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Boris Adamczewski, Anne Siegel and Wolfgang Steiner

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Valérie Berthé

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Discrete geometry and numeration

Valérie Berthé

Our aim is to show how discrete geometry and numeration systems interact in a natural way through the study of the most basic objects in discrete geometry, namely arithmetic discrete planes. Numeration systems are meant here in a wide sense as representation systems, which includes continued fraction expansions, classical positional number systems such as Ostrowski numeration, or else, some adic types of representations based on substitutions. In word combinatorics, Sturmian words and regular continued fractions are known to provide a very fruitful interaction between arithmetics, discrete geometry and symbolic dynamics. Sturmian words are infinite words over a two-letter alphabet which code irrational discrete lines. Most combinatorial properties of Sturmian words can be described in terms of the continued fraction expansion of the slope of the discrete line that they code. We want to extend this interaction to higher dimensions by working with multidimensional continued fraction algorithms. The idea is to find a suitable representation system for arithmetic discrete planes in S-adic terms, that is, by iterating a finite number of multidimensional substitutions. We then can deduce from this representation a generation method for discrete planes based on a numeration system associated with a multidimensional continued fraction. More precisely, we start with a multidimensional continued fraction algorithm formulated in terms of nonnegative unimodular matrices. We will illustrate this here with the Jacobi-Perron algorithm. We then interpret these unimodular matrices as incidence matrices of well-chosen substitutions. We finally use the formalism of generalized substitutions to associate with these unimodular matrices geometric maps acting on discrete planes.

1. S-adic expansion

A substitutions is a morphism of the free monoid that maps non-emtpy words onto non-empty words. Substitutions are widely used to generate infinite words (see e.g. [15, 14]). We consider here infinite words generated not only by iterating as usually a single substitution, but by taking successive compositions of an infinite sequence of substitutions as follows:

Définition 1. *A sequence $u \in \mathcal{A}^{\mathbb{N}}$ is said S-adic if there exist a finite set of substitutions $\mathcal{S}$ over an alphabet $\mathcal{D} = \{0, ..., d-1\}$, a morphism φ from $\mathcal{D}^\star$ to $\mathcal{A}^\star$, an infinite sequence of substitutions $(\sigma_n)_{n\geq 1}$ with values in $\mathcal{S}$ such that*

$$u = \lim_{n\to+\infty} \varphi \circ \sigma_1\sigma_2...\sigma_n(0).$$

For more on S-adic words, consider e.g. Chap. 11 in [14]. The terminology S-adic is inspired by Vershik [19] and enter the framework of arithmetic dynamics such as described in [18]. The aim of arithmetic dynamics is to provide explicit expansions of real numbers or of vectors which have a dynamical meaning in order to produce symbolic codings of dynamical systems which preserve their arithmetic structure.

Sturmian words are known to be S-adic. In particular, so-called characteristic Sturmian words are generated by the set $\mathcal{S} = \{\sigma_0, \sigma_1\}$ where $\sigma_0\colon 0 \mapsto 0, 1 \mapsto 10$ and $\sigma_1\colon 0 \mapsto 01,\ 1 \mapsto 1$ as

$$\lim_{n\to+\infty} \sigma_0^{a_1}\sigma_1^{a_2}\cdots\sigma_{2n}^{a_{2n}}\sigma_{2n+1}^{a_{2n+1}}(0)$$

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where the coefficients a_i are deduced from the continued fraction expansion of the slope. The S-adic expansion of general Sturmian words, which involve 4 substitutions, is governed by the Ostrowski expansion of the intercept. For more details, see [5, 13].

Having S-adic expansions allows us to define multidimensional continued fractions associated with some families of infinite words, like for instance the Arnoux-Rauzy words (for more details, see [14]). Conversely, we can start with a multidimensional continued fraction algorithm and associate with it an S-adic system. This is the viewpoint we take here and that will be developed in Section 2. We translate a continued fraction algorithm into S-adic terms. We then will give a multidimensional version of this process in Section 3.

2. S-adic expansions and continued fraction algorithms

In the multidimensional case, there is no canonical extension of the regular continued fraction algorithm. Several approaches have been proposed (see Brentjes [6] or Schweiger [17] for a summary). Here, we follow the definition of Lagarias [12], where multidimensional continued fraction algorithms produce sequences of matrices in $GL(d,\mathbb{Z})$.

Définition 2. *Let $X \subset \mathbb{R}^d$. A d-dimensional continued fraction map over X is a map $T : X \to X$ such that $T(X) \subset X$ and, for any $\mathbf{x} \in X$, there is $M(\mathbf{x}) \in GL(d,\mathbb{Z})$ satisfying:*

$$\mathbf{x} = M(\mathbf{x}).T(\mathbf{x}).$$

The associated continued fraction algorithm consists in iteratively applying the map T on a vector $\mathbf{x} \in X$. This yields the following sequence of matrices, called the continued fraction expansion of $\mathbf{x}$:

$$(M(T^n(\mathbf{x})))_{n\geq 1}.$$

If the matrices $M(\mathbf{x})$ have nonnegative entries, the algorithm is said nonnegative.

