ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

ROBERT BERMAN, JOHANNES SJÖSTRAND

Asymptotics for Bergman-Hodge kernels for high powers of complex line bundles

Tome XVI, n° 4 (2007), p. 719-771.

<http://afst.cedram.org/item?id=AFST_2007_6_16_4_719_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIV, n° 4, 2007
pp. 719–771

Asymptotics for Bergman-Hodge kernels for high powers of complex line bundles(*)

Robert Berman [1], Johannes Sjöstrand [2]

Abstract. — In this paper we obtain the full asymptotic expansion of the Bergman-Hodge kernel associated to a high power of a holomorphic line bundle with non-degenerate curvature. We also explore some relations with asymptotic holomorphic sections on symplectic manifolds.

Résumé. — Dans ce travail nous obtenons un développement asymptotique complet du noyau de Bergman-Hodge d'une puissance élevée d'un fibré en droites holomorphe à courbure non-dégenerée. Nous explorons aussi quelques relations avec des sections asymptotiquement holomorphes sur une variété symplectique.

Contents

(*) Reçu le 16 novembre 2005, accepté le 10 juillet 2006
(1) Department of Mathematics, Chalmers University of Technology, Eklandag. 86, SE-412 96 Göteborg.
robertb@math.chalmers.se
(2) CMLS, Ecole Polytechnique, FR-91128 Palaiseau cedex, UMR 7640, CNRS.
johannes@math.polytechnique.fr

1. Introduction

Let L be a Hermitian holomorphic line bundle over a compact complex Hermitian manifold X. Denote by Θ the curvature two-form of the canonical connection ∇ on L. By the Hodge theorem, the Dolbeault cohomology group $H^{0,q}(X,L)$ is isomorphic to the space $\mathcal{H}^{0,q}(X,L)$ of harmonic $(0,q)-$forms with values in L, i.e the null space of the Hodge Laplacian Δ_q. Denote by Π_q the corresponding Hodge projection, i.e. the orthogonal projection from $L^2(X,L)$ onto $\mathcal{H}^{0,q}(X,L)$. We will assume that Θ is non-degenerate of constant signature (n_-,n_+), i.e. the number of negative eigenvalues of Θ is n_- (the index of Θ). Then it is well-known, by the theorems of Kodaira and Hörmander, that $\mathcal{H}^{0,q}(X,L^k)$ is trivial when $q\neq n_-$, for a sufficiently high tensor power L^k. (See also [18].) We will study the asymptotics with respect to k of the corresponding Hodge projections $\Pi_{q,k}$ in the non-trivial case when $q=n_-$. The case when $n_-=0$, i.e. when L is a positive line bundle and $\Pi_{q,k}$ is the Bergman projection on the space of holomorphic sections with values in L^k, has been studied extensively before (compare the historical remarks below).

Let π_1 and π_2 be the projections on the first and the second factor of $X\times X$. Denote by K_k the Schwartz kernel of $\Pi_{q,k}$ (the subscripts k will be omitted in the sequel) with respect to the volume form ω_n on X induced by the Hermitian metric on X, so that K is a section of $\mathcal{L}(\pi_2^*(\Lambda^{0,q}(T^*X)\otimes L^k),\pi_1^*(\Lambda^{0,q}(T^*X)\otimes L^k))$.

Let t,s be local unitary sections of L over $\widetilde{X}$, $\widetilde{Y}$ respectively, where $\widetilde{X},\widetilde{Y}\subseteq X$. Then on $\widetilde{X}\times\widetilde{Y}$ we can write

$$K(x,y)=K_{t,s}(x,y;\frac{1}{k})t(x)^k s(y)^{*k},$$

where $K_{t,s}$ is a local section of $\mathcal{L}(\pi_2^*(\Lambda^{0,q}(T^*X)),\pi_1^*(\Lambda^{0,q}(T^*X)))$ so that for $x\in\widetilde{X}$, $u\in C_0^\infty(\widetilde{Y};\Lambda^{0,q}(T^*X\otimes L^k))$,

$$u(x)=t(x)^k\int_X K_{t,s}(x,y;\frac{1}{k})\left\langle u(y),s(y)^{*k}\right\rangle \omega_n(dy),$$

We say that a kernel

$$R(x,y)=R_{t,s}(x,y;\frac{1}{k})t(x)^k s(y)^{*k},$$

is *negligible* if

$$\partial_x^\alpha\partial_y^\beta R_{t,s}(x,y;\frac{1}{k})=\mathcal{O}_{\alpha,\beta,N}(k^{-N}),$$

locally uniformly on every compact set in $\widetilde{X} \times \widetilde{Y}$, for all multiindices α, β and all N in $\mathbb{N}$. Notice that this statement does not depend on the choice of t, s and on the local coordinates x, y.

Our main result tells us that K is negligible near every point (x_0, y_0) with $x_0 \neq y_0$ and that for (x, y) near a diagonal point (x_0, x_0)

$$K_{t,s}(x, y) = b(x, y; \frac{1}{k})e^{k\psi(x,y)} + R_{t,s}(x, y), \;\; R_{t,s} \text{ negligible,} \tag{1.1}$$

where ψ is smooth function with $\psi(x, x) = 0$, $\operatorname{Re}\psi(x, y) \backsim -|x - y|^2$ and

$$b(x, y; \frac{1}{k}) \sim k^n(b_0(x, y) + b_1(x, y)\frac{1}{k} + ...)$$

in $C^\infty(\text{neigh}(x_0, x_0); \mathcal{L}(\pi_2^*(\Lambda^{0,q}(T^*X)), \pi_1^*(\Lambda^{0,q}(T^*X)))$. Moreover, let $\widetilde{\mathcal{C}}$ be the graph of $\frac{1}{i}d\psi$ in $T^*\widetilde{X} \times T^*\widetilde{X}$ over the diagonal. Then $\widetilde{\mathcal{C}}$ locally represents the graph of the canonical connection ∇ of $L \otimes L^*$ over the diagonal in $X \times X$ and the semiclassical wave front of K. See Theorem 5.1 and the preceding explanations in Section 5 for a more precise local statement.

We will also explore some relations to the work [44] of B.Shiffman and S.Zelditch, where so called asymptotic holomorphic sections on symplectic manifolds are studied.

1.1. Overview

After locally fixing a unitary frame for L, we identify the Hodge Laplacian Δ_q acting on $(0, q)-$forms with values in L^k, with a local semiclassical differential operator (setting $h = 1/k$). Since the curvature form of L is assumed to be non-degenerate the characteristic variety Σ of Δ_q is symplectic. Modifying the approach in [40] we then construct associated local asymptotic heat kernels in Section 3 and investigate the limit when the time variable tends to infinity. In Section 4 it is shown that the limit operator is an asymptotic local projection operator. The complex canonical relation of the local projection operators is expressed in terms of the stable outgoing and incoming manifolds associated to Σ in Section 3. Assuming, in Section 5, that the number of negative eigenvalues of the curvature of L is equal to q everywhere on X we get a complete asymptotic expansion of the global projection operator Π_q. In Section 6 we investigate some relations to [44], where so called asymptotic holomorphic sections on symplectic manifolds are studied. We introduce a certain almost complex structure, closely related to the stable manifolds introduced in Section 3, making the curvature form of L positive. It is shown that for k sufficiently large the dimension of the null space of Δ_q coincides with the dimension of the corresponding space of asymptotically holomorphic sections (after a suitable twisting of L). In Section 7 the interplay between different complex structures is illustrated by homogeneous line bundles over flag manifolds.

1.2. Historical remarks

Most of the earlier results concern the positively curved case $n_- = 0$. G. Tian [49], followed by W. Ruan [43] and Z. Lu [32], computed increasingly many terms of the asymptotic expansion on the diagonal, using Tian's method of peak solutions. T. Bouche [11] also got the leading term using heat kernels.

S. Zelditch [51], D. Catlin [14] established the complete asymptotic expansion at $x = y$ by using a result of L. Boutet de Monvel, J. Sjöstrand [13] for the asymptotics of the Szegö kernel on a strictly pseudoconvex boundary (after the pioneering work of C. Fefferman [21]), here on the boundary of the unit disc bundle, and a reduction idea of L. Boutet de Monvel, V. Guillemin [13]. Scaling asymptotics away from the diagonal (roughly with a second order polynomial instead of ψ in (1.1) and corresponding more general amplitudes) was obtained by P. Bleher, B. Shiffman, S. Zelditch [6] and the asymptotics as in (1.1) by L. Charles [15], using again the reduction method. In the recent work [4] B. Berndtsson and the authors have worked out a short and direct proof for the asymptotics as in (1.1).

In more general situations, asymptotic expansions on the diagonal and in the scaling sense away from the diagonal were obtained by B. Shiffman, S. Zelditch [44] and X. Dai, K. Liu, X. Ma [17]. See also the works by X. Ma and G. Marinescu [34] for related spectral results and [35] for asymptotics on the diagonal.

Without a positive curvature assumption there have been fewer results. J.M. Bismut [5] used the heat kernel method in his approach to Demailly's holomorphic Morse inequalities. Using local holomorphic Morse inequalities [2], the leading asympotics of the Hodge projections were obtained by the first author in [3] without assuming that the curvature is non-degenerate. X. Ma has pointed out to us that the method and results of [17] can be extended to the case of non-positive holomorphic line bundles by using a spectral gap estimate from [34] and this was recently carried out in the preprint [36]. The result of Theorem 5.1 was announced in [47].

In this quick review, we omitted results away from the diagonal, since our work only concerns the asymptotics modulo $\mathcal{O}(k^{-\infty})$.

1.3. Why the heat kernel method?

Originally we thought about a direct semiclassical adaptation of the methods in [13] and both L. Boutet de Monvel and more recently the referee have suggested such an approach to us. For a long time our attempts in that direction were stalled by some algebraic problems in the case $n_- > 0$, and only recently (after finishing the present paper) did we get an idea about how to circumvent the algebraic difficulty.

We believe however that the heat kernel method has its own interest and is not really longer than the adaptation of [13]. Undoubtedly it can also be used to obtain the complete asymptotics of the inverse of Δ_q when $q \neq n_-$ and the partial inverse on the orthogonal of the kernel when $q = n_-$.

Acknowledgements. — The first author has been partially supported by a Marie Curie grant. The second author has benefitted from the hospitality of Chalmers and Gothenburg University in 2000–02. We express our gratitude to Bo Berndtsson for many stimulating discussions and for continued interest in this work. We have also benefitted from discussions with L. Boutet de Monvel L. Charles, X. Ma, H. Seppänen and G. Zhang, as well as with M. Shubin who suggested a similar problem to one of us in 1994. Finally, we thank the referee for several interesting remarks.

2. Holomorphic line bundles and the $\overline{\partial}$-complex, a review

Let L be a Hermitian holomorphic line bundle over X. Later, we shall use a local holomorphic non-vanishing section s. We write the point-wise norm of s as

$$|s|^2 = |s|_{h_1}^2 =: e^{-2\phi}.$$

The curvature form of L can be identified with the Levi form $\partial\overline{\partial}\phi$.

Add a Hermitian metric on $T^{1,0}X$:

$$H(\nu, \mu) = \sum H_{j,k}\nu_k\overline{\mu}_j, \text{ if } \nu = \sum \nu_j \frac{\partial}{\partial z_j},\ \mu = \sum \mu_j \frac{\partial}{\partial z_j}. \tag{2.1}$$

We have a natural duality between $T^*_{1,0}X$ and $T^{1,0}X$, satisfying

$$\langle dz_j, \frac{\partial}{\partial z_k}\rangle = \delta_{j,k},$$

so if $\omega = \sum \omega_k dz_k$, then $\langle \omega, \nu \rangle = \sum \omega_j \nu_j$. For each $x \in X$, we can choose $z_1, ..., z_n$ centered at x so that

$$H_{j,k}(x) = \delta_{j,k};\ H_x(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}) = \delta_{j,k}.$$

The metric H also determines a metric on $\Lambda^{0,q}(T^*X)$ such that in the special coordinates above, we have that

$$d\overline{z}_{j_1} \wedge ... \wedge d\overline{z}_{j_q},\ 1 \leqslant j_1 < j_2 < ... < j_q \leqslant n,$$

is an orthonormal basis of $\Lambda^{0,q}T_x^*X$. Then we have a natural metric also on $L \otimes \Lambda^{0,q}T^*X$.

Let us also fix some smooth positive integration density $m(dz)$ on X. (For instance, we can take $m(dz) = \omega_n(dz)$; the induced volume form.) Then we get a natural scalar product on

$$\mathcal{E}^{0,q}(L) = C^\infty(X; L \otimes \Lambda^{0,q}T^*X),$$

so if

$$\overline{\partial} :\ .. \to \mathcal{E}^{0,q}(L) \to \mathcal{E}^{0,q+1}(L) \to ..$$

is the $\overline{\partial}$ complex, then

$$\overline{\partial}^* :\ .. \leftarrow \mathcal{E}^{0,q}(L) \leftarrow \mathcal{E}^{0,q+1}(L) \leftarrow ...$$

is also a well-defined complex.

If ω is a 0,1-form, let $\omega^\rfloor : \Lambda^{0,q+1}T_x^*X \to \Lambda^{0,q}T_x^*X$ be the adjoint of left exterior multiplication $\omega^\wedge : \Lambda^{0,q}T_x^*X \to \Lambda^{0,q+1}T_x^*X$. Here we use the Hermitian inner product H^* on $\Lambda^{0,q}T_x^*X$ that is naturally obtained from H. Without that inner product, we can still define $\nu^\rfloor : \Lambda^{0,q+1}T_x^*X \to \Lambda^{0,q}T_x^*X$, when $\nu = \sum \nu_j \frac{\partial}{\partial \overline{z}_j}$ is a vector field of type 0,1, as the transpose of $\nu^\wedge : \Lambda^{0,q}T_xX \to \Lambda^{0,q+1}T_xX$. We have the standard identity,

$$\omega^\wedge \nu^\rfloor + \nu^\rfloor \omega^\wedge = \langle \omega, \nu \rangle \mathrm{id}.$$

In the present case we have the analogous identity,

$$\omega_1^\wedge \omega_2^\rfloor + \omega_2^\rfloor \omega_1^\wedge = H^*(\omega_1, \omega_2)\mathrm{id}, \tag{2.2}$$

when ω_1, ω_2 are (0,1)-forms. Notice also that $\omega_2^\rfloor$ depends anti-linearly on ω_2.

Let $e_1(z), ..., e_n(z)$ be an orthonormal frame for $\Lambda^{0,1}T^*X$. Let $Z_1(z), ..., Z_n(z)$ be the dual basis of $\Lambda^{0,1}TX$, so that on scalar functions,

$$\overline{\partial} = \sum_1^n e_j(z)^\wedge \otimes Z_j(z, \frac{\partial}{\partial \overline{z}}).$$

If $f(z)e_{j_1} \wedge ... \wedge e_{j_q}$ is a typical term in a general $(0,q)$-form, we get

$$\begin{aligned}
&\overline{\partial}(f(z)e_{j_1} \wedge ... \wedge e_{j_q}) \\
= \;& \sum_{j=1}^n Z_j(f)e_j^\wedge e_{j_1} \wedge ... \wedge e_{j_q} + \sum_{k=1}^q (-1)^{k-1} f(z)e_{j_1} \wedge .. \wedge (\overline{\partial} e_{j_k}) \wedge .. \wedge e_{j_q} \\
= \;& (\sum_{j=1}^n Z_j(f)e_j^\wedge)e_{j_1} \wedge ... \wedge e_{j_k} + (\sum_{j=1}^n (\overline{\partial} e_j)^\wedge e_j^\rfloor)(f(z)e_{j_1} \wedge ... \wedge e_{j_q}).
\end{aligned}$$

So for the given orthonormal frame we have the identification

$$\overline{\partial} \equiv \sum_{j=1}^n (e_j^\wedge \otimes Z_j + (\overline{\partial} e_j)^\wedge e_j^\rfloor) \tag{2.3}$$

and correspondingly

$$\overline{\partial}^* \equiv \sum_{j=1}^n (e_j^\rfloor \otimes Z_j^* + e_j^\wedge (\overline{\partial} e_j)^\rfloor),$$

where Z_j^* is the formal complex adjoint of Z_j in $L^2(m)$.

If s is a trivializing local holomorphic section of L, then s^k is a trivializing local section of L^k, and the corresponding metric h_k on L^k satisfies

$$|s^k|^2_{h_k} = |s|^{2k}_{h_1} = e^{-2k\phi(z)}.$$

Hence if

$$\begin{aligned}
\widetilde{\omega} &= s^k \omega \in \mathcal{E}^{0,q}(X; L^k), \\
\widetilde{w} &= s^k w \in \mathcal{E}^{0,q+1}(X; L^k),
\end{aligned}$$

we get for $\overline{\partial}$, $\overline{\partial}^*$, acting on $(0,q)$-forms with coefficients in L^k:

$$\begin{aligned}
\overline{\partial}(s^k \omega) &= s^k \sum_{j=1}^n (e_j^\wedge \otimes Z_j + (\overline{\partial} e_j)^\wedge e_j^\rfloor)\omega, \\
\overline{\partial}^*(s^k w) &= s^k \sum_{j=1}^n (e_j^\rfloor \otimes (Z_j^* + 2k\overline{Z}_j(\phi)) + e_j^\wedge (\overline{\partial} e_j)^\rfloor) w.
\end{aligned}$$

We next derive more symmetric representations for $\overline{\partial}$, $\overline{\partial}^*$ in spaces without exponential weights, by using the following local representation,

$$\widetilde{\omega} = (se^{\phi})^k \widehat{\omega} \in \mathcal{E}^{0,q}(X; L^k), \tag{2.4}$$

so that

$$\mathcal{E}^{0,q}(X) \ni \widehat{\omega} \mapsto (se^{\phi})^k \widehat{\omega} \in \mathcal{E}^{0,q}(X; L^k)$$

is locally unitary in view of the fact that $|s(x)e^{\phi(x)}|_{h_1(x)} = 1$:

$$\int |\widetilde{\omega}(x)|^2_{h_k(x)\otimes H} m(dx) = \int |\widehat{\omega}(x)|^2_{H(x)} m(dx). \tag{2.5}$$

Using (2.3), which makes sense directly on elements of $\mathcal{E}^{0,q}(X, L^k)$, we get

$$\overline{\partial}\widetilde{\omega} = (se^{\phi})^k \overline{\partial}_s \widehat{\omega}, \tag{2.6}$$

where,

$$\overline{\partial}_s \widehat{\omega} = \sum_{j=1}^n (e_j^{\wedge} \otimes (Z_j + kZ_j(\phi)) + (\overline{\partial} e_j)^{\wedge} e_j^{\rfloor}). \tag{2.7}$$

Now the formal adjoint of $\overline{\partial}_s$ for the scalar product given by the right hand side of (2.5) is

$$\overline{\partial}_s^* \widehat{w} = \sum_{j=1}^n (e_j^{\rfloor} \otimes (Z_j^* + k\overline{Z_j(\phi)}) + e_j^{\wedge} (\overline{\partial} e_j)^{\rfloor}), \tag{2.8}$$

where in view of the unitarity of the relation (2.4),

$$\overline{\partial}^* \widetilde{w} = (se^{\phi})^k \overline{\partial}_s^* \widehat{w}, \tag{2.9}$$

where

$$\widetilde{w} = (se^{\phi})^k \widehat{w}. \tag{2.10}$$

Now rewrite things semiclassically. Put

$$h = \frac{1}{k}, \tag{2.11}$$

$$h\overline{\partial}_s = \sum_{j=1}^n (e_j^{\wedge} \otimes (hZ_j + Z_j(\phi)) + h(\overline{\partial} e_j)^{\wedge} e_j^{\rfloor}), \tag{2.12}$$

$$h\overline{\partial}_s^* = \sum_{j=1}^n (e_j^{\rfloor} \otimes (hZ_j^* + \overline{Z_j(\phi)}) + he_j^{\wedge} (\overline{\partial} e_j)^{\rfloor}). \tag{2.13}$$

Here hZ_j is a semiclassical differential operator.

