Déviations pour les temps locaux d'auto-intersections

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Abstracts
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Abstracts

1. Mini-courses (5 h)

1.1. **Erwin Bolthausen: "On the volume of the intersection of two Wiener sausages".**

The lectures will give an overview of the attempts to derive the exact $x \to \infty$ behavior of

$$P\left(\left|R^{(1)} \cap R^{(2)}\right| \geq x\right)$$

where $R^{(1)}, R^{(2)}$ are the ranges of two independent infinite length random walks in dimension $d \geq 5$, and on the similar question for the Wiener sausage. The conjecture is that

$$P\left(\left|R^{(1)} \cap R^{(2)}\right| \geq x\right) = \exp\left[-c\,(d)\,x^{(d-2)/d}\right],$$

with a fairly explicitely known constant. Although, there is a fair amount of evidence for the conjecture, there still is no proof for it.

First lecture: The Khanin-Mazel-Sinai-Shlosman result [3] on the intersection of the ranges of two independent random walks. Detailed proof. Introduction into the results of the paper [1]. Outline of the Donser-Varadhan approach to the volume of the Wiener sausage.

Second lecture: Presentation of some of the key arguments in [1]: The large deviation result. The "Swiss cheese picture". The concentration of measure argument. Derivation of the upper bound.

Third lecture: Discussion of the analytical properties of the variational problem, including a proof of the important leakage of mass for $d \geq 5$. Introduction into the two-path problem analyzed in [2]: The upper bound.

Forth lecture: Continuation of the analysis of the two-path problem. Solution of the analytic variational problem leading to the variational formula for $c\,(d)$. Discussion of the open infinite time-horizon problem.

References

[1] van den Berg, M., Bolthausen, E., and den Hollander, F.: *Moderate deviations for the volume of the Wiener sausage.* Ann. Math. **153**, 355-406 (2001)
[2] van den Berg, M., Bolthausen, E., and den Hollander, F.: *On the volume of the intersection of two Wiener sausages.* Ann. Math. **159**, 741-783 (2004)
[3] Khanin, K.M., Mazel, A.E., Shlosman, S.B., and Sinai, Ya. G.: *Loop condensation effects in the behavior of random walks,* in "The Dynkin Festschrift" (M. Freidlin, ed.), Progr. Probab. **34**, Birkhäuser, Boston, 1994, pp. 167-184

1.2. **Xia Chen: "Large deviations for the intersection local times of Brownian motions and random walks".**

The sample path intersection has long been of interest to physicists and mathematicians. It presents a physically relevant model for real world phenomena such as random polymers and quantum field. On the other hand, its analysis has provided mathematical challenges. Thus studying the behavior of intersection local times and related functionals is both physically relevant and often requires a variety of new mathematical ideas. In this lecture we focus on the large deviation problems arising from this area.

The self-intersection local time

$$Q_n = \#\{(j,k);\quad 1 \leq j < k \leq n \text{ and } S(j) = S(k)\}$$

of the random walk $S(n)$ on $\mathbb{Z}^d$ measures the self-intersection of the random trajectory $\{S(1),\cdots,S(n)\}$; while the range defined as

$$R_n = \#\{S(1),\cdots,S(n)\}.$$

The self-intersection of a single random walk is closed related to the mutual intersection between two independent and identically distributed random walks $S(n)$ and $\widetilde{S}(n)$. The relevant quantities in mutual intersection are mutual intersection local time

$$I_n = \#\{(j,k)\in[1,n];\ S(j)=\widetilde{S}(k)\}$$

and the range intersection

$$J_n = \#\Big(\{S(1),\cdots,S(n)\}\cap\{\widetilde{S}(1),\cdots,\widetilde{S}(n)\}\Big).$$

The notion of the intersection local times can be extended from random walks to Brownian motions in the "sub-critical dimensions" (which mean different things for different type of intersections). Compared with their discrete counterparts, the Brownian intersection local times need a more technical setup. On the other hand, there are more analytical tools available to deal with continuous case. In the spirit of invariance principle, the study in the Brownian cases turns out to be a crucial step toward the later success in the discrete cases, in addition to the importance for its own sake.

This lecture presents some progresses recently made on the large deviations for sample path intersections and addresses the following topics:

1. Definition and renormalization of Brownian intersection local times.
2. Le Gall's moment identity and method of high moment asymptotics
3. LDP by Feynman-Kac formula.
4. Large deviations for Q_n, R_n, I_n, J_n and the relations among these results.
5. Unsolved problems.

References

Bolthausen, E. (1999). Large deviations and interacting random walks. *Lecture Notes in Math.* **1781** 1-124.

Chen, X. *Random Walk Intersections: Large Deviations and Related Topics.* Math. Surv. Mono. **157**, Providence 2009.

Le Gall, J-F. (1992). Some properties of planar Brownian motion. *École d'Été de Probabilités de Saint-Flour XX. 1990. Lecture Notes in Math* **1527** 111-235. Springer, Berlin.

