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An overview of some recent developments on integer-valued polynomials: Answers and Questions

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An overview of some recent developments on integer-valued polynomials: Answers and Questions

Jean-Luc Chabert

Abstract

The purpose of my talk is to give an overview of some more or less recent developments on integer-valued polynomials and, doing so, to emphasize that integer-valued polynomials really occur in different areas: combinatorics, arithmetic, number theory, commutative and non-commutative algebra, topology, ultrametric analysis, and dynamics. I will show that several answers were given to open problems, and I will raise also some new questions.

1. Introduction

First introduced in algebraic number theory at the beginning of the 20th century, the theory of integer-valued polynomials has been developed in the context of commutative algebra from the 1970s (*cf.* [67, Historical Introduction]). Shortly before the Second Meeting on Integer-Valued Polynomials [68], held in Marseille in 2000, Manjul Bhargava [9] introduced the notion of P-ordering, so useful for both the study of integer-valued polynomials and the construction of normal bases of ultrametric spaces of continuous functions. Once more, shortly before this Third Meeting [69], Bhargava [12] gave ingenious extensions of this notion in order to construct polynomial normal bases of several spaces of regular ultrametric functions. I will speak about these results and also about several other recent results, specially those that will not be considered by other participants.

In fact, the real aim of my talk is to show that integer-valued polynomials really occur in very different areas: combinatorics, arithmetic, number theory, commutative and non-commutative algebra, topology, ultrametric analysis, and dynamics. Without forgetting that the topic of integer-valued polynomials interests some mathematicians specialized in constructive mathematics (*cf.* Lombardi's contribution in these *Proceedings*) and others who initiate a category-theoretic approach to the subject (*cf.* Elliott's contribution in these *Proceedings*). I will take the opportunity of this overview to recall some questions (numbered from A to K) raised a few years ago in Cortona [31] and to speak about answers that were made during these last years. Of course, new questions appeared and some answers are already given by the participants during this meeting.

Notation. Recall that, if D is an integral domain with quotient field K and E is a subset of K, then $\mathrm{Int}(E, D)$ denotes the D-algebra of integer-valued polynomials on E:

$$\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}.$$

Usually, we write $\mathrm{Int}(D)$ instead of $\mathrm{Int}(D, D)$.

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2. Algebraic Number Theory: Pólya fields

In the field of algebraic number theory, I just want to recall the notion of Pólya field and some related questions. It is well known that the notion of integer-valued polynomials goes back to works of Pólya [57] and Ostrowski [53] that they published in 1919 in the *Journal für die Reine und Angewandte Mathematik*.

Let us recall that, for any fixed number field K with ring of integers $\mathcal{O}_K$, they considered the *integer-valued polynomials* on K, that is, the polynomials with coefficients in K which take integral values on the integers of K. These polynomials form an $\mathcal{O}_K$-algebra that now, following Robert Gilmer [42], we denote by $\mathrm{Int}(\mathcal{O}_K)$:

$$\mathrm{Int}(\mathcal{O}_K) = \{f \in K[X] \mid f(\mathcal{O}_K) \subseteq \mathcal{O}_K\}.$$

It is well known that the binomial polynomials:

$$\binom{X}{n} = \frac{X(X-1)\dots(X-n+1)}{n!} \quad (n \in \mathbb{N})$$

form a basis of the $\mathbb{Z}$-module:

$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

Although Pólya and Ostrowski did not know whether this $\mathcal{O}_K$-module is free or not, they asked whether this $\mathcal{O}_K$-module admits or not a *regular basis*, that is, a basis formed by a sequence of polynomials with one and only one polynomial of each degree, as in the classical case of $\mathrm{Int}(\mathbb{Z})$.

After these seminal works, nothing was done on this question of regular bases for sixty years until Zantema's work in 1982. He introduced the name of *Pólya field* for the number fields K such that the $\mathcal{O}_K$-module $\mathrm{Int}(\mathcal{O}_K)$ admits a regular basis, proved, for instance, that every cyclotomic field is a Pólya field, and characterized the quadratic number fields which are Pólya fields. In 1988, Van der Linden [61] began a similar study for function fields, largely completed only recently, in 2008 by David Adam [4] with respect to the analogs in positive characteristic of cyclotomic fields and of totally imaginary cyclic extensions.