PROPOSITION 2.1. — *Using the representation (2.4), we can identify the Hodge Laplacian with*

$$
\begin{aligned}
&\Delta = \qquad\qquad (2.14)\\
&(h\overline{\partial}_s)(h\overline{\partial}_s^*) + (h\overline{\partial}_s^*)(h\overline{\partial}_s) =\\
&\sum_{j=1}^{n} 1 \otimes (hZ_j^* + \overline{Z_j(\phi)})(hZ_j + Z_j(\phi))\\
&\qquad + \sum_{j,k} e_j^{\wedge} e_k^{\rfloor} \otimes [hZ_j + Z_j(\phi), hZ_k^* + \overline{Z_k(\phi)}]\\
&\qquad\qquad + \mathcal{O}(h)(hZ + Z(\phi)) + \mathcal{O}(h)(hZ^* + \overline{Z(\phi)}) + \mathcal{O}(h^2),
\end{aligned}
$$

where $\mathcal{O}(h)(hZ + Z(\phi))$ indicates a remainder term of the form $h\sum_k a_k(z)$ $(hZ_k + Z_k(\phi))$ with a_k smooth, matrix-valued, and similarly for the two other remainder terms in (2.14).

Proof. — We make a straightforward calculation.

$$
\begin{aligned}
&(h\overline{\partial}_s)(h\overline{\partial}_s)^* + (h\overline{\partial}_s)^*(h\overline{\partial}_s) =\\
&\sum_{1\leqslant j,k\leqslant n} \Big((e_j^{\wedge} \otimes (hZ_j + Z_j(\phi)))(e_k^{\rfloor} \otimes (hZ_k^* + \overline{Z_k(\phi)}))\\
&+ (e_k^{\rfloor} \otimes (hZ_k^* + \overline{Z_k(\phi)}))(e_j^{\wedge} \otimes (hZ_j + Z_j(\phi)))\\
&+ (e_j^{\wedge} \otimes (hZ_j + Z_j(\phi)))(he_k^{\wedge}(\overline{\partial}e_k)^{\rfloor}) + (he_k^{\wedge}(\overline{\partial}e_k)^{\rfloor})(e_j^{\wedge} \otimes (hZ_j + Z_j(\phi)))\\
&+ h((\overline{\partial}e_j)^{\wedge}e_j^{\rfloor})(e_k^{\rfloor} \otimes (hZ_k^* + \overline{Z_k(\phi)})) + (e_k^{\rfloor} \otimes (hZ_k^* + \overline{Z_k(\phi)}))h((\overline{\partial}e_j)^{\wedge}e_j^{\rfloor})\\
&+ h((\overline{\partial}e_j)^{\wedge}e_j^{\rfloor})he_k^{\wedge}(\overline{\partial}e_k)^{\rfloor} + he_k^{\wedge}(\overline{\partial}e_k)^{\rfloor}h((\overline{\partial}e_j)^{\wedge}e_j^{\rfloor})\Big).
\end{aligned}
$$

Using (2.2), we see that the sum of the first two terms inside the general term of the sum is equal to

$$
\begin{aligned}
&(e_j^{\wedge}e_k^{\rfloor} + e_k^{\rfloor}e_j^{\wedge}) \otimes ((hZ_k^* + \overline{Z_k(\phi)})(hZ_j + Z_j(\phi)))\\
&\qquad\qquad + e_j^{\wedge}e_k^{\rfloor}[hZ_j + Z_j(\phi), hZ_k^* + \overline{Z_k(\phi)}]\\
&= \delta_{j,k}(hZ_k^* + \overline{Z_k(\phi)})(hZ_k + Z_k(\phi)) + e_j^{\wedge}e_k^{\rfloor}[hZ_j + Z_j(\phi), hZ_k^* + \overline{Z_k(\phi)}].
\end{aligned}
$$

The proposition follows. □

Let q_j be the semiclassical principal symbol of $hZ_j + Z_j(\phi)$, that we shall write down more explicitly later, viewed as a function on the "real" cotangent space T^*X. (We refer to [42, 19] for standard terminology about semiclassical pseudodifferential operators, and to [26, 48] for the fact that the Weyl quantization permits to define the symbol of such an operator modulo $\mathcal{O}(h^2)$ even on a manifold.) The semiclassical principal symbol of Δ is

$$p_0 = 1 \otimes \sum_{j=1}^{n} \overline{q}_j q_j. \tag{2.15}$$

The semiclassical subprincipal symbol of Δ is a well-defined endomorphism of $\Lambda^{0,q}T^*X$ at every point $(x, \xi) \in \Sigma$ on the doubly characteristic manifold $\Sigma \subset T^*X$, given by $q_1 = ... = q_n = 0$. For an operator of the form $(hZ_k^* + \overline{Z_k(\phi)})(hZ_j + Z_j(\phi))$ this subprincipal symbol is given by $\frac{h}{2i}\{\overline{q}_k, q_j\}$ and the contribution from the double sum in (2.14) to the subprincipal symbol of Δ is

$$\frac{h}{i} \sum_{j,k} e_j^\wedge e_k^\rfloor \otimes \{\overline{q}_j, q_k\}.$$

Thus on Σ, we get the subprinicipal symbol of Δ:

$$hp_1 = h(1 \otimes \sum_j -\frac{1}{2i}\{q_j, \overline{q}_j\} + \sum_{j,k} e_j^\wedge e_k^\rfloor \frac{1}{i}\{q_j, \overline{q}_k\}). \tag{2.16}$$

Since p_1 is invariantly defined on Σ as well as the first sum, the double sum is also invariantly defined.

To compute further, we choose holomorphic coordinates $z_1, ..., z_n$, $z_j = x_j + iy_j$. We make the following fiberwise bijections between $\Lambda^{1,0}T^*X$, T^*X, $\Lambda^{0,1}T^*X$:

$$\sum_1^n \zeta_j dz_j \leftrightarrow \mathrm{Re}\,(\sum_1^n \zeta_j dz_j) \leftrightarrow \sum_1^n \overline{\zeta}_j d\overline{z}_j. \tag{2.17}$$

Writing

$$\zeta_j = \xi_j - i\eta_j,$$

we get

$$\mathrm{Re}\,(\sum \zeta_j dz_j) = \sum (\xi_j dx_j + \eta_j dy_j),$$

so in local coordinates, we have bijections between

$$(z, \zeta) \in \Lambda^{1,0}T^*X,\ (x, y; \xi, \eta) \in T^*X,\ (z, \overline{\zeta}) \in \Lambda^{0,1}T^*X.$$

The semiclassical symbol of $h\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(h\frac{\partial}{\partial x_j} + ih\frac{\partial}{\partial y_j})$ is $\frac{i}{2}(\xi_j + i\eta_j) = \frac{i}{2}\overline{\zeta}_j$. Hence the symbol of

$$h\frac{\partial}{\partial \overline{z}_j} + \frac{\partial \phi}{\partial \overline{z}_j} \text{ is } \frac{i}{2}\overline{\zeta}_j + \frac{\partial \phi}{\partial \overline{z}_j},$$

so in the coordinates (z, ζ), the equation for Σ becomes:

$$\overline{\zeta}_j = -\frac{2}{i}\frac{\partial \phi}{\partial \overline{z}_j},$$

or equivalently,

$$\zeta_j = \frac{2}{i}\frac{\partial \phi}{\partial z_j},\ j = 1, 2, .., n. \tag{2.18}$$

For later use we here compute the principal symbol q_j of $hZ_j + Z_j(\phi)$: Let the orthonormal frame $e_1, ..., e_n$ be given by

$$e_j(z) = \sum_k a_{j,k}(z) d\overline{z}_k,$$

and the corresponding dual basis $Z_1, ..., Z_n$ of $\Lambda^{0,1}T_z^*X$ by

$$Z_j = \sum_k b_{j,k}\frac{\partial}{\partial \overline{z}_k},$$

where the invertible matrices $(a_{j,k})$ and $(b_{j,k})$ are related by

$${}^t(b_{j,k})(a_{j,k}) = 1.$$

Then it follows from the calculations above that

$$q_j = \sum_k b_{j,k}(\frac{i}{2}\overline{\zeta}_k + \frac{\partial \phi}{\partial \overline{z}_k}). \tag{2.19}$$

Proposition 2.2. — *In the (z, ζ)-coordinates, the Poisson bracket $\{f, g\}$ of two C^1-functions f, g is given by*

$$\frac{1}{2}\{f, g\} = \frac{1}{2}H_f g = (\frac{\partial f}{\partial \zeta}\cdot\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \overline{\zeta}}\cdot\frac{\partial g}{\partial \overline{z}}) - (\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial \zeta} + \frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial g}{\partial \overline{\zeta}}) \tag{2.20}$$

Proof. — Consider the real canonical 1-form on T^*X:

$$\mathrm{Re}\,(\sum \zeta_j dz_j) = \sum(\xi_j dx_j + \eta_j dy_j).$$

Hence the real symplectic form becomes

$$d(\sum(\xi_j dx_j + \eta_j dy_j)) = \mathrm{Re}\,(\sum d\zeta_j \wedge dz_j) = \mathrm{Re}\,\sigma =: \omega,$$

where $\sigma = \sum d\zeta_j \wedge dz_j$. If f is a smooth real function on the real phase space, the corresponding Hamilton field H_f is given by

$$\langle \omega, t \wedge H_f\rangle = \langle t, df\rangle. \tag{2.21}$$

With $t = 2\mathrm{Re}\sum(a_j\frac{\partial}{\partial z_j} + b_j\frac{\partial}{d\zeta_j})$, the right hand side becomes

$$2\mathrm{Re}\sum(a_j\frac{\partial f}{\partial z_j} + b_j\frac{\partial f}{\partial \zeta_j}),$$

while the left hand side is equal to

$$\mathrm{Re}\,\langle\sigma, t\wedge H_f\rangle = \mathrm{Re}\sum(b_j\langle dz_j, H_f\rangle - a_j\langle d\zeta_j, H_f\rangle).$$

Varying t, we conclude that

$$\langle dz_j, H_f\rangle = 2\frac{\partial f}{\partial \zeta_j},\ \langle d\zeta_j, H_f\rangle = -2\frac{\partial f}{dz_j},$$

so

$$\frac{1}{2}H_f = (\frac{\partial f}{\partial\zeta}\cdot\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\cdot\frac{\partial}{\partial\zeta}) + (\frac{\partial f}{\partial\overline{\zeta}}\cdot\frac{\partial}{\partial\overline{z}} - \frac{\partial f}{\partial\overline{z}}\cdot\frac{\partial}{\partial\overline{\zeta}}).$$

In particular, we get (2.20) This expression now extends to the case when f, g are complex-valued functions which completes the proof. □

Of course (2.20) can also be obtained by straightforward calculation from

$$\{f,g\} = \frac{\partial f}{\partial\xi}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial\eta}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial\xi} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial\eta},\ \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial\overline{z}},\ \frac{\partial}{\partial y} = \frac{1}{i}(\frac{\partial}{\partial\overline{z}} - \frac{\partial}{\partial z}), \dots \tag{2.22}$$

Now return to the expressions (2.14), (2.15). If z_0 is a fixed point, we choose holomorphic coordinates $z_1, ..., z_n$ as above in such a way that $Z_j = \frac{\partial}{\partial\overline{z}_j}$, $e_j = d\overline{z}_j$ at z_0. Then $b_{j,k}(z_0) = \delta_{j,k}$ in (2.19) and at the corresponding point $\rho_0 = (z_0, \zeta_0) \in \Sigma$, we have

$$\{q_j, \overline{q}_k\}(\rho_0) = \{\frac{i}{2}\overline{\zeta}_j + \frac{\partial\phi}{\partial\overline{z}_j}, -\frac{i}{2}\zeta_k + \frac{\partial\phi}{\partial z_k}\}.$$

Applying (2.20), we now get

$$\frac{1}{2}\{q_j, \overline{q}_k\} = \frac{i}{2}\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k} + \frac{\partial^2\phi}{\partial z_k\partial\overline{z}_j}\frac{i}{2} = i\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}.$$

We rewrite this as

$$\frac{1}{2i}\{q_j, \overline{q}_k\} = \frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}, \tag{2.23}$$

and recognize here the coefficients of the Levi-matrix appearing also in $\overline{\partial}\partial\phi$.

PROPOSITION 2.3. — *Σ is symplectic at a point $(z_0;\xi_0,\eta_0)$ iff $(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k})(z_0)$ is non-degenerate. Indeed, if we identify $\Lambda^{1,0}T^*X$ and T^*X, by means of the first bijection in (2.17), then the real symplectic form ω becomes $\mathrm{Re}\,(\sum d\zeta_j\wedge dz_j)$ and its restriction to Σ can be identified with $\frac{2}{i}\overline{\partial}\partial\phi$.*

Proof. — With the above mentioned identification, Σ takes the form (2.18) which can be written more invariantly as

$$\zeta\cdot dz=\frac{2}{i}\partial\phi. \tag{2.24}$$

Hence,

$$\sigma_{|_\Sigma}=d\sum_1^n\frac{2}{i}\frac{\partial\phi}{\partial z_j}\wedge dz_j=\sum_{j=1}^n\sum_{k=1}^n\frac{2}{i}\frac{\partial^2\phi}{\partial\overline{z}_k\partial z_j}d\overline{z}_k\wedge dz_j=\frac{2}{i}\overline{\partial}\partial\phi.$$

This is a real form, so it is also the restriction to Σ of $\mathrm{Re}\,\sigma$ and it is non-degenerate precisely when $(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k})$ is (cf [45]). $\square$

Back to the general case, we recall the condition for having the apriori estimate

$$h\|u\|+\sum\|(hZ_j+Z_j(\phi))u\|+\sum\|(hZ_j^*+\overline{Z_j(\phi)})u\|\leqslant C\|\Delta_q u\|, \tag{2.25}$$

for $u\in C_0^\infty(\mathrm{neigh}\,(z_0);\Lambda^{0,q}T^*X)$.

PROPOSITION 2.4. — *(2.25) does not hold precisely when $n_-\leqslant q\leqslant n-n_+$, where (n_+,n_-) is the signature of $(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}(z_0))$.*

This is essentially well-known since the $\overline{\partial}$-estimates of L. Hörmander (see [27]), and in the context of more general hypoelliptic operators it was obtained in [46] in the non-degenerate symplectic case. The constant C in formula (2.25) is also closely related to the curvature term appearing in the Bochner-Kodaira-Nakano formula [24, 18]. The result will not be used explicitly since the heat equation method below will give enough control (and would allow to recover it easily, compare Proposition 3.1).

3. The associated heat equations

We work locally near a point $z_0\in X$, where

$$(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k})\text{ is non-degenerate of signature }(n_+,n_-), \tag{3.1}$$

so that the characteristic manifold Σ of Δ_q is symplectic. We review some results of A. Menikoff, J. Sjöstrand [40], [41] that apply to the present situation with minor changes:

In those works, we considered a scalar classical pseudodifferential operator with principal symbol p_0 vanishing to precisely the second order on a conic symplectic submanifold of T^*X. In the present work, we have a semi-classical differential operator with a leading symbol p_0 in (2.3) that we can view as scalar; $p_0 = \sum_1^n \overline{q}_j q_j$ and p_0 is no longer homogeneous, and Σ is no longer conic in the fiber variables.

In this section we consider the problem:

$$(h\partial_t + \Delta_q)u(t,x) = 0,\ u(0,x) = v(x). \tag{3.2}$$

We shall apply the standard WKB construction of an approximative solution operator and apply arguments from [40] together with a "Witten trick" to get additional properties to be used later. See Proposition 3.3 for the precise statement about the solution to (3.2). Following a standard idea, we will see how to reduce ourselves to the homogeneous situation (in the proof of Proposition 3.3). Since the non-scalar nature of the operator appears only in the subprincipal terms, it will only affect the transport equations which can be treated very much as in the scalar case. The really new feature is the exponential convergence of the heat parametrix when $t \to \infty$ in the case of $(0, n_-)$-forms.

We forget about most of the complex structure of X and work in some smooth local coordinates $x = (x_1, ..., x_{2n})$ defined on $\widetilde{X} \subset\subset X$. At least for small $t \geqslant 0$, we look for an approximate solution of (3.2) of the form $u(t,x) = U(t)v(x)$,

$$u(t,x) = \frac{1}{(2\pi h)^{2n}} \iint e^{\frac{i}{h}(\psi(t,x,\eta) - y\cdot\eta)} a(t,x,\eta;h) u(y) dy d\eta, \tag{3.3}$$

where a is a matrix-valued classical symbol of order 0:

$$a(t,x,\eta;h) \sim \sum_0^\infty a_k(t,x,\eta)h^k,\ a_{|t=0} = 1, \tag{3.4}$$

and ψ with $\operatorname{Im}\psi \geqslant 0$ should solve the eikonal equation,

$$i\partial_t\psi(t,x,\eta) + p_0(x, \psi_x'(t,x,\eta)) = 0 + \mathcal{O}((\operatorname{Im}\psi)^\infty),\ \psi_{|t=0} = x\cdot\eta. \tag{3.5}$$

The amplitude a is determined by a sequence of transport equations that will be reviewed later. (Here we follow the convention that $u = \mathcal{O}((\operatorname{Im}\psi)^\infty)$

means that $u = \mathcal{O}((\operatorname{Im}\psi)^N)$ for every $N \geqslant 0$, uniformly or locally uniformly depending on the context.)

