dr_\theta(z) - \int_{C^-_{\theta_2^+}(R)} f_2(z)e^{sr_\theta(z)}dr_\theta(z) + \right.$$

$$\left. + \int_R^\infty (f_2(z) - f_1(z))e^{sr_\theta(z)}dr_\theta(z)\right]$$

Proof. — From (5.7) we deduce that

$$\phi_{1,\theta}(s) = \tilde{\phi}^{[1]}_{1,\theta}(s) - \tilde{\phi}^{[2]}_{1,\theta}(se^{2\pi i}) + (\tilde{\phi}^{[2]}_{1,\theta} - \tilde{\phi}^{[1]}_{1,\theta})(se^{2\pi i}),\ \arg s = 0$$

The integral representation for $\phi_{1,\theta}$ now follows from (5.5) and (5.6) (extended to $0 \leqslant \arg s < \pi$ and $\pi < \arg s \leqslant 2\pi$, respectively). It is easily seen that the first two terms in this representation grow at most exponentially as $s \to \infty$, $\arg s = 0$. From (5.8) and lemma 5.1 (ii) it follows that the third term satisfies a growth condition of the form (5.4). □

The estimate on the growth of $\phi_{1,\theta}$ shows that it can be "accelerated from level 1 to level 1^+" by means of a *weak acceleration operator* $\mathcal{A}_{\theta',\theta}$, where $\theta' \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ (cf. [3, 4]). $\mathcal{A}_{\theta',\theta}$ is an integral operator (generalizing the operators $\mathcal{A}_\theta$ defined in [6, 7]) with kernel $A_{\theta',\theta}$:

$$A_{\theta',\theta}(t,s) = \frac{1}{2\pi i}\int_U e^{t\psi_{\theta'}(z) - sr_\theta(z)}d\psi_{\theta'}(z),\ t > 0,\ s > 0 \tag{5.9}$$

where U denotes a contour consisting of the half line from $-\infty - i\delta$ to $\delta - i\delta$, the directed segment from $\delta - i\delta$ to $\delta + i\delta$ and the half line from $\delta + i\delta$ to $-\infty + i\delta$, with $\delta > 0$. Applying the saddle point method to the integral on the right-hand side of (5.9) we find that, as $\frac{s}{t} \to \infty$, $A_{\theta',\theta}$ behaves as

$$A_{\theta',\theta}(t,s) = \{\frac{e^{\frac{s}{t} - 1 - i(\theta' - \theta) + O(t/s)}}{2\pi t^3}\}^{\frac{1}{2}} s e^{-te^{\frac{s}{t} - 1 - i(\theta' - \theta) + O(t/s)}} \tag{5.10}$$

The *weak accelerate* $\mathcal{A}_{\theta',\theta}(\phi)$ of a function ϕ satisfying an appropriate growth condition, is defined by

$$\mathcal{A}_{\theta',\theta}(\phi)(t) = \int_0^\infty A_{\theta',\theta}(t,s)\phi(s)ds, \quad t > 0 \tag{5.11}$$

PROPOSITION 5.5. — *Under the conditions of proposition 5.4, the weak accelerate $\mathcal{A}_{\theta',\theta}(\phi_{1,\theta})(t)$ of the Borel transform $\phi_{1,\theta}$ of (f_1, f_2) with respect to $r_\theta(z)$, defined by (5.11), exists for all $\theta' \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ and all $t \in (0, \omega)$.*

Proof. — According to proposition 5.4, $\phi_{1,\theta}$ satisfies a growth condition of the form (5.4). With (5.10) it follows that $\mathcal{A}_{\theta',\theta}(\phi_{1,\theta})(t)$ exists for all $t \in (0,\omega)$. □

In [7] we have proved the following proposition ([7, Proposition 2.11]).

Proposition 5.6. — *Let $\theta_1 < \theta_2$, $I = [\theta_1, \theta_2]$ and $R > R_I$. Suppose that f is an analytic function on $\widetilde{D}_I(R)$, with the following properties*
(i) for every $\epsilon > 0$,

$$\sup_{z\in\widetilde{D}_I(R)} |f(z)|e^{-\epsilon|z|\log|z|} < \infty$$

(ii) f grows at most subexponentially as $z \to \infty$ on $C^+_{\theta_1}(R) \cup C^-_{\theta_2}(R)$.
Then, for any $\theta \in (\theta_1, \theta_2)$, the function $\tilde{\phi}_{1^+,\theta}$ defined by

$$\tilde{\phi}_{1^+,\theta}(t) = \frac{1}{2\pi i}\int_R^\infty f(z)e^{t\psi_\theta(z)}d\psi_\theta(z),\ Re\ t < 0$$

