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Complex Pisot Numeration Systems

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Complex Pisot Numeration Systems

Maki Furukado and Shunji Ito

1. Background

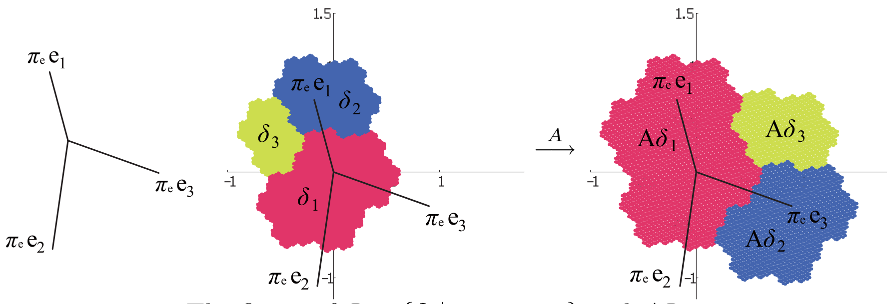
Starting from an unimodular Pisot substitution, it is well-known that we can constuct the numeration system. ([R], [A-I], [I-R], [F-F-I-W], etc.)

Let σ be $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ and the incidence matrix M_σ of σ is $M_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
Then, $A = M_\sigma^{-1}$ satisfies the complex Pisot condition, i.e. the eigenvalues of A satisfy $\lambda\,(= \lambda_1)$, $\overline{\lambda}\,(= \lambda_2) \in \mathbb{C}\backslash\mathbb{R}$ and $|\lambda| = \left|\overline{\lambda}\right| > 1 > |\lambda_3|$. The family of compact sets $\mathcal{P} = \{\delta_1, \delta_2, \delta_3\}$ of the A-invariant expanding plane P_e, which is given by the projection method, i.e.

$$\delta_i = \mathrm{cl}\left(\{\pi_e f\,(s_1 s_2 \ldots s_{k-1}) \mid \exists k \in \mathbb{N}, s_k = i\}\right),$$

where $w = s_1 s_2 \ldots = \lim_{n\to\infty} \sigma^n\,(1)$ is the fixed point of σ, satisfies the following set equations:

$$A\delta_1 = \delta_1 \cup \delta_2 \cup \delta_3,\ \ A\delta_2 = \delta_1 + \pi_e \boldsymbol{e}_3,\ \ A\delta_3 = \delta_2 + \pi_e \boldsymbol{e}_3.$$



The figure of $\mathcal{P} = \{\delta_i \mid i = 1, 2, 3\}$ and $A\mathcal{P}$.

Moreover, we know that we get the above family $\mathcal{P}$ of compact sets by using the 2-dimensional extension $E_2\,(\theta)$ where $\theta = \overleftarrow{\sigma}^{-1}$ ([E]).

Then we have the question: starting from an automorphism instead of a substitution, can we construct the numeration system?

For simplicity, we discuss the case of the complex Pisot number λ with degree 3 under some assumptions ([H-F-I]). The discussion in the case of degree 4 can be found in [A-F-H-I].

2. Notation

Assumption 2.1. (1) $\lambda \in \mathbb{C}\backslash\mathbb{R}$ *is a* complex Pisot number *with degree* 3*, i.e.* λ *is the algebraic integer of the minimal polynomial:*

$$p_\mp\,(x) = x^3 - ax^2 - bx \pm 1,\ \ a, b \in \mathbb{Z}$$

whose roots $\lambda\,(= \lambda_1)$, $\overline{\lambda}\,(= \lambda_2)$*,* λ_3 *satisfy* $|\lambda| = \left|\overline{\lambda}\right| > 1 > |\lambda_3|$;

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This is the joint work with Masaki Hama.

(2) *A is the 3×3 integer matrix such as* $A_{\mp} := \begin{bmatrix} 0 & 0 & \mp1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$ *whose characteristic polynomial is given by $p_{\mp}(x)$ respectively.*

Then, A is called the *companion Pisot matrix* of λ.

Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ be the eigenvectos of A with respect to λ_i, $i = 1, 2$, and let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ be

$$\boldsymbol{v}_1 := \frac{\boldsymbol{u}_2 + \boldsymbol{u}_1}{2}, \ \boldsymbol{v}_2 := \frac{\boldsymbol{u}_2 - \boldsymbol{u}_1}{2i}.$$

The A-expanding plane P_e of A is written by $P_e = \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ and the space $\mathbb{R}^3$ is decomposed by P_e and the A-contractive line $P_c : \mathbb{R}^3 = P_e \oplus P_c$. Then, let us define the projection $\pi_e : \mathbb{R}^d \to P_e$ along P_c by $\pi_e \boldsymbol{x} = x_1$ for $\boldsymbol{x} = x_1 + x_2 \in \mathbb{R}^d$ where $x_1 \in P_e$ and $x_2 \in P_c$.

