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Minkowski sums and Brownian exit times(*)

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Abstract. — If C is a domain in $\mathbf{R}^n$, the Brownian exit time of C is denoted by T_C. Given domains C and D in $\mathbf{R}^n$ this paper gives an upper bound of the distribution function of T_{C+D} when the distribution functions of T_C and T_D are known. The bound is sharp if C and D are parallel affine half-spaces. The paper also exhibits an extension of the Ehrhard inequality.

Résumé. — Si C est un domaine de $\mathbf{R}^n$, le temps de sortie brownien de C est noté T_C. Donnant domaines C et D de $\mathbf{R}^n$ cet article montre une borne supérieure pour la fonction de répartition de T_{C+D} quand les fonctions de répartition de T_C et T_D sont connues. En plus l'article exhibe une généralisation de l'inégalité d'Ehrhard.

1. Introduction

Throughout $W = (W(t))_{t\geqslant 0}$ denotes Brownian motion in $\mathbf{R}^n$ and if C is a domain in $\mathbf{R}^n$,

$$T_C = T_C^W = \inf\{t > 0;\ W(t) \notin C\}$$

is called the exit time from C. Below the notation $P_x[\cdot]$ or $E_x[\cdot]$ indicates that Brownian motion starts at the point x at time zero.

The main aim of this paper is to prove an inequality of the Brunn-Minkowski type for distribution functions of Brownian exit times from domains in $\mathbf{R}^n$, such that equality occurs for parallel affine half-spaces. Here perhaps the most interesting point is the fact that the set of all Brownian paths $\{W(\omega);\ T_C > t\}$ is not an affine half-space if C is an affine half-space in $\mathbf{R}^n$. Recall that affine half-spaces often turn out to be extremals for Gaussian measures (see Ehrhard [5] and Carlen, Kerce [3]). In connection with

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our main result Theorem 1.1, the Bachelier formula for the distribution of the maximum of real-valued Brownian motion (see e.g. Karatzas and Shreve [6] , p. 96) plays an important part. Finally, to make comparisons with the Ehrhard inequality (Ehrhard [4] , Borell [2]) we find it natural to extend this inequality to more general linear combinations of sets.

Let us continue by giving some more definitions. First the so called vector sum or Minkowski sum of two subsets A and B of $\mathbf{R}^n$ equals

$$A + B = \{x + y;\ x \in A \text{ and } y \in B\}.$$

Moreover, if $\alpha > 0$, the dilation $\alpha A = \{\alpha x;\ x \in A\}$.

In [1] I use a method based on the maximum principle for elliptic differential equations to prove the following inequality for expected Brownian exit times. Suppose C and D are bounded domains in $\mathbf{R}^n$ and $x \in C$, $y \in D$. Then

$$\sqrt{E_{x+y}\left[T_{C+D}\right]} \geqslant \sqrt{E_x\left[T_C\right]} + \sqrt{E_y\left[T_D\right]}. \tag{1.1}$$

Here equality occurs in many interesting cases. First recall that

$$E_x\left[T_C\right] = \int_0^\infty P_x\left[T_C > t\right] dt.$$

Therefore by the scaling property of Brownian motion

$$\sqrt{E_{\alpha x}\left[T_{\alpha C}\right]} = \alpha\sqrt{E_x\left[T_C\right]},\ \alpha > 0$$

and it follows that equality occurs in (1.1) if C is convex and $D \times \{y\} = \lambda(C \times \{x\}) + (a, a)$ for appropriate $\lambda > 0$ and $a \in \mathbf{R}^n$. In this paper we will use a method similar to those in my papers [1] and [2] to prove inequalities of the Brunn-Minkowski type for distribution functions of Brownian exit times.

If H is an open affine half-space in $\mathbf{R}^n$, the Bachelier formula for the distribution of the maximum of real-valued Brownian motion yields

$$P_x\left[T_H > t\right] = \Psi\left(\frac{d(x, H^c)}{\sqrt{t}}\right),\ t > 0,\ x \in H$$

where $d(x, H^c) = \min_{y \notin H} \mid x - y \mid$ and

$$\Psi(r) = 2\int_0^r \exp\left(-\frac{\lambda^2}{2}\right)\frac{d\lambda}{\sqrt{2\pi}},\ 0 \leqslant r \leqslant \infty.$$