can be continued analytically to the sector $0 < \arg t < 2\pi$ and is continuous on the sector $0 \leqslant \arg t \leqslant 2\pi$, and the function $\phi_{1^+,\theta}$ defined on the half line $\arg t = 0$, by :

$$\phi_{1^+,\theta}(t) = \tilde{\phi}_{1^+,\theta}(t) - \tilde{\phi}_{1^+,\theta}(te^{2\pi i})$$

is quasi-analytic on $(0,\infty)$ [3]*. It can be represented by the integral*

$$\phi_{1^+,\theta}(t) = \frac{1}{2\pi i}\int_{\partial\widetilde{D}_I(R)} f(z)e^{t\psi_\theta(z)}d\psi_\theta(z)$$

where we integrate in the direction of increasing $Im\ z$. Moreover, if $f(z) = O(z^\mu)$ as $z \to \infty$, uniformly on $\widetilde{D}_I(R)$, where $\mu < 0$, then

$$f(z) = \int_0^\infty \phi_{1^+,\theta}(t)e^{-t\psi_\theta(z)}dt \tag{5.12}$$

Definition 5.7. — *We call the function $\phi_{1^+,\theta}$ the Borel transform of f with respect to the variable $\psi_\theta(z)$ and denote it by $\mathcal{B}_{1^+,\theta}(f)$.*

Theorem 5.8. — *Let $I_i = [\theta_i^-, \theta_i^+]$, $i \in \{1,2\}$, such that $\theta_i^- < \theta_i^+$ for $i = 1,2$, $\theta_1^- < \theta_2^+ - \pi$ and $\theta_1^+ < \theta_2^-$. Let $I_{12} := (\theta_1^+, \theta_2^-)$, R a sufficiently large positive number and $D_{12} := D_{I_{12}}(R)$. Suppose we are given functions f_i, analytic on $\widetilde{D}_{I_i}(R)$ and $O(z^\mu)$ as $z \to \infty$, uniformly on $\widetilde{D}_{I_i}(R)$, $i = 1,2$, where $\mu < 0$, with the property that*

$$f_2 - f_1 \in b^{1^+}_{\theta'',\omega}(D_{12})$$

---

(3) More precisely, for all $0 < a < b$, the restriction of $\phi_{1^+,\theta}$ to $[a,b]$ belongs to ${}^1D[a,b]$

where $\theta'' \in I_{12}$ *and* $\omega > 0$*. Let* $i \in \{1,2\}$*,* $\theta' \in (\theta_i^-, \theta_i^+)$*,* $\theta \in (\theta' - \frac{\pi}{2}, \theta' + \frac{\pi}{2}) \cap (\theta_1^- + \frac{\pi}{2}, \theta_2^+ - \frac{\pi}{2})$*, and let* $\phi_{1,\theta}$ *denote the Borel transform of* (f_1, f_2) *with respect to the variable* $r_\theta(z)$*. Then the weak accelerate* $\mathcal{A}_{\theta',\theta}(\phi_{1,\theta})(t)$ *coincides with* $\mathcal{B}_{1+,\theta'}(f_i)(t)$ *for sufficiently small positive values of* t*.*

Proof. — The proof is similar to that of Theorem 4.5 in [7], but the presence of the variable $r_\theta(z)$ requires some additional technicalities. Let $\theta \in (\theta_1^- + \pi/2, \theta_2^+ - \pi/2)$ and $f_{21} := f_2 - f_1$. According to proposition 5.4, $\phi_{1,\theta}$ can be represented as follows

$$\phi_{1,\theta}(s) = \frac{1}{2\pi i}\left[\int_{C^+_{\theta_1^-}(R)} f_1(z)e^{sr_\theta(z)}dr_\theta(z) - \int_{C^-_{\theta_2^+}(R)} f_2(z)e^{sr_\theta(z)}dr_\theta(z) + \right.$$

$$\left. + \int_R^\infty f_{21}(z)e^{sr_\theta(z)}dr_\theta(z)\right] \tag{5.13}$$