Lemma 2.2. *Put $\lambda_1 = a + bi$, $a, b \in \mathbb{R}$, then* $A[\boldsymbol{v}_1\boldsymbol{v}_2] = [\boldsymbol{v}_1\boldsymbol{v}_2]\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Definition 2.3 (Complex Pisot numeration system)**.** *For a complex Pisot number λ, if we can find the finite family of compact sets $\mathcal{P} = \{\gamma_j\}_{j\in\{1,2,3\}}$ of P_e with the finite integer vector sequence $\left\{f_k^{(j)}\right\}_{1\le k\le l_j}$, $f_k^{(j)} \in \mathbb{Z}^d$ and the finite index sequence $\left\{V_k^{(j)}\right\}_{1\le k\le l_j}$, $V_k^{(j)} \in \{1, 2, 3\}$ satisfying*

(N1) $\mu_e(\gamma_j) > 0$, $\mathrm{cl}(\mathrm{int}(\gamma_j)) = \gamma_j$, *and* $\mu_e(\partial\gamma_j) = 0$
where μ_e is the Lebesgue measure on P_e, $\mathrm{int}(A)$ *is the interior of the set A,* $\mathrm{cl}(A)$ *is the closure of the set A, and $\partial\gamma_j := \gamma_j \backslash \mathrm{int}(\gamma_j)$;*

(N2) *for each $j \in \{1, 2, 3\}$, the set equation holds:*
$A\gamma_j = \bigcup_{k=1}^{l_j} \left(\gamma_{V_k^{(j)}} + \pi_e f_k^{(j)}\right)$ *(disjoint)*
where $\pi_e : \mathbb{R}^3 \to P_e$ is the projection along P_c, "$\bigcup_k A_k$ (disjoint)" means that $\mathrm{int}(A_k) \cap \mathrm{int}(A_{k'}) = \emptyset$ *if $k \neq k'$;*

(N3) $\bigcup_{j\in\{1,2,3\}} \gamma_j$ *(disjoint)* ,

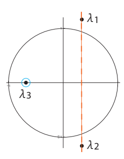
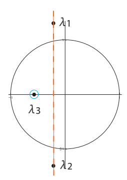
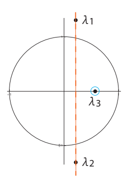
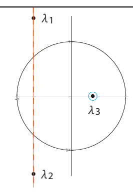
then, we say that the pair $(A, \mathcal{P})$ is the complex Pisot numeration system *of λ.*

The reason why we call $(X, \mathcal{P})$ the complex Pisot numeration system is that $\boldsymbol{x} \in \bigcup_{i=1,2,3} X_i$ is written as $\boldsymbol{x} = \sum_{n=1}^{\infty} A^{-n}\left(\pi_e f_{k_{n-1}}^{(j_{n-1})}\right)$ by the integer vector sequence $\left(f_{k_0}^{(j_0)}, f_{k_1}^{(j_1)}, \ldots, f_{k_{n-1}}^{(j_{n-1})}, \ldots\right)$ satisfying $x = x_0 \in \gamma_{j_0}$, $x_n = Ax_{n-1} - \pi_e f_{k_{n-1}}^{(j_{n-1})}$.

Question 1. *When a complex Pisot number λ is given, how do we obtain the complex Pisot numeration system? In other words, how do we obtain the finite family of compact sets $\mathcal{P} = \{\gamma_j\}_{j=1,2,3}$ of P_e satisfying (N1), (N2), (N3)?*

3. Results

First, we classify the distribution of λ_i, $i = 1, 2, 3$ of A into the following four types:

<table>
<tr><td colspan="2">$p_-(x)$</td><td colspan="2">$p_+(x)$</td></tr>
<tr><td>type 1</td><td>type 2</td><td>type 3</td><td>type 4</td></tr>
<tr><td>
</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td colspan="2">$\lambda_3 < 0$</td><td colspan="2">$\lambda_3 > 0$</td></tr>
<tr><td colspan="2">$a, b \in \mathbb{Z}$ satisfy
$(1)_-$ $-a + b < 0$
$(2)_-$ $\begin{cases} a^2 + 3b \le \text{ or} \\ a^2 + 3b > 0 \\ \Rightarrow 27 - 4a^3 - 18ab - a^2b^2 - 4b^3 > 0. \end{cases}$</td><td colspan="2">$a, b \in \mathbb{Z}$ satisfy
$(1)_+$ $a + b < 0$
$(2)_+$ $\begin{cases} a^2 + 3b \le \text{ or} \\ a^2 + 3b > 0 \\ \Rightarrow 27 + 4a^3 + 18ab - a^2b^2 - 4b^3 > 0. \end{cases}$</td></tr>
<tr><td>$Re(\lambda_1) > 0$</td><td>$Re(\lambda_1) < 0$</td><td>$Re(\lambda_1) > 0$</td><td>$Re(\lambda_1) < 0$</td></tr>
<tr><td>$a \ge 0$</td><td>$a < 0$</td><td>$a > 0$</td><td>$a \le 0$</td></tr>
</table>