The main aim of this paper is to prove the following

THEOREM 1.1. — *Suppose C and D are domains in $\mathbf{R}^n$ and $f{:}C \to [0,1]$, $g{:}D \to [0,1]$, and $h{:}C+D \to [0,1]$ continuous functions such that*

$$\Psi^{-1}(h(x+y)) \geqslant \Psi^{-1}(f(x)) + \Psi^{-1}(g(y)),\ x \in C,\ y \in D. \tag{1.2}$$

Then, if $x \in C$, $y \in D$, and $t > 0$,

$$\begin{aligned} &\Psi^{-1}(E_{x+y}\,[h(W(t)); T_{C+D} > t]) \\ &\geqslant \Psi^{-1}(E_x\,[f(W(t)); T_C > t]) + \Psi^{-1}(E_y\,[g(W(t)); T_D > t]). \end{aligned} \tag{1.3}$$

In particular,

$$\Psi^{-1}(P_{x+y}\,[T_{C+D} > t]) \geqslant \Psi^{-1}(P_x\,[T_C > t]) + \Psi^{-1}(P_y([T_D > t]). \tag{1.4}$$

Equality occurs in (1.4) *if C and D are parallel affine half-spaces.*

It is not obvious to the author that Theorem 1.1 implies (1.1).

Next we introduce some additional definitions. Below F denotes a real, separable Fréchet space and γ a centered Gaussian measure on F, that is γ is a Borel probability measure on F such that each bounded linear functional on F has a centered Gaussian distribution. The Borel field in F is denoted by $\mathcal{B}(F)$. The definitions of Minkowski sums and dilations of subsets of F are as in the special case $F = \mathbf{R}^n$.

If

$$\Phi(r) = \int_{-\infty}^{r} \exp(-\frac{\lambda^2}{2})\frac{d\lambda}{\sqrt{2\pi}},\ \ -\infty \leqslant r \leqslant \infty,$$

and $0 < \theta < 1$, my paper [2] proves the so called Ehrhard inequality

$$\Phi^{-1}(\gamma(\theta A + (1-\theta)B)) \geqslant \theta\Phi^{-1}(\gamma(A)) + (1-\theta)\Phi^{-1}(\gamma(B))$$

for all $A, B \in \mathcal{B}(F)$. As in the Latała paper [8] we here follow the convention that $\infty - \infty = -\infty + \infty = -\infty$.

The following result is slightly more informative than the Ehrhard inequality.

THEOREM 1.2. — *Suppose $\alpha, \beta > 0$ are given. Then the inequality*

$$\Phi^{-1}(\gamma(\alpha A + \beta B) \geqslant \alpha\Phi^{-1}(\gamma(A)) + \beta\Phi^{-1}(\gamma(B)) \tag{1.5}$$

is valid for all $A, B \in \mathcal{B}(F)$ *if*

$$\alpha + \beta \geqslant 1 \text{ and } \mid \alpha - \beta \mid \leqslant 1. \tag{1.6}$$

Moreover, if γ *is not a Dirac measure at origin and* (1.5) *is valid for all* $A, B \in \mathcal{B}(F)$ *then* (1.6) *holds.*

Equality occurs in (1.5) *if* A *and* B *are parallel affine half-spaces. If, in addition,* $\alpha + \beta = 1$ *equality occurs in* (1.5) *if* A *is convex and* $B = A$.

To comment on a certain relation between Theorems 1.1 and 1.2 we denote by $\mathcal{C}([0,\infty[\,;\mathbf{R}^n)$ the space of all continuous maps of $[0,\infty[$ into $\mathbf{R}^n$ equipped with its standard locally convex topology of uniform convergence on compact subintervals of $[0,\infty[$. Furthermore, we will have the picture that the identity map on $\mathcal{C}([0,\infty[\,;\mathbf{R}^n)$ gives a representation of Brownian motion in $\mathbf{R}^n$ relative to Wiener measure on $\mathcal{C}([0,\infty[\,;\mathbf{R}^n)$. Now using the shorthand notation $W_x(t) = x + W(t)$ it follows that

$$[W_{\alpha x+\beta y}(s) \in \alpha C + \beta D,\ 0 < s \leqslant t]$$

$$\supseteq \alpha\,[W_x(s) \in C,\ 0 < s \leqslant t] + \beta\,[W_y(s) \in D,\ 0 < s \leqslant t]\,,\ \alpha, \beta > 0$$