According to the general theory in [37, 38], this equation can be solved locally, provided that we also denote by p_0 an almost holomorphic extension. The general theory also tells us that $U(t)$ is associated to a canonical transformation,

$$\kappa_t = \exp(-itH_{p_0}). \tag{3.6}$$

(Here κ_t depends slightly on the choice of almost holomorphic extension of p_0, so $\kappa_t(\rho)$ is well-defined only up to $|\operatorname{Im}\rho|^\infty$. In [38] we also made the assumption that $p_0(x,\xi)$ is positively homogeneous of degree 1 in ξ, but as noticed for instance in [40] and will be reviewed in the proof of Proposition 3.3, one can easily reduce the general case to the homogeneous one, by adding a variable x_0 and consider the homogeneous symbol $\xi_0 p(x,\xi/\xi_0)$, then restrict the results to $\xi_0 = 1$.)

So far, we only used the non-negativity of (the real part of) p_0. Now we use that $p_0 \sim \operatorname{dist}(\cdot,\Sigma)^2$. It follows that

$$\psi(t,x,\eta) = x\cdot\eta + \mathcal{O}(t\operatorname{dist}(x,\eta;\Sigma)^2), \tag{3.7}$$

$$\operatorname{Im}\psi(t,x,\eta) \sim t\operatorname{dist}(x,\eta;\Sigma)^2, \tag{3.8}$$

for $0 \leqslant t \leqslant t_0$, and $t_0 > 0$ fixed. Correspondingly, we have

$$\kappa_{t|_\Sigma} = \mathrm{id}, \tag{3.9}$$

$$\begin{aligned}&\text{When } t>0,\ \kappa_t \text{ is a strictly positive canonical}\\ &\text{transformation with graph}\,(\kappa_t)\cap(T^*X)^2 = \operatorname{diag}(\Sigma\times\Sigma).\end{aligned} \tag{3.10}$$

Recall that a positive canonical transformation is strictly positive if the graph κ intersects $T^*X\times T^*X$ cleanly along a smooth submanifold. Thanks to these simplifying features, all essential properties of ψ and κ_t are captured by their Taylor expansions at $t=0$ and at Σ.

In [40] it was shown that (3.5) can be solved for all $t \geqslant 0$, and that we have,

$$\operatorname{Im}\psi(t,x,\eta) \sim \operatorname{dist}(x,\eta;\Sigma)^2, \tag{3.11}$$

uniformly for $t \geqslant 1$, that (3.9), (3.10) remain valid for all $t > 0$, and finally that there exists a smooth function $\psi(\infty,x,\eta)$, well-defined mod $\mathcal{O}(\operatorname{dist}(x,\eta;\Sigma)^\infty)$ such that for all k,α:

$$\partial_t^k\partial_{x,\eta}^\alpha(\psi(t,x,\eta)-\psi(\infty,x,\eta)) = \mathcal{O}(e^{-t/C}), \tag{3.12}$$

uniformly on $[0,+\infty[\times\Sigma$. (In [41] we also established asymptotic expansions when $t\to\infty$ in terms of exponentials in t. We do not need those improved results here.) Here we have locally uniformly on $\widetilde{X}\times\mathbf{R}^{2n}$:

$$\psi(\infty,x,\eta)=x\cdot\eta+\mathcal{O}(\mathrm{dist}\,(x,\eta;\Sigma)^2),\ \mathrm{Im}\,\psi(\infty,x,\eta)\sim\mathrm{dist}\,(x,\eta;\Sigma)^2. \quad (3.13)$$

Further, the canonical relation C_∞ generated by the phase $\psi(\infty,x,\eta)-y\cdot\eta$ is strictly positive with

$$C_\infty\cap(T^*X\times T^*X)=\mathrm{diag}\,(\Sigma\times\Sigma), \quad (3.14)$$

and C_∞ can be described in the following way:

There are two almost holomorphic manifolds $J_+,J_-\subset T^*X^{\mathbf{C}}$ (where the latter set is the almost complexification of T^*X) intersecting T^*X cleanly along Σ, with the following properties:

$$\begin{aligned}&\mathrm{codim}_{\mathbf{C}}J_\pm=n,\ J_\pm\subset p_0^{-1}(0),\\&J_\pm\text{ are involutive and }J_-=\overline{J}_+,\\&\frac{1}{i}\sigma(t,\overline{t})>0,\ \forall t\in T_\rho(J_+)\setminus T_\rho(\Sigma^{\mathbf{C}}),\ \rho\in\Sigma.\end{aligned} \quad (3.15)$$

Here the involutivity of J_+ (and similarly for J_-) means that J_+ is given by the equations $\widetilde{q}_1=...=\widetilde{q}_n=0$, where $d\widetilde{q}_1,...,d\widetilde{q}_n$ are $\mathbf{C}$-linearly independent and $\{\widetilde{q}_j,\widetilde{q}_k\}=0$ on J_+. Further the complexification $\Sigma^{\mathbf{C}}$ is contained in J_+ and $H_{\widetilde{q}_1},...,H_{\widetilde{q}_n}$ span $T_\rho J_+/T_\rho\Sigma^{\mathbf{C}}$. The positivity property above is equivalent to the fact that the Hermitian matrix $(\frac{1}{i}\{\widetilde{q}_j,\overline{\widetilde{q}}_k\})$ is positive definite. In terms of $J_\pm$, we can describe the limiting canonical relation C_∞ as $\{(\rho,\mu)\in J_+\times J_-;$ the n-dimensional bicharacteristic leaves through ρ, μ of J_+ and J_- respectively, intersect $\Sigma_+^{\mathbf{C}}$ at the same point $\}$.

Finally we can also view C_∞ as the limit of $C_t=\mathrm{graph}\,(\kappa_t)$, when $t\to+\infty$, where the convergence is exponentially fast (in the sense of Taylor expansions at $\mathrm{diag}\,(\Sigma\times\Sigma)$). We can also view J_+, J_- as the stable outgoing and incoming manifolds respectively, for the H_{-ip}-flow, near the fixed point set $\Sigma^{\mathbf{C}}$. Let us also add that $J_\pm$ are uniquely determined and that in the case $n_+=n$, we can take $\widetilde{q}_j=q_j$.

Next we consider the behaviour of a in (3.3), (3.4), where we recall that a_0, a_1, ... are successively determined by a sequence of transport equations. Following [40] this can be done in the following way, where we take some advantage of the fact that we work in the Weyl quantization. (See also appendix b of [26].): Formally, with $\psi=\psi(t,\cdot,\eta)$, $P=\Delta_q$ and with the exponent w indicating that we take the h-Weyl quantization, we get

$$\begin{aligned} e^{-i\psi} \circ P \circ e^{i\psi/h} &= P(x, \psi'_x(x) + \xi; h)^w + \mathcal{O}(h^2) = \\ & p(x, \psi'_x) + hp_1(x, \psi'_x) + \frac{1}{2}(hD_x \circ p'_\xi(x, \psi'_x) \\ & +p'_\xi(x, \psi'_x) \circ hD_x) + \mathcal{O}(h^2) = \\ & p(x, \psi'_x) + hp_1(x, \psi'_x) + \frac{h}{i} p'_\xi(x, \psi'_x) \cdot \frac{\partial}{\partial x} \\ & + \frac{h}{2i} \mathrm{div}\,(p'_\xi(x, \psi'_x) \cdot \frac{\partial}{\partial x}) + \mathcal{O}(h^2), \end{aligned}$$

where the "$\mathcal{O}(h^2)$" refers to the action on symbols and p_1 is the subprincipal symbol. This gives the first transport equation for a_0:

$$(\nu + \frac{1}{2}\mathrm{div}\,(\nu) + p_1)a_0 = 0,$$

where

$$\nu = \frac{\partial}{\partial t} - ip'_\xi(x, \psi'_x) \cdot \frac{\partial}{\partial x}.$$

The higher transport equations for a_j, $j \geqslant 1$, are of the form

$$\nu(a_j) = F_j(t, x, a_0, ..., a_{j-1}).$$

Then if $a(t, x, \eta; h) \sim \sum_0^\infty a_j(t, x, \eta)h^j$ in $C^\infty([0, +\infty[\times \widetilde{X} \times \mathbf{R}^{2n})$, we have

$$(h\partial_t + \Delta_q)(e^{\frac{i}{h}\psi(t,x,\eta)} a(t, x, \eta; h)) = \mathcal{O}(h^\infty)$$

locally uniformly on $[0, +\infty[\times \widetilde{X} \times \mathbf{R}^{2n}$ and similarly for the derivatives.

The discussion on page 69 in [40] shows that $\mathrm{div}\,(\nu) \to \frac{1}{2}\widetilde{\mathrm{tr}}\, F$ exponentially fast on Σ, where $\widetilde{\mathrm{tr}}\, F = \sum f_j$, and F is the fundamental matrix of p ie the linearization of H_p at the point of Σ and has the spectrum $\sigma(F) = \{\pm if_j\}$, $f_j \geqslant 0$. In the further discussion of the transport equations the only new feature is that p_1 is now a square matrix rather than a scalar, and whenever we needed a lower bound on $\mathrm{Re}\, p_1$, we now need a lower bound on the set of real parts of the eigenvalues of p_1. Proposition 2.2 in [40] becomes

Proposition 3.1. — *Let $\lambda \in C(\Sigma; \mathbf{R})$ satisfy*

$$\lambda(x, \eta) < \frac{1}{2}\widetilde{\mathrm{tr}}\, F(x, \eta) + \inf \mathrm{Re}\, \sigma(p_1(x, \eta)),\ (x, \eta) \in \Sigma.$$

Then for every compact set $K \subset \Sigma$, $j \in \mathbf{N}$ and $(\gamma, \alpha, \beta) \in \mathbf{N}^{1+2n+2n}$, we have

$$|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t, x, \eta)| \leqslant C_{j,\alpha,\beta,\gamma} e^{-t\lambda(x,\eta)},\ (x, \eta) \in K,\ t \geqslant 0.$$

We are therefore interested in whether

$$\frac{1}{2}\widetilde{\mathrm{tr}}\, F + \inf \mathrm{Re}\, \sigma(p_1) > 0 \text{ on } \Sigma \tag{3.16}$$

or not. Now

$$p = \sum_1^n \overline{q}_j q_j, \quad H_p = \sum (\overline{q}_j H_{q_j} + q_j H_{\overline{q}_j}).$$

At a given point $\rho_0 \in \Sigma$, we choose the basis $H_{q_1}, ..., H_{q_n}, H_{\overline{q}_1}, ..., H_{\overline{q}_n}$ for $T_{\rho_0}(T^*X)^{\mathbf{C}}/\Sigma^{\mathbf{C}}$, and compute the linearization of H_p:

$$H_p(\rho_0 + \sum t_k H_{q_k} + \sum s_k H_{\overline{q}_k}) = \\ \mathcal{O}((t,s)^2) + \sum_{j,k} t_k \{q_k, \overline{q}_j\} H_{q_j} + \sum_{j,k} s_k \{\overline{q}_k, q_j\} H_{\overline{q}_j}.$$

So the matrix F_p of the linearization is expressed in the basis above by

$$\frac{1}{i} F_p = \begin{pmatrix} (\frac{1}{i}\{q_k, \overline{q}_j\}) & 0 \\ 0 & (\frac{1}{i}\{\overline{q}_k, q_j\}) \end{pmatrix},$$

where we recall (2.23). Let $\mu_1, ..., \mu_n$ be the eigenvalues of $(\partial_{\overline{z}_j}\partial_{z_k}\phi)$, with $\mu_j > 0$ for $1 \leqslant j \leqslant n_+$ and $\mu_j < 0$ for $n_+ + 1 \leqslant j \leqslant n$. Then

$$(i^{-1}\{q_k, \overline{q}_j\}) = {}^t(i^{-1}\{q_j, \overline{q}_k\}) \text{ has the eigenvalues } 2\mu_1, ..., 2\mu_n,$$

and

$$(i^{-1}\{\overline{q}_k, q_j\}) = -{}^t(i^{-1}\{q_j, \overline{q}_k\}) \text{ has the eigenvalues } -2\mu_1, ..., -2\mu_n.$$

Hence the non-vanishing eigenvalues of F_p are $\pm 2i\mu_1, ..., \pm 2i\mu_n$, and

$$\frac{1}{2}\widetilde{\mathrm{tr}}\, F_p = \mu_1 + ... + \mu_{n_+} - \mu_{n_++1} - ... - \mu_n. \tag{3.17}$$

For the first term in (2.16), we get

$$\sum_j -\frac{1}{2i}\{q_j, \overline{q}_j\} = -\frac{1}{2i}\mathrm{tr}\,(\{q_j, \overline{q}_k\}) = -\sum_1^n \mu_j. \tag{3.18}$$

We can also compute the eigenvalues of the matrix part of the subprincipal symbol appearing in (2.16) and in the subsequent remark about invariance. We choose holomorphic coordinates such that at the given point z_0: $Z_j = \partial_{\overline{z}_j}$, $e_j = d\overline{z}_j$ and moreover $(i^{-1}\{q_j, \overline{q}_k\})$ is diagonalized, equal to

$$\begin{pmatrix} 2\mu_1 & 0 & .. & 0 \\ 0 & 2\mu_2 & .. & 0 \\ .. & & .. & .. \\ 0 & 0 & .. & 2\mu_n \end{pmatrix}.$$

Then

$$\sum_{j,k}\frac{1}{i}\{q_j,\overline{q}_k\}e_j^\wedge e_k^\rfloor=\sum_j 2\mu_j e_j^\wedge e_j^\rfloor.$$

On $(0,q)$-forms, the eigenvalues are the numbers

$$2(\mu_{j_1}+\mu_{j_2}+...+\mu_{j_q}),\ \text{for } 1\leqslant j_1<j_2<...<j_q\leqslant n.$$

From (3.17), (2.16) and the other calculations we get

$$p_1+\frac{1}{2}\widetilde{\operatorname{tr}}\,F=-2\sum_{n_++1}^{n}\mu_j+\sum_{j,k}\frac{1}{i}\{q_j,\overline{q}_k\}e_j^\wedge e_k^\rfloor,$$

which on the space of $(0,q)$-forms has the eigenvalues

$$-2\sum_{n_++1}^{n}\mu_j+2(\mu_{j_1}+..+\mu_{j_q}),\ 1\leqslant j_1<...<j_q\leqslant n.$$

We see that on Σ

$$\inf\sigma(p_1+\frac{1}{2}\widetilde{\operatorname{tr}}\,F)\begin{cases}=0, & q=n_-\\ >0, & q\neq n_-\end{cases}. \tag{3.19}$$

This is the answer to the question (3.16) and Proposition 3.1 then shows that when $q\neq n_-$, there exists a constant $C>0$ such that

$$|\partial_t^k\partial_{x,\eta}^\alpha a_j(t,x,\eta)|\leqslant C_{k,\alpha,j}e^{-t/C},\ t\geqslant 0,\ (x,\eta)\in\Sigma, \tag{3.20}$$

while in the case $q=n_-$, we have for every $\epsilon>0$:

$$|\partial_t^k\partial_{x,\eta}^\alpha a_j(t,x,\eta)|\leqslant C_{k,\alpha,j,\epsilon}e^{\epsilon t},\ t\geqslant 0,\ (x,\eta)\in\Sigma. \tag{3.21}$$

We also notice from [40], that (3.20) and (3.21) respectively hold also when the initial condition in (3.4) is replaced by $a_{|_{t=0}}=b$ for any classical symbol $b(x,\eta;h)\sim\sum_0^\infty b_j(x,\eta)h^j$.

Using the particular structure of the problem, we will next show

Proposition 3.2. — *Consider the case $q=n_-$ and let a be the symbol in (3.3), (3.4). Then there exist $C>0$ and a classical symbol*

$$a^\infty(x,\eta;h)\sim\sum_0^\infty a_j^\infty(x,\eta)h^j,$$

such that

$$|\partial_t^k\partial_{x,\eta}^\alpha(a_j(t,x,\eta)-a_j^\infty(x,\eta))|\leqslant C_{k,\alpha,j}e^{-t/C},\ t\geqslant 0,\ (x,\eta)\in\Sigma. \tag{3.22}$$

Proof. — a is determined by the initial condition in (3.4) and the fact that

$$(h\partial_t + \Delta_q)(e^{\frac{i}{h}\psi(t,x,\eta)}a(t,x,\eta;h)) = \mathcal{O}(h^\infty), \tag{3.23}$$

locally uniformly in t, and similarly for the derivatives. Let $Z_\phi := h\overline{\partial}_s$ be given in (2.12) , so that Z_ϕ^* is given by (2.13). Then we have the intertwining properties,

$$\Delta_{q+1}Z_\phi = Z_\phi\Delta_q,\ \Delta_{q-1}Z_\phi^* = Z_\phi^*\Delta_q. \tag{3.24}$$

Combining this with (3.23), we get

$$(h\partial_t + \Delta_{q-1})(Z_\phi^*(e^{\frac{i}{h}\psi}a)) = \mathcal{O}(h^\infty), \tag{3.25}$$

$$(h\partial_t + \Delta_{q+1})(Z_\phi(e^{\frac{i}{h}\psi}a)) = \mathcal{O}(h^\infty). \tag{3.26}$$

Now

$$Z_\phi^*(e^{\frac{i}{h}\psi}a) = e^{\frac{i}{h}\psi}\widetilde{a},\ Z_\phi(e^{\frac{i}{h}\psi}a) = e^{\frac{i}{h}\psi}\widehat{a}, \tag{3.27}$$

where $\widetilde{a}$, $\widehat{a}$ are classical symbols of order 0 in h, and combining this with (3.25), (3.26), we see that (3.20) applies to $\widetilde{a}$, $\widehat{a}$. Now, $\Delta_q = Z_\phi^*Z_\phi + Z_\phi Z_\phi^*$, so

$$\Delta_q(e^{\frac{i}{h}\psi}a) = e^{\frac{i}{h}\psi}b \tag{3.28}$$

where $b \sim \sum_0^\infty b_j(t,x,\eta)h^j$ and the b_j satisfy (3.20).

Combining this with (3.23), we see that

$$h\partial_t(e^{\frac{i}{h}\psi}a) = e^{\frac{i}{h}\psi}c, \tag{3.29}$$

where c $(= -b + \mathcal{O}(h^\infty))$ has the same properties as b. But

$$c = h\partial_t a + i(\partial_t\psi)a,$$

so if we combine (3.12), (3.21) with the fact that c satisfies (3.20), we get

$$|\partial_t^k\partial_{x,\eta}^\alpha\partial_t a_j(t,x,\eta)| \leqslant C_{k,\alpha,j}e^{-t/C},\ t \geqslant 0,\ (x,\eta) \in \Sigma, \tag{3.30}$$

for $k \geqslant 1, \alpha \in \mathbb{N}^{4n}$. From this we get (3.22). $\square$

We introduce the semiclassical Sobolev space

$$H^s(\mathbf{R}^{2n}) = \{u \in \mathcal{S}'(\mathbf{R}^{2n});\ \langle hD_x\rangle^s u \in L^2\},\ s \in \mathbf{R},$$

with the h-dependent norm $\|u\|_{H^s} = \|\langle hD_x\rangle^s u\|$. Here $\langle hD_x\rangle = (1+(hD)^2)^{1/2}$. From this, we form $H^s_{\mathrm{comp}}(X)$, $H^s_{\mathrm{loc}}(X)$ in the usual way, when X is a smooth paracompact manifold, as well as $H^s(X)$, when X is compact. On the Fréchet space $H^s_{\rm loc}(X)$, we have natural h-dependent semi-norms, so it makes sense to say that $u=u_h$ is $\mathcal{O}(h^{N_0})$ in $H^s_{\rm loc}$.

