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ABSOLUTELY-p-SUMMING OPERATORS IN $\mathcal{L}_r$-SPACES

by

Albrecht PIETSCH

The purpose of this paper is to give a uniform presentation of all known results about absolutely-p-summing operators in $\mathcal{L}_r$-spaces.

§ 1. ABSOLUTELY-p-SUMMING OPERATORS (cf. [11]).

Let E and F be Banach spaces. We denote by $\mathcal{L}(E,F)$ the set of all bounded linear operators from E into F. An operator $T \in \mathcal{L}(E,F)$ is called absolutely-p-summing ($1 \leq p < \infty$) if there exists a constant $\rho \geq 0$ such that for every finite set of elements $x_1, \ldots, x_m \in E$ the inequality

$$\{\sum_i \|T x_i\|^p\}^{1/p} \leq \rho \sup_{\|a\| \leq 1} \{\sum_i |\langle x_i, a\rangle|^p\}^{1/p}$$

holds. The set $\mathcal{P}_p(E,F)$ of all absolutely-p-summing operators $T \in \mathcal{L}(E,F)$ is a Banach space with norm defined by

$$\pi_p(T) := \inf \rho .$$

It is convenient to put

$$\mathcal{P}_\infty(E,F) := \mathcal{L}(E,F) \quad \text{and} \quad \pi_\infty(T) := \|T\|$$

If $1 \leq p \leq q < \infty$ then

$$\mathcal{P}_p(E,F) \subset \mathcal{P}_q(E,F) \quad \text{and} \quad \pi_p(T) \geq \pi_q(T) .$$

§ 2. THE $\mathcal{L}_r$-SPACES (cf. [8]).

In the following let l_r^n be the Banach space of all n-dimensional real vectors $x = (\xi_i)$ with the norm

$$\|x\|_r := \{\sum_i |\xi_i|^r\}^{1/r} \quad \text{for } 1 \leq r < \infty \quad \text{and} \quad \|x\|_\infty := \sup_i |\xi_i| \; .$$

A real Banach space E is called an $\mathcal{L}_r$-space if for every finite set of elements $x_1,\dots,x_m \in E$ there exist operators $A \in \mathcal{L}(E,l_r^n)$ and $X \in \mathcal{L}(l_r^n,E)$ such that

$$\|x_i - X\,A\,x_i\| \leq 1 \quad \text{for } i=1,\dots,m \quad \text{and} \quad \|X\|\,\|A\| \leq c_E \; ,$$

where the constant $c_E \geq 1$ depends only on E. We note that our definition is a slightly weaker than the definition of J. Lindenstrauss and A. Pełczynski.

All function spaces $L_r(S,\Sigma,\mu)$ are of type $\mathcal{L}_r$. The operators A and X can be constructed as follows. Given $x_1,\dots,x_m \in L_r(S,\Sigma,\mu)$ we find step functions $x_1^o,\dots,x_m^o \in L_r(S,\Sigma,\mu)$ such that $\|x_i - x_i^o\|_r \leq 1/2$. Then there exist disjoint subsets $S_1,\dots,S_n \in \Sigma$ with $\mu(S_i) > 0$ such that the step functions $x_1^o,\dots,x_m^o$ are linear combinations of the corresponding characteristic functions $f_1,\dots,f_n$. Now we define the operators A and X by

$$A\,x := (\mu(S_k)^{-1/r'} \int_S x(s)\, f_k(s)\, d\mu(s))$$

and

$$X(\xi_k) := \sum_k \xi_k\, \mu(S_k)^{-1/r}\, f_k \; .$$

Then we have

$$\|A\| = \|X\| = 1$$

and since $X\,A\,f_k = f_k$ the estimate

$$\|x_i - X\,A\,x_i\| \leq \|x_i - x_i^o\| + \|X\,A\,x_i^o - X\,A\,x_i\| \leq 1$$

holds.

Now we show that it is possible to reduce the considerations of absolutely-p-summing operators in $\mathcal{L}_r$-spaces to finite dimensional l_r^n-spaces.

Proposition : The following statements are equivalent :

(1) There exists a constant $c_{rs,pq} > 0$ such that

$$\pi_p(T) \leq c_{rs,pq}\, \pi_q(T) \quad \text{for all} \quad T \in \mathcal{L}(l_r^n, l_s^n) \text{ and } n = 1,2,\ldots .$$

(2) For every $\mathcal{L}_r$-space L_r and every $\mathcal{L}_s$-space L_s the inclusion

$$\mathfrak{P}_q(L_r, L_s) \subset \mathfrak{P}_p(L_r, L_s)$$

holds.

Proof : (1) → (2) Let $T \in \mathfrak{P}_q(E,F)$ and $x_1,\ldots,x_m \in E$. Then for all $\varepsilon > 0$ there exist $A \in \mathcal{L}(E, l_r^n)$, $X \in \mathcal{L}(l_r^n, E)$, $B \in \mathcal{L}(F, l_s^n)$, and $Y \in \mathcal{L}(l_s^n, F)$ such that

$$\|x_i - X A x_i\| \leq \varepsilon , \ \|T x_i - Y B T x_i\| \leq \varepsilon, \ \|X\| \, \|A\| \leq c_E, \text{ and } \|Y\| \, \|B\| \leq c_F.$$

Then

$$\|Tx_i\| \leq \|T x_i - Y B T x_i\| + \|Y B T x_i - Y B T X A x_i\| + \|Y B T X A x_i\|$$

$$\leq \varepsilon(1 + c_F \|T\|) + \|Y B T X A x_i\| ,$$

$$\{\sum_i \|Y B T X A x_i\|^p\}^{1/p} \leq \pi_p(Y B T X A) \sup_{\|a\| \leq 1} \{\sum_i |\langle x_i, a\rangle|^p\}^{1/p} ,$$

and

$$\pi_p(Y B T X A) \leq \|Y\| \, \pi_p(B T X) \, \|A\| \leq c_{rs,pq} \|Y\| \, \pi_q(B T X) \, \|A\|$$

$$\leq c_{rs,pq} \|Y\| \, \|B\| \, \pi_q(T) \, \|X\| \, \|A\| \leq c_{rs,pq}\, c_E\, c_F\, \pi_q(T).$$