Going back to characteristic zero, very recent works by Amandine Leriche extend Zantema's results [48] and answer positively [49] to the following question (*cf.* Leriche's contribution in these *Proceedings*):

Question I. *Is every number field contained in a Pólya field?*

This question, in some sense, is similar to the embedding problem of a number field in another number field with class number one. From my point of view, there is still an interesting question:

Question I*. *Is there a natural definition for what could be called the* Pólya closure *of a number field?*

Of course, both questions may be raised in positive characteristic. In the context of algebraic number theory, let us recall another question which in some sense is linked to the previous one:

Question F. *Are there number fields* (*distinct from* $\mathbb{Q}$) *which are Newtonian fields?*

Recall that the notion of Newtonian field is stronger than those of Pólya field: not only there are regular bases $\{f_n\}_{n\geq 0}$, but also bases of a particular type since the f_n's are of the form $\alpha_n \prod_{k=0}^{n-1}(X-u_k)$ for some sequence $\{u_n\}_{n\geq 0}$ of integers of K (analogously to the sequence of integers for $\mathbb{Z}$). More generally, about the determination of Newtonian sequences in number fields or function fields, see [3], [4], [20], [63] and Adam and Cahen's contribution in these *Proceedings*.

3. Commutative algebra: counterexamples

Introduced in number theory, the study of integer-valued polynomials has greatly flourished and developed in the context of commutative algebra. Now, I just want to show that rings of integer-valued polynomials are a great source of counterexamples.

Clearly, a domain which is the intersection of rank-one valuation domains is completely integrally closed. Krull raised the question of the converse and, in 1942, Nakayama answered negatively, yet with an *ad hoc* counterexample rather complicated. There is a more natural example: for every prime number p, the ring

$$\mathrm{Int}(\mathbb{Z}_p) = \{f \in \mathbb{Q}_p[X] \mid f(\mathbb{Z}_p) \subseteq \mathbb{Z}_p\}$$

of integer-valued polynomials on the p-adic integers is a completely integrally closed domain, which is not an intersection of rank-one valuation domains [26].

Along the line of simple or natural examples, the case of $\mathrm{Int}(\mathbb{Z})$ is exemplary. Indeed, in algebra the natural constructions lead most often to Noetherian rings, as is generally the case with polynomial rings, unlike functional spaces in analysis. However, the ring $\mathrm{Int}(\mathbb{Z})$ provides a natural algebraic example of non-Noetherian ring.

Being non-Noetherian, the ring $\mathrm{Int}(\mathbb{Z})$ is not a Dedekind domain, but it is a two-dimensional Prüfer domain. More generally, for every number field K, the ring $\mathrm{Int}(\mathcal{O}_K)$ is a Prüfer domain ([26] and [17]). As new examples of Prüfer domains, the rings of integer-valued polynomials were used in 2002 by Brewer and Klinger [16] to test whether every two-dimensional Prüfer domain has the stacked basis property, and in particular raised the following question:

Question K. *Does the two-dimensional Prüfer domain* $\mathrm{Int}(\mathbb{Z})$ *have the stacked basis property?*

Let us explain why this question is interesting. It is well known that the structure theorem for finitely generated abelian groups may be extended to free finitely generated module over a principal ideal domain, and more generally, to modules over a Dedekind domain. One may ask whether there exists such a generalization for modules over Prüfer domains.

Definition 3.1. [41, V§4] The Prüfer domain D is said to have *the stacked basis property* if, for every free D-module M with finite rank m and for every finitely generated submodule N of M with rank $n \leq m$, there exist rank-one projective D-modules $P_1, \ldots, P_m$ and nonzero ideals $\mathfrak{I}_1, \ldots, \mathfrak{I}_n$ of D such that:

$$M = P_1 \oplus \cdots \oplus P_m \quad , \quad N = \mathfrak{I}_1 P_1 \oplus \cdots \oplus \mathfrak{I}_n P_n$$

and

$$\mathfrak{I}_{j+1} \subseteq \mathfrak{I}_j \quad \text{for} \quad 1 \leq j \leq n-1.$$

One knows that the one-dimensional Prüfer domains have the stacked basis property. Brewer and Klinger proved that, for every prime p, the two-dimensional Prüfer domain $\mathrm{Int}(\mathbb{Z}_p)$ has the stacked basis property, but they could not conclude in the global case. This question is both ways interesting: if the answer is no, then $\mathrm{Int}(\mathbb{Z})$ would be the first example of a Prüfer domain which does not have the stacked basis property, if it yes, it would then be another interesting property of this strange ring $\mathrm{Int}(\mathbb{Z})$.

This problem may be translated in terms of matrices [41, V.Thm 4.8] and was recently reduced to a question concerning matrices with only two columns [29]:

Question K*. *Does there exists, for every matrix* $B \in \mathfrak{M}_{m\times 2}(\mathrm{Int}(\mathbb{Z}))$ *with unit content, a matrix* $C \in \mathfrak{M}_{2\times 2}(\mathrm{Int}(\mathbb{Z}))$ *such that* BC *has unit content and all* 2×2 *minors of* BC *are zero.*

There are other conjectures in commutative algebra concerning the rings of integer-valued polynomials. For instance, with respect to Prüfer domains, one knows that, if V is a rank-one valuation domain with quotient field K, then the ring $\mathrm{Int}(E, V) = \{f \in K[X] \mid f(E) \subseteq V\}$ is a Prüfer domain provided E is a precompact subset of K (that is, the completion $\widehat{E}$ of E is compact) [23, Thm. 4.1]. But does the converse hold [23]? More precisely,