We now return to our local coordinate patch $\widetilde{X}\subset X$, and define

$$U(t)u(x)=\frac{1}{(2\pi h)^{2n}}\iint e^{\frac{i}{h}(\psi(t,x,\eta)-y\cdot\eta)}a(t,x,\eta;h)u(y)dyd\eta, \tag{3.31}$$

with ψ, $a\sim\sum_0^\infty a_j(t,x,\eta)h^j$ constructed as above. More precisely, we can choose $a,a_j\in C^\infty([0,\infty[\times\widetilde{X}\times\mathbf{R}^{2n})$ with the following properties:

$$\partial_t^k\partial_x^\alpha\partial_\eta^\beta a_j=\begin{cases}\mathcal{O}_{j,\alpha,\beta,K}(1)e^{-t/C}, q\neq n_-\\ \mathcal{O}_{j,\alpha,\beta,K,\epsilon}(1)e^{\epsilon t},\ \ q=n_-\end{cases},\ (x,\eta)\in K\subset\subset\widetilde{X}\times\mathbf{R}^{2n},\ \epsilon>0, \tag{3.32}$$

$$\partial_t^k\partial_x^\alpha\partial_\eta^\beta(a-\textstyle\sum_0^{N-1}h^ja_j)=h^N\begin{cases}\mathcal{O}_{k,\alpha,\beta,K}(1)e^{-t/C}, q\neq n_-\\ \mathcal{O}_{k,\alpha,\beta,K,\epsilon}(1)e^{\epsilon t},\ \ q=n_-\end{cases}, \tag{3.33}$$
$$(x,\eta)\in K\subset\subset\widetilde{X}\times\mathbf{R}^{2n},\ \epsilon>0.$$

Moreover, in the case when $q=n_-$, we have $a(\infty,x,\eta;h)\sim\sum_0^\infty a_j(\infty,x,\eta)h^j$ in $C^\infty(\widetilde{X}\times\mathbf{R}^{2n})$, such that

$$\partial_t^k\partial_x^\alpha\partial_\eta^\beta(a_j(t,x,\eta)-a_j(\infty,x,\eta))=\mathcal{O}_{k,\alpha,\beta,K}(1)e^{-t/C} \tag{3.34}$$

and similarly for $a(t,x,\eta;h)-a(\infty,x,\eta;h)$. We also arrange so that

$$a(0,x,\eta;h)=1. \tag{3.35}$$

The construction of ψ,a can be extended in the natural way to the elliptic region $|\eta|\gg 1$, and here it all boils down to Taylor expanding in t. We quickly review a way of treating this standard heat evolution problem by a simple dilation argument. (The reader may skip this and go directly to Proposition 3.3.) If $\Delta_q=P(x,hD_x;h)$ (say with $P(x,\xi;h)$ denoting the Weyl symbol for our local coordinates) then in the problem (3.2), we let $\lambda\gg 1$ and make the change of time variable $s=\lambda t$, so that $\lambda^{-1}\partial_t=\partial_s$. Then dividing (3.2) by λ^2, we get the new evolution equation

$$(\widetilde{h}\partial_s+\widetilde{P}(x,\widetilde{h}D_x,\frac{1}{\lambda};\widetilde{h}))u=0,\ \widetilde{h}=h/\lambda, \tag{3.36}$$

where

$$\widetilde{P}(x,\xi,\frac{1}{\lambda};\widetilde{h})=\frac{1}{\lambda^2}P(x,\lambda\xi;h). \tag{3.37}$$

Recall here that $P(x,\xi;h) = p(x,\xi) + hp_1(x,\xi) + h^2p_2(x)$, where p, p_1, p_2 are polynomials in ξ of degree 2, 1 and 0 respectively. If we decompose into homogeneous polynomials,

$$\begin{aligned} p(x,\xi) &= p^2(x,\xi) + p^1(x,\xi) + p^0(x), \\ p_1(x,\xi) &= p_1^1(x,\xi) + p_1^0(x), \end{aligned}$$

then we know that $p^2(x,\xi)$ is elliptic; $p^2(x,\xi) \backsim |\xi|^2$, and

$$\begin{aligned} \widetilde{P}(x,\xi,\frac{1}{\lambda};\widetilde{h}) &= (p^2(x,\xi) + \frac{1}{\lambda}p^1(x,\xi) + \frac{1}{\lambda^2}p^0(x)) \\ &\quad + \frac{h}{\lambda}(p_1^1(x,\xi) + \frac{1}{\lambda}p_1(x)) + (\frac{h}{\lambda})^2 p_2(x). \end{aligned} \tag{3.38}$$

If λ is sufficiently large, then $p^2(x,\xi)$ is dominating in the region $|\xi| \backsim 1$, and we can construct WKB-solutions to (3.36) $\mathrm{mod}\, \mathcal{O}(\widetilde{h}^\infty)$ with all the derivatives, of the form

$$e^{i\widetilde{\psi}(s,x,\widetilde{\eta},\frac{1}{\lambda})/\widetilde{h}}\widetilde{a}(s,x,\widetilde{\eta},\frac{1}{\lambda};\widetilde{h}), \tag{3.39}$$

with

$$\widetilde{\psi}_{|s=0} = x\cdot\widetilde{\eta},\ |\widetilde{\eta}| \backsim 1,\ \widetilde{a}_{|s=0} = 1.$$

We are now in the elliptic region and it suffices to solve the eikonal equation and the transport equations to infinte order at $s=0$, since $\mathrm{Im}\,\widetilde{\psi} \backsim s$.

If $\eta = \lambda\widetilde{\eta}$, $|\widetilde{\eta}| \backsim 1$, then, at least formally, (3.39) is just the WKB solution $e^{\frac{i}{h}\psi(t,x,\eta)}a(t,x,\eta;h)$ of the original problem (3.2) with $\psi_{|_{t=0}} = x\cdot\eta$, $a_{|_{t=0}} = 1$, so we can choose

$$\begin{aligned} \psi(t,x,\eta) &= \lambda\widetilde{\psi}(\lambda t, x, \frac{\eta}{\lambda}, \frac{1}{\lambda}) \\ a(t,x,\eta;h) &= \widetilde{a}(\lambda t, x, \frac{\eta}{\lambda}, \frac{1}{\lambda}; \frac{h}{\lambda}), \end{aligned}$$

where $\lambda \backsim |\eta|$. Now $\mathrm{Im}\,\lambda\widetilde{\psi}(\lambda t, x, \frac{\eta}{\lambda}, \frac{1}{\lambda}) \backsim \lambda^2 t$ for $0 \leqslant \lambda t \ll 1$ and we get

$$e^{\frac{i}{h}\psi(t,x,\eta)} = \mathcal{O}((\frac{h}{\lambda})^\infty),\ \text{when } \lambda t \geqslant (h/\lambda)^{1-\delta},$$

for any fixed $\delta > 0$.

The above discussion indicates how to take care of the uninteresting elliptic region. A more complete (and more tedious) treatment could be given for example by combining the above scaling argument with a dyadic decomposition in ξ-space. We observe that a satisfies the symbol estimates

$$\partial_t^k\partial_x^\alpha\partial_\eta^\beta a = \mathcal{O}(\langle\eta\rangle^{k-|\beta|}).$$

PROPOSITION 3.3. — *Modulo a standard reduction to homogeneous non-semiclassical theory (see the proof), $U(t)$ is a Fourier integral operator of order 0 with complex phase in the sense of [37], associated to the canonical transformation κ_t.*

$U(t)$ is $\mathcal{O}(1)e^{-t/C}$ and $\mathcal{O}_\epsilon(1)e^{\epsilon t}$, $\forall\epsilon>0$: $H^s_{\rm comp}(\widetilde{X})\to H^s_{\rm loc}(\widetilde{X})$, in the cases $q\neq n_-$ and $q=n_-$ respectively.

We have

$$(h\partial_t+\Delta_q)U(t)=\mathcal{O}(h^\infty)\begin{cases} e^{-t/C}, \\ \mathcal{O}_\epsilon(1)e^{\epsilon t},\ \epsilon>0\end{cases} :\ H^{s-N}_{\rm comp}\to H^{s+N}_{loc},$$

in the cases

$$\begin{cases} q\neq n_-, \\ q=n_-, \end{cases}$$

for all $s\in\mathbf{R}$, $N\geqslant 0$.

Proof. — The statement could be proved directly, but it is perhaps more convenient to use the classical theory of Fourier integral operators with complex phase ([37]). The standard trick to get a reduction to that situation is by adding a variable x_0 and to relate semiclassical objects (without a tilde) to non-semiclassical objects (with a tilde) in the following way:

For functions we relate the semiclassical ones;
$u(x)$, to $\widetilde{u}(x_0,x)=e^{ix_0/h}u(x)$.

We relate a semiclassical Fourier integral operator

$$Fu(x)=\iint e^{\frac{i}{h}\phi(x,y,\theta)}a(x,y,\theta;h)u(y)dyd\theta$$

to a standard (microlocally defined) Fourier integral operator

$$\widetilde{F}\widetilde{u}(x_0,x)=\iiiint_{\theta_0>0} e^{i(\phi(x,y,\theta)\theta_0+(x_0-y_0)\theta_0)}a(x,y,\theta;\frac{1}{\theta_0})\widetilde{u}(y_0,y)\frac{dy_0}{2\pi}dyd\theta_0d\theta,$$

so that

$$\widetilde{F}(e^{\frac{i}{h}x_0}u(x))=e^{\frac{i}{h}x_0}Fu(x).$$

Here, we require that $\operatorname{Im}\phi\geqslant 0$, so that the same holds for

$$\widetilde{\phi}=\phi(x,y,\theta)\theta_0+(x_0-y_0)\theta_0.$$

Let $C_\phi=\{(x,y,\theta);\,\phi'_\theta(x,y,\theta)=0\}$ and recall that ϕ is non-degenerate if $d\phi'_{\theta_1},\ldots,d\phi'_{\theta_N}$ are linearly independent at every point of C_ϕ. Then it easy to see that ϕ is non-degenerate iff $\widetilde{\phi}$ is, and we have

$$C_{\widetilde{\phi}}=\{(x_0,y_0,\theta_0;x,y,\theta);\,(x,y,\theta)\in C_\phi,\ x_0=y_0-\phi(x,y,\theta)\}$$

We assume (which is the case for $U(t)$) that we are in the non-degenerate case. Then we introduce the corresponding canonical relation

$$\Lambda_\phi = \{(x, \phi'_x; y, -\phi'_y);\ (x, y, \theta) \in C_\phi\}.$$

Then for $\widetilde{\phi}$, we have

$$\begin{aligned}\Lambda_{\widetilde{\phi}} \quad = \quad & \{(x_0, \xi_0, x, \xi; y_0, \eta_0, y, \eta);\ x_0 = y_0 - \phi(x, y, \theta), \\ & \qquad\qquad \xi_0 = \eta_0 = \theta_0,\ (x, \frac{\xi}{\theta_0}; y, \frac{\eta}{\theta_0}) \in \Lambda_\phi\}.\end{aligned}$$

The corresponding relation between the evolution equations is that

$$(h\partial_t + P(x, hD_x))u = \mathcal{O}(h^\infty) \Leftrightarrow (\partial_t + D_{x_0} P(x, D_{x_0}^{-1} D_x))\widetilde{u} = 0 \text{ microlocally},$$

when $\widetilde{u}(t, x_0, x) = e^{ix_0/h} u(t, x)$. This is coherent with the two other correspondances above, let us just check the geometric one: The canonical transformations associated to $U(t)$, and the solution operator $\widetilde{U}(t)$ of the second evolution problem are denoted by κ_t and $\widetilde{\kappa}_t$ respectively, so that $\widetilde{\kappa}_t$ is obtained by integrating the system:

$$i\dot{x}_0 = \partial_{\xi_0}\widetilde{p},\ i\dot{\xi}_0 = -\partial_{x_0}\widetilde{p},\ i\dot{x} = \partial_\xi \widetilde{p},\ i\dot{\xi} = -\partial_x \widetilde{p}, \tag{3.40}$$

with

$$\widetilde{p}(x_0, x; \xi_0, \xi) = \xi_0 p(x, \xi_0^{-1}\xi),$$

while the corresponding evolution problem giving κ_t is

$$i\dot{x} = \partial_\xi p,\ i\dot{\xi} = -\partial_x p. \tag{3.41}$$

Now (3.40) becomes

$$\begin{aligned}& i\dot{x}_0 = p(x, \xi/\xi_0) - p'_\xi(x, \xi/\xi_0) \cdot (\xi/\xi_0),\ i\dot{\xi}_0 = 0, \\ & i\dot{x} = p'_\xi(x, \xi/\xi_0),\ i\dot{\xi}/\xi_0 = -p'_x(x, \xi/\xi_0),\end{aligned}$$

which reduces to κ_t after restriction to $\xi_0 = 1$.

To get the second statement, we observe that $\psi(t, \xi, \eta) \to \psi(\infty, x, \eta)$ and that the corresponding canonical relation κ_∞ is strictly positive with real part being the identity relation on Σ. The statement then follows by the description of our operators after conjugation by an FBI-Bargmann transform as in [39].

The proof of the third statement is straightforward. $\square$

In the remainder of this section, we assume that $q \neq n_-$

Proposition 3.4. — *We have*

$$[\Delta_q, U(t)] = \mathcal{O}(h^N)e^{-t/C},\ H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}}, \tag{3.42}$$

for all $s \in \mathbf{R}$ and all $N \geqslant 0$.

Proof. — Using the theory of [37], we see that

$$[\Delta_q, U(t)]u(x) = \frac{1}{(2\pi h)^{2n}} \iint e^{\frac{i}{h}(\psi(t,x,\eta)-y\cdot\eta)} b(t,x,\eta;h)u(y)dyd\eta + R(t)u(x), \tag{3.43}$$

where $R(t) = \mathcal{O}(h^\infty)e^{-t/C} : H^{-\infty}_{\mathrm{comp}} \to H^\infty_{\mathrm{loc}}$, and $b(t,x,\eta;h) \sim \sum_0^\infty b_j(t,x,\eta)h^j$ satisfies (3.32), (3.33) in a region with η bounded and $\partial_t^k\partial_x^\alpha\partial_\eta^\beta b = \mathcal{O}(\langle\eta\rangle^{2+k-|\beta|})$ in a region where t is bounded. Further, we have

$$[\Delta_q, U(0)] = 0, \tag{3.44}$$

$$(h\partial_t + \Delta_q)[\Delta_q, U(t)] = \mathcal{O}(h^\infty)e^{-t/C}. \tag{3.45}$$

From (3.44) we conclude that $b_j(0,\eta) = 0$ and from (3.45) we see that b_j satisfy the same transport equations as a_j, and hence

$$b_j = \mathcal{O}(t^\infty),\ b_j = \mathcal{O}(e^{-t/C}\mathrm{dist}\,(\cdot,\Sigma)^\infty),$$

where we restrict the attention to a region with η bounded for simplicity. From this we deduce (3.42). □

Combining the last two propositions, we get

$$h\partial_t U(t) + U(t)\Delta_q = \mathcal{O}(h^\infty)e^{-t/C} : H^{-\infty}_{\mathrm{comp}}(\widetilde{X}) \to H^\infty_{\mathrm{loc}}(\widetilde{X}). \tag{3.46}$$

From this we get a two-sided parametrix for Δ_q:

Theorem 3.5. — *We recall that we work with the assumption $q \neq n_-$. Put*

$$E = \frac{1}{h}\int_0^\infty U(t)dt. \tag{3.47}$$

Then

$$E = \mathcal{O}(h^{-1}) : H^s_{\mathrm{comp}} \to H^s_{\mathrm{loc}}, \tag{3.48}$$

for every $s \in \mathbf{R}$, and

$$\Delta_q E - 1,\ E\Delta_q - 1 = \mathcal{O}(h^\infty) : H^{-\infty}_{\mathrm{comp}} \to H^\infty_{\mathrm{loc}}. \tag{3.49}$$

Proof. — The first estimate follows from the second statement in Proposition 3.3. Further,

$$\Delta_q E = \frac{1}{h}\int_0^\infty -h\partial_t U(t)dt + \frac{1}{h}\int_0^\infty (h\partial_t + \Delta_q)U(t)dt.$$

Here the first integral is equal to 1, since $U(0) = 1$, and the second integral is $\mathcal{O}(h^\infty) : H^{-\infty}_{\mathrm{comp}} \to H^\infty_{\mathrm{loc}}$ by the last part of Proposition 3.3. The proof of (3.49) is similar except that we use (3.46) instead. □

4. Π as a local projection on $\mathcal{N}(\Delta_q)$ mod $\mathcal{O}(h^\infty)$

In this section we continue to work in a connected open subset where the curvature $\overline{\partial}\partial\phi$ is non-degenerate of signature (n_+, n_-) and we restrict the attention to $(0, q)$-forms, with $q = n_-$.

Recall that $U(t)$, defined by (3.3), is well-defined mod $\mathcal{O}(h^\infty)$ as an operator: $H^s_{\mathrm{comp}} \to H^s_{\mathrm{loc}}$ for $t \geqslant 0$ and as an operator: $H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}}$, for $t \geqslant t_0$ for all $t_0 > 0$. Put

$$\Pi u = \frac{1}{(2\pi h)^{2n}} \iint e^{\frac{i}{h}(\psi(\infty,x,\eta) - y\cdot\eta)} a(\infty, x, \eta; h) u(y) dy d\eta, \tag{4.1}$$

so that Π is well-defined mod $\mathcal{O}(h^\infty)$ as an operator $H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}}$, for all $s \in \mathbf{R}$, $N \geqslant 0$. Then by Proposition 3.2, and (3.12), we have

$$U(t) = \Pi + V(t),\ V(t) = \mathcal{O}(e^{-t/C}) : H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}},\ t \geqslant t_0,\ \forall\, t_0 > 0. \tag{4.2}$$

To see this one can introduce $U_s(t)$ with phase $(1-s)\psi(t, x, \eta) + s\psi(\infty, x, \eta)$ and amplitude $(1 - s)a(t, x, \eta) + sa(\infty, x, \eta; h)$, $0 \leqslant s \leqslant 1$, and show that $\partial_s U_s(t)$ satisfies the estimate in (4.2).