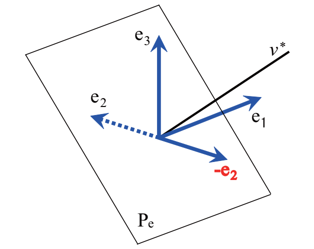
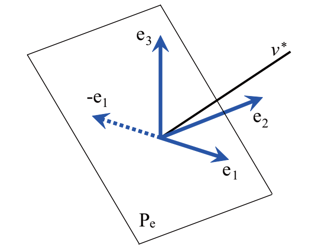
Let $\boldsymbol{v}^*$ and $\boldsymbol{v}$ be the left and right-eigenvectors of λ_3: $\boldsymbol{v}^* A = \lambda_3 \boldsymbol{v}^*$, $A\boldsymbol{v} = \lambda_3 \boldsymbol{v}$, then, we see that $\boldsymbol{v}^* = \left[1, \lambda_3, \lambda_3^2\right]$, $\boldsymbol{v} = {}^t\left[\mp\frac{1}{\lambda_3}, \lambda_3 - a, 1\right]$.

Lemma 3.1. *The expanding plane P_e can be characterized by using $\boldsymbol{v}^*$:*

$$P_e = \{\boldsymbol{x} = (x_1, x_2, x_3) \mid \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle = 0\}$$

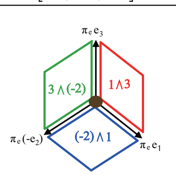
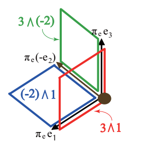
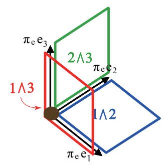
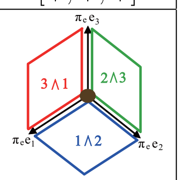
where $\langle \boldsymbol{y}, \boldsymbol{z} \rangle$ means the inner product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

From the signature of $\boldsymbol{v}^*$, we have the following:

	type 1	type 2	type 3	type 4
sgn $\boldsymbol{v}^*$	$[+,-,+]$		$[+,+,+]$	
Distribution of $\{\boldsymbol{e}_i\}$, $i = 1, 2, 3$				

For $\boldsymbol{x} \in \mathbb{R}^3$, $i, j \in \{\pm 1, \pm 2, \pm 3\}$, let $(\boldsymbol{x}, i \wedge j)$ be the 2-dimensional positive oriented unit face $i \wedge j$ located at $\boldsymbol{x} \in \mathbb{Z}^3$, i.e.

$$(\boldsymbol{x}, i \wedge j) := \left\{\boldsymbol{x} + \lambda\left(\operatorname{sgn}(i)\right)\boldsymbol{e}_{|i|} + \mu\left(\operatorname{sgn}(j)\right)\boldsymbol{e}_{|j|} \;\middle|\; \boldsymbol{x} \in \mathbb{Z}^3, 0 \le \lambda, \mu \le 1\right\}.$$

	type 1	type 2	type 3	type 4
sgn $\boldsymbol{v}$	$[+,-,+]$	$[+,+,+]$	$[+,-,+]$	$[+,+,+]$
				
	V_1	V_2	V_3	V_4
The unit faces	$(-\boldsymbol{e}_2, 1\wedge 2)$ $(\boldsymbol{0}, 1\wedge 3)$ $(-\boldsymbol{e}_2, 2\wedge 3)$	$(-\boldsymbol{e}_2, 1\wedge 2)$ $(\boldsymbol{0}, 3\wedge 1)$ $(-\boldsymbol{e}_2, 2\wedge 3)$	$(\boldsymbol{0}, 1\wedge 2)$ $(\boldsymbol{0}, 1\wedge 3)$ $(\boldsymbol{0}, 2\wedge 3)$	$(\boldsymbol{0}, 1\wedge 2)$ $(\boldsymbol{0}, 3\wedge 1)$ $(\boldsymbol{0}, 2\wedge 3)$

By using the 2-dimensional positive oriented unit faces V_t, $t = 1, 2, 3, 4$, we define