As f_{21} decreases exponentially of order 1 on D_{12} and $e^{sr_\theta(z)}$ grows at most subexponentially as $z \to \infty$ on $C^-_{\theta_1^+}(R) \cup C^+_{\theta_2^-}(R)$, we have, for $s > 0$,

$$\int_R^\infty f_{21}(z)e^{sr_\theta(z)}dr_\theta(z) = \int_{C^-_{\theta_1^+}(R)} f_{21}(z)e^{sr_\theta(z)}dr_\theta(z) =$$

$$= \int_{C^+_{\theta_2^-}(R)} f_{21}(z)e^{sr_\theta(z)}dr_\theta(z)$$

Suppose that $\theta' \in (\theta_1^-, \theta_1^+)$ and $\theta \in (\theta' - \frac{\pi}{2}, \theta' + \frac{\pi}{2}) \cap (\theta_1^- + \frac{\pi}{2}, \theta_2^+ - \frac{\pi}{2})$, the other case is analogous. Let $\theta^* \in (\theta - \frac{\pi}{2}, \theta')$, $\theta^{**} \in (\theta', \min\{\theta_1^+, \theta + \frac{\pi}{2}\})$, $R' > 0$ and let $\tilde{C}_{\theta'}(R')$ denote the contour consisting of $C^+_{\theta^*}(R')$ and $C^-_{\theta^{**}}(R')$, described in the direction of increasing Im z. As Re $\psi_{\theta'}(z) = R' \log R' + (\tilde{\theta} - \theta')\text{Im } z$ for all $z \in C_{\tilde{\theta}}(R')$ and $\tilde{\theta} \in \mathbb{R}$, Re $\psi_{\theta'}(z) \to -\infty$ as $z \to \infty$ on $C^+_{\tilde{\theta}}(R')$ if $\tilde{\theta} < \theta'$, or $z \to \infty$ on $C^-_{\tilde{\theta}}(R')$ if $\tilde{\theta} > \theta'$. Thus, we can deform the path of integration in (5.9) so as to obtain

$$A_{\theta',\theta}(t,s) = \frac{1}{2\pi i}\int_{\tilde{C}_{\theta'}(R')} e^{t\psi_{\theta'}(z) - sr_\theta(z)}d\psi_{\theta'}(z),\ t > 0,\ s > 0 \tag{5.14}$$

Let $M > 0$, $t > 0$ and let

$$I_M^-(t) := \int_0^M ds \int_{C^-_{\theta_1^+}(R)} dr_\theta(\zeta) f_{21}(\zeta)e^{sr_\theta(\zeta)}A_{\theta',\theta}(t,s)$$

For each $M > 0$ we can replace the path of integration $\tilde{C}_{\theta'}(R')$ in (5.14) by $\tilde{C}_{\theta'}(R_M)$, where R_M is a suitable number $> R$, to be specified later on. Changing the order of integration we find

$$I_M^-(t) = \int_{\tilde{C}_{\theta'}(R_M)} d\psi_{\theta'}(z) e^{t\psi_{\theta'}(z)} \int_{C^-_{\theta_1^+}(R)} dr_\theta(\zeta) \frac{f_{21}(\zeta)(e^{M(r_\theta(\zeta)-r_\theta(z))}-1)}{2\pi i(r_\theta(\zeta)-r_\theta(z))}$$

On $C^-_{\theta_1^+}(R)$ we have, for any $\theta'' \in \mathbb{R}$,

$$\mathrm{Re}\ \psi_{\theta''}(\zeta) = R\log R - (\theta'' - \theta_1^+)\mathrm{Im}\ \zeta$$

As $f_{21} \in b^{1^+}_{\theta'',\omega}(D_{12})$ for some $\theta'' \in (\theta_1^+, \theta_2^-)$, there exist positive numbers c and C such that, for all $\zeta \in C^-_{\theta_1^+}(R)$,

$$|f_{21}(\zeta)| \leqslant Ce^{c\mathrm{Im}\ \zeta} \tag{5.15}$$

Furthermore, it can be shown that there exists a positive constant δ such that, for all $\zeta \in C_{\theta_1^+}(R)$ and all $z \in D_{\theta_1^+}(R+1)$, the following inequality holds

$$|r_\theta(\zeta) - r_\theta(z)| \geqslant \frac{\delta}{\log|z|} \tag{5.16}$$

provided R is sufficiently large. Now consider the integral

$$\tilde{I}_M^-(t) := \int_{\tilde{C}_{\theta'}(R_M)} d\psi_{\theta'}(z) e^{t\psi_{\theta'}(z)} \int_{C^-_{\theta_1^+}(R)} dr_\theta(\zeta) \frac{f_{21}(\zeta)e^{M(r_\theta(\zeta)-r_\theta(z))}}{2\pi i(r_\theta(\zeta)-r_\theta(z))}$$