Consequently,

$$\{\sum_i \|T x_i\|^p\}^{1/p} \leq$$

$$\varepsilon(1 + c_F \|T\|) m^{1/p} + c_{rs,pq} c_E c_F \pi_q(T) \sup_{\|a\| \leq 1} \{\sum_i |\langle x_i, a\rangle|^p\}^{1/p}.$$

If $\varepsilon \to 0$, we obtain

$$\pi_p(T) \leqslant c_{rs,pq} c_E c_F \pi_q(T) \quad \text{and} \quad T \in \mathfrak{P}_p(E,F).$$

(2) → (1). Since the sequence space l_r, resp. l_s, is of type $\mathcal{L}_r$, resp. $\mathcal{L}_s$, we have

$$\mathfrak{P}_q(l_r, l_s) \subset \mathfrak{P}_p(l_r, l_s) \ .$$

Consequently, by the closed graph theorem there exists a constant $c_{rs,pq} > 0$ such that

$$\pi_p(T) \leqslant c_{rs,pq} \pi_q(T) \quad \text{for all} \quad T \in \mathfrak{P}_q(l_r, l_s).$$

Let us consider the operators

$$Q_n(\xi_1, \dots, \xi_n, \dots) := (\xi_1, \dots, \xi_n)$$

and

$$J_n(\xi_1, \dots, \xi_n) := (\xi_1, \dots, \xi_n, 0, \dots) \ .$$

Then for every $T \in \mathcal{L}(l_r^n, l_s^n)$ since $T = Q_n(J_n \, T \, Q_n) J_n$ we have

$$\pi_p(T) \leqslant \pi_p(J_n \, T \, Q_n) \leqslant c_{rs,pq} \pi_q(J_n \, T \, Q_n) \leqslant c_{rs,pq} \pi_q(T).$$

§ 3. HISTORICAL REMARKS.

The first result about absolutely-p-summing operators in $\mathcal{L}_r$-spaces goes back to A. Grothendieck [5] who showed in 1956 that all bounded linear operators from an $\mathcal{L}_1$-space into an $\mathcal{L}_2$-space are absolutely-1-summing. A simplified proof of this important results was given by J. Lindenstrauss and A. Pełczynski [8].

In 1967, A. Pełczynski [9] and A. Pietsch [11] proved that the absolutely-p-summing operators in Hilbert spaces coincide with the Hilbert-Schmidt operators. This proof used Chintchin's inequality for Rademacher functions. Finally, D.J.H. Garling [3] determined the exact value of the π_p-norm of diagonal operators in l_2.

Important progress was made in 1969, when L. Schwartz [13], [14], [15] remarked, in his theory of p-radonifying operators, that it is possible to use in place of Rademacher functions general sequences of independant and equidistributed random variables. By his method S. Kwapien [7] and P.Saphar [12] proved the fundamental theorems on absolutely-p-summing operators in $\mathcal{L}_r$-spaces.

§ 4. A PROBABILITY LEMMA.

For $1 \leqslant s \leqslant 2$ let μ_s be the probability measure on the real line which is uniquely determined by its characteristic function

$$e^{-|\alpha|^s} = \int_{\mathbb{R}} e^{i\alpha\beta} \, d\,\mu_s(\beta) \; .$$

If $1 \leqslant s < 2$ and $1 \leqslant p < s$ or if $s = 2$ and $1 \leqslant p < \infty$ then the moments

$$c_{sp} : = \{ \int_{\mathbb{R}} |\beta|^p \, d\,\mu_s(\beta) \}^{1/p} > 0$$

exist (cf. [4]).
Let μ_s^n be the n-dimensional product measure of μ_s then the following probability lemma holds. It was used in functional analysis at first by J. Bretagnolle, D. Dacunha-Castelle and J. D. Krivine [1].

<u>Lemma</u> : <u>If $y \in \mathbb{R}^n$ then</u>

$$\{ \int_{\mathbb{R}^n} | < y, b > |^p d\, \mu_s^n(b) \}^{1/p} = c_{sp} \, \|y\|_s .$$

<u>Proof</u> : We consider on the probability space $[\mathbb{R}^n, \mu_s^n]$ the independant random variables

$$f_i(b) : = \beta_i \quad \text{for} \quad i = 1, \ldots, n.$$

Then

$$\hat{f}_i \, (\alpha) = e^{-|\alpha|^s} \; ,$$

where $\hat{f}_i$ is the characteristic function of f_i.

Consequently, the random variable

$$f(b) : = < y, \, b > = \sum_i \eta_i \, f_i(b)$$

has the characteristic function

$$\hat{f}(\alpha) = e^{-\|y\|_s^s \; |\alpha|^s} \; .$$

The same characteristic function corresponds to the random variable

$$\varphi(\beta) : = \|y\|_s \; \beta$$

which is defined on the probability space $[\mathbb{R},\mu_s]$.
Therefore, the two random variables f and φ are equidistributed and we have

$$\int_{\mathbb{R}^n} |\langle y,b\rangle|^p \, d\,\mu_s^n(b) = \int_{\mathbb{R}} |\beta|^p d\mu_s(\beta) \; \|y\|_s^p \, .$$

§ 5. ABSOLUTELY-p-SUMMING OPERATORS IN l_r^n-SPACES.

We begin the central part of this paper with some few lemmata.

Lemma 1 : Let $T \in \mathfrak{L}(E,l_s^n)$. If $1 < s < 2$ and $1 \leqslant p < s$ or if $s = 2$ and $1 \leqslant p < \infty$ then

$$\pi_p(T) \leqslant c_{sp}^{-1} \{ \int_{\mathbb{R}^n} \|T' b\|^p d\,\mu_s^n(b) \}^{1/p} \, .$$

Proof : It follows from the probability lemma that if $x_1,\dots,x_m \in E$ then

$$\{ \sum_i \|T x_i\|_s^p \}^{1/p} = c_{sp}^{-1} \{ \int_{\mathbb{R}^n} \sum_i |\langle T x_i \, b\rangle|^p d\,\mu_s^n(b)\}^{1/p}$$

$$\leqslant c_{sp}^{-1} \{ \int_{\mathbb{R}^n} \|T'b\|^p d\,\mu_s^n(b) \}^{1/p} \sup_{\|a\| \leqslant 1} \{ \sum_i |\langle x_i,a\rangle|^p \}^{1/p} \, .$$