Question L. *Let* V *be a rank-one valuation domain and* E *be a subset of* V*. To what extend is the precompactness of* E *necessary for* $\mathrm{Int}(E, V)$ *to be Prüfer?*

Another question concerns the dimension of $\mathrm{Int}(D)$ where D denotes any integral domain. Recall that, for every integral domain D, we have ([18] and [37]):

$$\dim(D[X]) - 1 \leq \dim(\mathrm{Int}(D)) \leq \dim_v(D) + 1,$$

where $\dim_v(D)$ denotes the valuative dimension of D, that is,

$$\dim_v(D) = \sup\{\dim V \mid D \subseteq V \subseteq K, \ \ V \text{ valuation domain}\},$$

or equivalently,

$$\dim_v(D) = \lim_{n\to+\infty} \dim(D[X_1, \ldots, X_n]) - n.$$

It is conjectured that the answer to the following question is positive.

Question M. *Does the following inequality hold*

$$\dim(\mathrm{Int}(D)) \leq \dim D[X]\,?$$

For many references on this question, see [38].

4. Non-commutative Algebras: Quaternions and Matrices

Integer-valued polynomials have been introduced also in non-commutative algebra. G. Gerboud was the first one to consider integer-valued polynomials on quaternions. He begun his study in the 1990's for the ring $\mathbb{H}$ formed by the Hurwitz quaternions. But things seems to be really difficult and the most interesting results that he obtained are about factorials (factorials will be defined in the next section) like this one:

$$n!_{\mathbb{H}} \neq (1) \Leftrightarrow n \geq 4 \quad \text{and} \quad 4!_{\mathbb{H}} = \left(\frac{1+i}{2}\right).$$

Thus, we raised the question:

Question A. *Describe the integer-valued polynomials on quaternions.*

Very recently, Nicholas Werner [62] published a fine and extensive study on this topic, but far from being complete. Let us recall some of his results. For every subring D of $\mathbb{R}$, let $\mathcal{Q}D$ denote the ring of quaternions with coefficients in D :

$$\mathcal{Q}D = \{a + bi + cj + dk \mid a, b, c, d \in D\}.$$

He considers the set:

$$\text{Int}(\mathcal{Q}\mathbb{Z}) = \{f \in \mathcal{Q}\mathbb{Q}[X] \mid f(\mathcal{Q}\mathbb{Z}) \subseteq \mathcal{Q}\mathbb{Z}\}.$$

Obviously, this set is a left and right $\mathcal{Q}\mathbb{Z}$-module. In fact, Werner shows that this a ring and this is not obvious because generally :

$$f(w)g(w) \neq (fg)(w) \quad \text{for } f, g \in \mathcal{Q}\mathbb{Q}[X],\ w \in \mathcal{Q}\mathbb{Z}.$$

In the case of integers, the *Fermat polynomial*:

$$F_p(X) = \frac{1}{p}(X^p - X)$$

is clearly integer-valued on $\mathbb{Z}$. The analogous result for quaternions is that the *quaternionic Fermat polynomial*:

$$\mathcal{Q}F_p(X) = \frac{1}{p}(X^{p^2} - X)(X^p - X)$$

belongs to $\text{Int}(\mathcal{Q}\mathbb{Z})$. Although Werner proved that it is not possible to have p in the denominator of an integer-valued polynomial on $\mathcal{Q}\mathbb{Z}$ with degree less than the degree of $\mathcal{Q}F_p$, the following question remains open:

Question A*. *Find a system of generators of the $\mathcal{Q}\mathbb{Z}$-module* $\text{Int}(\mathcal{Q}\mathbb{Z})$.

Let us look now at the spectrum. In the case of $\text{Int}(\mathbb{Z})$, one knows [26] that the prime ideals lying over a prime number p are the ideals

$$\mathfrak{p}_\alpha = \{f \in \text{Int}(\mathbb{Z}) \mid f(\alpha) \in p\mathbb{Z}_p\} \quad \text{where } \alpha \in \mathbb{Z}_p.$$

Very few is known for $\text{Int}(\mathcal{Q}\mathbb{Z})$. For instance, following Werner, for an odd prime number p and for an element a of $\mathcal{Q}\mathbb{Z}$, the subset

$$\mathfrak{p}_a = \{f \in \text{Int}(\mathcal{Q}\mathbb{Z}) \mid f(a), f(-iai), f(-jaj), f(-kak) \in p\mathcal{Q}\mathbb{Z}\}$$

is an ideal, but the fact that $\mathfrak{p}_a$ is a prime ideal depends on a. So that, the following question is wide open:

Question A**. *Characterize the spectrum of* $\text{Int}(\mathcal{Q}\mathbb{Z})$.