With (5.15) and (5.16) we have

$$|\tilde{I}_M^-(t)| \leqslant C' \int_{\tilde{C}_{\theta'}(R_M)} |d\psi_{\theta'}(z)| \log|z| e^{t\mathrm{Re}\ \psi_{\theta'}(z) - M\rho_\theta(z)} \int_{C^-_{\theta_1^+}(R)} |dr_\theta(\zeta)| e^{c\mathrm{Im}\ \zeta + M\rho_\theta(\zeta)}$$

where C' is a positive constant, independent of M, provided $R_M \geqslant R+1$. A straightforward computation shows that

$$\min_{z\in\tilde{C}_{\theta'}(R')} (\rho_\theta(z) - \log\log|z|) = R'(1+o(1)) \text{ as } R' \to \infty$$

For all $z \in C^+_{\theta^*}(R_M)$ we have $\mathrm{Re}\ \psi_{\theta'}(z) = R_M \log R_M + (\theta^* - \theta')\mathrm{Im}\ z$, while, for all $z \in C^-_{\theta_1^+}(R_M)$, $\mathrm{Re}\ \psi_{\theta'}(z) = R_M\log R_M + (\theta_1^+ - \theta')\mathrm{Im}\ z$. As $\theta^* < \theta' < \theta^{**}$, there exists $\alpha \in (0,1)$ and a positive number K, independent of M, such that, for every $t > 0$,

$$\int_{\tilde{C}_{\theta'}(R_M)} |d\psi_{\theta'}(z)| \log|z| e^{t\mathrm{Re}\ \psi_{\theta'}(z) - M\rho_\theta(z)} \leqslant \frac{K}{t} e^{tR_M\log R_M - \alpha M R_M}$$

provided R_M is sufficiently large. In order to derive an estimate for $\int_{C^-_{\theta_1^+}(R)} |dr_\theta(\zeta)| e^{c\mathrm{Im}\,\zeta + M\rho_\theta(\zeta)}$, we put $\mathrm{Im}\,\zeta = x$ and $\mathrm{Re}\,\zeta = \rho(x)$ for all $\zeta \in C^-_{\theta_1^+}(R)$. Let $\zeta(x) = \rho(x) + ix$, $\phi(x) = \arg \zeta(x)$ and $r(x) = |\zeta(x)|$. From the relation

$$\rho(x) \log |\zeta(x)| = (\arg \zeta(x) + \theta_1^+)x + R \log R$$

we deduce that

$$\rho_\theta(\zeta(x)) = \frac{(\phi(x) + \theta_1^+)x + R\log R}{\log r(x)} - \theta \frac{x \log r(x) - \rho(x)\phi(x)}{(\log r(x))^2 + \phi(x)^2} \leqslant$$

$$\leqslant \frac{(\pi + |\theta_1^+| + |\theta|)|x| + R\log R}{\log r(x)}\Big(1 + \frac{|\theta|\pi}{(\log r(x))^2}\Big) \leqslant \frac{a_{\theta,R}|x|}{\log |x|}$$

where $a_{\theta,R}$ is a positive constant and $x \leqslant -e$. Hence it follows that, for sufficiently large M,

$$\sup_{x\in(-\infty,-e)} M\rho_\theta(\zeta(x)) + cx \leqslant e^{Ma_{\theta,R}/c}$$

From the above considerations we conclude that, for all sufficiently large M,

$$\int_{C^-_{\theta_1^+}(R)} |dr_\theta(\zeta)| e^{c\mathrm{Im}\,\zeta + M\rho_\theta(\zeta)} \leqslant e^{b_{\theta,R} e^{Ma_{\theta,R}/c}}$$

where $b_{\theta,R}$ is a positive constant, independent of M. Let $\beta > \frac{a_{\theta,R}}{c}$. Choosing $R_M = e^{\beta M}$ we find that $\lim_{M\to\infty} \tilde{I}^-_M(t) = 0$ for all $t \in (0, \alpha/\beta)$. Consequently, the integral

$$I_{21}(t) := \int_0^\infty ds \int_R^\infty dr_\theta(\zeta) f_{21}(\zeta) e^{sr_\theta(\zeta)} A_{\theta',\theta}(t,s)$$

converges for sufficiently small $t > 0$, and equals

$$I_{21}(t) = \int_{\tilde{C}_{\theta'}(R')} d\psi_{\theta'}(z) e^{t\psi_{\theta'}(z)} \int_{C^-_{\theta_1^+}(R)} dr_\theta(\zeta) \frac{f_{21}(\zeta)}{2\pi i(r_\theta(\zeta) - r_\theta(z))}$$