Proposition 4.1. — *We have in the sense of operators:* $H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}}$,

$$\Delta_q \Pi \equiv \Pi\Delta_q \equiv 0 \mod \mathcal{O}(h^\infty), \tag{4.3}$$

$$\Pi^* - \Pi \equiv 0 \mod \mathcal{O}(h^\infty), \tag{4.4}$$

$$[\Pi, V(t)] = \mathcal{O}(e^{-t/C} h^\infty). \tag{4.5}$$

Proof. — We know from (3.27), (3.20) that

$$Z_\phi(e^{\frac{i}{h}\psi(t,x,\eta)} a(t, x, \eta; h)) = e^{\frac{i}{h}\psi(t,x,\eta)} b(t, x, \eta; h),$$

where $b = \mathcal{O}(e^{-t/C})$. Write this as

$$\mathcal{O}(e^{-t/C}) = b = e^{-\frac{i}{h}\psi(t,x,\eta)} \circ Z_\phi \circ e^{\frac{i}{h}\psi(t,x,\eta)}(a(t,x,\eta;h)) =: Z_{\phi,\psi,\eta} a(t,x,\eta;h).$$

Here

$$Z_{\phi,\psi,\eta} = Z_{\phi,\psi_\infty,\eta} + \mathcal{O}(e^{-t/C}),$$

in the sense of Taylor expansions of the coefficients in the h-asymptotic expansions at Σ, by (3.12). We conclude that

$$Z_{\phi,\psi_\infty,\eta} a_\infty = \mathcal{O}(e^{-t/C}) + \mathcal{O}(h^\infty),$$

but here the left hand side is independent of t and hence

$$Z_{\phi,\psi_\infty,\eta} a_\infty = \mathcal{O}(h^\infty). \tag{4.6}$$

This means that

$$Z_\phi(e^{\frac{i}{h}\psi_\infty} a_\infty) = \mathcal{O}(h^\infty). \tag{4.7}$$

Similarly,

$$Z_\phi^*(e^{\frac{i}{h}\psi_\infty} a_\infty) = \mathcal{O}(h^\infty). \tag{4.8}$$

Hence,

$$\Delta_q(e^{\frac{i}{h}\psi_\infty} a_\infty) = \mathcal{O}(h^\infty), \tag{4.9}$$

which implies that

$$\Delta_q \Pi = \mathcal{O}(h^\infty) : \, H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}}. \tag{4.10}$$

$[\Delta_q, U(t)]$ is an h-Fourier integral operator of the same type as $U(t)$. We have

$$\partial_t[\Delta_q, U(t)] + \Delta_q[\Delta_q, U(t)] = [\Delta_q, \partial_t U(t) + \Delta_q U(t)] = \mathcal{O}(h^\infty)$$

in the sense of such operators and using also that $[\Delta_q, U(0)] = 0$, we get $[\Delta_q, U(t)] = \mathcal{O}(h^\infty)$ in the sense of such operators and hence $[\Delta_q, U(t)] = \mathcal{O}(h^\infty)$ as an operator: $H^{s+2}_{\mathrm{comp}} \to H^s_{\mathrm{loc}}$ for $t \geqslant 0$ and $H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}}$, for $t \geqslant t_0 > 0$. It follows that $[\Pi, \Delta_q] = \mathcal{O}(h^\infty) : H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}}$ and together with (4.10), this gives (4.3).

Next we see that in the sense of h-Fourier integral operators:

$$\partial_t U^* + \Delta_q U^* \equiv (\partial_t U + U\Delta_q)^* \equiv (\partial_t U + \Delta_q U)^* \equiv 0.$$

Hence $\partial_t(U^* - U) + \Delta_q(U^* - U) \equiv 0$, $(U^* - U)(0) = 0$, so by considering again the transport equations, we get $U^* \equiv U$. It follows that $\Pi^* \equiv \Pi$, so we have (4.4).

Consider $[U(t), U(s)]$, $0 \leqslant s < \infty$ (after introducing a cutoff near the diagonal to make our operators properly supported without affecting any other properties). This commutator is obviously a Fourier integral operator associated to κ_{t+s}. For $s = 0$, we have

$$[U(0), U(t)] = [1, U(t)] = 0.$$

Moreover, since Δ_q commutes with $U(t)$:

$$(h\partial_s + \Delta_q)[U(s), U(t)] \equiv [(h\partial_s + \Delta_q)U(s), U(t)] \equiv [0, U(t)] = 0.$$

From considering the transport equations for the amplitude of $[U(s), U(t)]$ with the phase $\psi(t+s, x, \eta) - y \cdot \eta$, we see that $[U(s), U(t)] \equiv 0$. Letting $s \to \infty$, we get $[\Pi, U(t)] \equiv 0$ and (4.5) follows. $\square$

For $\operatorname{Re} z < 0$, we put

$$R(hz) = -\frac{1}{h}\int_0^\infty e^{tz} U(t)dt = \mathcal{O}(h^{-1}) : H^s_{\mathrm{comp}} \to H^s_{\mathrm{loc}}. \tag{4.11}$$

Then modulo $\mathcal{O}(h^\infty) : H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}}$ we have,

$$\begin{aligned} \Delta_q R(hz) &= -\frac{1}{h}\int_0^\infty e^{tz}\Delta_q U(t)dt \equiv \frac{1}{h}\int_0^\infty e^{tz} h\partial_t U(t)dt \\ &= -\int_0^\infty \partial_t(e^{tz})U(t)dt - 1 = hzR(hz) - 1. \end{aligned}$$

We also have $R(hz)\Delta_q \equiv \Delta_q R(hz)$, so we get

$$(hz - \Delta_q)R(hz) \equiv R(hz)(hz - \Delta_q) \equiv 1. \tag{4.12}$$

In order to extend to a domain, $\operatorname{Re} z < 1/(2C)$, we first rewrite (4.11) as

$$R(hz) = -\frac{1}{h}\int_0^\infty e^{tz}(\Pi + V(t))dt = \frac{1}{hz}\Pi - \frac{1}{h}\int_0^\infty e^{tz} V(t)dt,$$

and for

$$\operatorname{Re} z < 1/(2C),\ |z| \geqslant h^{N_0}, \tag{4.13}$$

with $N_0 > 0$ arbitrarily large but fixed, we define

$$R(hz) = \frac{1}{hz}\Pi - \frac{1}{h}\int_0^\infty e^{tz}V(t)dt = \mathcal{O}(|hz|^{-1} + h^{-1}) :\ H^s_{\mathrm{comp}} \to H^s_{\mathrm{loc}}. \tag{4.14}$$

Then this is a holomorphic extension of $R(hz)$, defined by (4.11). It is therefore no surprise that (4.12) remains valid (even though we cannot appeal to unique holomorphic extension, since we work with errors that are $\mathcal{O}(h^\infty)$): Use that $(h\partial t+\Delta_q)V(t)=\mathcal{O}(h^\infty e^{-t/C})$:$H^{s-N}_{\rm comp}\to H^{s+N}_{\rm loc}$ (cf (4.2), (4.3)) $V(0)=1-\Pi$, to get mod $\mathcal{O}(h^\infty): H^{s-N}_{\rm comp}\to H^{s+N}_{\rm loc}$,

$$\begin{aligned}\Delta_q R(hz) &\equiv -\frac{1}{h}\int_0^\infty e^{tz}\Delta_q V(t)dt\equiv\int_0^\infty e^{tz}\partial_t V(t)dt\\ &= \Pi-1-z\int_0^\infty e^{tz}V(t)dt=\Pi-1-hz\frac{1}{h}\int_0^\infty e^{tz}V(t)dt\\ &= \Pi-1+hz(R(hz)-\frac{1}{hz}\Pi)=hzR(hz)-1,\end{aligned}$$

so indeed we have (4.12) for z in the region (4.13).

PROPOSITION 4.2. — *We have*

$$\Pi=\frac{1}{2\pi i}\int_{|z|=r}R(z)dz, \tag{4.15}$$

if $h^{N_0}\leqslant r\leqslant 1/(2C)$. Moreover,

$$\Pi^2\equiv\Pi. \tag{4.16}$$

In order for (4.16) to make sense, we have multiplied the distribution kernel of $U(t)$ by a cutoff near the diagonal in order to make all the operators properly supported without changing any of their other properties.

Proof. — (4.15) is immediate from (4.14), since the last term in (4.14) is holomorphic in $|z|<1/(2C)$. To prove (4.16), we follow the standard procedure and establish first an approximate version of the resolvent identity when $h^{N_0}\leqslant|z|,\,|w|\leqslant h/(2C)$, modulo $\mathcal{O}(h^\infty)$: $H^{s-N}_{\rm comp}\to H^{s+N}_{\rm loc}$,

$$R(z)-R(w)\equiv R(z)(w-z)R(w)\equiv R(w)(w-z)R(z). \tag{4.17}$$

Write

$$(z-\Delta_q)-(w-\Delta_q)=(z-w),$$

and apply $R(z)R(w)$. Then (4.17) follows.

Using (4.17), we write

$$\begin{aligned}\Pi^2 &= (\frac{1}{2\pi i})^2\int_{|z|=r_1}\int_{|w|=r_2}R(z)R(w)dwdz\\ &\equiv (\frac{1}{2\pi i})^2\int_{|z|=r_1}\int_{|w|=r_2}(w-z)^{-1}R(z)dwdz\\ &\quad +(\frac{1}{2\pi i})^2\int_{|w|=r_2}\int_{|z|=r_1}(z-w)^{-1}R(w)dzdw.\end{aligned}$$

Choose $h^{N_0} \leqslant r_1 < r_2 \leqslant h/(2C)$. In the second integral, we first integrate with respect to z and get 0. In the first integral, we first integrate in w and get

$$\frac{1}{2\pi i}\int_{|z|=r_1} R(z)dz \equiv \Pi. \qquad \square$$

The next result together with (4.3), (4.5), (4.16) says that in an approximate sense Π is the orthogonal projection onto the kernel of P and that $1-\Pi$ is approximately the orthogonal projection onto the range of P:

Theorem 4.3. — *For $h^{N_0} \leqslant r \leqslant h/(2C)$, put*

$$E = -\frac{1}{2\pi i}\int_{|z|=r} \frac{1}{z}R(z)dz = \mathcal{O}(\frac{1}{h}) : H^s_{\mathrm{comp}} \to H^s_{\mathrm{loc}}. \tag{4.18}$$

Then modulo $\mathcal{O}(h^\infty) : H^{s-N}_{\mathrm{comp}} \to H^{s+N}_{\mathrm{loc}}$,

$$1 \equiv \Pi + \Delta_q E \equiv \Pi + E\Delta_q. \tag{4.19}$$

Proof. — Since $E\Delta_q \equiv \Delta_q E$, we only have to prove the first relation in (4.19):

$$\begin{aligned}
\Delta_q E &= -\frac{1}{2\pi i}\int_{|z|=r} \frac{1}{z}\Delta_q R(z)dz \\
&= -\frac{1}{2\pi i}\int_{|z|=r} \frac{1}{z}(\Delta_q - z)R(z)dz - \frac{1}{2\pi i}\int_{|z|=r} R(z)dz \\
&\equiv \frac{1}{2\pi i}\int_{|z|=r} \frac{1}{z}dz - \Pi = 1 - \Pi. \qquad \square
\end{aligned}$$

From the discussion around (3.7)–(3.14), we recollect that uniformly for $t \geqslant t_0 > 0$:

$$\psi(t,x,\eta) = x\cdot\eta + \mathcal{O}(\mathrm{dist}\,(x,\eta;\Sigma)^2), \tag{4.20}$$

$$\mathrm{Im}\,\psi(t,x,\eta) \sim \mathrm{dist}\,(x,\eta;\Sigma)^2. \tag{4.21}$$

The complex stationary phase method ([37]) then permits us to carry out the η-integration in (3.31), (4.1), to get

Theorem 4.4. — *For every $t_0 > 0$, we have uniformly for $t \geqslant t_0$*

$$U(t)u(x) = h^{-n}\int e^{\frac{i}{h}\widetilde{\psi}(t,x,y)} b(t,x,y;h)u(y)m(dy) + R(t)u(x), \tag{4.22}$$

$$\Pi u(x) = h^{-n} \int e^{\frac{i}{h}\widetilde{\psi}(\infty,x,y)} b(\infty, x, y; h)u(y)m(dy) + R(\infty)u(x), \quad (4.23)$$

where

$$b(t, x, y; h) \sim \sum_0^\infty b_j(t, x, y; h)h^j, \quad (4.24)$$

$$R(t)u(x) = \int r(t, x, y; h)u(y)m(dy), \quad (4.25)$$

$$\partial^\alpha_{(t,x,y)} r = \mathcal{O}(h^\infty), \quad (4.26)$$

$$\operatorname{Im} \widetilde{\psi}(t, x, y) \sim |x - y|^2, \;\; \widetilde{\psi}(t, y, x) = -\overline{\widetilde{\psi}}(t, x, y), \quad (4.27)$$

$$\operatorname{graph} \kappa_t = \{(x, \partial_x\widetilde{\psi}(t, x, y); y, -\partial_y\widetilde{\psi}(t, x, y));\ (x, y) \in \operatorname{neigh}(\operatorname{diag}(\widetilde{X}\times\widetilde{X}))\}, \quad (4.28)$$

$$\partial^\alpha_{t,x,y}(\widetilde{\psi}(t, x, y) - \widetilde{\psi}(\infty, x, y))_{|y=x} = \mathcal{O}(e^{-t/C}) \quad (4.29)$$

and similarly for b_j*.*

5. The global null-projection

We first recollect what we have done locally. Let s be a local non-vanishing holomorphic section of L, defined on $\widetilde{X} \subset X$. Write $|s(x)|^2 = e^{-2\phi(x)}$, and recall that we have the unitary map

$$\begin{array}{rcl} \mathcal{E}^{0,q}(\widetilde{X}) \ni u & \mapsto & \widetilde{u} = (se^\phi)^k u \in \mathcal{E}^{0,q}(\widetilde{X}; L^k) \\ \overline{\partial}_s & \mapsto & h\overline{\partial} \\ \Delta_q & \mapsto & \widetilde{\Delta}_q, \end{array} \quad (5.1)$$

where $\widetilde{\Delta}_q = h\overline{\partial}h\overline{\partial}^* + h\overline{\partial}^* h\overline{\partial}$ is the Hodge Laplacian on $\mathcal{E}^{0,q}(\widetilde{X}; L^k)$. Assume the curvature is non-degenerate with $n_- = q$ on X. In Section 4 we constructed an approximate resolvent for Δ_q for z in the domain (4.13) and an approximate null-projection of the form

$$\Pi u(x) = h^{-n} \int e^{\psi(x,y)/h} b(x, y; h)u(y)m(dy), \;\; h = 1/k, \quad (5.2)$$

where our new ψ is related to $\widetilde{\psi}(\infty, x, y)$ in (4.23) by

$$\psi(x, y) = i\widetilde{\psi}(\infty, x, y), \quad (5.3)$$

so that (4.27), (4.28) give

$$\operatorname{Re} \psi(x, y) \sim -|x - y|^2, \;\; \psi(y, x) = \overline{\psi(x, y)} \quad (5.4)$$

$$d_x \frac{1}{i}\psi(x,y) \in J_+,\ -d_y \frac{1}{i}\psi(x,y) \in J_-. \tag{5.5}$$

When $x = y$, this implies that

$$d_x i^{-1}\psi(x,y) = -d_y i^{-1}\psi(x,y) \in \Sigma.$$

We also know from the construction that

$$\psi(x,x) = 0. \tag{5.6}$$

On the other hand, we know that Σ is given by $\mathrm{Re}\,\xi dx = \mathrm{Re}\,\frac{2}{i}\frac{\partial\phi}{\partial x}dx$ (using the notations of Section 2 but writing x, ξ instead of z, ζ), so we get for $x = y$:

$$d_x \frac{1}{i}\psi(x,y) = \mathrm{Re}\,\frac{2}{i}\frac{\partial\phi}{\partial x}dx = \frac{1}{i}\frac{\partial\phi}{\partial x}dx - \frac{1}{i}\frac{\partial\phi}{\partial\overline{x}}d\overline{x}.$$

Hence for $x = y$:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial x},\ \frac{\partial\psi}{\partial\overline{x}} = -\frac{\partial\phi}{\partial\overline{x}},\ \frac{\partial\psi}{\partial y} = -\frac{\partial\phi}{\partial x},\ \frac{\partial\psi}{\partial\overline{y}} = \frac{\partial\phi}{\partial\overline{x}}. \tag{5.7}$$

Since Π is selfadjoint modulo $\mathcal{O}(h^\infty)$, we also have

$$b(x,y;h)^* = b(y,x;h), \tag{5.8}$$

where the $*$ indicates that we take the complex adjoint of

$$b(x,y;h) :\ \Lambda^{0,q}T_y^*X \to \Lambda^{0,q}T_x^*X.$$

In terms of

$$\widetilde{u} = (se^\phi)^k u,\ \widetilde{v} = (se^\phi)^k v \in \mathcal{E}^{0,q}(\widetilde{X}; L^k),$$

we get from $v = \Pi u$, that $\widetilde{v} = \widetilde{\Pi}\widetilde{u}$, with

$$\widetilde{v} = h^{-n}\int e^{\psi(x,y)/h}\widetilde{b}(x,y;h)\widetilde{u}(y)m(dy), \tag{5.9}$$

where the "symbol"

$$\widetilde{b}(x,y;h) = (s(x)e^{\phi(x)})^k b(x,y;h)(s(y)e^{\phi(y)})^{-k} \tag{5.10}$$

maps

$$L_y^k \otimes \Lambda^{0,q}T_y^*X \to L_x^k \otimes \Lambda^{0,q}T_x^*X \tag{5.11}$$

and satisfies (5.8), now in the sense of maps as in (5.11). Notice that though se^ϕ is normalized, the "symbol" $\widetilde{b}$ may contain oscillations, contrary to the true symbol b.

Let s_1 be a second non-vanishing local holomorphic section of L with $|s_1| = e^{-2\phi_1}$, so that $s_1 e^{\phi_1}$ is normalized. In the intersection of the domains of definition, we have

$$s_1 e^{\phi_1} = s e^{\phi} e^{ig},$$

with g real and $\phi_1 - \phi$ pluriharmonic. We then have the local representation

$$\widetilde{u} = (s_1 e^{\phi_1})^k u_1, \ \widetilde{v} = (s_1 e^{\phi_1})^k v_1,$$

and a null-projection that is unitarily equivalent to the one in (5.2):

$$\Pi_1 u_1(x) = h^{-n} \int e^{\psi_1(x,y)/h} b_1(x,y;h) u_1(y) m(dy), \ h = 1/k. \tag{5.12}$$

Since the heat parametrix constructed in Section 3 is unique $\mathrm{mod}\, \mathcal{O}(h^\infty)$: $H^{-\infty}_{\mathrm{comp}} \to H^{\infty}_{\mathrm{loc}}$, we have the corresponding facts for the local resolvents and null-projections, so (5.12) necessarily leads to the same relation (5.9), and we can also relate Π, Π_1 more directly, by writing

$$u_1 = \left(\frac{s e^{\phi}}{s_1 e^{\phi_1}}\right)^k u = e^{-ikg} u = e^{-ig/h} u,$$

to get

$$\Pi_1 = e^{-ig/h} \circ \Pi \circ e^{ig/h},$$

so

$$b_1 = b, \ \psi_1(x,y) = \psi(x,y) - ig(x) + ig(y). \tag{5.13}$$

In particular, $\mathrm{Re}\, \psi(x,y)$ does not depend on the choice of local holomorphic section s. The argument above gives a clear idea about the asymptotic behaviour of the kernel of the projection onto the space of q-harmonic forms. To justify this idea we shall consider the global resolvents.