- the set of unit faces S_t of P_e: $S_t := \left\{(\boldsymbol{x}, i\wedge j) \;\middle|\; \begin{array}{l} \boldsymbol{x} \in \mathbb{Z}^3,\ \{i,j,k\} = \{1,2,3\},\ i \wedge j \in V_t, \\ \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle \ge 0,\ \langle \boldsymbol{x} - \boldsymbol{e}_k, \boldsymbol{v}^* \rangle < 0 \end{array}\right\}$;
- the family of finite sets of unit faces $\mathcal{G}_t$: $\mathcal{G}_t := \left\{\sum_{\mu \in \Lambda} (\boldsymbol{x}, i \wedge j)_\mu \;\middle|\; \#\Lambda < +\infty,\ (\boldsymbol{x}, i\wedge j)_\mu \in S_t\right\}$;
- the stepped plane $\mathscr{S}_t$ of P_e: $\mathscr{S}_t := \bigcup_{(\boldsymbol{x}, i\wedge j) \in S_t} (\boldsymbol{x}, i \wedge j)$.

The stepped plane $\mathscr{S}_t$ is the surface generated by 2-dimensional positive oriented unit faces. We know that the projection $\pi^* : \mathscr{S}_t \to P_e$ along $\boldsymbol{v}^*$ is bijective and there are not any lattice points except $\boldsymbol{0}$ between P_e and $\mathscr{S}_t$.

For the companion matrix A_- (A_+), let us choose the automorphism θ_- (θ_+) on the free group $F\langle 1, 2, 3\rangle$ whose incidence matrix is A_- (A_+) respectively as follows:

$$\theta_- : \begin{array}{ccc} 1 & \to & 2 \\ 2 & \to & 3 \\ 3 & \to & 3^a 1^{-1} 2^b \end{array}, \quad \theta_+ : \begin{array}{ccc} 1 & \to & 2 \\ 2 & \to & 3 \\ 3 & \to & 1 2^b 3^a. \end{array}$$

Using the automorphism θ, let us define the 2-dimensional extension $E_2(\theta) : \mathcal{G}_t \to \mathcal{G}_t$ as follows:

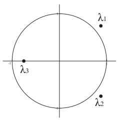
$$\begin{aligned}
E_2(\theta)(\mathbf{0}, i \wedge j) &:= (\mathbf{0}, \theta(i) \wedge \theta(j)) := \sum_{\substack{1 \le k \le l_i \\ 1 \le l \le l_j}} \left(f\left(P_k^{(i)}\right) + f\left(P_l^{(j)}\right), W_k^{(i)} \wedge W_l^{(j)} \right) \\
E_2(\theta)(\boldsymbol{x}, i \wedge j) &:= A\boldsymbol{x} + E_2(\theta)(\mathbf{0}, i \wedge j) \\
E_2(\theta)\left(\sum_{\mu} (\boldsymbol{x}, i \wedge j)_{\mu} \right) &:= \sum_{\mu} \left(E_2(\theta)(\boldsymbol{x}, i \wedge j)_{\mu} \right)
\end{aligned}$$

where $f : F\langle 1,2,3 \rangle \to \mathbb{Z}^3$ is the homomorphism satisfying

$$f(\varepsilon) = \mathbf{0},\ f(i) = \boldsymbol{e}_i,\ \theta(i) = W_1^{(i)} W_2^{(i)} \dots W_{l_i}^{(i)},$$

$P_k^{(i)}$ is the prefix of $W_k^{(i)}$, i.e. $P_k^{(i)} = W_1^{(i)} \dots W_{k-1}^{(i)}$, and $\boldsymbol{y} + (\mathbf{0}, i \wedge j) = (\boldsymbol{y}, i \wedge j)$. We will show the behavior of $E_2(\theta)$ on some examples.

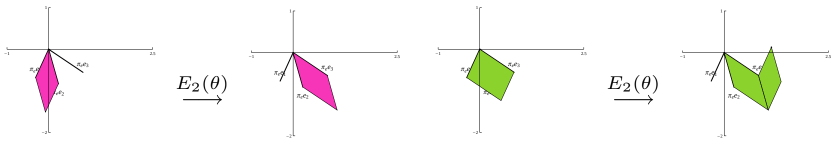
3.1. **The results on type 1 and type 4.** Let us consider the case $(a,b) = (1,0)$ of type 1 as an example, i.e. $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and its characteristic polynomial is $x^3 - x^2 + 1$. The distribution of the eigenvalues of A is