Lemma 2 : Let $T \in \mathfrak{L}(l_{s'}^n,F)$. If $1 < s < 2$ and $1 \leqslant p < s$ or if $s = 2$ and $1 \leqslant p < \infty$ then

$$c_{sp}^{-1} \{ \int_{\mathbb{R}^n} \|T x\|^p \, d\,\mu_s^n(x) \}^{1/p} \leqq \pi_p(T).$$

Proof : The main theorem of absolutely-p-summing operators (cf. [8], [11]) implies that there exists a measure μ on the closed unit ball U_s^n of l_s^n such that

$$\|T x\| \leqslant \{ \int_{U_s^n} | \langle x,a\rangle|^p d\,\mu(a) \}^{1/p} \text{ for all } x \in E \text{ and } \mu(U_s^n)^{1/p} = \pi_p(T).$$

Therefore, it follows from the probability lemma that

$$\{ \int_{\mathbb{R}} \|T x\|^p d\,\mu_s^n(x)\}^{1/p} \leqslant \{ \int_{\mathbb{R}^n} \int_{U_s^n} |\langle x,a\rangle|^p d\,\mu(a) \, d\,\mu_s^n(x)\}^{1/p}$$

$$\leqslant \int_{U_s^n} c_{sp}^p \|a\|_s^p \, d\,\mu(a) \}^{1/p}$$

$$\leqslant c_{sp}\pi_p(T).$$

Now we obtain the following lemma, which was proved by S. Kwapien [7], immediately.

Lemma 3 : Let $T \in \mathcal{L}(E, l_s^n)$. If $1 < s < 2$ and $1 \leqslant p \leqslant q < s$ or if $s = 2$ and $1 \leqslant p \leqslant q < \infty$ then

$$\pi_p(T) \leqslant c_{sq}\, c_{sp}^{-1}\, \pi_q(T') .$$

In particular,

$$\pi_p(T) \leqslant \pi_p(T') .$$

Proof : Applying lemma 1 to $T \in \mathcal{L}(E, l_s^n)$ and lemma 2 to $T' \in \mathcal{L}(l_{s'}^n, E')$ we obtain

$$\pi_p(T) \leqslant c_{sp}^{-1} \{ \int_{\mathbb{R}^n} \|T' b\|^p \, d\mu_s^n(b) \}^{1/p}$$

$$\leqslant c_{sp}^{-1} \{ \int_{\mathbb{R}^n} \|T' b\|^q \, d\mu_s^n(b) \}^{1/q} \leqslant c_{sq} c_{sp}^{-1} \pi_q(T').$$

Remark (of C. Sunyack) : Let $T \in \mathcal{L}(l_{s'}^n, l_s^n)$. Then by lemma 3

$$\pi_p(T) = \pi_p(T') ,$$

and from the inequality in the proof of lemma 3 we obtain the equality

$$\pi_p(T) = c_{sp}^{-1} \{ \int_{\mathbb{R}^n} \|T' b\|^p d\mu_s^n(b) \}^{1/p} = c_{sp}^{-1} \{ \int_{\mathbb{R}^n} \|T x\|^p \, d\mu_s^n(x) \}^{1/p} .$$

In particular, if I_n is the identity operator of the Hilbert space l_2^n then (cf. [3])

$$\pi_p(I_n) = \left(\frac{\Gamma(\frac{n+p}{2})\, \Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})\, \Gamma(\frac{1+p}{2})} \right)^{1/p}$$

The next lemma was proved by J. S. Cohen [2] and P. Saphar [12].

Lemma 4 : Let $T \in \mathcal{L}(E, l_s^n)$. Then

$$\pi_s(T) \leqslant \pi_s(T').$$

Proof : If $e_1, \ldots, e_n$ are the usual unit vectors we have

$$\|T x\|_s = \{ \sum_k |\langle T x, e_k \rangle|^s \}^{1/s} = \{ \sum_k |\langle x, T' e_k \rangle|^s \}^{1/s}$$

and

$$\{\sum_k \|T'e_k\|^s\}^{1/s} \leqslant \pi_s(T') \sup_{\|y\|_s \leqslant 1} \{\sum_k |\langle y,e_k\rangle|^s\}^{1/s} = \pi_s(T').$$

Consequently, if $x_1,\ldots,x_m \in E$ then

$$\{\sum_i \|T x_i\|_s^s\}^{1/s} = \{\sum_{ik} |\langle x_i,T'e_k\rangle|^s\}^{1/s}$$

$$\leqslant \{\sum_k \|T'e_k\|^s\}^{1/s} \sup_{\|a\| \leqslant 1} \{\sum |\langle x_i,a\rangle|^s\}^{1/s}$$

$$\leqslant \pi_s(T') \sup_{\|a\| \leqslant 1} \{\sum_i |\langle x_i,a\rangle|^s\}^{1/s} .$$

The proofs of the following propositions are obtained by different combinations of lemma 3 and 4.

<u>Proposition 1</u> : <u>Let</u> $T \in \mathcal{L}(l_r^n,l_s^n)$. <u>If $2 < r < \infty$, $1 \leqslant s < 2$, and $1 \leqslant p < r'$ then</u>

$$\pi_1(T) \leqslant c_{r'p}\, c_{r'1}^{-1}\, \pi_p(T)$$

<u>Proof</u> : Applying lemma 3 to $T' \in \mathcal{L}(l_{s'}^n,l_{r'}^n)$ we obtain

$$\pi_1(T') \leqslant c_{r'p}\, c_{r'1}^{-1}\, \pi_p(T) .$$

On the other hand by lemma 3 in the case $1 < s < 2$, and by lemma 4 in the case $s = 1$,

$$\pi_1(T) \leqslant \pi_1(T') .$$

<u>Theorem 1</u> (P. Saphar [12]) : <u>Let</u> $T \in \mathcal{L}(l_r^n, F)$. <u>If $2 < r < \infty$ and $1 \leqslant p < r'$ then</u>

$$\pi_1(T) \leqslant c_{r'p}\, c_{r'1}^{-1}\, \pi_p(T) .$$

<u>Proof</u> : Without loss of generality we may assume that the Banach space F has the extension property. Consequently (cf. [10]), for all $\varepsilon > 0$ there exists a factorization

$$T : l_r^n \xrightarrow{A} l_\infty^m \xrightarrow{D} l_p^m \xrightarrow{Y} F$$

such that

$$\|A\| \leqslant 1, \quad \|Y\| \leqslant 1, \text{ and } \pi_p(D) \leqslant \pi_p(T) + \varepsilon .$$