And also:

Question A***. *Same questions than Questions A^* and A^{**} replacing $\mathcal{Q}\mathbb{Z}$ by the ring $\mathbb{H}$ of Hurwitz quaternions.*

There are other possible studies in the non-commutative case: for instance, the integer-valued polynomials on matrices whose study has been undertaken by Sophie Frisch [40] (*cf.* Frisch's related contribution in these *Proceedings*).

5. Combinatorics: Bhargava's factorials

Bhargava's factorials associated to a subset of $\mathbb{Z}$ lead us to combinatorics. He introduced these factorials at the end of the last century [11]. One way to define them is the following: if

$$\mathrm{Int}(S,\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(S) \subseteq \mathbb{Z}\}$$

denotes the ring of integer-valued polynomials on S, then the n-th factorial ideal of S is the generator of the ideal formed be the denominators of the integer-valued polynomials on S with degree n, that is,

$$(n!_S) = \{d \in \mathbb{Z} \mid df \in \mathbb{Z}[X] \ \ \forall f \in \mathrm{Int}(S,\mathbb{Z}), \deg(f) = n\}.$$

For the subset $\mathbb{Z}$ itself, we get the classical factorials:

$$n!_{\mathbb{Z}} = n!.$$

For the subset $\mathbb{P}$ formed by the prime numbers, we have [32]:

$$n!_{\mathbb{P}} = \prod_{p\in\mathbb{P}} p^{\sum_{k\geq 0}\left[\frac{n-1}{(p-1)p^k}\right]} \quad (n>0)$$

and the sequence of factorials $\{n!_{\mathbb{P}}\}_{n\geq 1}$ begins with

$$1, 2, 24, 48, 5760, 11520, ...$$

In 2000, Bhargava [11] noticed that there are strong links between this sequence and products of the denominators δ_n of $\frac{B_n}{n}$ where B_n denotes the n-th Bernoulli number:

$$(2m+1)!_{\mathbb{P}} = 2^{2m} \prod_{1\leq k\leq m} \delta_k \text{ and } (2m+2)!_{\mathbb{P}} = 2(2m+1)!_{\mathbb{P}}.$$

Question B. *Explain these links between the $n!_{\mathbb{P}}$'s and the B_n's.*

In a recent paper [27], we prove the following formula:

$$\left(-\frac{\ln(1-x)}{x}\right)^m = 1 + \frac{m}{2}x + \frac{m(3m+5)}{24}x^2 + \frac{m(m^2+5m+6)}{48}x^3 + \dots$$

which shows links between $n!_{\mathbb{P}}$ and the denominators of the *Bernoulli polynomials*. For other formulas using $n!_{\mathbb{P}}$ and the Bernoulli numbers, see Bencherif's contribution in these *Proceedings*.

Moreover, looking at *The On-Line Encyclopedia of Integer Sequences*, we see that $(n+1)!_{\mathbb{P}}$ is also the least common multiple of the orders of all finite subgroups of $GL_n(\mathbb{Q})$ (*cf.* Minkovski [52] (1887) and Schur [58] (1905)). And, following Johnson [44], this sequence also occurs in Algebraic Topology: as the denominators of the Laurent polynomials forming a regular basis for K*K, the Hopf algebroid of stable cooperations for complex K-theory. For other links to algebraic topology, see for instance [59] and Clarke's contribution in these *Proceedings*.

For other computations in combinatorics concerning Bhargava's factorials, see [15], [45], [46], and Johnson's contribution in these *Proceedings*. Here is another interesting result due to Mingarelli generalizing a property of the number e.

Proposition 5.1. [51, Theorem 3.3] *For every infinite subset S of $\mathbb{Z}$, denoting by e_S the sum*

$$e_S = \sum_{k=0}^{\infty} \frac{1}{k!_S},$$

then e_S is an irrational number.

Let us end this section with another elementary question:

Question M. *Characterize the sequences of integers $\{k_n\}_{n\geq 0}$ which are sequences of factorials $\{n!_S\}_{n\geq 0}$ for some subsets S of $\mathbb{Z}$.*

6. Topology: the polynomial closure

Let D be an integral domain with quotient field K. Let us recall:

Definition 6.1. [19] Let S be a subset of K.

(1) The *D-polynomial closure* of S is the largest subset $\overline{S}$ of K such that:

$$\forall f \in K[X] \quad f(S) \subseteq D \Rightarrow f(\overline{S}) \subseteq D.$$

(2) The subset S is said *D-polynomially closed* if $\overline{S} = S$.

The most recent works on polynomial closure are due to M.H. Park [55] and F. Tartarone [60]. It is but natural to ask the following:

Question N. *Is the polynomial closure a topological closure?*

In other words, when are the D-polynomially closed subsets of K the closed subsets for some topology on K?