and, hence, by (1.5),

$$\Phi^{-1}(P_{\alpha x+\beta y}\,[T_{\alpha C+\beta D} > t]) \geqslant \alpha\Phi^{-1}(P_x\,[T_C > t]) + \beta\Phi^{-1}(P_y\,[T_D > t]) \tag{1.7}$$

for all reals α and β satisfying (1.6). The inequality (1.4) is not weaker than the inequality (1.7) with $\alpha = \beta = 1$ since

$$\Psi(\Psi^{-1}(a) + \Psi^{-1}(b)) \geqslant \Phi(\Phi^{-1}(a) + \Phi^{-1}(b)) \tag{1.8}$$

for all $0 \leqslant a, b \leqslant 1$, which follows from the fact that there is equality in (1.4) when C and D are parallel affine half-spaces. In fact, strict inequality holds in (1.8) for all $0 \leqslant a < 1$, $0 \leqslant b < 1$, such that $(a, b) \neq (0, 0)$. To see this, suppose $0 \leqslant a \leqslant 1$, $0 \leqslant b < 1$ and let $f(a)$ be the difference of the members on the left-hand and right-hand side of (1.8). If $b = 0$, then $f(a) = a > 0$ for all $0 < a < 1$. Next suppose $0 < b < 1$. The function f is continuous and $f(0) = b$ and $f(1) = 0$. Furthermore, if $0 < a < 1$,

$$f'(a) = \exp(-\Psi^{-1}(a)\Psi^{-1}(b) - \frac{1}{2}(\Psi^{-1}(b))^2)$$

$$- \exp(-\Phi^{-1}(a)\Phi^{-1}(b) - \frac{1}{2}(\Phi^{-1}(b))^2).$$

If $\frac{1}{2} \leqslant b < 1$, then $\Phi^{-1}(b) \geqslant 0$ and since $\Psi^{-1}(y) > \max(0, \Phi^{-1}(y))$ if $0 < y < 1$ we have that $f'(a) < 0$ for all $0 < a < 1$. Thus f is strictly decreasing and we get $f(a) > 0$ if $0 \leqslant a < 1$. Next suppose $0 < b < \frac{1}{2}$. Then $\Phi^{-1}(b) < 0$ and it follows that f' is decreasing on $]0,1[$. Thus f is concave and, accordingly from this, $f(a) > 0$ if $0 \leqslant a < 1$.

It does not seem to exist any natural counterpart of the inequality (1.4) for linear combinations of sets as in (1.7). For example, the inequality

$$\Psi^{-1}(P_{\frac{1}{2}x+\frac{1}{2}y}\left[T_{\frac{1}{2}C+\frac{1}{2}D} > t\right]) \geqslant \frac{1}{2}\Psi^{-1}(P_x\left[T_C > t\right]) + \frac{1}{2}\Psi^{-1}(P_y([T_D > t])$$

is not true in general. In fact, if that was the case we use the concavity of Ψ to get

$$P_{\frac{1}{2}x+\frac{1}{2}y}\left[T_{\frac{1}{2}C+\frac{1}{2}D} > t\right] \geqslant \frac{1}{2}P_x\left[T_C > t\right] + \frac{1}{2}P_y([T_D > t]\,.$$

Now if C is convex and $D = C$ we integrate over $0 \leqslant t < \infty$ and have that the expected exit time $E_x\left[T_C\right]$ is a concave function of $x \in C$, which is wrong for the plane domain $\left\{z \in \mathbf{C};\, 0 < \arg z < \frac{\pi}{4} \text{ and } \mid z \mid < 1\right\}$ (see my paper [1], Example 3.1).

By passing note that if $\alpha = \beta = 1$ in (1.5) and the function Φ^{-1} is everywhere replaced by Ψ^{-1} the resulting inequality is false since, otherwise, (1.8) would be an equality for all $0 < a, b < 1$.

Next suppose $F = \mathbf{R}^n$ and $\gamma = \gamma_n$ in Theorem 1.2. Then if $C \in \mathcal{B}(\mathbf{R}^n)$ is convex and $A = B = C$ and $\alpha = \beta = \frac{r}{2} \geqslant \frac{1}{2}$ in (1.5) we get the following result by Sudakov and Tsirelson [9] of independent interest.