2. Lectures (45 min)

2.1. **Frank den Hollander: "On the collision local time of two transient random walks".** In Birkner, Greven and den Hollander (2010), a quenched large deviation principle (LDP) was established for the empirical process of words obtained by cutting an i.i.d. sequence of letters into words according to a renewal process. This LDP can be applied to prove that the radius of convergence of the moment generating function of the collision local time of two independent copies of a symmetric and strongly transient random walk on $\mathbb{Z}^d$ with $d \geq 1$, both starting from the origin, strictly increases when we condition on one of the random walks, both in discrete time and in continuous time. The same is expected to hold when the random walk is transient but not strongly transient. The presence of these gaps implies the existence of an intermediate phase for the long-time behaviour of a class of coupled branching processes, interacting diffusions, respectively, directed polymers in random environments.

2.2. **Michiel van den Berg: "Asymptotics of the heat exchange and some conjectures of M. V. Berry".** Let K be a compact subset in Euclidean space $\mathbb{R}^m$, and let $E_K(t)$ denote the total amount of heat in $\mathbb{R}^m \backslash K$ at time t, if K is kept at fixed temperature 1 for all $t \geq 0$, and if $\mathbb{R}^m \setminus K$ has initial temperature 0. For two disjoint compact subsets K_1 and K_2 we define the heat exchange $H_{K_1,K_2}(t) = E_{K_1}(t) + E_{K_2}(t) - E_{K_1 \cup K_2}(t)$. We obtain the leading asymptotic behaviour of $H_{K_1,K_2}(t)$ as $t \to 0$ under mild regularity conditions on K_1 and K_2. We discuss some conjectures of M. V. Berry and show how the renormalised heat content of a region D in $\mathbb{R}^m$ determines the length of the shortest closed periodic geodesic in D.

2.3. **Yvan Velenik: "Percolation with a line of defects".** We consider the Bernoulli bond percolation process on the d-dimensional lattice, with occupation probability p for all edges, except those along one line (say, the first coordinate axis), for which the occupation probability is p'.

We assume that p is below the critical value, $p < p_c(d)$. We first prove that the probability that two vertices x and y along the line are connected decays exponentially fast in $|y - x|$ for any value of $p' < 1$. Denote by $\xi_{p,p'}$ the corresponding rate of exponential decay and write $\xi_p = \xi_{p,p}$. Since $\xi_{p,p'} = \xi_p$, for all $p' \leq p$, we can define $p'_c(p, d) = \sup\{p' : \xi_{p,p'} = \xi_p\}$.

We prove that, in dimensions 1, 2 and 3, $p'_c = p$, while in higher dimensions $p < p'_c < 1$. We also analyze the behavior of $\xi_p - \xi_{p,p'}$ as p' decreases toward p in dimensions 1, 2 and 3.

Finally, we derive sharp asymptotics of pure exponential form for the probability of connecting two distant vertices along the line, valid for all $p < p_c$ and all $p' > p'_c$.

Results of the type above are well-known for effective models, in which the central object (here the cluster connecting two distant vertices along the line) is replaced by the path of a random walk and the pinning potential by a suitable exponential functional of the local time at the origin. As far as we know, our results constitute the first analysis of such problems beyond the framework of effective models (apart from the exact computations in the 2D Ising model that triggered the study of these effective models).

Based on joint work with Sacha Friedli and Dmitry Ioffe.

3. Short talks

3.1. **Anne-Laure Basdevant: "Vertex reinforced jump processes on Galton-Watson trees".** A vertex reinforced jump process on a tree is a continuous time random walk which jumps to a neighbouring vertex with rate proportional to the local time at that vertex plus a constant. In this talk, we give a criterion to determine whether the walk is recurrent or transient depending on the parameters of the model.

3.2. **Achim Klenke: "Multiple Non-Intersection Exponents of Planar Brownian Motion".** Consider the paths of three planar Brownian motions killed when they leave the unit circle. It is well known that all three paths intersect (i.e., they produce triple points) if all motions are started at the origin. What is the probability p_R that the paths do *not* intersect when started at random positions at the boundary of a small circle with radius R?

It is shown that

$$p_R \approx R^{\xi} \qquad \text{as } R \downarrow 0,$$

where ξ is the so-called non-intersection exponent. The exact numerical value is unknown and is estimated via a Monte-Carlo simulation. For a slightly more general setup with different numbers of Brownian motions, we obtain rigorous inequalities, precise results and/or approximate values via simulations.

The situation is quite different (in terms of the quality of results) from the one with double points. Here, the corresponding exponents for the non-intersection of m Brownian motions of one type with n Brownian motions of the other type are known exactly due to work of Lawler, Werner and Schramm. (Joint work with Peter Mörters.)

3.3. **Alexandre Gaudillière: "Fluctuation for internal DLA".** We show logarithmic and sub-logarithmic fluctuations with respect to a spherical asymptotic shape for internal DLA in dimension two and larger than or equal to three respectively.