On the full manifold X we know that the Hodge Laplacians $\widetilde{\Delta}_{q-1}$, $\widetilde{\Delta}_{q+1}$ have no spectrum below h/C for some $C > 0$ (as could easily be proved using Theorem 3.5) and by a standard argument, we conclude that the spectrum of $\widetilde{\Delta}_q$ below h/C is reduced to $\{0\}$. For z in a set (4.13) we can glue together the local operators $R(z)$ of Section 4 to an operator $\widetilde{R}(z)$ (or rather we first glue together the locally unique heat kernels to a global one and then define $\widetilde{R}(z)$ as in (4.14)) in such a way that

$$(z - \widetilde{\Delta}_q)\widetilde{R}(z) \equiv \widetilde{R}(z)(z - \widetilde{\Delta}_q) \equiv 1 \ \mathrm{mod}\, \mathcal{O}(h^\infty) : \ H^{-\infty}(X) \to H^{\infty}(X). \tag{5.14}$$

Here we define the Sobolev spaces $H^s(X) = H^s(X, L)$ of sections of L^k with $h = 1/k$ in a straightforward way from the local representations (5.1), by means of coverings and partitions of unity. The choice of such coverings and partitions will affect the H^s-norm only up to an equivalence that is uniform in k.

Since $\widetilde{\Delta}_q$ is an elliptic operator in the classical sense, we know on the other hand that for z in the set (4.13),

$$(z-\widetilde{\Delta}_q)^{-1}=\mathcal{O}(h^{-N_0-1}):\, H^s\to H^{s+2} \tag{5.15}$$

for all $s\in\mathbf{R}$, so combining this with (5.14), we get

$$(z-\widetilde{\Delta}_q)^{-1}\equiv \widetilde{R}(z) \bmod \mathcal{O}(h^\infty):H^{-\infty}\to H^\infty. \tag{5.16}$$

Notice that the distribution kernel of an operator which is $\mathcal{O}(h^\infty):H^{-\infty}\to H^\infty$ is $\mathcal{O}(h^\infty)$ together with all its derivatives. On the other hand, the approximate global projection $\widetilde{\Pi}$ discussed earlier in this section satisfies (cf. Proposition 4.2)

$$\widetilde{\Pi}\equiv\frac{1}{2\pi i}\int_{|z|=r}\widetilde{R}(z)dz \bmod \mathcal{O}(h^\infty):H^{-\infty}\to H^\infty, \tag{5.17}$$

while the true nullspace projection of $\widetilde{\Delta}_q$,

$$\Pi_0:1_{\{0\}}(\widetilde{\Delta}_q) \tag{5.18}$$

satisfies

$$\Pi_0=\frac{1}{2\pi i}\int_{|z|=r}(z-\widetilde{\Delta}_q)^{-1}dz. \tag{5.19}$$

Combining (5.16), (5.17), (5.19), (5.8), we get the main result of this work:

THEOREM 5.1. — *Let L be a Hermitian holomorphic line bundle over a compact complex manifold X and fix a positive smooth measure $m(dx)$ on X, so that the Hodge Laplacian $\widetilde{\Delta}_q=\widetilde{\Delta}_{q,k}=\overline{\partial}^*\overline{\partial}+\overline{\partial}\overline{\partial}^*$ is well-defined on $(0,q)$-forms with coefficients in L^k, $k\in\mathbf{N}$. Assume the curvature of L has constant signature (n_-,n_+) with $n_-+n_+=n:=\dim X$. Then for $k\gg 1$, the null-space of $\widetilde{\Delta}_{q,k}$ is reduced to 0 when $q\neq n_-$.*

In the case $q=n_-$, let s be a non-vanishing holomorphic section of L on the open subset $\widetilde{X}$, so that (5.1) gives a unitary map between $(0,q)$-forms on $\widetilde{X}$ and $(0,q)$-forms on $\widetilde{X}$ with coefficients in L^k. If Π_0 denotes the orthogonal projection onto the null-space of $\widetilde{\Delta}_q$, we put $\Pi_{0,s}u=(se^\phi)^{-k}\Pi_0(se^\phi)^k u$, $u\in L^2(\widetilde{X},\Lambda^{0,q}T^*\widetilde{X})$. Then the distribution kernel of $\Pi_{0,s}$ is of the form

$$K_{\Pi_{0,s}}(x,y)=h^{-n}e^{\psi(x,y)/h}\widetilde{b}(x,y;h)+r(x,y;h),\;\; h=1/k, \tag{5.20}$$

with ψ, $\widetilde{b}$ as in (5.9), (5.3) , (4.23), (5.5), (5.6), (5.7), (5.8) and where $\partial^\alpha_{x,y} r = \mathcal{O}(h^\infty)$ for all α.

Remark 5.2. — Theorem 5.1 also holds for the more general situation of $(0,q)$-forms with values in $L^k \otimes E$, where E is a rank r holomorphic Hermitian vector bundle over X. Indeed, locally E is isomorphic to the trivial holomorphic vector bundle $\mathbf{C}^r \times X$ with a Hermitian metric γ. The local expression (2.14) for Δ_q then still holds if the operators $hZ_j + Z_j(\phi)$ and their adjoints are tensored by I_r, the identity matrix on $\mathbf{C}^r$. This follows from the fact that the Hermitian metric γ on E is independent of $h = k^{-1}$. Moreover, globally there is still a spectral gap for the same reason (as is well-known), giving an asymptotic expansion as before. For example, if μ_n is a general volume form on X and ω_n is the one induced by the given Hermitian metric on X, then the function μ_n/ω_n defines a Hermitian metric on the trivial line bundle E.

6. Change of complex structure

In this section we will investigate some relations to [44] (see also [12]). Let us first recall the setting in [44]. Assume given a symplectic manifold (X,ω) such that $\pi\omega$ represents an integral cohomology class. Then there exists a Hermitian line bundle L over X with a unitary connection ∇ whose curvature satisfies $\frac{i}{2}\Theta = \omega$ (compare Section 6.2 for notation). Take an almost complex structure J' on X (i.e. $J' \in \mathrm{End}(TX)$, $J'^2 = -I$) such that

$$\begin{array}{lrcl} (i) & \omega(J'v, J'w) & = & \omega(v,w) \\ (ii) & \omega(v, J'v) & > & 0 \end{array} \tag{6.1}$$

for all v, w in TX. We decompose

$$TX \otimes \mathbf{C} = T^{1,0}(X,J') \oplus T^{0,1}(X,J'),$$

so that $J' \otimes \mathbf{C} = i \oplus -i$ (then 6.1 means that ω is a positive $(1,1)-$form with respect to J'). Then we get an operator $\nabla^{0,1} := \overline{\partial}_{J'}$ acting on sections of L. Furthermore, a Riemannian metric g is said to be compatible with J' if

$$g(J'v, J'v) = g(v,v), \tag{6.2}$$

i.e. g corresponds to the real part of a Hermitian metric on $T^{1,0}(X,J')$.

In [44] Shiffman and Zelditch, motivated by the work [20] of Donaldson, define a sequence of spaces imitating $H^0(X, L^k)$ in the usual integrable case. A naive choice would be the kernel of $\overline{\partial}_{J'}$ acting on L^k, but if J'

is non-integrable then these spaces are too small. Instead, Shiffman and Zelditch, following Boutet de Monvel and Guillemin [12], introduce a sequence of spaces of so called *asymptotically almost holomorphic sections.* The main result in the present section (theorem 6.5) says that the dimension of the null-space of Δ_q, studied in the previous sections, coincides with the dimension of a space of asymptotically almost holomorphic sections. The latter space is defined with respect to a new almost complex structure on the original complex manifold X. It would be very interesting to know if this correspondence could be extented to the level of Bergman kernels in a suitable sense, in particular in view of the results in [35] on lower order terms of generalized Bergman kernels. It should finally be pointed out that in [44] the analysis is reduced to the homogenous theory in [12] by adding a varible dual to k, i.e. by embedding X in the unit circle bundle in L^* (this is a global version of the reduction used in proposition 3.3). But since we work directly in a semiclassical inhomogenous setting we have developped some of the material in [44] from our point of view.

6.1. The pair (J, J')

We now return to the situation in the previous chapters, i.e. we take L to be a Hermitian line bundle which is also holomorphic over (X, J) where J denotes the integrable complex structure. Then it has a canonical connection ∇ (see Section 6.2). The curvature Θ of ∇ is assumed to be of signature $(n_-, n_+) = (q, n-q)$ and we will call q the *index* of Θ. Hence, $\omega := \frac{i}{2}\Theta$ is not positive with respect to J, unless $q = 0$. However, given a Hermitian metric H on $T^{1,0}(X, J)$ as in Section 2 (so that its real part corresponds to g in (6.2)) we can define an almost complex structure J' making ω positive, in the following way. Split the real tangent bundle TX as

$$TX = (TX)_- \oplus (TX)_+ \tag{6.3}$$

according to the positive and negative eigenspaces of $\omega(\cdot, J\cdot)$ with respect to the metric g. Then J splits as $J_+ \oplus J_-$ by restriction. Now define J' by the splitting

$$J' = (-J_-) \oplus J_+. \tag{6.4}$$

Then, clearly, $\omega(v, J'v) > 0$. Equivalently, let e^i be a local frame for $T^{*0,1}(X, J)$, orthonormal with respect to H, such that

$$\Theta = \sum_i \lambda_i \overline{e^i} \wedge e^i, \tag{6.5}$$

where $\lambda_i < 0$ for $i \leqslant q$ and $\lambda_i > 0$ for $i > q$. Let $e'^i = \overline{e^i}$ for $i \leqslant q$ and $e'^i = e^i$ for $i > q$. Then $T^{*0,1}(X, J')$ is spanned by all e'^i and $\Theta = |\lambda_i| \overline{e'^i} \wedge e'^i$

satisfies (6.1). The canonical connection ∇ on the Hermitian line bundle L induced by J now gives an operator $\nabla^{0,1} := \overline{\partial}_{J'}$ (decomposing with respect to J').

In the sequel X and X' will denote the almost complex manifolds (X, J) and (X, J') respectively and in general a prime on an object will indicate that it is defined with respect to the almost complex structure J'.

Remark 6.1. — Even though the pair (ω, J') fits into the setup of [44] it should be pointed out that the Riemannian metric $\omega(v, J'w)$ on X was used in [44], but we will use the the Riemannian metric g induced by the given Hermitian metric H instead. It should be pointed out that the results in this paper are independent of the metric, but the metric may be important in a more refined study involving Bergman kernels. Also, in [44] the asymptotics of projection operators acting on L^k were studied, but as mentioned there, the case $L^k \otimes E$ where E is a complex vector bundle is similar. In Section 6.5 we will study $L^k \otimes E$ for a certain complex line bundle $E = K^{-}_{X'}$.

6.2. Connections and commutation relations

Let us first recall some basic facts about connections [50],[24]. A *connection* ∇ on a complex line bundle L over a real manifold X is an operator

$$\nabla : C^\infty(X; L) \to C^\infty(X; L \otimes T^*X)$$

satisfying Leibniz rule: $\nabla(fs) = df \otimes s + f\nabla s$ for f a function and s a section of L. Given a vector field v on X the contracted operator ∇_v on sections of L is called the covaraint derivative along v. The curvature two-form Θ of ∇ can be defined by

$$\Theta(v, w) = [\nabla_v, \nabla_w] - \nabla_{[v,w]}, \tag{6.6}$$

where v and w are vector fields on X. If L has a Hermitian metric $\langle \cdot, \cdot \rangle$, then a connection ∇ is called *unitary* if

$$d \langle s, t \rangle = \langle \nabla s, t \rangle + \langle s, \nabla t \rangle \tag{6.7}$$

and if L is a holomorphic line bundle over a complex manifold X, then ∇ is called *holomorphic* if

$$\nabla^{0,1} = \overline{\partial} \tag{6.8}$$

i.e. $\nabla^{0,1} s = 0$ if s is a holomorphic section. There is a unique unitary holomorphic connection (see below) on a Hermitian holomorphic line bundle L. If (X, J) is only an almost complex manifold, any given connection ∇ defines an operator $\overline{\partial}_J := \nabla^{0,1}$ acting on sections with values in L, but there is no canonical operator $\overline{\partial}_J$ on L. In the following we will only consider unitary connections ∇ on L over an almost complex manifold (X, J).

The local situation is as follows. Let t be a local unitary trivializing section of L and let A be the local one form defined by $\nabla t = A \otimes t$. Note that ∇ is unitary (i.e. (6.7) holds) precisely when A is an imaginary one form. Now we get the local representation $\nabla = d + A$, i.e.

$$\nabla(fs) = (d + Af)s, \tag{6.9}$$

and the curvature two-form Θ of ∇ is locally given by

$$\Theta = dA. \tag{6.10}$$

If $\widehat{t} = e^{ig}t$ is another unitary frame for L over U, then, using Leibniz rule, the corresponding one form is given by

$$\widehat{A} = A + idg, \tag{6.11}$$

confirming that the curvature two-form (6.10) is independent of the local frame. Take local dual orthonormal frames Z_i and e_i for $T^{0,1}(X, J)$ and $T^{*0,1}(X, J)$, respectively as in Section 2. Splitting $\nabla = \nabla^{1,0} + \nabla^{0,1}$ we may then write

$$\nabla = \sum_i (\overline{e^i}\nabla_{\overline{i}} + e^i\nabla_i), \tag{6.12}$$

where $\nabla_i := \nabla_{Z_i}$ and $\nabla_{\overline{i}}$ are the corresponding covariant derivatives along Z_i and $\overline{Z}_i$ respectively. Let us now consider some local commutation relations. Write

$$\begin{array}{rcl} [Z_i, \overline{Z_j}] & = & \sum_p (a_{ij}^p Z_p - \overline{a_{ji}^p Z_p}) \\ [Z_i, Z_j] & = & \sum_p (f_{ij}^p Z_p + \overline{N_{ij}^p Z_p}) \end{array}, \tag{6.13}$$

where the bracket denotes the commutator between the corresponding differential operators. Then N_{ij}^p is identically zero precisely when $T^{1,0}(X, J)$ is closed under the bracket, which in turn is equivalent to J being integrable [27, 44]. In general the N_{ij}^p define the so called Nijenhuis tensor of the almost complex structure J. Now, using formulas (6.6) and (6.13) we get the following commutation relations:

$$\begin{array}{rcl} [\nabla_i, \nabla_{\overline{j}}] & = & \Theta(Z_i, \overline{Z_j}) + \sum_p (a_{ij}^p \nabla_p - \overline{a_{ji}^p} \nabla_{\overline{p}}) \\ [\nabla_i, \nabla_j] & = & \sum_p (f_{ij}^p \nabla_p + \overline{N_{ij}^p} \nabla_{\overline{p}}), \end{array} \tag{6.14}$$

where we have used that Θ vanishes on $T^{0,1}X \otimes T^{0,1}X$, by the assumption (6.1) (i) on J.

Let us now specialize to our original situation (compare Section 6.1), where J is integrable and Θ has index q and take a local frame e^i diagonalizing Θ. Recall (Section 2) that $|s|^2 = e^{-2\phi}$ where s is a local holomorphic trivializing section of L so that $t :=: e^{\phi}s$ is a local unitary section. Then if ∇ denotes a connection satisfying (6.7) and (6.8) we see that ∇ is unique since the local one form A is given by

$$A = -\partial\phi + \overline{\partial}\phi, \;\; \Theta = dA = 2\partial\overline{\partial}\phi \tag{6.15}$$

Indeed, the assumption (6.8) gives as in Section 2 that $\nabla^{0,1}$ is locally represented (with respect to t) by

$$\overline{\partial} + \overline{\partial}\phi = \sum_i e^i(Z_i + Z_i\phi)$$

and since A is imaginary (by (ii)) we get (6.15). Moreover, we get that $\Theta_{ij} = \lambda_i\delta_{ij}$ and $N^p_{ij} = 0$ in (6.14). Next, introducing the complex structure J' defined above corresponds to letting

$$\nabla_i^{'} = \nabla_{\overline{i}}, \; i \leqslant q \quad \text{and} \;\; \nabla_i^{'} = \nabla_i, \; i > q \tag{6.16}$$

since the decomposition (6.12) of ∇ changes in the corresponding way.

6.3. Symbols and ideals

We will now consider an arbitrary almost complex structure J again and replace L by L^k and consider semiclassical symbols as in Section 2 (setting $h = k^{-1}$). The discussion will be local on U, given a unitary trivializing section t over U and dual orthonormal frames Z_i and e_i for $T^{0,1}(U, J)$ and $T^{*0,1}(U, J)$, respectively. Any given connection ∇ on L induces a connection on L^k, that we also denote by ∇ (i.e. locally A in (6.9) is replaced by kA). We denote by σ the semiclassical principal symbol map (compare the discussion about semiclassical principal symbols following the proof of proposition 2.1) and let

$$q_i := \sigma(h\nabla_i)$$

in terms of the decomposition (6.3) (i.e. q_i is the principal symbol of the ith component of $h\overline{\partial}_J$). We will call $\mathcal{J} = (q_1, ..., q_n)$ the *symbol ideal* of $\overline{\partial}_J$. Since ∇ is unitary, i.e. it satifies (6.7), integration by parts gives

$$\sigma(-h\nabla_{\overline{i}}) = \sigma(h\nabla_i^*) = \overline{q_i}, \tag{6.17}$$

also using the general fact that $\sigma(D^*) = \overline{\sigma(D)}$ in the last equality. Recall the following general relation between the operator bracket and the Poisson bracket:

$$\sigma[D_1, D_2] = -ih\{\sigma(D_1), \sigma(D_2)\},$$

Hence, the commutator relations (6.14) become:

$$\begin{array}{rcl} i\{q_i, \overline{q_j}\} & = & \Theta(Z_i, \overline{Z_j}) + \sum_p (a_{ij}^p q_p - \overline{a_{ji}^p}\overline{q_p}) \\ i\{q_i, q_j\} & = & \sum_p (f_{ij}^p q_p + \overline{N_{ij}^p}\overline{q_p}), \end{array} \tag{6.18}$$

Let now J and J' be as in Section 6.1. Note that when ∇ is the canonical connection determined by J, the local expression (6.15) shows that

$$h\nabla_i = hZ_i + hZ_i\phi$$

and note that (6.18) is consistent with the formula (2.23). Now (6.18) and (6.17) give $q_i' = -\overline{q_i}$ for $i \leqslant q$ and $q_i' = q_i$ for $i > q$. In particular, the zero varieties in T^*U defined by the symbol ideals $\mathcal{J}$ and $\mathcal{J}'$ coincide and are equal to the real characteristic variety $\Sigma = \{p_{0:} := \sigma(\Delta_q) = 0\}$, where p_0 is as in formula (2.15). In Section 3 the local almost holomorphic manifold J^+ in the almost complexification of T^*X was introduced. It corresponds to a local ideal $\mathcal{J}^+$ of local smooth functions on the symplectic manifold T^*X such that for all f in $\mathcal{J}^+$

$$\overline{\partial}\widetilde{f}_{J^+}(x) = \mathcal{O}(\mathrm{Im}x)^\infty,$$

where $\widetilde{f}$ denotes an almost holomorphic extension of f from T^*X. The properties of J^+, reviewed in (3.15), when formulated in terms of the ideal $\mathcal{J}^+$, can be stated as the following lemma [12],[44],[40], where $\mathcal{I}_\Sigma$ denotes the ideal of elements in $\mathcal{C}^\infty(X, \mathbf{C})$ vanishing on Σ.