Now it follows by proposition 1 that

$$\pi_1(T) \leqslant \|Y\| \, \pi_1(D\,A) \leqslant c_{r'p} \, c_{r'1}^{-1} \, \pi_p(D\,A)$$
$$\leqslant c_{r'p} \, c_{r'1}^{-1} [\, \pi_p(T) + \varepsilon \,] .$$

<u>Proposition 2</u> (S. Kwapien [7]) : <u>Let</u> $T \in \mathfrak{L}(l_r^n, l_2^n)$. <u>If $1 < r \leqslant \infty$ then</u>

$$\pi_1(T) \leqslant c_{2r'} \, c_{21}^{-1} \, \pi_{r'}(T).$$

<u>Proof</u> : Applying lemma 3 to $T \in \mathfrak{L}(l_r^n, l_2^n)$ and lemma 4 to $T' \in \mathfrak{L}(l_2^n, l_{r'}^n)$ we obtain

$$\pi_1(T) \leqslant c_{2r'} \, c_{21}^{-1} \, \pi_{r'}(T') \quad \text{and} \quad \pi_{r'}(T') \leqslant \pi_{r'}(T) .$$

The case $r = 1$, which is not dealt with in proposition 2, is identical with the fundamental theorem of A. Grothendieck [5].

<u>Proposition 2^G</u> : <u>Let</u> $T \in \mathfrak{L}(l_1^n, l_2^n)$. <u>Then</u>

$$\pi_1(T) \leqslant c_G \, \|T\| .$$

<u>Remark</u> : If the constant c_G is the best possible then

$$\pi/2 \leqslant c_G \leqslant \sinh \pi/2.$$

<u>Theorem 2</u> (S. Kwapien)[7]: <u>Let</u> $T \in \mathfrak{L}(l_r^n, F)$. <u>If $r = 1$, resp. $1 < r \leqslant 2$, then</u>

$$\pi_1(T) \leqslant c_G \, \pi_2(T), \ \underline{\text{resp.}} \ \pi_1(T) \leqslant c_{2r'} \, c_{21}^{-1} \, \pi_2(T) .$$

<u>Proof</u> : For all $\varepsilon > 0$ there exists a factorization

$$T : l_r^n \xrightarrow{A} l_\infty^m \xrightarrow{D} l_2^m \xrightarrow{Y} F$$

such that

$$\|A\| \leqslant 1, \ \|Y\| \leqslant 1, \text{ and } \pi_2(D) \leqslant \pi_2(T) + \varepsilon .$$

Now it follows by proposition 2 that

$$\pi_1(T) \leqslant \|Y\| \, \pi_1(D\,A) \leqslant c_{2r'} \, c_{21}^{-1} \, \pi_{r'}(D\,A)$$
$$\leqslant c_{2r'} \, c_{21}^{-1} \, \pi_2(D\,A) \leqslant c_{2r'} \, c_{21}^{-1} [\pi_2(T) + \varepsilon \,] .$$

The proof in the case $r = 1$ is the same.

<u>Proposition 3</u> : <u>Let</u> $T \in \mathcal{L}(l_2^n, l_s^n)$. <u>If $1 \leqslant s \leqslant p < \infty$ then</u>

$$\pi_s(T) \leqslant c_{2p}\, c_{2s}^{-1}\, \pi_p(T).$$

<u>Proof</u> : Applying lemma 4 to $T \in \mathcal{L}(l_2^n, l_s^n)$ and lemma 3 to $T' \in \mathcal{L}(l_{s'}^n, l_2^n)$ we obtain

$$\pi_s(T) \leqslant \pi_s(T') \quad \text{and} \quad \pi_s(T') \leqslant c_{2p}\, c_{2s}^{-1}\, \pi_p(T).$$

<u>Theorem 3</u> (S. Kwapien [7]) : <u>Let</u> $T \in \mathcal{L}(E, l_s^n)$. <u>If $1 \leqslant s \leqslant 2$ and $2 \leqslant p < \infty$ then</u>

$$\pi_2(T) \leqslant c_{2p}\, c_{2s}^{-1}\, \pi_p(T) .$$

<u>Proof</u> : If $x_1, \ldots, x_m \in E$ we define the operator $X \in \mathcal{L}(l_2^m, F)$ by

$$X(\xi_i) := \sum_i \xi_i x_i .$$

Then

$$\| X \| = \sup_{\|a\| \leqslant 1} \{ \sum_i | < x_i, a > |^2 \}^{1/2} .$$

Consequently, by proposition 3 we have

$$\{ \sum_i \| T x_i \|^2 \}^{1/2} = \{ \sum_i \| T X e_i \|^2 \}^{1/2}$$

$$\leqslant \pi_2(T X) \sup_{\|f\|_2 \leqslant 1} \{ \sum_i | < e_i, f > |^2 \}^{1/2}$$

$$\leqslant \pi_s(T X) \leqslant c_{2p}\, c_{2s}^{-1}\, \pi_p(T X)$$

$$\leqslant c_{2p}\, c_{2s}^{-1}\, \pi_p(T) \sup_{\|a\| \leqslant 1} \{ \sum_i | < x_i, a > |^2 \}^{1/2} .$$

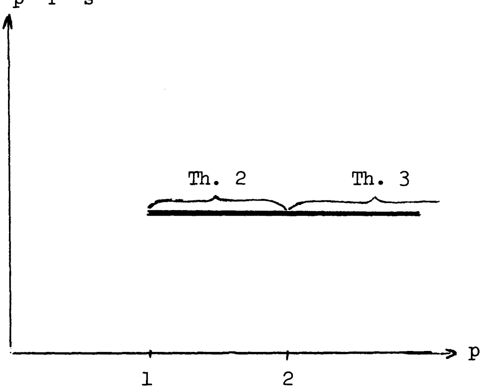
Because of symmetry it seems very probable that we have :