Corollary 1.1. — *Suppose $C \in \mathcal{B}(\mathbf{R}^n)$ is convex and H an open affine half-space in $\mathbf{R}^n$ such that*

$$\gamma_n(C) = \gamma_n(H).$$

Then

$$\gamma_n(rC) \geqslant \gamma_n(rH) \quad \text{if } r \geqslant 1.$$

Stated otherwise,

$$\Phi^{-1}(\gamma_n(rC)) \geqslant r\Phi^{-1}(\gamma_n(C)) \quad \text{if } r \geqslant 1.$$

See also Yurinsky´s book [10] and the early paper by Landau and Shepp [7], which shows Corollary 1.1 in the special case $\gamma_n(C) \geqslant \frac{1}{2}$.

The present paper is organized as follows. In Section 2 we prove Theorem 1.1. Finally, Section 3 is devoted to a (partly sketchy) proof of Theorem 1.2.

2. Proof of Theorems 1.1

By monotone convergence there is no loss of generality to assume that C and D are finite unions of open cubes with edges parallel to the coordinate axes. In addition, given $\delta \in]0,1[$, we may assume $f{:}C \to [0,\delta]$ and $g{:}D \to [0,\delta]$ are continuous functions with compact supports in C and D, respectively. Finally, there is no loss of generality to assume that h possesses compact support in $C+D$ and

$$h : C + D \to \left[0, \Psi(2\Psi^{-1}(\delta))\right].$$

Now for each $q = f, g, h$, set

$$u_q(t,x) = E_x\left[q(W(t));\ T_{\text{Dom } q} > t\right]$$

for every $t \geqslant 0$ and every x belonging to the closure of Dom q, where Dom q denotes the domain of definition of q. Moreover, set

$$U_q = \Psi^{-1}(u_q)$$

and introduce the continuous function

$$V(t,x,y) = U_h(t,x+y) - U_f(t,x) - U_g(t,y)$$

defined for all $t \geqslant 0$ and $x \in \bar{C}$, $y \in \bar{D}$. We will prove that $V(t,x,y) \geqslant 0$, which implies (1.3).

The construction shows that $V(0,x,y) \geqslant 0$ for all $x \in \bar{C}$, $y \in \bar{D}$. Furthermore, if $x \in C$ and $y \in \partial D$, then $V(t,x,y) \geqslant 0$ if and only if $u_h(t,x+y) \geqslant u_f(t,x)$. The latter inequality is obvious since

$$\begin{aligned} u_f(t,x) &= E\left[f(x+W(t)); x+W(s) \in C,\ 0 < s \leqslant t\right] \\ &\leqslant E\left[h(x+W(t)+y);\ x+W(s)+y \in C+y,\ 0 < s \leqslant t\right] \\ &\leqslant E\left[h(x+W(t)+y);\ x+W(s)+y \in C+D,\ 0 < s \leqslant t\right] = u_h(t,x+y). \end{aligned}$$

In a similar way, it follows that $V(t,x,y) \geqslant 0$ if $x \in \partial C$ and $y \in D$. In the next step we will show that $V(t,x,y)$ is a solution of a certain parabolic differential equation and the non-negativity of $V(t,x,y)$ then follows from the maximum principle.

Recall that $\Psi(a) = 2\Phi(a) - 1$, $0 \leqslant a \leqslant \infty$, so that $\Psi'(a) = 2\varphi(a)$, $0 \leqslant a < \infty$, where $\varphi(a) = \Phi'(a)$ if $a \in \mathbf{R}$. Moreover, if $q = f, g, h$ we have in the interior of Dom u_q that

$$\frac{\partial u_q}{\partial t} = \frac{1}{2}\Delta u_q$$

and, as $u_q = \Psi(U_q)$,

$$\frac{\partial u_q}{\partial t} = 2\varphi(U_q)\frac{\partial U_q}{\partial t},$$
$$\nabla u_q = 2\varphi(U_q)\nabla U_q$$

and

$$\Delta u_q = 2\varphi(U_q)(\Delta U_q - U_q \mid \nabla U_q \mid^2)$$

Thus

$$\frac{\partial U_q}{\partial t} = \frac{1}{2}\Delta U_q - \frac{1}{2}U_q \mid \nabla U_q \mid^2 .$$

To simplify notation, from now on let

$$\xi = (t, x), \;\; \eta = (t, y), \;\text{ and } \varsigma = (t, x + y)$$

so that, if $t > 0$, $x \in C$ and $y \in D$,

$$\nabla_x V = (\nabla U_h)(\varsigma) - (\nabla U_f)(\xi),$$
$$\nabla_y V = (\nabla U_h)(\varsigma) - (\nabla U_g)(\eta),$$
$$\Delta_x V = (\Delta U_h)(\varsigma) - (\Delta U_f)(\xi),$$
$$\Delta_y V = (\Delta U_h)(\varsigma) - (\Delta U_g)(\eta)$$

and

$$\sum_{1\leqslant i\leqslant n} \frac{\partial^2 V}{\partial x_i \partial y_i} = (\Delta U_h)(\varsigma).$$