As noticed by P.-J.Cahen [19], if such a polynomial topology exists, then the domain D has to be local. In view of our question, let us recall the p-adic analog of the Stone-Weierstrass approximation theorem:

Proposition 6.2. [47] *Let K be a valued field, that is, a field endowed with a rank-one valuation v. For every compact subset S of K, $K[X]$ is dense in the Banach space $\mathcal{C}(S, K)$ of continuous functions from S to K for the uniform convergence topology.*

It follows from this proposition that the compact subsets of K are polynomially closed. Moreover, it is easy to verify that the polynomially closed subsets are topologically closed and that the polynomial closure of every non-bounded subset is K. As a consequence, if the valued field K is locally compact (namely, if K is a local field, that is, a field K endowed with a discrete valuation v which is complete for the corresponding topology and with a finite residue field), the polynomially closed subsets are K itself and the compact subsets of K, and hence, the polynomial closure is a topological closure (which corresponds to the topology of the Alexandrov compactification of K). It was then a natural question to ask whether this is also true for any valued field K.

S. Frisch [39] proved that the polynomial closure correspond to a topology when the valuation of the valued field K is discrete whatever the cardinality of the residue field. Very recently, by means of a generalization of the notion of pseudo-convergence introduced by Ostrowski [54], we were able to extend the result to all valued fields:

Proposition 6.3. [28] *In any valued field, the polynomial closure (corresponding to the valuation domain) is a topological closure. This polynomial topology is the topology spanned by the complements of the closed balls.*

7. Entire functions: Pólya's theorem in characteristic p

In 1915, Pólya [56] proved that every entire function f on $\mathbb{C}$ such that

$$f(\mathbb{N}) \subseteq \mathbb{Z} \quad \text{and} \quad \limsup_{r\to+\infty} \frac{\ln |f|_r}{r} < \ln 2\,, \quad \text{where } |f|_r = \sup_{|z|\leq r} |(f(z)|,$$

is a polynomial (and hence, belongs to $\mathrm{Int}(\mathbb{Z})$.) Morever, the fonction 2^z shows that the bound $\ln 2$ is sharp.

Replacing the subset $\mathbb{N}$ by $\mathbb{Z}$, Pólya obtained also the same result with the bound $\ln\left(\frac{3+\sqrt{5}}{2}\right)$ instead of $\ln 2$. In 1933, Gel'fond gave an analogous result for the subset $\{q^n \mid n \in \mathbb{N}\}$ where q is an integer ≥ 2. In 1980, Gramain [43] extended Pólya's theorem to entire functions f such that $f(\mathcal{O}_K) \subseteq \mathcal{O}_K$ where K denotes an imaginary quadratic field. Very recently, Ably [1] generalized Gramain's result. But the recent results in which we are interested here concern analogs in positive characteristic.

Let us first fix some notation. Let $\mathbb{F}_q[T]$ be a finite field with q elements. Let $\mathbb{F}_q((T^{-1}))$ be the completion of $\mathbb{F}_q(T)$ for the $\frac{1}{T}$-adic valuation and let Ω be the completion of an algebraic closure of $\mathbb{F}_q((T^{-1}))$ for the extension v of the $\frac{1}{T}$-adic valuation to this algebraic closure. We know that Ω

is not only complete but also algebraically closed. The degree with respect to $\frac{1}{T}$ which is defined on $\mathbb{F}_q(T)$ extends to Ω letting $\deg(z) = -v(z)$.

The ring $\mathbb{F}_q[T]$ is the analog of the ring $\mathbb{Z}$, the field $\mathbb{F}_q(T)$ is the analog of $\mathbb{Q}$, $\mathbb{F}_q((T^{-1}))$ the analog of $\mathbb{R}$, and Ω the analog of $\mathbb{C}$.

A first result along that line was given in 1997 by Mireille Car [24] for the set $\mathbb{F}_q[T]$. Car's bound was improved first in 2003 by Delamette [35], then in 2004 by David Adam whose bound is known to be optimal thanks to an example previously given by Car.

Proposition 7.1. [2] *Let f be an entire function on Ω such that*

$$f(\mathbb{F}_q[T]) \subseteq \mathbb{F}_q[T] \quad \textit{and} \quad \limsup_{r\to+\infty} \frac{M(f,r)}{q^r} < \frac{1}{e\ln q}$$

where

$$M(f,r) = \sup_{\deg(z)\leq r} \deg(f(z)).$$

Then, f is a polynomial, and hence, f belongs to $\mathrm{Int}(\mathbb{F}_q[T])$. *Moreover, the bound $\frac{1}{e\ln q}$ is sharp.*

Replacing $\mathbb{F}_q[T]$ by the subset S formed by the powers of a non-constant polynomial, Adam [2] obtained also an optimal result. It is worth noticing the proofs are obtained thanks to a good choice of bases for the $\mathbb{F}_q[T]$-modules $\mathrm{Int}(\mathbb{F}_q[T])$ and $\mathrm{Int}(S, \mathbb{F}_q[T])$ (*cf.* [25]).