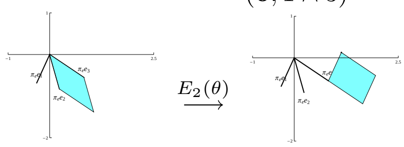
. Then the automorphism $\theta : \begin{array}{ccc} 1 & \to & 2 \\ 2 & \to & 3 \\ 3 & \to & 31^{-1} \end{array}$ is determined and $E_2(\theta)$ is given by

$$\begin{aligned}
E_2(\theta)(\mathbf{0}, 1 \wedge 2) &= (\mathbf{0}, \theta(1) \wedge \theta(2)) = (\mathbf{0}, 2 \wedge 3) \\
E_2(\theta)(\mathbf{0}, 1 \wedge 3) &= \left(\mathbf{0}, 2 \wedge 31^{-1}\right) = (\mathbf{0}, 2 \wedge 3) + \left(\boldsymbol{e}_3, 2 \wedge 1^{-1}\right) \overset{(*)}{=} (\mathbf{0}, 2 \wedge 3) + ((\boldsymbol{e}_3 - \boldsymbol{e}_1), 1 \wedge 2) \\
E_2(\theta)(\mathbf{0}, 2 \wedge 3) &= \left(\mathbf{0}, 3 \wedge 31^{-1}\right) = \left(\boldsymbol{e}_3, 3 \wedge 1^{-1}\right) \overset{(*)}{=} ((\boldsymbol{e}_3 - \boldsymbol{e}_1), 1 \wedge 3)
\end{aligned}$$

where $(*)$ means the rearrangement.



$(\mathbf{0}, 1 \wedge 2)$ $(\mathbf{0}, 1 \wedge 3)$



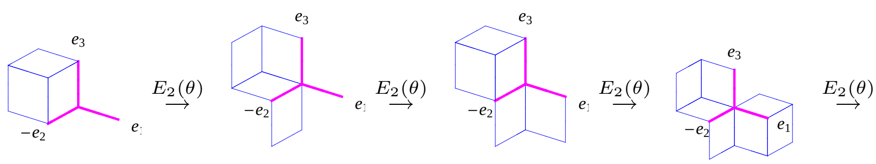
$(\mathbf{0}, 2 \wedge 3)$

We can see that $E_2(\theta)$ generates the patches whose orientations are positive in general on type 1 and we obtain the following propositons on type 1.

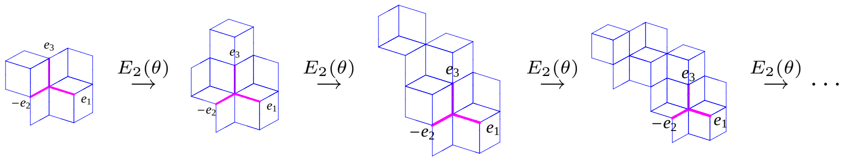
Proposition 3.2. *On type 1, we consider*

$$\mathcal{U}_1 := (\boldsymbol{x} - \boldsymbol{e}_2, 1 \wedge 2) + (\boldsymbol{x}, 1 \wedge 3) + (\boldsymbol{x} + \boldsymbol{e}_1, 2 \wedge 3)$$

where $\boldsymbol{x}$ *is the solution of* $\boldsymbol{x} + \boldsymbol{e}_3 - \boldsymbol{e}_2 = A\boldsymbol{x} + a\boldsymbol{e}_3 - \boldsymbol{e}_1$. *Then,* $E_2(\theta)^2(\mathcal{U}_1) \succ \mathcal{U}_1$.



$\mathcal{U}_1$



The figure of $E_2(\theta)^n(\mathcal{U}_1)$, $n = 0, 1, 2, \ldots, 7$ in the case of $(a, b) = (1, 0)$ of type 1.

Let us define

$$\begin{aligned} \mathcal{T}_1 &:= \left\{ \pi_e(\boldsymbol{x}, i \wedge j) \;\middle|\; (\boldsymbol{x}, i \wedge j) \in E_2(\theta)^{2n} \mathcal{U}_1(\boldsymbol{x}) \right\}, \\ \gamma_{i \wedge j} &:= \lim_{n \to \infty} A^{-n} \pi_e E_2(\theta)^n (\boldsymbol{x}_{i \wedge j}, i \wedge j) \quad \textit{for } (\boldsymbol{x}_{i \wedge j}, i \wedge j) \in \mathcal{U}_1(\boldsymbol{x}). \end{aligned}$$

Then,

(1) $\mathcal{T}_1$ *is the* quasi-periodic polygonal *tiling of* P_e *and* $\mathcal{T}_1 = \mathcal{T}_1^\circ - \pi_e \boldsymbol{x}$ *where* $\mathcal{T}_1^\circ := \{\pi_e(\boldsymbol{x}, i \wedge j) \mid (\boldsymbol{x}, i \wedge j) \in S_1\}$;
(2) $\mathcal{P}_1 := \{\gamma_{i \wedge j} \mid i \wedge j \in V_1\}$, *then* $(A, \mathcal{P}_1)$ *is the complex Pisot numeration system;*
(3) $\widehat{\mathcal{T}_1} := \{\pi_e \boldsymbol{x} + \gamma_{i \wedge j} \mid \pi_e(\boldsymbol{x}, i \wedge j) \in \mathcal{T}_1\}$ *is a self-similar tiling of* P_e.