Lemma 6.2. — *There exists a unique positive Poisson ideal $\mathcal{J}^+$ with respect to Σ containing p_0. That is, there exists a unique ideal $\mathcal{J}^+ \subset \mathcal{I}_\Sigma$ with common zero set Σ satisfying (i) $\mathcal{J}^+$ is closed under the Poisson bracket and (ii) there are generators q_i of $\mathcal{J}^+$ such that the matrix $\frac{1}{i}\{q_i, \overline{q_j}\}$ is positive definite on Σ and $p_0 \in \mathcal{J}^+$.*

Note that by (6.18) the ideal $\mathcal{J}$ fails to satisfy the positivity condition (*ii*) above, since Θ is assumed to have index q. On the other hand, the positive ideal $\mathcal{J}'$ only satisfies condition (*i*) mod $\mathcal{I}_\Sigma$ (compare Proposition 6.5). By the uniqueness of $\mathcal{J}^+$, we then deduce that $\mathcal{J}^+ = \mathcal{J}'$ mod $\mathcal{I}_\Sigma^2$. In fact, the ideal $\mathcal{J}^+$ can be constructed from $\mathcal{J}'$ by induction with respect to N on the vanishing order $\mathcal{I}_\Sigma^N$ [44] so that $\mathcal{J}^+$ is unique mod $\mathcal{I}_\Sigma^N$ for each N.

Remark 6.3. — The uniqueness mod $\mathcal{I}_\Sigma^2$ in Lemma 6.2 is equivalent to the well-known fact that given a Riemannian metric g on a symplectic manifold (X, ω) there is a unique almost complex structure J such that (6.2) and (6.1) hold. The point is that given (X, ω, g, J) we get a map

$$(T^{0,1}X, J) \to \mathcal{J}^+/\mathcal{I}_\Sigma^2, \;\; Z_i \mapsto \sigma(h\nabla_{Z_i}) =: q_i$$

By (6.18) the conditions (6.1) on ω correspond to the conditions in Lemma 6.2 on the Poisson brackets when restricted to Σ and the condition (6.2) on g corresponds to the condition on $p_0 = \sigma(\Delta)$. However, Lemma 6.2 applies to a more general situation where p_0 is a general function on a symplectic manifold Y (replacing T^*X with its usual symplectic form) vanishing to second order on $\Sigma := \{p_0 = 0\}$. Then one gets a complex structure on the normal bundle $TY/T\Sigma$ of Σ in Y [12].

One final

Remark 6.4. — In Section 3 J^+ was only locally defined, but it corresponds to a global submanifold of the almost complexification of the affine bundle AX defined in Section 8.

6.4. J' is generically non-integrable

We will call a function f on T^*X *fiber affine* if it is affine on each fixed fiber of T^*X. Equivalently, f is fiber affine if it is the semiclassical principal symbol of a first order $h-$differential operator on X. Consider $\mathbf{C}^2$ with its standard complex structure and metric and let L be the trivial holomorphic line bundle with fiber metric ϕ. We will also assume that the index of the curvature $\Theta = 2\partial\overline{\partial}\phi$ is one.

Proposition 6.5. — *The almost complex structure J' is non-integrable for generic fiber metrics ϕ. More precisely, J' is non-integrable if*

$$\frac{\partial^3\phi}{\partial^2 z_1 \partial\bar{z}_2} \neq 0 \tag{6.19}$$

at 0. *In particular, the ideal $\mathcal{J}^+$ has no fiber affine generators then.*

Proof. — We will identify Θ with a Hermitian matrix: $\Theta_{ij} := \Theta(\frac{\partial}{\partial\bar{z}_i}, \frac{\partial}{\partial z_j}) = -2\frac{\partial^2\phi}{\partial z_j \partial\bar{z}_i}$ with respect to the standard orthogonal frame $\frac{\partial}{\partial\bar{z}_i}$ and we may assume that $\Theta(0)$ is diagonal. Denote by Z_i an orthonormal frame diagonalizing Θ close to $z = 0$, i.e. $D_{ij} := \Theta(Z_i, \overline{Z_j}) =: -\delta_{ij}\lambda_i$. Equivalently, $Z_i = U\frac{\partial}{\partial\bar{z}_i}$ where the matrix valued function U satisfies

$$(i) \quad U^*U \quad = \quad I \quad (ii) \quad U^*\Theta U \quad = \quad D \; , \tag{6.20}$$

denoting by U^* the Hermitian adjoint $\overline{U}^t$. By the definition (6.4) of J' and the subsequent discussion we may take $Z'_1 = \overline{Z_1}$ and $Z'_2 = Z_2$. In particular, J' is non-integrable if a^1_{21}, defined with respect to J in (6.13), is non-vanishing at the origin. Now observe that at $z = 0$,

$$-a^1_{21} = \left\langle [\overline{Z_1}, Z_2], Z_1 \right\rangle = (\frac{\partial}{\partial z_1} u_{21})(0) \tag{6.21}$$

Indeed, $Z_1 = \frac{\partial}{\partial \bar{z}_1}$ at $z = 0$ and when calculating

$$[\overline{Z_1}, Z_2] = [\overline{u_{11}} \frac{\partial}{\partial z_1} + \overline{u_{12}} \frac{\partial}{\partial z_2}, u_{21} \frac{\partial}{\partial \bar{z}_1} + u_{22} \frac{\partial}{\partial \bar{z}_2}],$$

we can use Leibniz rule for the bracket to expand the right hand side and get terms of the form

$$\overline{(u_{11}}[\frac{\partial}{\partial z_1}, u_{21}]) \frac{\partial}{\partial \bar{z}_1} + ...$$

But since, $u_{ij}(0) = \delta_{ij}$ the other term proportional to $\frac{\partial}{\partial \bar{z}_1}$ vanishes at $z = 0$, proving (6.21). Hence, we just have to show that $(\frac{\partial}{\partial z_1} u_{21})(0) \neq 0$, if (6.19) holds. To this end, apply $\frac{\partial}{\partial z_1}$ to (6.20) and use that $U(0) = I$ and $\Theta(0) = D$ to get at $z = 0$

$$(i) \ \frac{\partial}{\partial z_1}(U^*) = -\frac{\partial}{\partial z_1} U, \quad (ii) \ \frac{\partial}{\partial z_1} \Theta + [D, \frac{\partial}{\partial z_1} U] = \frac{\partial}{\partial z_1} D. \tag{6.22}$$

In particular,

$$\frac{\partial}{\partial z_1} \Theta_{21} - (\lambda_1 - \lambda_2) \frac{\partial}{\partial z_1} u_{21} = 0,$$

i.e. at $z = 0$ we have

$$\frac{\partial}{\partial z_1} u_{21} = 2 \frac{\partial^3 \phi}{\partial^2 z_1 \partial \bar{z}_2} / (\lambda_2 - \lambda_1)$$

By (6.21) this proves first part of the proposition about the non-integrability of J'.

The second part is a direct consequence of the first part, by the way $\mathcal{J}^+$ is constructed in [12]. Indeed, by the uniqueness property mod $\mathcal{I}^2_\Sigma$ in Lemma 6.2 we have, since $\mathcal{J}^+$ is assumed to be generated by fiber affine functions, that $\mathcal{J}^+ = (q'_1, q'_2, ..., q'_n)$. But then the assumption that $\mathcal{J}^+$ is a Poisson ideal forces $N'^p_{ij} = 0$ in the relations corresponding to (6.18) for the almost complex structure J'. But this contradicts the first part of the proposition. $\square$

6.5. Partial Serre duality

In this section J and J' (and X and X') will be as in Section 6.1. Moreover, we will consider globally defined symbol ideals etc (compare Remark 6.4). Denote by $\mathcal{H}^{0,q}(X, L^k)$ the global null space of Δ_q and denote in this section by Π_X^q the orthogonal projection on $\mathcal{H}^{0,q}(X, L^k)$. In simple cases, e.g. when X is a product of complex curves and L is the product of pulled back line bundles one can show that that, for k sufficiently large, any element α in $\mathcal{H}^{0,q}(X, L^k)$ may be written locally as

$$\alpha = fe^1 \wedge ... \wedge e^q$$

with respect to a frame as in formula (6.5). Moreover f is holomorphic with respect to a new integrable complex structure of the form J'. In fact, this follows from "partial Serre duality", i.e. Serre duality along the negative directions of the line bundle in the product case. In this section we will show that a version of this phenomenon, with J' possibly non-integrable, persists for general X.

Denote by K_X the canonical line bundle on $X = (X, J)$, i.e. K_X is the holomorphic line bundle $\Lambda^{n,0}(T^*X, J)$ (considering (T^*X, J) as a holomorphic vector bundle). The splitting (6.3) then induces a decomposition of complex line bundles

$$K_X = K_X^- \otimes K_X^+ \tag{6.23}$$

where $K_X^- := \Lambda^{q,0}((T^*X)_-, J)$ and $K_X^+ := \Lambda^{n-q,0}((T^*X)_+, J)$. Note that the bundles $K_X^\pm$ are not holomorphic in general. Now

$$K_{X'}^- := \overline{K_X^-} = \Lambda^{q,0}((T^*X)_-, J')$$

is a complex line bundle on X' with a connection induced by the canonical connection on (TX, J) determined by the metric g and the complex structure J. Given a sufficiently large integer k the complex line bundle $L^k \otimes K_{X'}^-$ over X' has positive curvature with respect to J' and fits into the setup in the beginning of Section 6. Denote by $H^0(X', L^k \otimes K_{X'}^-)$ the space of *asymptotically almost holomorphic sections* defined in [44] (see Remark 6.1) We will just recall that $H^0(X', L^k \otimes K_{X'}^-)$ is defined as the range of a global projection operator $\Pi_{X'}^0$, which is a Fourier integral operator with complex phase and its canonical relation can be described in the following way. Let Σ' be the real characteristic variety of $\overline{\partial}_{J'}$ and let $\mathcal{J}'^+$ be the ideal obtained from Lemma 6.2 applied to Σ'. Then the canonical relation may be written as $\mathcal{C}' = \mathcal{J}'^+ \times_{\Sigma'} \overline{\mathcal{J}'^+}$, which is to be interpreted in terms of bicharacteristic strips as in the expression for C_∞ (the canonical relation of Π_X^q) in Section 3. But since $\Sigma' = \Sigma$, as observed in Section 6.3, the uniqueness in Lemma

6.2 gives $\mathcal{J}'^{+} = \mathcal{J}^{+}$ and hence $\mathcal{C}' = \mathcal{C}_\infty$. The construction in [44] actually only determines $\Pi^0_{X'}$ mod $\mathcal{O}(k^{-\infty})$ i.e. the asymptotics of its distribution kernel is only determined up to terms of order $\mathcal{O}(k^{-\infty})$. But this means that the dimension of $H^0(X', L^k \otimes K^-_{X'})$ is independent of the construction for k sufficiently large. We will now prove the following

Theorem 6.6. — *Assume that the index of Θ is q. Then, for k sufficiently large,*

$$\dim \mathcal{H}^{0,q}(X, L^k) = \dim H^0(X', L^k \otimes K^-_{X'}). \tag{6.24}$$

Furthermore, if K_X has a square root $K_X^{1/2}$, then

$$\dim \mathcal{H}^{0,q}(X, L^k \otimes K_X^{1/2}) = \dim H^0(X', L^k \otimes K_{X'}^{1/2}), \tag{6.25}$$

where the right hand side is defined using the induced connection on $K_{X'}^{1/2}$.

Proof. — In the proof we will identify $TX := (TX, J)$ with $T^{1,0}(X, J)$ as complex vector bundles, so that $\Lambda^{r,0}(T^*X, J)$ is identified with $\Lambda^r(T^*X)$ (in particular $K_X = \Lambda^n(T^*X)$ and similarly for $TX' := (TX, J')$. Let us first prove (6.24). Observe that for k sufficiently large, the left hand side of (6.24) is given by

$$\dim \mathcal{H}^{0,q}(X, L^k) = (-1)^q \int_X Td(TX) \wedge e^{kc^1(L)}, \tag{6.26}$$

where $Td(TX)$ is the Todd class of the complex vector bundle (TX, J). Indeed, for any line bundle L the Riemann-Roch theorem [24],[23] applied to the complex $(\mathcal{E}^{0,*}(X, L^k), \overline{\partial})$ gives that the alternating sum of the dimensions of the spaces $\mathcal{H}^{0,j}(X, L^k)$ is given by the right hand side in (6.26). Moreover, if L has index q, then the dimensions of all $\mathcal{H}^{0,j}(X, L^k)$ such that $j \neq q$ vanish for k sufficiently large (as follows from Proposition 2.4) giving (6.26). Similarly, it was shown in [12] that the right hand side of (6.24) is given by

$$\dim H^0(X', L^k \otimes K^-_{X'}) = \int_{X'} Td(TX') \wedge e^{kc^1(L) + c^1(K^-_{X'})}, \tag{6.27}$$

now using the Todd class of the complex vector bundle (TX, J'). Using (6.26) and (6.27) and the fact that $[X'] = (-1)^q[X]$ as integration currents (since the orientation depends on the almost complex structure) it is enough to show that

$$Td(TX) \wedge e^{kc^1(L)} = Td(TX') \wedge e^{kc^1(L) + c^1(K^-_{X'})}. \tag{6.28}$$

to prove the theorem. To this end, we first recall the following basic properties of the Todd class. Let F be a complex line bundle and E_1 and E_2 complex vector bundles over a real manifold X. Then

$$\begin{array}{lrcl} (i) & Td(F) & = & c^1(F)/(1-e^{-c^1(F)}) \\ (ii) & Td(E_1 \oplus E_2) & = & Td(E_1) \wedge Td(E_2) \end{array}, \tag{6.29}$$

where the expression in (i) is to be interpreted as a formal power series in $c^1(F)$, yielding a polynomial in $c^1(F)$, since $c^1(F)^j$ vanishes if $j > n$. In fact, by the "splitting principle" the properties (6.29) determine Td uniquely [10]. Next, we will show that the following universal identity holds

$$Td(\overline{E}) \wedge e^{c^1(E)} = Td(E). \tag{6.30}$$

To prove a universal identity between characteristic classes it is, by the "splitting principle" enough to prove it when E is a direct sum of line bundles over a manifold Y. Moreover, by (6.29) (ii) and the multiplicativity of e^{c^1} we may then assume that E is a line bundle. By (6.29) (i) the identity (6.30) is then equivalent to the function identity

$$\frac{x}{1-e^{-x}} = \frac{-x}{1-e^{-(-x)}} \cdot e^x$$

which clearly holds. Let us now finish the proof of the identity (6.28). By the definition of J' the splitting (6.3) gives

$$TX' = \overline{TX_-} \oplus TX_+$$

as complex vector bundles. Substituting this into the right hand side of (6.28) and using the multiplicative property (6.29) (ii) we see that it is enough to show that

$$Td((TX)_-) = Td(\overline{(TX)_-}) \wedge e^{c^1(K^-_{X'})}.$$

Finally, since $c^1(K^-_{X'}) := c^1(\Lambda^q(\overline{(T^*X)_-}) = c^1((TX)_-)$, the identity (6.28) follows from the identity (6.30) applied to $E = (TX)_-$. This finishes the proof of (6.24). To prove (6.25), note that the previous argument also shows that (6.24) remains true after replacing L^k by $L^k \otimes F$ in both sides of (6.24), where F is a complex vector bundle. In particular, letting $F = K_X^{1/2}$ we get, using the decomposition (6.12), that $F \otimes K^-_{X'}$ is given by

$$((K^-_X)^{1/2} \otimes (K^+_X)^{1/2}) \otimes (K^-_X)^{-1} = K^{-1/2}_{X^-} \otimes K^{1/2}_{X^+} = K^{1/2}_{X'}$$

where we have used that $\overline{E} \simeq E^* := E^{-1}$ for any complex line bundle E. This proves (6.25). $\square$

Remark 6.7. — To prove the second part of the previous theorem one could also use that any almost complex structure whose canonical line bundle has a square root determines a spin structure on X. Then the use of the Riemann-Roch theorem may be replaced by the index theorem for the correponding Dirac operator. In this context it is well-known that the index only depends on the induced orientation of the real manifold X. See [23].

7. Examples: Flag manifolds

In this section we will recall (without giving proofs) the construction of flag manifolds and their homogeneous line bundles, emphasizing the complex analytical aspects. It turns out that the new almost complex structures J' (defined by (6.4)) in this context are actually integrable and we show that Theorem 6.6 corresponds to a weak version of the Borel-Weil-Bott theorem. For general references on flag manifolds see [7][29][22][1]. See also [31] and [28] where they are also studied from an asymptotic point of view.