<u>Theorem 4</u> (<u>CONJECTURE</u>) : <u>Let</u> $T \in \mathcal{L}(E, l_s^n)$. <u>If $2 < s < p \leqslant q < \infty$ then, with a constant</u> $c_{s,pq} > 0$,

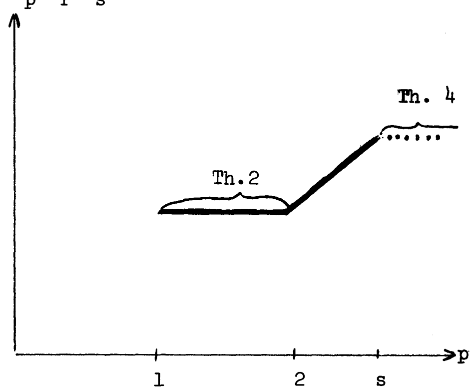
$$\pi_p(T) \leqslant c_{s,pq}\, \pi_q(T) .$$

Finally, we illustrate the results in the following diagrams where the ordinate is a symbolic measure of the largeness of $\mathcal{P}_p(L_r,L_s)$.

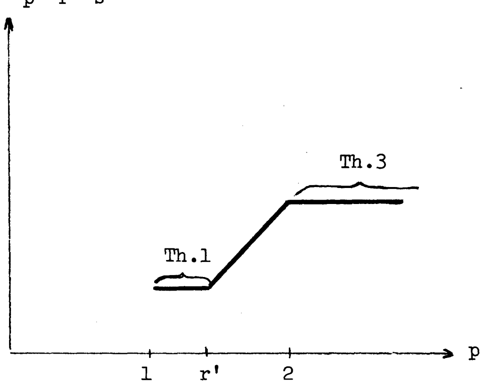
$\mathcal{P}_p(L_r,L_s)$, $1 \leqslant r \leqslant 2$, $1 \leqslant s \leqslant 2$



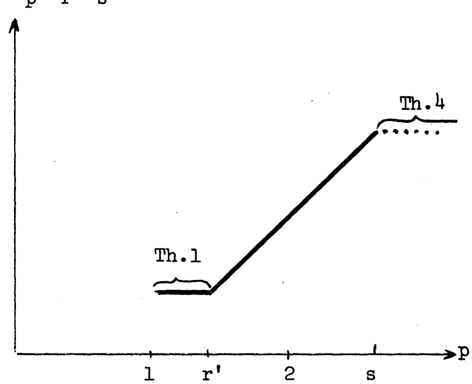
$\mathcal{P}_p(L_r,L_s)$, $1 \leqslant r \leqslant 2$, $2 \leqslant s \leqslant \infty$



$\mathcal{P}_p(L_r,L_s)$, $2 \leqslant r \leqslant \infty$, $1 \leqslant s \leqslant 2$



$\mathcal{P}_p(L_r,L_s)$, $2 \leqslant r \leqslant \infty$, $2 \leqslant s \leqslant \infty$



Remarks :

(1) If the spaces L_r and L_s are infinite dimensional then "/" means that $\mathcal{P}_p(L_r,L_s)$ is strictly increasing

(2) $\mathcal{P}_p(L_r,L_s)$ depends continuously on p if and only if it is constant since B. Maurey proved, assuming approximations property, the following results.

If $\mathcal{P}_p(E,F) = \bigcap_{\varepsilon>0} \mathcal{P}_{p+\varepsilon}(E,F)$ then $\mathcal{P}_p(E,F) = \mathcal{P}_{p+\varepsilon}(E,F)$ for $0 < \varepsilon < \varepsilon_o$.

If $\mathcal{P}_p(E,F) = \bigcup_{\varepsilon>0} \mathcal{P}_{p-\varepsilon}(E,F)$ then $\mathcal{P}_p(E,F) = \mathcal{P}_{p-\varepsilon}(E,F)$ for $0 < \varepsilon < \varepsilon_o$.

§ 6. THE ν_p-NORM (cf. [10], [23], [24]).

In the following let us assume that at least one of the Banach spaces E and F has finite dimension. Then every operator $T \in \mathfrak{L}(E,F)$ can be represented in the form

$$T = \sum_i < x_i \; a_i > y_i \quad \text{for all} \quad x \in E$$

with $a_1,\dots,a_n \in E'$ and $y_1,\dots,y_n \in F$. Now the ν_p-norm is defined by

$$\nu_p(T) := \inf\left[\{\sum_i \|a_i\|^p\}^{1/p} \sup_{\|b\| \leqslant 1} \{\sum_i |<y_i,b>|^{p'}\}^{1/p'}\right],$$

$1 < p < \infty$, where the infimum is taken over all possible representations. In the case $p = 1$ and $p = \infty$ we put

$$\nu_1(T) := \inf\left[\sum_i \|a_i\| \, \|y_i\|\right]$$

and

$$\nu_\infty(T) := \inf\left[\sup_i \|a_i\| \sup_{\|b\| \leqslant 1} \sum |<y_i,b>| \right].$$

It follows from the well-known relations

$$\pi_p(T) = \sup\{|\text{trace}(ST)| : S \in \mathfrak{L}(F,E),\ \nu_{p'}(S) \leqslant 1\} \text{ for all } T \in \mathfrak{L}(E,F)$$

and

$$\nu_{p'}(S) = \sup\{|\text{trace}(ST)| : T \in \mathfrak{L}(E,F),\ \pi_p(T) \leqslant 1 \quad \text{for all} \quad S \in \mathfrak{L}(F,E)$$

that the inequalities

$$\pi_p(T) \leqslant c\ \pi_q(T) \quad \text{for all } T \in \mathfrak{L}(E,F)$$

and

$$\nu_{q'}(S) \leqslant c\ \nu_{p'}(S) \text{ for all } S \in \mathfrak{L}(F,E)$$

are equivalent.

We have

$$\pi_p(T) \leqslant \nu_p(T) \quad \text{for all} \quad T \in \mathfrak{L}(E,F),$$

and in the case p = 2,

$$\pi_2(T) = \nu_2(T) \quad \text{for all} \quad T \in \mathcal{L}(E,F) .$$

If at least one of the Banach spaces E and F has the extension property then also the equation

$$\pi_p(T) = \nu_p(T) \quad \text{for all} \quad T \in \mathcal{L}(E,F)$$

is valid. On the other side A. Pełczynski [21] has shown that there exists no constant $c > 0$ such that for every bounded linear operator T between arbitrary finite dimensional Banach spaces the inequality

$$\nu_p(T) \leqslant c\, \pi_p(T)$$

holds.