Analogous results are obtained on type 4.

Proposition 3.3. *On type 4, we consider*

$$\mathcal{U}_4 := (\mathbf{0}, 1 \wedge 2) + (\mathbf{0}, 3 \wedge 1) + (\mathbf{0}, 2 \wedge 3).$$

Then, $E_2(\theta)(\mathcal{U}_4) \succ \mathcal{U}_4$. *Let us define*

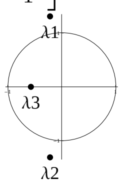
$$\begin{aligned} \mathcal{T}_4 &:= \{\pi_e(\boldsymbol{x}, i \wedge j) \mid (\boldsymbol{x}, i \wedge j) \in E_2(\theta)^n \mathcal{U}_4\}, \\ \gamma_{i \wedge j} &:= \lim_{n \to \infty} A^{-n} \pi_e E_2(\theta)^n (\boldsymbol{x}_{i \wedge j}, i \wedge j) \quad \textit{for } (\boldsymbol{x}_{i \wedge j}, i \wedge j) \in \mathcal{U}_4. \end{aligned}$$

Then,

(1) $\mathcal{T}_4$ *is the* quasi-periodic polygonal *tiling of* P_e *and* $\mathcal{T}_4 = \mathcal{T}_4^\circ$ *where* $\mathcal{T}_4^\circ := \{\pi_e(\boldsymbol{x}, i \wedge j) \mid (\boldsymbol{x}, i \wedge j) \in S_4\}$;
(2) $\mathcal{P}_4 := \{\gamma_{i \wedge j} \mid i \wedge j \in V_4\}$, *then* $(A, \mathcal{P}_4)$ *is the complex Pisot numeration system;*
(3) $\widehat{\mathcal{T}_4} := \{\pi_e \boldsymbol{x} + \gamma_{i \wedge j} \mid \pi_e(\boldsymbol{x}, i \wedge j) \in \mathcal{T}_4\}$ *is a self-similar tiling of* P_e.

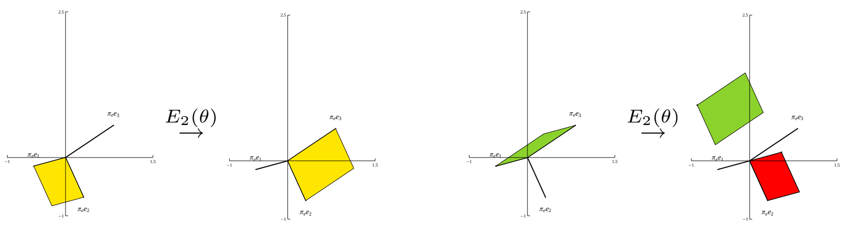
Remark 3.4. *We see that the numeration system produced from the type 4 is the same as the numeration system produced from the unimodular Pisot substitution.*

3.2. **The results on type 2.** Let us observe the case $(a, b) = (-1, -2)$ of type 2 as an example, i.e. $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ and its characteristic polynomial is $x^3 + x^2 + 2x + 1$. The eivenvalues of A is distributed as

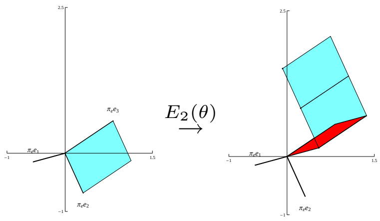


, then the automorphism $\theta : \begin{array}{lcl} 1 & \mapsto & 2 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 1^{-1}2^{-1}2^{-1}3 \end{array}$ is determined and $E_2(\theta)$ is given by