Thus introducing the differential operator

$$\mathcal{E} = \frac{1}{2}\left\{\Delta_x - \sum_{1\leqslant i\leqslant n} \frac{\partial^2}{\partial x_i \partial y_i} + \Delta_y\right\}$$

we get

$$\mathcal{E}V = \frac{1}{2}\left\{(\Delta U_h)(\varsigma) - (\Delta U_f)(\xi) - (\Delta U_g)(\eta)\right\}.$$

Note here that the quadratic form

$$Q(r_1, ..., r_n, s_1, ..., s_n) = \sum_{1\leqslant i\leqslant n} r_i^2 - \sum_{1\leqslant i\leqslant n} r_i s_i + \sum_{1\leqslant i\leqslant n} s_i^2$$

is positive semi-definite. From the above

$$\mathcal{E}V = \frac{\partial U_h}{\partial t}(\varsigma) + \frac{1}{2}U_h(\varsigma) \mid (\nabla U_h)(\varsigma) \mid^2$$

$$-\frac{\partial U_f}{\partial t}(\xi) - \frac{1}{2}U_f(\xi) \mid (\nabla U_f)(\xi) \mid^2$$
$$-\frac{\partial U_g}{\partial t}(\eta) - \frac{1}{2}U_g(\eta) \mid (\nabla U_g)(\eta) \mid^2$$

or

$$\mathcal{E}V = \frac{\partial V}{\partial t} + \Omega(t,x,y)$$

with

$$\Omega(t,x,y) = \frac{1}{2}U_h(\varsigma) \mid (\nabla U_h)(\varsigma) \mid^2 - \frac{1}{2}U_f(\xi) \mid (\nabla U_f)(\xi) \mid^2$$
$$-\frac{1}{2}U_g(\eta) \mid (\nabla U_g)(\eta) \mid^2 .$$

Here

$$\mid (\nabla U_f)(\xi) \mid^2 = \mid (\nabla U_h)(\varsigma) \mid^2$$
$$+ \sum_{1 \leqslant i \leqslant n} \left\{ \frac{\partial U_f}{\partial x_i}(\xi) + \frac{\partial U_h}{\partial x_i}(\varsigma) \right\} \left\{ \frac{\partial U_f}{\partial x_i}(\xi) - \frac{\partial U_h}{\partial x_i}(\varsigma) \right\}$$

and

$$\mid (\nabla U_g)(\eta) \mid^2 = \mid (\nabla U_h)(\varsigma) \mid^2$$
$$+ \sum_{1 \leqslant i \leqslant n} \left\{ \frac{\partial U_g}{\partial x_i}(\eta) + \frac{\partial U_h}{\partial x_i}(\varsigma) \right\} \left\{ \frac{\partial U_g}{\partial x_i}(\eta) - \frac{\partial U_h}{\partial x_i}(\varsigma) \right\}.$$

From the above equations it follows that

$$\Omega(t,x,y) = \frac{1}{2} \mid (\nabla U_h)(\varsigma) \mid^2 V - b(t,x,y) \cdot \nabla_{(x,y)} V$$

for an appropriate function $b(t,x,y)$. Moreover,

$$\mathcal{E}V + b(t,x,y) \cdot \nabla_{(x,y)} V = \frac{\partial V}{\partial t} + \frac{1}{2} \mid (\nabla U_h)(\varsigma) \mid^2 V.$$

The non-negativity of $V(t,x,y)$ now follows from the maximum principle. For completeness we give a direct proof here. Let $T \in]0,\infty[$ be fixed. We know that the function $V(t,x,y)$ is non-negative on $(\{0\}\times(C\times D))\cup([0,T]\times\partial(C\times D))$. Therefore, if $V(t,x,y) < 0$ at some point $(t,x,y) \in [0,T]\times(\bar{C}\times\bar{D})$ there exists a strictly positive number ε such that the function $\varepsilon t + V(t,x,y)$ possesses a strictly negative minimum in $[0,T]\times(\bar{C}\times\bar{D})$ at a certain point $\varsigma_0 = (t_0,x_0,y_0) \in]0,T]\times(C\times D)$. But then

$$V(\varsigma_0) < 0,\ \frac{\partial V}{\partial t}(\varsigma_0) \leqslant -\varepsilon,\ \nabla_{(x,y)} V(\varsigma_0) = 0,\ \text{and } \mathcal{E}V(\varsigma_0) \geqslant 0.$$

and we have got a contradiction. This proves the inequality (1.3).