Problem. : If $1 \leqslant r, s \leqslant \infty$ and $1 < p < \infty$, does there exists a constant $c_{rsp} > 0$ such that

$$\nu_p(T) \leqslant c_{rsp}\, \pi_p(T) \quad \text{for all} \quad T \in \mathcal{L}(l_r^n, l_s^n) \ ?$$

Now we prove further results by duality.

Theorem 1* : Let $T \in \mathcal{L}(E, l_s^n)$. If $2 < s < p \leqslant \infty$ then

$$\nu_p(T) \leqslant c_{s'p'}\, c_{s'1}^{-1}\, \nu_\infty(T) .$$

Proof : If $2 < s < \infty$ and $1 \leqslant p' < s'$ then by theorem 1 we have

$$\pi_1(S) \leqslant c_{s'p'} c_{s'1}^{-1}\, \pi_{p'}(S) \quad \text{for all} \quad S \in \mathcal{L}(l_s^n, E).$$

Consequently, there holds the dual inequality

$$\nu_p(T) \leqslant c_{s'p'}\, c_{s'1}^{-1}\, \nu_\infty(T) \quad \text{for all} \quad T \in \mathcal{L}(E, l_s^n).$$

Theorem 2* : Let $T \in \mathcal{L}(E, l_s^n)$. If $s = 1$, resp. $1 < s \leqslant 2$, then

$$\nu_2(T) \leqslant c_G \nu_\infty(T), \text{ resp. } \nu_2(T) \leqslant c_{2s'} c_{21}^{-1}\, \nu_\infty(T)$$

Theorem 3* : Let $T \in \mathcal{L}(l_r^n, F)$. If $1 \leqslant r \leqslant 2$ and $1 < p \leqslant 2$ then

$$\nu_p(T) \leqslant c_{2p'}\, c_{21}^{-1}\, \nu_2(T)$$

Theorem 4* (CONJECTURE) : Let $T \in \mathcal{L}(l_r^n, F)$. If $2 < r < \infty$ and $1 < p < q < r'$ then, with a constant $c_{r,pq} > 0$,

$$\nu_p(T) \leqslant c_{r,pq} \nu_q(T)$$

Finally, we formulate some spacial cases of theorem 1* and 2*.

Proposition 4 (S. Kwapien [7]) : Let $T \in \mathcal{L}(l_\infty^n, l_s^n)$. If $2 < s < p < \infty$ then

$$\nu_p(T) \leqslant c_{s'p'} \, c_{s'1}^{-1} \, \|T\|$$

Proposition 5 (J. Lindenstrauss and A. Pełczynski [8]) : Let $T \in \mathcal{L}(l_\infty^n, l_s^n)$. If $s = 1$, resp. $1 < s \leqslant 2$, then

$$\nu_2(T) \leqslant c_G \, \|T\| \text{ , resp. } \nu_2(T) \leqslant c_{2s'} \, c_{21}^{-1} \, \|T\| \, .$$

Proof : The results follow from the fact that

$$\nu_\infty(T) = \|T\| \quad \text{for all } T \in \mathcal{L}(l_\infty^n, F)$$

Remark : It is easy to prove the following stronger form of

Lemma 4 : Let $T \in \mathcal{L}(E, l_s^n)$. Then

$$\nu_s(T) \leqslant \pi_s(T') \, .$$

One can obtain further results by using this inequality.

§ 7. IDENTITY OPERATORS IN l_r^n-SPACES.

Let I_n be the identity operator from l_r^n into l_s^n. We define the limit order $\lambda_I(r,s,\pi_p)$ to be the infimum of all real numbers λ for which there exists a constant $c_{rs,p} > 0$ such that the inequality

$$\pi_p(I_n : l_r^n \to l_s^n) \leqslant c_{rs,p} n^\lambda$$

for all $n = 1,2,\ldots$ holds. The limit order $\lambda_I(r,s,\nu_p)$ is defined in the same way.

Historical remark : The π_p - and ν_p-norm of the identity operator from l_r^n into itself was determined or estimated by D.J.H. Garling and Y. Gordon (cf. [16] ,[17] , [18]). In the cases ν_∞ and π_1 the first result was proved by B. Grünbaum [19] and D. Rutovitz [22]. A. Tong [26] has given necessary and sufficient conditions for a diagonal operator from l_r into l_s to be nuclear (cf. also L. Schwartz [25]).

Lemma 5 : If $\alpha + \beta \leqslant 1$,

$$\lambda_I(r,s,\pi_p) \leqslant \alpha \ \underline{\text{and}}\ \lambda_I(s,r,\nu_{p'}) \leqslant \beta$$

then

$$\lambda_I(r,s,\pi_p) = \alpha \ \underline{\text{and}}\ \lambda_I(s,r,\nu_{p'}) = \beta$$

Proof : Since

$$n = \text{trace } (I_n) \leqslant \pi_p(I_n : l_r^n \to l_s^n)\ \nu_{p'}(I_n : l_s^n \to l_r^n)$$

we have

$$1 \leqslant \lambda_I(r,s,\pi_p) + \lambda_I(s,r,\nu_{p'}) \leqslant \alpha+\beta = 1 .$$

Consequently, identity holds.