Pólya's theorem was extended in 1946 by Pisot to almost integer-valued functions. Analogous results were obtained in positive characteristic by Adam and Hirata-Kohno [6] in 2006.

8. Ultrametric analysis (1^{st} part): polynomial approximation of continuous functions

In this section, we still talk about power series, but more specifically about p-adic power series. Mahler gave an explicit description of the p-adic approximation of continuous functions by polynomials (see Proposition 6.2) when $K = \mathbb{Q}_p$ and $S = \mathbb{Z}_p$:

Proposition 8.1. [50] *Every continuous function $\varphi \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$ may be written as a binomial series:*

$$\varphi(x) = \sum_{n\geq 0} a_n \binom{x}{n} \quad \textit{where } a_n \in \mathbb{Q}_p \textit{ and } v_p(a_n) \to \infty.$$

Moreover,

$$\inf_{x\in\mathbb{Z}_p} v_p(\varphi(x)) = \inf_{n\in\mathbb{N}} v_p(a_n)$$

One says that the sequence formed by the binomial polynomials $\left\{\binom{x}{n} \mid n \geq 0\right\}$ is a *normal basis* of the Banach space $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$.

Mahlher's result has been extended by replacing the field $\mathbb{Q}_p$ by any local field K, so that K is locally compact, and $\mathbb{Z}_p$ by a compact subset S of K. The first extension was made by Amice [7] but with a condition on the compact subset (it must be regular in some sense defined by her). It was then proved by Bhargava and Kedlaya [14] for any compact subset S of K, thanks to Bhargava's notion of v-ordering. Finally, Bhagarva and Kedlaya's result was extended [22] to every compact subset S of any valued field K (the valuation needs not be discrete, nor the residue field be finite).

In order to explain these extensions of Mahler's result, let us recall the notion of v-ordering introduced by Bhargava:

Definition 8.2. [9] A *v-ordering* of S is any sequence $\{a_n\}_{n\geq 0}$ of elements of S such that:

$$\forall n \geq 1 \quad v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) = \inf_{x\in S} v\left(\prod_{k=0}^{n-1}(x - a_k)\right).$$

Such a sequence always exists when v is discrete or when S is compact and may be contructed inductively (the first element being arbitrarily choosen).

Assuming that $\{a_n\}_{n\geq 0}$ is a v-ordering of S, we define the *generalized binomials* on S, as follows:

$$\binom{X}{n}_S = \prod_{k=0}^{n-1} \frac{X-a_k}{a_n-a_k} \quad (n\geq 0).$$

The sequence of the generalized binomials is then a basis of the V-module $\mathrm{Int}(S,V)$ of integer-valued polynomials on S. In fact it is a normal basis of the Banach space of continuous functions $\mathcal{C}(S,K)$:

Proposition 8.3. [22] *Let S be any compact subset of any valued field K and assume that $\{a_n\}_{n\geq 0}$ is a v-ordering of S. Then, every $\varphi \in \mathcal{C}(S,K)$ can uniquely be expanded in the form:*

$$\varphi = \sum_{n\geq 0} b_n \binom{X}{n}_S \quad \textit{with } b_n\in K \textit{ and } v(b_n)\to\infty.$$

Moreover,

$$\inf_{v\in V} v(\varphi(x)) = \inf_{n\in\mathbb{N}} v(b_n).$$

I recalled the previous results (essentially from the end of the last century) in order to introduce the next section.

9. Ultrametric analysis (2^{nd} part): polynomial approximation of regular functions

During the second meeting on integer-valued polynomials which held in Marseille in 2000, Manjul Bhargava suggested possible extensions of the notion of v-ordering in order to study some spaces of analytic functions. However, it is only shortly before this third meeting that he published in the *Journal of the American Mathematical Society* a very interesting paper entitled *On p-orderings, integer-valued polynomials and ultrametric analysis* where he gave ingenious extensions of the notion of v-ordering.

He is interested in two kinds of regular functions on a compact subset S of a local field K. First, the Banach space of locally analytic functions of order h, that is, functions whose restrictions on balls of radius e^{-h} may be written as power series. Julie Yeramian ([64] and [65]) begun this study in the case where S is the valuation domain V of K. Bhargava studied the general case of a compact subset S.

In fact, I will talk on the second kind of regular functions as it seems to me to be the more interesting one: the Banach space $\mathcal{C}^r(S,K)$ of r-times continuously differentiable functions on S. Before doing that, I have to precise what is a r-times continuously differentiable function in the ultrametric case.