Consider for instance the Jacobi-Perron algorithm. Its projective version is defined on the unit square $[0,1) \times [0,1)$ by:

$$(\alpha,\beta) \mapsto \left(\frac{\beta}{\alpha} - \lfloor\frac{\beta}{\alpha}\rfloor, \frac{1}{\alpha} - \lfloor\frac{1}{\alpha}\rfloor\right) = (\{\beta/\alpha\},\{1/\alpha\}).$$

Its linear version is defined on the positive cone $X = \{(a,b,c) \in \mathbb{R}^3 | 0 \leq a, b < c\}$ by:

$$T(a,b,c) = (a_1,b_1,c_1) = (b - \lfloor b/a\rfloor a, c - \lfloor c/a\rfloor a, a).$$

Set $B = \lfloor b/a\rfloor a$, $C = \lfloor c/a\rfloor$. One has

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & B \\ 0 & 1 & C \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$$

The Jacobi-Perron algorithm is thus a unimodular nonnegative continued fraction algorithm.

Assume we are given a set of nonnegative matrices in $SL_3(\mathbb{N})$ which describes a continued fraction algorithm. Let us interpret these nonnegative unimodular matrices as transpose of incidence matrices of well-chosen substitutions. The incidence matrix $M = (m_{i,j})$ of a substitution σ is defined as $m_{i,j} = |\sigma(j)|_i$, where $|w|_i$ stands for the number of occurrences of the letter i in w. The fact that we take the transpose of these matrices will be explained in Section 3. Let us stress the fact that this choice is highly non-canonical.

We come back to the Jacobi-Perron algorithm as an illustration. The iteration of the algorithm yields a sequence of digits $(B_n, C_n)_{n\geq 1}$, with admissibility conditions

$$0 \leq B_n \leq C_n,\ C_n \geq 1, \quad \text{if } B_n = C_n \text{ then } B_{n+1} \neq 0. \tag{1}$$

Let B, C be nonnegative integers and C be positive. We set

$$\mathbf{M}_{B,C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & B & C \end{pmatrix}.$$

We consider the three-letter alphabet substitution $\sigma_{B,C}$ with incidence matrix $M_{B,C}$:

$$\sigma_{B,C}(1) = 3, \qquad \sigma_{B,C}(2) = 13^B, \qquad \sigma_{B,C}(3) = 23^C.$$

This choice for the substitution $\sigma_{B,C}$ yields convergence for the underlying S-adic process. Indeed, for any sequence of Jacobi-Perron digits $(B_n, C_n)_{n\geq 1}$, with $0 \leq B_n, C_n$ and $C_n \geq 1$ for all n, $\lim_{n\to+\infty} \sigma_{B_1,C_1}\sigma_{B_2,C_2}...\sigma_{B_n,C_n}(1)$ is easily seen to exist and to coincide with the limit of the same sequence of iterations starting from the letter 2 or 3. We use the fact that $\sigma_{B,C}\circ\sigma_{B,C'}\circ\sigma_{B'',C''}$ has a primitive incidence matrix and that the image of any letter starts by itself (provided that $CC'C'' \neq 0$).

We thus define the Jacobi-Perron S-adic system as the set of infinite words generated in an S-adic way based on the set of substitutions $\mathcal{S} = \{\sigma_{B,C} \mid B, C \in \mathbb{N},\ C \geq 1,\ B \leq C\}$. First, note that this set is infinite but the Jacobi-Perron algorithm can be made easily into an additive algorithm yielding a finite set of substitutions. Second, we have to restrict ourselves to the admissibility conditions (1) for the applications using the arithmetic properties of Jacobi-Perron algorithm.

3. Multidimensional substitutions

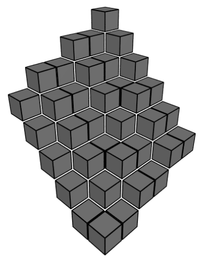
We now want to translate these S-adic systems associated with multidimensional continued fractions into the formalism of generalized substitutions due to [1]. Before recalling this formalism, we first introduce the notion of a discrete approximation of a hyperplane, according to [16]. We here approximate a hyperplane of normal vector $\mathbf{x}$.