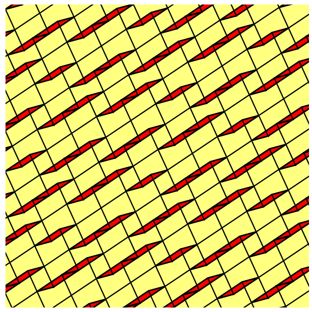
$$\begin{aligned} E_2(\theta)(\mathbf{0}, 1 \wedge 2) &:= (\mathbf{0}, 2 \wedge 3) \\ E_2(\theta)(\mathbf{0}, 3 \wedge 1) &:= (-\boldsymbol{e}_1, 2 \wedge 1) + (-\boldsymbol{e}_1 - 2\boldsymbol{e}_2 - \boldsymbol{e}_3, 2 \wedge 3) \\ E_2(\theta)(\mathbf{0}, 2 \wedge 3) &:= (-\boldsymbol{e}_1, 1 \wedge 3) + (-\boldsymbol{e}_1 - \boldsymbol{e}_2, 2 \wedge 3) + (-\boldsymbol{e}_1 - 2\boldsymbol{e}_2, 2 \wedge 3) \end{aligned}$$



$(\mathbf{0}, 1 \wedge 2)$ $(\mathbf{0}, 1 \wedge 3)$



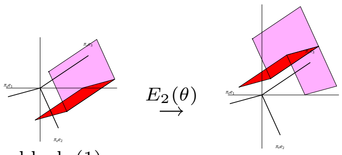
$(\mathbf{0}, 2 \wedge 3)$



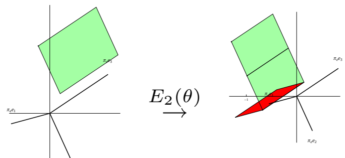
$E_2(\theta)^{12}(\mathbf{0}, 2 \wedge 3)$

In the case of type 2, $E_2(\theta)$ generates the patches including the negative faces. How shall we treat the negative faces?

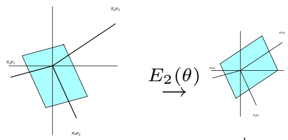
Let us introduce the blocking method on type 2 and we see that the blocking method works well on type 2 in general.



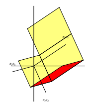
block (1) $\quad$ $(-\boldsymbol{e}_2 + \text{ block (1)}) + (\boldsymbol{e}_3 + \text{ block (3)})$



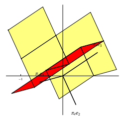
block (2) $\quad$ $-\boldsymbol{e}_2 - \boldsymbol{e}_3 +$ (block (1) + block (2))



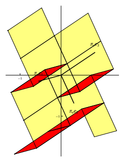
block (3) $\quad$ $\boldsymbol{e}_1 + \boldsymbol{e}_2 +$ block (2)



$\mathcal{U}_2$



$E_2(\theta)\mathcal{U}_2$



$E_2(\theta)^2\mathcal{U}_2$
(period=2)

Proposition 3.5. *On type 2, let us consider* $\mathcal{U}_2$*:*

$$\mathcal{U}_2 := \textit{ block (1)} + \textit{ block (2)} + \textit{ block (3)}$$

where

$$\begin{aligned}
\textit{block (1)} &:= (\boldsymbol{x} + \boldsymbol{e}_2, 1 \wedge 3) + (\boldsymbol{x}, 2 \wedge 3) + \sum_{k=1}^{-a-1} (\boldsymbol{x} - k\boldsymbol{e}_2, 2 \wedge 3);\\
\textit{block (2)} &:= (\boldsymbol{x} - \boldsymbol{e}_2, 2 \wedge 3);\\
\textit{block (3)} &:= (\boldsymbol{x}, 1 \wedge 2)
\end{aligned}$$

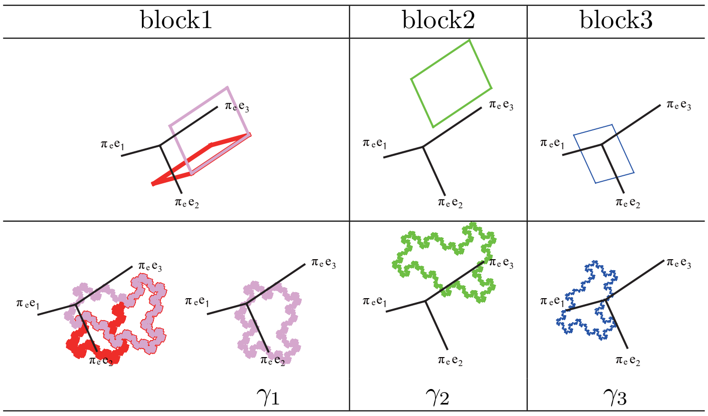
and $\boldsymbol{x}$ *is the solution of* $\boldsymbol{x} = A^2\boldsymbol{x} - \boldsymbol{e}_1 - \boldsymbol{e}_2$*. Then,* $E_2(\theta)$ *satisfies* $E_2(\theta)^2\mathcal{U}_2 \succ \mathcal{U}_2$.