For every $f: S\to K$, define

$$\Phi(f) : S\times S\setminus\Delta \to K \quad \text{where } \Delta = \{(x,x)\mid x\in S\}$$

by

$$\Phi(f)(x,y) = \frac{f(x)-f(y)}{x-y}.$$

One says that $f\in\mathcal{C}^1(S,K)$ if $\Phi(f)$ may be extended continuously to $S\times S$. Then, in particular, $f'(x) = \Phi(f)(x,x)$ for every $x\in S$. More generally, one defines finite differences of order r, $\Phi^r(f)$, by induction on r. For instance,

$$\Phi^2(f)(x,y,z) = \frac{\Phi(f)(x,y)-\Phi(f)(x,z)}{y-z}$$

($\Phi^2(f)$ is defined on $S^3\setminus\Delta$ where $\Delta = \{(x,y,z)\mid x=y \text{ or } y=z \text{ or } z=x\}$.) Then, $f\in\mathcal{C}^2(S,K)$ means that $\Phi^2(f)$ may be extended continuously to S^3. And so on.

If we restrict to polynomials, we obtain polynomials that are integer-valued together with their finite differences, and may define the following sets (in fact rings):

$$\mathrm{Int}^{\{1\}}(S,V) = \{f\in\mathrm{Int}(S,V)\mid \Phi(f)\in\mathrm{Int}(S^2,V)\}$$

$$\mathrm{Int}^{\{2\}}(S,V) = \{f\in\mathrm{Int}^{\{1\}}(S,V)\mid \Phi^2(f)\in\mathrm{Int}(S^3,V)\}$$

ad so on ...

Barsky [8] gave bases of the V-module $\mathrm{Int}^{\{1\}}(V)$. But, we did not have any formulas for bases of $\mathrm{Int}^{\{r\}}(S,V)$ for $r \geq 2$ or $S \neq V$.

With respect to this notion, we must notice that we considered here finite differences instead of divided differences [as Barsky], and that the corresponding rings of integer-valued polynomials differ for $r \geq 2$ [13].

Bhargava had the brilliant idea of introducing the following notion:

Definition 9.1. (Bhargava [12]). A *r-removed v-ordering* of S is a sequence $\{a_n\}_{n\geq 0}$ of elements of S such that : the first $r+1$ elements $a_0, a_1, \ldots, a_r$ being arbitrarily chosen, a_n is recursively defined, for $n > r$, by the condition

$$\inf_{0\leq i_1<\ldots<i_r<n} v\left(\prod_{0\leq k<n,\, k\neq i_1,\ldots,i_r}(a_n - a_k)\right) = \inf_{x\in S} \quad \inf_{0\leq i_1<\ldots<i_r<n} v\left(\prod_{0\leq k<n,\, k\neq i_1,\ldots,i_r}(x - a_k)\right).$$

Such a sequence is not unique but, as for v-orderings, at each step, this minimum does not depend on the elements $a_0, a_1, \ldots, a_{n-1}$. We denote this minimum by $w_S^{\{r\}}(n)$. If we choose an element π such that $v(\pi) = 1$, we define the *r-removed factorials* by

$$n!_S^{\{r\}} = \pi^{w_S^{\{r\}}(n)} \quad (n \geq 0).$$

Finally, if $\{a_n\}$ is an r-removed v-ordering, we may define an *r-removed generalized binomial polynomial* by

$$\binom{X}{n}_S^{\{r\}} = \frac{(X-a_0)(X-a_1)\ldots(X-a_{n-1})}{n!_S^{\{r\}}}.$$

Proposition 9.2. [12, Theorems 7 and 17] *Let S be a compact subset of a local field K. If $\{a_n\}_{n\geq 0}$ is an r-removed v-ordering of S, then the corresponding r-removed generalized binomial polynomials $\binom{X}{n}_S^{\{r\}}$ form:*
– a basis of the V-module $\mathrm{Int}^{\{r\}}(S,V)$,
– a normal basis of the Banach space $\mathcal{C}^r(S,K)$.

Question O. *For particular subsets of valued fields, are there explicit formulas that give the r-removed factorials?*

For instance, are there formulas like Legendre's formula for the r-removed factorials of $\mathbb{Z}_p$? Moreover, if there are such formulas, are there global formulas for the r-removed factorials of $\mathbb{Z}$:

$$n!_{\mathbb{Z}}^{\{r\}} = \prod_{p\in\mathbb{P}} p^{w_p^{\{r\}}(n)} \quad ?$$

For a quick answer concerning $\mathbb{Z}_p$, see Johnson's contribution in these *Proceedings*. For a partial answer to Question O, see [33].