Définition 3. *Let $\mathbf{x}$ be a strictly positive vector in $\mathbb{R}^d$ and let $h \in \mathbb{R}$. The (standard) arithmetic discrete plane $\mathcal{P}_{(\mathbf{x},h)}$ is defined as*

$$\mathcal{P}_{(\mathbf{x},h)} = \{\mathbf{y} \in \mathbb{Z}^d \mid 0 < \langle \mathbf{x}, \mathbf{y}\rangle + h \leq ||\mathbf{x}||_1\}.$$

The stepped plane $\mathfrak{P}_{(\mathbf{x},h)}$ is defined as the union of the facets of integer translates of unit cubes whose set of integer vertices equals $\mathcal{P}_{(\mathbf{x},h)}$

This definition is illustrated below.



Arithmetic discrete planes are classic objects in discrete geometry that are widely studied. We recover Sturmian words when $d = 2$ by using the Freeman code which codes in the stepped line $\mathfrak{P}_{(\mathbf{x},h)}$ horizontal steps and vertical steps by two different letters. Note that there is no multidimensional continued fraction algorithm that occurs in a natural way in the study of arithmetic discrete planes.

Our aim is to represent in an S-adic way the arithmetic discre plane $\mathcal{P}_{(\mathbf{x},h)}$. We first need to be able to define a multidimensional version of the notion of substitution.

We have two methods at our disposal. We can work by induction of the underlying $\mathbb{Z}^2$-action by translations on $\mathbb{R}/\mathbb{Z}$ such as described in [2], or else, we can use the formalism of generalized substitutions introduced in [1]. By choosing a suitable interval of induction of $\mathbb{R}/\mathbb{Z}$, both notions are proved to coincide (see [2]).

The formalism introduced by [1] works as follows. It is defined by duality with respect to a geometric extension of the notion of substitution to unions of segments in $\mathbb{R}^d$ located at integers points. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denote the canonical basis of $\mathbb{R}^3$. For $\mathbf{x} \in \mathbb{R}^3$, let $(\mathbf{x}, 1^*)$, $(\mathbf{x}, 2^*)$, $(\mathbf{x}, 3^*)$ stand for the following facets of integer translates of the unit hypercube:

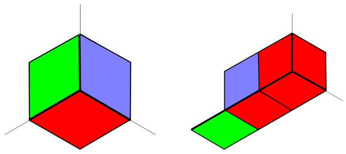
$$\begin{aligned}(\mathbf{x}, 1^*) &= \{\mathbf{e}_1 + \lambda\mathbf{e}_2 + \mu\mathbf{e}_3,\ (\lambda,\mu) \in [0,1[^2\}\\ (\mathbf{x}, 2^*) &= \{\mathbf{e}_2 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_3,\ (\lambda,\mu) \in [0,1[^2\}\\ (\mathbf{x}, 3^*) &= \{\mathbf{e}_3 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2,\ (\lambda,\mu) \in [0,1[^2\}.\end{aligned}$$

A substitution σ is said unimodular if the determinant of its incidence matrix equals ± 1. We define the abelianization map $\mathbf{l} : \mathcal{A}^\star \to \mathbb{N}^d$, $\mathbf{l}(w) = {}^t(|w|_1, |w|_2, \cdots, |w|_d)$ where $|w|_i$ stands for the number of occurrences of the letter i in the word w.

Let σ be a unimodular substitution. The map E_1^* acts on unions of facets in a morphic way with repect to the union and is defined on facets as

$$E_1^*(\sigma)(\mathbf{x}, i^*) = \sum_{k \in \mathcal{A}} \sum_{P,\, \sigma(k) = PiS} (\mathbf{M}_\sigma^{-1} (\mathbf{x} - \mathbf{l}(P)), k^*).$$

We illustrate this definition with the image by $E_1^*(\sigma_{1,2})$ of the upper unit cube



One key property of generalized substitutions is that they map arithmetic discrete planes onto arithmetic discrete planes [1, 8]. Indeed, let σ be a unimodular substitution with incidence matrix M. One has

$$E_1^*(\sigma)\mathcal{P}_{(\mathbf{x},h)} = \mathcal{P}_{({}^tM\mathbf{x},h)}. \tag{2}$$

This explains why we have chosen in Section 2 to take substitutions having as incidence matrix the transpose of the matrices produced by a multidimensional continued fraction algorithm.