Let K be a compact semi-simple real Lie group and take a maximal connected Abelian subgroup T of K (i.e. a *maximal torus* of K). The $K-$homogenous manifold $X := K/T$ is called a flag manifold. Recall that the complexification of the Lie algebra $\mathfrak{k}$ of K decomposes as

$$\mathfrak{k}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \bigoplus_{\alpha \in \Delta} E_\alpha \tag{7.1}$$

diagonalizing the adjoint action of $\mathfrak{t}$ on $\mathfrak{k}$ (acting by the Lie bracket). The label α of the eigen space E_α is called a *root* and it defines a non-zero element of $\mathfrak{t}^*_{\mathbf{C}}$:

$$[t, \overline{Z}_\alpha] = (\alpha, t)\overline{Z}_\alpha$$

for any element $\overline{Z}_\alpha$, called a *root vector*, of the *root space* E_α. From (7.1) and a consistent choice of positive roots Δ_+ one gets a decomposition at the identity element e of K :

$$T_e X \otimes \mathbf{C} \cong \Big(\bigoplus_{\alpha \in \Delta_+ \bigsqcup -\Delta_+} E_\alpha \Big) =: T_e^{1,0} X \oplus T_e^{0,1} X \tag{7.2}$$

inducing an invariant integrable complex structure on X. In fact, exponentiating the $(1,0)$ part of (7.2) expresses X as a holomorphic quotient,

$$X := K/T \backsimeq G/B, \tag{7.3}$$

where B is a *Borel group* in the complexification G of K. We fix a Hermitian invariant metric on K/T making the decomposition (7.2) orthogonal.

The Hermitian holomorphic line bundles on X may be identified with the *weight lattice* in $\mathfrak{t}^*$, i.e. the elements λ of $\mathfrak{t}^*$ that exponentiate to characters on the torus T, with values in $U(1)$. To see this, recall that in general a hermitian line bundle over a manifold X can be considered as the vector bundle associated to a principal $U(1)-$bundle over X. In our situation K is a principal $T-$bundle over $X(=K/T)$ and since λ induces a homomorphism of the fiber T into $U(1)$ it determines a Hermitian line bundle L_λ over X. The curvature two form Θ_λ of L_λ is determined by

$$\Theta_\lambda(\overline{Z}_\alpha, Z_\beta) = \delta_{\alpha\beta}c_\alpha \langle \lambda, \alpha\rangle \tag{7.4}$$

using the Killing form $\langle\cdot,\cdot\rangle$ on $\mathfrak{t}^*_{\mathbf{C}}$, where $\overline{Z}_\alpha$ is a normalized root vector in E_α and c_α is a certain positive number. Formula (7.4) shows that the index of Θ_λ is equal to the *index* of λ, where the latter is defined as the number of positive roots α such that $\langle\lambda, \alpha\rangle < 0$. The hyper planes $\ker \alpha$ divide $\mathfrak{t}^*$ into so called *Weyl chambers* and the index is constant for all λ in the interior of a chamber. *In the following we will assume that the curvature of L_λ is non-degenerate i.e. that λ is in the interior of a chamber.*

Example 7.1. — Let $K = SU(n+1)$. Then $T = U(1)^n$ and $G = SL(n+1, \mathbf{C})$ with B the subgroup of upper triangular matrices and X is the manifold of all complete flags in $\mathbf{C}^{n+1}$, i.e. the set of all $n-$tuples of linear subspaces $(V_1, ...V_n)$ such that $V_i \subsetneq V_{i+1}$ For example, if $n = 1$, then $X = \mathbf{P}^1$ and the conjugate complex manifold $\overline{\mathbf{P}^1}$ is obtained by letting B be defined by lower triangular matrices. Moreover, $\mathfrak{t} = i\mathbf{R}$, the weight lattice is, under proper normalization, $i\mathbf{Z}$ and the Weyl chambers are the positive and negative half-axes. The element im corresponds to the line bundle $\mathcal{O}(m)$, whose sections are the homogoneous polynomials of degree m. If $n = 2$, then X may be identfied with the three dimensional manifold

$$(Z, W) \in \mathbf{P}^2 \times \mathbf{P}^2 \; : \; Z_0W_0 + Z_1W_1 + Z_2W_2 = 0, \tag{7.5}$$

in terms of homogenous coordinates and the action of $SU(3)$ is given by the action

$$(A; (Z, W)) \mapsto (AZ, (A^t)^{-1}W).$$

The weight lattice is now $i\mathbf{Z}^2$ and there are six Weyl chambers. This follows from the representation theory of $SU(3)$ but using the realization (7.5) it is straight forward to see that all line bundles on X are obtained as $\pi_1^*(\mathcal{O}(m)) \otimes \pi_2^*(\mathcal{O}(n))$ in terms of the projections on the factors in (7.5). Moreover, by homogenity it is, using the fiber metric induced by the Fubini-Study metric, enough to calculate the index at a given point. Then one sees that there are six chambers determined by linear conditions on m and n.

7.1. Change of complex structure - The Weyl group

The *Weyl group* is the group generated by the reflections in the hyper planes $\ker\alpha$ determined by the roots. It preserves the weight lattice and acts transitively and simply on the set of Weyl chambers. In particular, if λ has index q there is an element w of the Weyl group such that $w(\lambda)$ is positive. Dualy, the action of the Weyl group may be interpreted as a change of the complex structure J. Indeed, since

$$\langle w(\lambda), \alpha\rangle > 0 \Leftrightarrow \left\langle \lambda, w^{-1}(\alpha)\right\rangle > 0 \tag{7.6}$$

the weight $w(\lambda)$ is in the positive Weyl chamber if and only if the line bundle L_λ is positive with respect to J_w, where J_w is the complex structure determined by the positive roots $w^{-1}(\alpha)$. Hence, L_λ determines a unique invariant complex structure on X, making L_λ positive. More concretely, assume that the positive roots α_i are labeled so that

$$\langle \lambda, \alpha_i\rangle < 0,\ i \leqslant q,\ \ \langle \lambda, \alpha_i\rangle > 0,\ i > q$$

This means that the functional defined by λ is positive precisely on the subset

$$\{-\alpha_1, ..., -\alpha_q, \alpha_{q+1}, ...\} \tag{7.7}$$

of the set the roots. By (7.6) this set must then be the image of the positive roots under w^{-1} (which is known to permute the roots). Furthermore, the action of w induces an isomorphism of holomorphic line bundles:

$$\begin{array}{ccc} L_\mu & \to & L_{w(\mu)} \\ \downarrow & & \downarrow \\ G/B_w & \to & G/B \end{array}, \tag{7.8}$$

where G/B_w is the holomorphic quotient corresponding to (X, J_w). The point is that w can be identified with an element of K, acting on G by the adjoint action.

7.2. The Borel-Weil-Bott theorem

Theorem 6.6 applied to a homogenous line bundle L over the homogenous complex manifold X (that can be represented as in (7.3)) gives, with $\rho := \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$:

COROLLARY 7.2. — *Assume that the weight λ is in the interior of a Weyl chamber and that it has index q. Then, after replacing λ by a sufficiently large multiple,*

$$\dim H^q(G/B, L_\lambda) = \dim H^0(G/B_w, L_{\lambda+\rho-w^{-1}(\rho)}). \tag{7.9}$$

Equivalently, fixing the complex structure J on K/T :

$$\dim H^q(K/T, L_\lambda) = \dim H^0(K/T, L_{w(\lambda+\rho)-\rho}) \tag{7.10}$$

Proof. — Assume that L_λ has index q and let J' be the new invariant almost complex structure determined by (6.4). Then J' is an almost complex structure such that L_λ is positive with respect to J' and so is the complex structure J_w determined by w in the Weyl group as explained in Section 7.1. By the uniqueness in Remark 6.3 we have $J' = J_w$. Hence, Theorem 6.6 gives (7.9), but with the line bundle $L_\lambda \otimes K^-_{X'}$ in the right hand side. To see that $L_\lambda \otimes K^-_{X'} = L_{\lambda+\rho-w^{-1}(\rho)}$ note that, given the ordering of the positive roots in Section 7.1,

$$K^-_{X'} \leftrightarrow \sum_{i=1}^{q} \alpha_i = \rho - w^{-1}(\rho) \tag{7.11}$$

Indeed, from the definition (7.2) of the complex structure J on K/T

$$T^{1,0}X = \bigoplus_{\alpha\in\Delta_+} L_\alpha,$$

where L_α is the line bundle corresponding to the root α, giving

$$K_X = \Lambda^n(T^{*1,0}X) \simeq \bigotimes_{\alpha\in\Delta_+} L_{-\alpha} = L_{-2\rho}$$

and a similar argument gives the first correspondence in (7.11). Finally, since the image of the positive roots under w^{-1} is given by (7.7),

$$\rho - w^{-1}(\rho) = \frac{1}{2}(\sum_{i=1}^{q}\alpha_i + \sum_{i=q+1}^{n}\alpha_i) - \frac{1}{2}(\sum_{i=1}^{q}-\alpha_i + \sum_{i=q+1}^{n}\alpha_i) = \sum_{i=1}^{q}\alpha_i$$

Now the induced isomorphism (7.8) applied to $\mu = \lambda+\rho-w^{-1}(\rho)$ proves (7.10). □

The previous corollary (in the formulation (7.10)) is a weak version of Bott's generalization of the Borel-Weil theorem [8],[9]. The Borel-Weil-Bott theorem may also be proved using Lie algebra cohomology [30][52].

8. Appendix: The affine bundle AX

We will define an affine bundle AX over X with symplectic form Ω so that the global sections of AX are the unitary connections of the Hermitian line bundle L over X. Given an open set U and a local unitary frame t for L over U we identify (AU, Ω) with $(T^*U, dp \wedge dx)$ in terms of the usual coordinates (x, p) on T^*U. If $\widehat{t} = e^{ig}t$ is another unitary section the two identifications are assumed to be related by

$$(x, \widehat{p}_1, ..., \widehat{p}_n) = (x, p_1 - \frac{\partial}{\partial x_1} g, ..., p_n - \frac{\partial}{\partial x_n} g),$$

Hence, $\Omega = dx \wedge dp$ is a globally well-defined symplectic two-form on AX. Given a global connection ∇ represented by $d + A$ with respect to the frame t the transformation property (6.11) now shows that $(x, iA_1(x), .., iA_n(x))$ defines a global section of AX.

Notice that the local characteristic variety Σ in Proposition 2.3 corresponds globally to the graph in AX of the canonical connection ∇ on the Hermitian holomorphic line bundle L. Indeed, by (2.17) and (2.24) we get locally on Σ

$$pdx = \mathrm{Re}(\tfrac{2}{i}\partial\phi) = i(-\partial\phi + \overline{\partial}\phi)$$

By (6.15) the right hand side equals iA, where A is the local one form associated to ∇ with respect to $t = e^{\phi}s$.

Finally, for comparison with [25][16] observe that any given unitary connection ∇ on L induces a global isomorphism

$$\Phi_{\nabla} : AX \leftrightarrow T^*X, \ \ (x, p) \mapsto (x, p_1 - iA_1, ...), \tag{8.1}$$

The map is defined using local frames t as above and it maps the graph of the section of AX corresponding to ∇ to the zero-section in T^*X. We get that

$$(\Phi_{\nabla}^{-1})^*(\Omega) = d(-\gamma) + \pi^*(-i\Theta),$$

where γ is the tautological $1-$form on T^*X and $\pi^*(-i\Theta)$ is the normalized curvature of ∇ pulled back from X. The bundle AX may also be defined by symplectic reduction of $(T^*Y, d(-\gamma)$ where Y is the unit circle bundle in L^* (compare the proof of Theorem 2.3 in [44]).

Bibliography

[1] Baston R.J., Eastwood M.G.. — The Penrose transform. Its interaction with representation theory, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1989. xvi+213 pp

[2] Berman R.. — Bergman kernels and local holomorphic Morse inequalities, Math. Z. 248(2), p. 325–344 (2004).

[3] Berman R.. — Super Toeplitz operators on holomorphic line bundles, J. Geom. Anal. 16(1), p. 1-22 (2006).

[4] Berndtsson B., Berman R., Sjöstrand J.. — Asymptotics of Bergman kernels, arXiv.org/abs/math.CV/050636

[5] Bismut J.M.. — Demailly's asymptotic Morse inequalities, a heat equation proof, J. Funct. Anal. 72, p. 263–278 (1987).

[6] Bleher P., Shiffman B., Zelditch S.. — Universality and scaling of correlations between zeros on complex manifolds, Invent. Math. 142(2), p. 351–395 (2000).

[7] Borel A., Hirzebruch F.. — Characteristic classes and homogeneous spaces I. Amer. J. Math. 80, p. 458–538 (1958).

[8] Bott R.. — Homogeneous vector bundles, Ann. of Math. 66(2), p. 203–248 (1957).

[9] Bott R.. — On induced representations, The mathematical heritage of Hermann Weyl (Durham, NC, 1987), p. 1–13, Proc.Sympos. Pure Math. 48, Amer. Math. Soc., Providence, RI, 1988.

[10] Bott R., Tu L.. — Differential forms in algebraic topology, Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982

[11] Bouche T.. — Convergence de la métrique de Fubini Study d'un fibré linéaire positif, Ann. Inst. Fourier 40(1), p. 117-130 (1990).

[12] Boutet de Monvel L., Guillemin V.. — The spectral theory of Toeplitz operators, Annals of Mathematics Studies, 99. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981.

[13] Boutet de Monvel L., Sjöstrand J.. — Sur la singularité des noyaux de Bergman et de Szegö, Astérisque 34-35 , p. 123–164 (1976).

[14] Catlin D.. — The Bergman kernel and a theorem of Tian, Analysis and geometry in several complex variables (Katata, 1997), 1–23, Trends in Math. Birkhäuser, Boston, MA, 1999.

[15] Charles L.. — Berezin-Toeplitz operators, a semi-classical approach, CMP 239, p.1–28 (2003).

[16] Charles L.. — Aspects semi-classiques de la quantification géometrique, Ph.D. thesis, Universite Paris IX-Dauphine (2000).

[17] Dai X., Liu K., Ma X.. — On the asymptotic expansion of Bergman kernel, J. Diff. Geom. 72(1), p. 1–41 (2006).

[18] Demailly J.P., Peternell T., Schneider M.. — Holomorphic line bundles with partially vanishing cohomology, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 165–198, Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan, 1996.)

[19] Dimassi M., Sjöstrand J.. — Spectral asymptotics in the semi-classical limit, London Math. Soc. Lecture Notes Series 268, Cambridge Univ. Press 1999.

[20] Donaldson S.K.. — Symplectic submanifolds and almost-complex geometry, J. Differential Geom. 44(4), p. 666-705 (1996).

[21] FEFFERMAN C.. — The Bergman kernel and biholomorphic mappings of pseudoconvex domains, Invent. Math. 26, p. 1–65 (1974).

[22] FULTON W., HARRIS J.. — Representation theory. A first course, Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991. xvi+551 pp.

[23] GILKEY P.. — Invariance theory, the heat equation, and the Atiyah-Singer index theorem, Second edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[24] GRIFFITHS P., HARRIS J.. — Principles of algebraic geometry. Reprint of the 1978 original, Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[25] GUILLEMIN J., URIBE A.. — On the de Haas-van Alphen effect, Asymptotic Anal. 6(3), p. 205–217 (1993).

[26] HELFFER B., SJÖSTRAND J.. — Semiclassical analysis for Harper's equation. III. Cantor structure of the spectrum, Mém. Soc. Math. France (N.S.) No. 39, p. 1–124 (1989).

[27] HÖRMANDER L.. — An introduction to complex analysis in several variables, van Nostrand, (1966), 1967.

[28] KARABEGOV A.V.. — Pseudo-Kähler quantization on flag manifolds, Comm. Math. Phys. 200(2), p. 355–379(1999).

[29] KIRILLOV A.A.. — Lectures on the orbit method, Graduate Studies in Mathematics, 64. American Mathematical Society, Providence, RI, 2004. xx+408 pp

[30] KOSTANT B.. — Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. of Math. 74(2), p. 329–387 (1961).

[31] KURONYA A.. — Asymptotic cohomological functions on projective varieties, (arXiv.org/abs/math.AG/0501491)

[32] LU Z.. — On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch, Am. J. Math. 122(2), p. 235–273 (2000).

[33] LU Z., TIAN G.. — The log term of Szegö kernel, Duke Math. J. 125(2), p. 351-387 (2004).

[34] MA X., MARINESCU G.. — The spin^c Dirac operator on high tensor powers of a line bundle, Math. Z. 240(3), p. 651–664 (2002).

[35] MA X., MARINESCU G.. — Generalized Bergman kernels on symplectic manifolds, C. R. Math. Acad. Sci. Paris 339(7)(2004), 493-498, (arXiv.org/abs/math.DG/0411559).

[36] MA X., MARINESCU G.. — The first coefficients of the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator, International J. Math. 17(6) (2006), 737-759.

[37] MELIN A., SJÖSTRAND J.. — Fourier integral operators with complex valued phase functions, Springer LNM, 459.

[38] MELIN A., SJÖSTRAND J.. — Fourier integral operators with complex phase functions and parametrix for an interior boundary value problem, CPDE, 1(4), p. 313–400 (1976).

[39] MELIN A., SJÖSTRAND J.. — Determinants of pseudodifferential operators and complex deformations of phase space, Methods and Appl. of Anal. 9(2), p. 177–238 (2002).

[40] MENIKOFF A., SJÖSTRAND J.. — On the eigenvalues of a class of hypoelliptic operators, Math. Ann. 235, p. 55–85 (1978).

[41] MENIKOFF A., SJÖSTRAND J.. — The eigenvalues of hypoelliptic operators III, the non-semibounded case, J. d'Analyse Math. 35, p. 123–150 (1979).

[42] Robert D.. — Autour de l'approximation semi-classique, Progress in Mathematics, 68. Birkhäuser Boston, Inc., Boston, MA, 1987.

[43] Ruan W.. — Canonical coordinates and Bergman metrics, Comm. Anal. Geom. 6, p. 589–631 (1998).

[44] Shiffman B., Zelditch Z.. — Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds, J. Reine Angew. Math. 544, p. 181–222 (2002).

[45] Sjöstrand J.. — Singularités analytiques microlocales, Astérisque 95 (1982).

[46] Sjöstrand J.. — Parametrices for pseudodifferential operators with multiple characteristics, Ark. Mat. 12, p. 85–130 (1974).

[47] Sjöstrand J.. — Asymptotics for Bergman kernels for high powers of complex line bundles, based on joint work with B. Berndtsson and R. Berman, Sém. équations aux dérivées partielles, Ecole Polytechnique, 2004–2005, exposé no 23 (17.5.2005), http://www.math.polytechnique.fr/seminaires/seminaires-edp/2004-2005/sommaire2004-2005.html

[48] Sjöstrand J., Zworski M.. — Quantum monodromy and semi-classical trace formulae, J. Math. Pures Appl. 81(1), no. 1, p. 1–33 (2002).

[49] Tian G.. — On a set of polarized Kähler metrics, J. Diff. Geom. 32, p. 99–130 (1990).

[50] Wells R.O.. — Differential analysis on complex manifolds, Graduate texts in mathematics 65, Springer 1980.

[51] Zelditch S.. — Szegö kernels and a theorem of Tian, IMRN 1998(6), p. 317–331.

[52] Zierau R.. — Representations in Dolbeault cohomology, Representation theory of Lie groups (Park City, UT, 1998), 91–146, IAS/Park City Math. Ser., 8, Amer. Math. Soc., Providence, RI, 2000