The inequality (1.4) follows by choosing $f = g = h = 1$ in (1.3). Furthermore, the distribution function of Brownian exit time from an affine half-space shows that equality occurs in (1.4) if C and D are parallel affine half-spaces. This completes the proof of Theorem 1.1.

3. A sketchy proof of Theorem 1.2

Let

$$d\gamma_n(x) = \exp(-\frac{1}{2} \mid x \mid^2)\frac{dx}{\sqrt{2\pi}^n}$$

be the canonical Gauss measure in $\mathbf{R}^n$.

First suppose $\alpha, \beta > 0$ and

$$\Phi^{-1}(\gamma_n(\alpha A + \beta B)) \geqslant \alpha\Phi^{-1}(\gamma_n(A)) + \beta\Phi^{-1}(\gamma_n(B))$$

for all $A, B \in \mathcal{B}(\mathbf{R}^n)$. We claim that (1.6) holds.

To see this suppose $C \in \mathcal{B}(\mathbf{R}^n)$ is convex, symmetric, and $0 < \gamma_n(C) < \frac{1}{2}$. Then

$$\Phi^{-1}(\gamma_n((\alpha + \beta)C)) \geqslant \alpha\Phi^{-1}(\gamma_n(C)) + \beta\Phi^{-1}(\gamma_n(C))$$

Now, if $\alpha + \beta < 1$ it follows that $(\alpha + \beta)C \subseteq C$ and we get

$$\Phi^{-1}(\gamma_n(C)) \geqslant \alpha\Phi^{-1}(\gamma_n(C)) + \beta\Phi^{-1}(\gamma_n(C))$$

or

$$0 \geqslant (\alpha + \beta - 1)\Phi^{-1}(\gamma_n(C))$$

which is a contradiction. On the other hand if $\mid \alpha - \beta \mid > 1$ we get a contradiction as follows. Depending on symmetry there is no loss of generality to assume that $\beta - \alpha > 1$. Then

$$\mathbf{R}^n \setminus C \supseteq \alpha C + \beta(\mathbf{R}^n \setminus C)$$

and we get

$$\Phi^{-1}(\gamma_n(\mathbf{R}^n \setminus C)) \geqslant \alpha\Phi^{-1}(\gamma_n(C)) + \beta\Phi^{-1}(\gamma_n(\mathbf{R}^n \setminus C))$$

or

$$-\Phi^{-1}(\gamma_n(C)) \geqslant \alpha\Phi^{-1}(\gamma_n(C)) - \beta\Phi^{-1}(\gamma_n(C))$$

since $\Phi^{-1}(1-y) = -\Phi^{-1}(y)$ for all $0 < y < 1$. Thus

$$0 > (\alpha + 1 - \beta)\Phi^{-1}(\gamma_n(C))$$

which is a contradiction.

To prove that (1.6) implies that (1.5) is valid for all $A, B \in \mathcal{B}(F)$ there is no loss of generality to assume $F = \mathbf{R}^n$ and $\gamma = \gamma_n$. Most parts of the proof may be arranged in a similar way as the proof of Theorem 1.1 above and, moreover, we may proceed almost in the same manner as in my proof of Ehrhard's inequality [2] (replace the pair $(\theta, 1-\theta)$ by (α, β) and replace the differential operator

$$\mathcal{E} = \frac{1}{2}\left\{\Delta_x + 2\sum_{1\leqslant i\leqslant n}\frac{\partial^2}{\partial x_i \partial y_i} + \Delta_y\right\}$$

by the differential operator

$$\mathcal{E}_{new} = \frac{1}{2}\left\{\Delta_x + \frac{1-\alpha^2-\beta^2}{\alpha\beta}\sum_{1\leqslant i\leqslant n}\frac{\partial^2}{\partial x_i \partial y_i} + \Delta_y\right\}.$$

Finally, note that if, $\alpha, \beta > 0$, the differential operator $\mathcal{E}_{new}$ is semi-elliptic if and only if $\alpha + \beta \geqslant 1$ and $|\alpha - \beta| \leqslant 1$). The details are omitted here.

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