Let us come back to the Jacobi-Perron algorithm. We assume $h = 0$ for the sake of clarity and use the notation $\mathcal{P}_{(a,b,c)}$ for $\mathcal{P}_{((a,b,c),0)}$. Let $(a, b, c) \in X$ and let $(a_n, b_n, C_n) = T^n(a, b, c)$ for all n. We deduce from (2) that

$$E_1^*(\sigma_{B,C})(\mathcal{P}_{(a_1,b_1,c_1)}) = \mathcal{P}_{(a,b,c)}.$$

Assume now we are given the sequence of Jacobi-Perron digits $(B_n, C_n)_{n\geq 1}$. One has

$$E_1^*(\sigma_{(B_1,C_1)}) \circ E_1^*(\sigma_{(B_2,C_2)}) \cdots \circ E_1^*(\sigma_{(B_n,C_n)})(\mathcal{P}_{(a_n,b_n,c_n)}) = \mathcal{P}_{(a,b,c)}.$$

We now use the fact that the upper unit cube $\mathcal{U} = (\mathbf{0}, 1^*) + (\mathbf{0}, 2^*) + (\mathbf{0}, 3^*)$ belongs to every arithmetic discrete plane with parameter $h = 0$. We thus deduce that

$$E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U}) \subset \mathcal{P}_{(a,b,c)}.$$

The question is now whether the patterns $E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U})$ generate the whole plane $\mathcal{P}_{(a,b,c)}$, that is, whether

$$\lim_{n\to\infty} E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U}) = \mathcal{P}_{(a,b,c)}.$$

If yes, then we get an S-adic type representation of the arithmetic discrete plane $\mathcal{P}_{(a,b,c)}$. Note that this question is a substitutive counterpart of the Finiteness Property in beta-numeration, which states that every $x \in \mathbb{Z}[1/\beta] \cap [0, 1)$ has a finite β-expansion. For more details, see [4].

This question has been answered in the Jacobi-Perron case in [10]. Indeed, it has been proved in [10] that there exists a finite set of facets $\mathcal{V}$ with $\mathcal{U} \subset \mathcal{V}$ such that if there exists n such that for all k

$$B_{n+3k} = C_{n+3k},\ C_{n+3k+1} - B_{n+3k+1} \geq 1,\ B_{n+3k+2} = 0,$$

then the sequence of patterns $E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{V})$ generates the whole plane $\mathcal{P}_{(a,b,c)}$. Otherwise, the sequence of patterns $E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U})$ generates the whole plane $\mathcal{P}_{(a,b,c)}$. This result is proved thanks a tedious graph study which is not likely to be easily handled for other multidimensional continued fraction algorithms such as Brun algorithm.

Question Assume we are given a nonnegative unimodular continued fraction algorithm T. Characterize the expansions of vectors $\mathbf{x}$ for which the associated generalized S-adic system generates the whole arithmetic discrete plane $\mathcal{P}_{\mathbf{x}}$ starting from the unit cube $\mathcal{U}$.

Consider now a vector $\mathbf{x}$ for which the associated generalized S-adic system generates the whole arithmetic discrete plane $\mathcal{P}_{\mathbf{x}}$. It is natural to consider the topological porperties of the shape of the corresponding patterns. Let us recall the following result of [7]. Let σ be an invertible three-letter substitution (by invertible, we mean that σ considered as a morphism of the free group is an automorphism). The boundary of $E_1^*(\sigma)(\mathcal{U})$ is given by the mirror image of the inverse of σ.

We thus can apply this result to get a description of the boudary of the generating patterns and to deduce various topological properties such as connectedness, dislikeness etc.

Let us stress that we are not only able to substitute, i.e., to replace facets of hypercubes by unions of facets, but also to desubstitute, i.e., to perform the converse operation, by using the algebraic property $E_1^*(\sigma) = E_1^*(\sigma^{-1})$. For more details see [3] which deals with Brun algorithm. This is used in [9], where original algorithms for both digital plane recognition and digital plane generation problems. The discrete plane recognition problem can be stated as follows: given a set of points in $\mathbb{Z}^d$, does there exist an arithmetic discrete plane that contains it? This question is classical and central in the field of discrete geometry for the segmentation of discrete surfaces and for polyhedrization issues, for instance. Indeed, numerous applications can be derived in image analysis and synthesis, volume modeling, pattern recognition, etc.

Note that we have not taken yet into account the parameter h of Definition 3 in our discussion. By considering a skew product of the Jacobi-Perron algorithm, it is possible to define a numeration system playing the role of Ostrowski's numeration which allows us to define an S-adic expansion for $\mathcal{P}_{(\mathbf{x},h)}$.

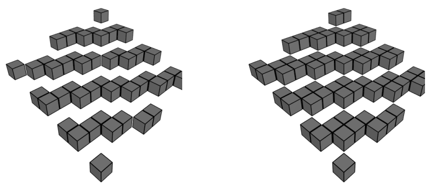
Let us end with a nice question issued from discrete geometry.

Question Find the smallest width ω for which the plane

$$\mathcal{P}_{\vec{x},h,\omega} = \{\mathbf{y} \in \mathbb{Z}^d \mid 0 < \langle \vec{x}, \vec{y} \rangle + h \leq \omega\}$$

is connected (either edge connected or vertex connected).

The case of rational parameters has been solved in [11]. Examples of a disconnected arithmetic discrete plane (on the left) and of a vertex connected one (on the right) are depicted below.



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