Furthermore, it is easily verified that, for all $t > 0$,

$$I_1(t) := \int_0^\infty ds \int_{C^+_{\theta_1^-}(R)} dr_\theta(\zeta) f_1(\zeta) e^{sr_\theta(\zeta)} A_{\theta',\theta}(t,s)$$

$$= -\int_{\tilde{C}_{\theta'}(R')} d\psi_{\theta'}(z) e^{t\psi_{\theta'}(z)} \int_{C^+_{\theta_1^-}(R)} dr_\theta(\zeta) \frac{f_1(\zeta)}{2\pi i(r_\theta(\zeta) - r_\theta(z))}$$

$$= -\int_{\tilde{C}_{\theta'}(R')} d\psi_{\theta'}(z) e^{t\psi_{\theta'}(z)} \int_{C^+_{\theta^*}(R)} dr_\theta(\zeta) \frac{f_1(\zeta)}{2\pi i (r_\theta(\zeta) - r_\theta(z))}$$

and

$$I_2(t) := \int_0^\infty ds \int_{C^-_{\theta_2^+}(R)} dr_\theta(\zeta) f_2(\zeta) e^{sr_\theta(\zeta)} A_{\theta',\theta}(t,s)$$

$$= -\int_{\tilde{C}_{\theta'}(R')} d\psi_{\theta'}(z) e^{t\psi_{\theta'}(z)} \int_{C^-_{\theta_2^+}(R)} dr_\theta(\zeta) \frac{f_2(\zeta)}{2\pi i (r_\theta(\zeta) - r_\theta(z))}$$

$$= -\int_{\tilde{C}_{\theta'}(R')} d\psi_{\theta'}(z) e^{t\psi_{\theta'}(z)} \int_{C^-_{\theta_1^+}(R)} dr_\theta(\zeta) \frac{f_2(\zeta)}{2\pi i (r_\theta(\zeta) - r_\theta(z))}$$

In view of (5.13) we have

$$\mathcal{A}_{\theta',\theta}(\phi_{1,\theta})(t) = \frac{1}{2\pi i}(I_1(t) + I_{21}(t) - I_2(t))$$

Inserting the above expressions for $I_1(t)$, $I_{21}(t)$ and $I_2(t)$ into the right-hand side of this identity we find

$$\mathcal{A}_{\theta',\theta}(\phi_{1,\theta})(t) = -\frac{1}{2\pi i}\int_{\tilde{C}_{\theta'}(R')} d\psi_{\theta'}(z) e^{t\psi_{\theta'}(z)} \int_{\tilde{C}_{\theta'}(R)} dr_\theta(\zeta) \frac{f_1(\zeta)}{2\pi i (r_\theta(\zeta) - r_\theta(z))}$$

Applying Cauchy's theorem we get

$$\mathcal{A}_{\theta',\theta}(\phi_{1,\theta})(t) = \frac{1}{2\pi i}\int_{\tilde{C}_{\theta'}(R')} f_1(z) e^{t\psi_{\theta'}(z)} d\psi_{\theta'}(z)$$

and this proves that, for sufficiently small $t > 0$, $\mathcal{A}_{\theta',\theta}(\phi_{1,\theta})$ coincides with the Borel transform of f_1 (cf. proposition 5.6). □

From Theorem 3.2, Proposition 5.6 and Theorem 5.8 one easily deduces the following result (cf. also Remark 5.3 and [7, Remark 4.6]).

THEOREM 5.9. — *Assume that the conditions of Theorem 3.2 are satisfied and that, in addition, $|I| > \pi$. Then the solutions f_1 and f_2 are accelero-sums of $\hat{f}$, i.e.*

$$f_i(z) = y_0 + \int_0^\infty \mathcal{A}_{\theta_i',\theta_i}(\phi_{1,\theta_i})(t) e^{-t\psi_{\theta_i'}(z)} dt, \quad Re\ \psi_{\theta_i'}(z) \geqslant c_{\theta_i'},\ i = 1, 2 \tag{5.17}$$

where y_0 is the zero order term of $\hat{f}$, $\phi_{1,\theta_i} = \mathcal{B}_{1,\theta_i}((f_1, f_2))$, $\theta_i' \in I_i$, $\theta_i \in (\theta_i' - \frac{\pi}{2}, \theta_i' + \frac{\pi}{2}) \cap (\theta_1^- + \frac{\pi}{2}, \theta_2^+ - \frac{\pi}{2})$ and $c_{\theta_i'}$ is some sufficiently large positive number, $i = 1, 2$.

This theorem shows that, if the conditions of Theorem 3.2 are satisfied and $|I| > \pi$, then the functions f_1 and f_2 are characterized by their asymptotic properties *alone* (independently of the equation they satisfy).

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