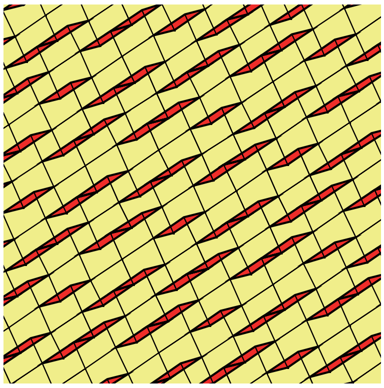
Moreover,

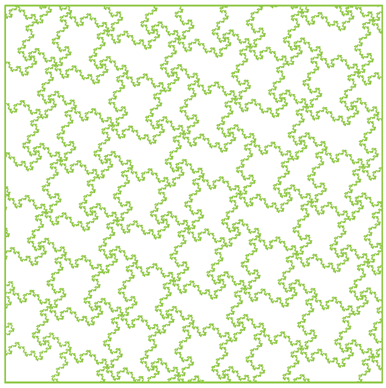
(1) $\mathcal{C}_2 := \left\{ \pi_e(\boldsymbol{y} + \text{ block } (i)) \;\middle|\; \begin{array}{l} (\boldsymbol{y} + \text{ block } (i)) \in E_2(\theta)^n(\text{block } (j)), \text{block } (j) \in \mathcal{U}_2, \\ \text{for some } j \in \{1,2,3\} \end{array} \right\}$ is the covering of P_e;

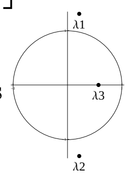
(2) $\gamma_i := \lim_{n\to\infty} A^{-2n}\pi_e E_2(\theta)^{2n}$(block (i)), then $(A, \mathcal{P}_2)$, $\mathcal{P}_2 = \{\gamma_1, \gamma_2, \gamma_3\}$ is the complex Pisot numeration system;

(3) $\widehat{\mathcal{C}_2} := \{\pi_e\boldsymbol{y} + \gamma_i \mid \pi_e(\boldsymbol{y} + \text{ block } (i)) \in \mathcal{C}_2\}$ is a *self-similar* tiling of P_e.



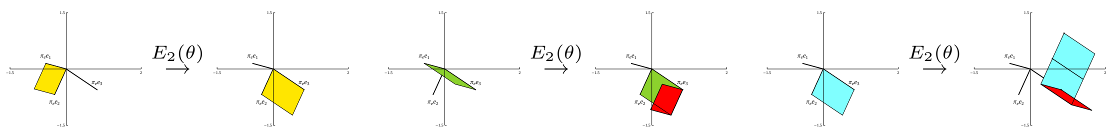




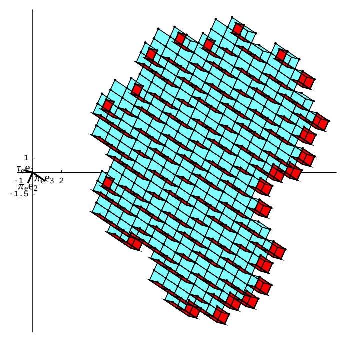
3.3. **The results on type 3.** Let us give an example in the case of $(a,b)=(1,-2)$ of type 3. $A=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ and its characteristic polynomial is x^3-x^2+2x-1. The distribution of the eigenvalues of A is 

. Then the automorphism $\theta : \begin{array}{ccc} 1 & \mapsto & 2 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 312^{-1}2^{-1} \end{array}$ is determined and $E_2(\theta)$ is given by

$$\begin{aligned} E_2(\theta)(\mathbf{0},1\wedge 2) &:= (\mathbf{0},2\wedge 3) \\ E_2(\theta)(\mathbf{0},1\wedge 3) &:= (\mathbf{0},2\wedge 3)+(\boldsymbol{e}_3,2\wedge 1) \\ E_2(\theta)(\mathbf{0},2\wedge 3) &:= (\boldsymbol{e}_3,3\wedge 1)+(\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3,2\wedge 3)+(\boldsymbol{e}_1-2\boldsymbol{e}_2+\boldsymbol{e}_3,2\wedge 3) \end{aligned}$$



In this case, $E_2(\theta)$ also generates patches including the *negative* unit faces. On this example, the blocking method doesn't seem to work well. But on the stepped plane, we can find θ' with P. Arnoux by the private discussion whose incidence matrix is same as the companion matrix and we succeed in finding the tiling substitution $\theta^* : \mathcal{G}_3 \to \mathcal{G}_3$, $\theta^* \neq E_2(\theta)$. However, it is unclear whether this method can apply all of type 3 in general.



The figure of of $E_2(\theta)^{10}(\mathbf{0},2\wedge 3)$.

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