Lemma 6 :

$$\lambda_I(r,s, \|\cdot\|) \leqslant \begin{cases} 1/s - 1/r & \underline{\text{if}} \quad r \geqslant s \\ 0 & \underline{\text{if}} \quad r \leqslant s \end{cases} .$$

Proof : The result follows from the well-known inequality

$$\|I_n : l_r^n \to l_s^n\| \leqslant \begin{cases} n^{1/s - 1/r} & \text{if} \quad r \geqslant s \\ 1 & \text{if} \quad r \leqslant s \end{cases} .$$

Lemma 7 :

$$\lambda_I(1,\infty,\nu_1) \leqslant 0 .$$

Proof : If $e = (\varepsilon_i)$ ranges over the set of all n-dimensional vectors with $\varepsilon_i = \pm 1$ then the identity operator I_n has the representation

$$I_n\, x = 2^{-n} \sum_e < x,e > e \quad \text{for all } x \in l_1^n .$$

Consequently,

$$\nu_1(I_n : l_1^n \to l_\infty^n) \leqslant 1 .$$

Lemma 8 : If $1 < p \leqslant \infty$ then

$$\lambda_I(1,2,\nu_p) \leqslant 0 .$$

Proof : We represent the identity operator I_n in the form

$$I_n x = 2^{-n} \sum_e < x,e > e \quad \text{for all } x \in l_1^n .$$

Then

$$\{ \sum_e \| e \|_\infty^p \}^{1/p} = 2^{n/p} .$$

On the other hand, it follows from Littlewood's inequality (cf. [20]) that

$$\sup_{\|b\|_2 \leq 1} \{ \sum_e | < e,b > |^{p'} \}^{1/p'} \leq 2^{n/p'} c_{p'} .$$

Therefore,

$$\nu_p(I_n : l_1^n \to l_2^n) \leq c_{p'} .$$

Lemma 9 :

$$\lambda_I(1, 2, \pi_1) \leq 0 .$$

Proof : From Littlewood's inequality we have

$$\|x\|_2 \leq c_L 2^{-n} \sum_e | <x, e> | .$$

Consequently, if $x_1,\ldots,x_m \in l_1^n$

$$\sum_i \| x_i \|_2 \leq c_L \sup_{\|a\|_\infty \leq 1} \sum | < x_i, a > | ,$$

and therefore,

$$\pi_1(I_n : l_1^n \to l_2^n) \leq c_L .$$

Remark : Lemma 9 follows also from proposition 2^G.

Lemma 10 :

$$\lambda_I(\infty,p,\nu_p) \leq 1/p$$

Proof : If $e_1,\ldots, e_n$ are the usual unit vectors we can represent the identity operator I_n in the form

$$I_n x = \sum_i < x,e_i > e_i \text{ for all } x \in l_\infty^n .$$

Since

$$\{\sum_i \|e_i\|_1^p\}^{1/p} = n^{1/p} \quad \text{and} \quad \sup_{\|a\|_{p'}\leq 1} \{\sum_i |\langle e_i,a\rangle|^{p'}\}^{1/p'} = 1$$

we obtain

$$\nu_p(I_n : l^n \to l^n_p) \leq n^{1/p} .$$

Lemma 11 : If $1 \leq s < 2$ then

$$\lambda_I(s',s,\pi_1) \leq 1/s$$

Proof : In the case $s = 1$ the result follows from lemma 10. Now we assume $1 < s < 2$. Then there exists ε with $0 < \varepsilon < s-1$. By lemma 3 and 10 we obtain

$$\begin{aligned}\pi_1(I_n : l^n_{s'} \to l^n_s) &\leq c_{s\,s-\varepsilon}\, c^{-1}_{s1}\, \pi_{s-\varepsilon}(I_n : l^n_{s'} \to l^n_s) \\ &\leq c_{s\,s-\varepsilon}\, c^{-1}_{s1} \| I_n : l^n_{s'} \to l^n_\infty\| \pi_{s-\varepsilon}(I_n : l^n_\infty \to l^n_{s-\varepsilon}) \|I_n : l^n_{s-\varepsilon} \to l^n_s\| \\ &\leq c_{s\,s-\varepsilon}\, c^{-1}_{s1}\, n^{1/(s-\varepsilon)} .\end{aligned}$$

Consequently

$$\lambda_I(s',s,\pi_1) \leq 1/(s-\varepsilon)$$

The result follows since ε can be made as small as we please.

Remark : It should be possible to determine the exact asymptotic behaviour of $\pi_p(I_n : l^n_{s'} \to l^n_s)$ as n tends to infinity by using the relation

$$\pi_p(I_n : l^n_{s'} \to l^n_s) = c^{-1}_{sp} \{ \int_{R^n} \|x\|^p_s \, d\,\mu^n_s(x) \}^{1/p} , \quad 1 \leq p < s .$$

The limit orders $\lambda_I(r,s,\|\cdot\|)$ and $\lambda_I(s,r,\nu_1)$

By lemma 6 we have

$$(1) \qquad \lambda_I(r,s,\|\cdot\|) \leq \begin{cases} 1/s - 1/r & \text{if } r \geq s \\ 0 & \text{if } r \leq s \end{cases} .$$

On the other hand it follows from lemma 6, 7 and 10 that

$$\lambda_I(s,r,\nu_1) \leq \lambda_I(s,1,\|\cdot\|) + \lambda_I(1,\infty,\nu_1) + \lambda_I(\infty,r,\|\cdot\|) \leq 1/s' + 1/r$$

and

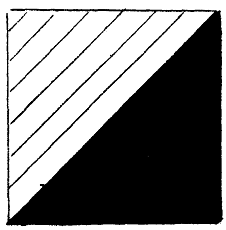
$$\lambda_I(s,r,\nu_1) \leqslant \lambda_I(s,\infty,\|\cdot\|) + \lambda_I(\infty,1,\nu_1) + \lambda_I(1,r,\|\cdot\|) \leqslant 1 \quad .$$

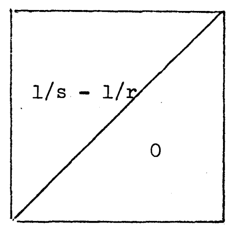
In each case, choosing the best result we obtain

$$(1^*) \qquad \lambda_I(s,r,\nu_1) \leqslant \begin{cases} 1/s' + 1/r & \text{if} \quad r \geqslant s \\ 1 & \text{if} \quad r \leqslant s \end{cases} \quad .$$

Finally, lemma 5 implies that identity holds in (1) and (1*). In what follows we illustrate our results with pairs of diagrams in the unit square with coordinates 1/r and 1/s. In the left hand diagram we plot the level curves of $\lambda_I(r,s,\pi_p)$. In the right hand diagrams we indicate the algebraic expression for $\lambda_I(r,s,\pi_p)$.