10. Dynamical systems

First, I will show the links between dynamical systems, v-orderings and ergodicity. For every prime p, the successor function σ on $\mathbb{Z}$:

$$\sigma : k \in \mathbb{Z} \mapsto k+1 \in \mathbb{Z}$$

may be extended by continuity to the ring of p-adic integers $\mathbb{Z}_p$ and this extension $\sigma_p : \mathbb{Z}_p \to \mathbb{Z}_p$ is an isometry of $\mathbb{Z}_p$ for the p-adic valuation v_p. This isometry σ_p is ergodic with respect to the Haar measure μ_p on $\mathbb{Z}_p$. Recall that μ_p is characterized by $\mu_p(B(x,p^{-r})) = \frac{1}{p^r}$. Recall also that to say that σ_p is ergodic for μ_p means first that the map σ_p preserves μ_p:

$$\forall S \subset \mathbb{Z}_p \qquad [\mu_p(\sigma_p(S)) = \mu_p(S)]$$

and secondly that

$$\forall S \subset \mathbb{Z}_p \quad [(\sigma_p(S) = S) \Rightarrow (\mu_p(S) = 0 \text{ or } 1)]\,.$$

Moreover, one knows (for instance, see [34]) that every isometry τ of $\mathbb{Z}_p$ which is ergodic with respect to the Haar measure μ_p is conjugate to σ_p, that is, there exists an isometry $h : \mathbb{Z}_p \to \mathbb{Z}_p$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\sigma_p} & \mathbb{Z}_p \\ h\big\downarrow & & \big\downarrow h \\ \mathbb{Z}_p & \xrightarrow{\tau} & \mathbb{Z}_p \end{array}$$

These results may be generalized by replacing the field $\mathbb{Q}_p$ of p-adic numbers by a local field or, more generally, by any complete valued field K. We may also replace the ring $\mathbb{Z}_p$ of p-adic integers by a compact subset S of K whose factorial ideals satisfy a formula very similar to the Legendre's formula for classical factorials:

$$(*) \qquad v(n!_S) = n\gamma_0 + \sum_{k\geq 1} \left[\frac{n}{q_{\gamma_k}}\right](\gamma_k - \gamma_{k-1})$$

where $\{\gamma_k\}_{k\geq 0}$ denotes the increasing sequences or real numbers γ such that $\gamma = v(x-y)$ for some $x, y \in S$, and where q_{γ_k} denotes the cardinality of the set of residue classes of S modulo the ideal $\{x \mid v(x) \geq \gamma\}$. This sets generalize the Legendre sets introduced by Evrard and Fares [36], which themselves generalize Amice's regular compact sets [7]. We then have the following assertions:

Proposition 10.1. [30] *For every compact subset S of a complete valued field K which satisfies $(*)$, there exist:*
– a measure μ on S characterized by

$$\forall x \in S \; \forall k \geq 0 \quad \mu(\{y \in S \mid v(x-y) \geq \gamma_k\}) = \frac{1}{q_{\gamma_k}}$$

– sequences $\mathbf{b} = \{b_n\}_{n\geq 0}$ of elements of S such that, for each $k \geq 0$, the sequence $\{b_n\}_{n\geq k}$ is a v-ordering of S.

Such a sequence $\mathbf{b} = \{b_n\}$ may be considered as defining a successor function in S that can be extended to an isometry $\sigma_{\mathbf{b}}$ of S. Then,
1- the isometry $\sigma_{\mathbf{b}}$ is ergodic for the measure μ of S,
2- every isometry of S which is ergodic for μ is associated to such a sequence,
3- all these isometries are conjugate.

Now, I end with a curious example. First recall the following (and last) question:

Question E. *Find natural subsets of $\mathbb{Z}$ which admit simultaneous orderings, that is, sequences which are p-orderings for every prime p.*

One knows the following examples:

$$\mathbb{Z}, \{n\}_{n\geq k}, \{n\}_{n\leq k}, \{q^n\}_{n\geq 0}, \{n^2\}_{n\geq 0}, \left\{\frac{n(n+1)}{2}\right\}_{n\geq 0}$$

and all the subsets that one can deduce from them by a linear function. I ask for natural subsets because it is always possible to construct ad hoc subsets by a recursive choice of the elements. Dynamical systems provide an answer:

Proposition 10.2. [5] *Let $f \in \mathrm{Int}(\mathbb{Z})$ and consider the dynamical system $(\mathbb{Z}, f)$. Then, for every $x \in \mathbb{Z}$, the orbit $\{x, f(x), f(f(x)), \ldots, f^n(x), \ldots\}$ admits a simultaneous ordering, namely, the sequence $\{f^n(x)\}_{n\geq 0}$ itself, equivalently:*

$$\forall n \leq m \quad \prod_{j=0}^{n-1} \frac{f^m(X) - f^j(X)}{f^n(X) - f^j(X)} \in \mathrm{Int}(\mathbb{Z}).$$

Application. The orbit of 3 under the iteration of the quadratic polynomial $f(X) = X^2 - 2X + 2$ admits a simultaneous ordering. This orbit is the set

$$\{F_n = 2^{2^n} + 1 \mid n \geq 0\}$$

formed by the Fermat numbers.

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