$\underline{\lambda_I(r,s,\|\cdot\|)}$





<u>The limit orders</u> $\lambda_I(r,s,\pi_2)$ <u>and</u> $\lambda_I(s,r,\nu_2)$

By lemmas 6, 9 and 10 we have

$$\lambda_I(r,s,\pi_2) \leqslant \lambda_I(r,\infty,\|\cdot\|) + \lambda_I(\infty,2,\pi_2) + \lambda_I(2,s,\|\cdot\|)$$

$$\leqslant 0 + 1/2 + \begin{cases} 1/s - 1/2 & \text{if} \quad 1 \leqslant s \leqslant 2 \\ 0 & \text{if} \quad 2 \leqslant s \leqslant \infty \end{cases}$$

and

$$\lambda_I((r,s,\pi_2) \leqslant \lambda_I(r,1,\ \|\cdot\|) + \lambda_I(1,2,\pi_2) + \lambda_I(2,s,\ \|\cdot\|)$$

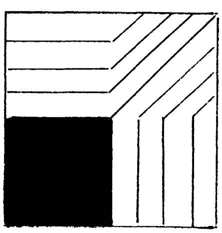
$$\leqslant 1/r' + 0 + \begin{cases} 1/s - 1/2 & \text{if} \quad 1 \leqslant s \leqslant 2 \\ 0 & \text{if} \quad 2 \leqslant s \leqslant \infty \end{cases} .$$

Consequently,

$$\lambda_I(r,s,\pi_2) \leqslant \begin{cases} 1/r' + 1/s - 1/2 & \text{if} \quad 1 \leqslant r \leqslant 2,\ 1 \leqslant s \leqslant 2, \\ 1/s & \text{if} \quad 2 \leqslant r \leqslant \infty,\ 1 \leqslant s \leqslant 2, \\ 1/r' & \text{if} \quad 1 \leqslant r \leqslant 2,\ 2 \leqslant s \leqslant \infty, \\ 1/2 & \text{if} \quad 2 \leqslant r \leqslant \infty,\ 2 \leqslant s \leqslant \infty. \end{cases} \tag{2}$$

Since $\lambda_I(s,r,\nu_2) = \lambda_I(s,r,\pi_2)$ it follows from lemma 5 that identity holds in (2).

$\underline{\lambda_I(r,s,\pi_2)}$



$\frac{1}{s}$	$\frac{1}{r'}+\frac{1}{s}-\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{r'}$

<u>The limit orders</u> $\lambda_I(r,s,\pi_p)$ <u>and</u> $\lambda_I(s,r,\nu_{p'})$ <u>with</u> $1 \leqslant p < 2$

Since by theorem 2 and 2* for $1 \leqslant r \leqslant 2$ we have

$$\lambda_I(r,s,\pi_p) = \lambda_I(r,s,\pi_2) \quad \text{and} \quad \lambda_I(s,r,\nu_{p'}) = \lambda_I(s,r,\nu_2)$$

in the following we need only consider the case $2 < r \leqslant \infty$.

By lemma 6 and 11 we obtain

$$\lambda_I(r,s,\pi_p) \leq \lambda_I(r,s',\|\cdot\|) + \lambda_I(s',s,\pi_p) \leq 1/s \quad \text{if } r \leq s' \text{ and } 1 \leq s \leq 2,$$

and

$$\lambda_I(r,s,\pi_p) \leq \lambda_I(r,r',\pi_p) + \lambda_I(r',s,\|\cdot\|) \leq 1/r' \quad \text{if } r' \leq s \text{ and } 1 \leq r' \leq 2.$$

On the other hand it follows from lemma 6 and 10 that

$$\lambda_I(r,s,\pi_p) \leq \lambda_I(r,\infty,\|\cdot\|) + \lambda_I(\infty,p,\pi_p) + \lambda_I(p,s,\|\cdot\|)$$

$$\leq 0 + 1/p + \begin{cases} 1/s - 1/p & \text{if } p \geq s, \\ 0 & \text{if } p \leq s . \end{cases}$$

In each case, choosing the best result we obtain

$$\text{(3)} \qquad \lambda_I(r,s,\pi_p) \leq \begin{cases} 1/s & \text{if } p' \leq r \leq \infty, & -1 \leq s \leq p, \\ 1/p & \text{if } p' \leq r \leq \infty, & p \leq s \leq \infty, \\ 1/s & \text{if } 2 \leq r \leq p', & 1 \leq s \leq r', \\ 1/r' & \text{if } 2 \leq r \leq p', & r' \leq s \leq \infty . \end{cases}$$

By lemma 6 and 11

$$\lambda_I(s,r,\nu_{p'}) \leq \lambda_I(s,\infty,\|\cdot\|) + \lambda_I(\infty,p',\nu_{p'}) + \lambda_I(p',r,\|\cdot\|)$$

$$\leq 0 + 1/p' + \begin{cases} 1/r - 1/p' & \text{if } p' \geq r, \\ 0 & \text{if } p' \leq r . \end{cases}$$

Moreover,

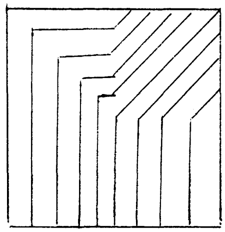
$$\lambda_I(s,r,\nu_{p'}) \leq \lambda_I(s,r,\nu_2) = 1/s' \quad \text{if } 1 \leq s \leq 2.$$

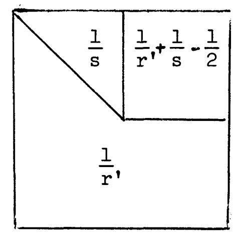
Consequently,

$$\text{(3*)} \qquad \lambda_I(s,r,\nu_{p'}) \leq \begin{cases} 1/s' & \text{if } p' \leq r \leq \infty, & 1 \leq s \leq p, \\ 1/p' & \text{if } p' \leq r \leq \infty, & p \leq s \leq \infty, \\ 1/s' & \text{if } 2 \leq r \leq p', & 1 \leq s \leq r', \\ 1/r & \text{if } 2 \leq r \leq p', & r' \leq s \leq \infty . \end{cases}$$

Finally, lemma 5 implies that identity holds in (3) and (3*).

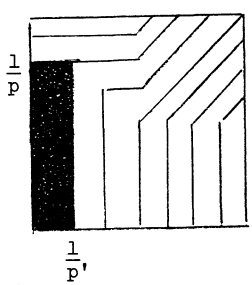
$\underline{\lambda_I(r,s,\pi_1)}$

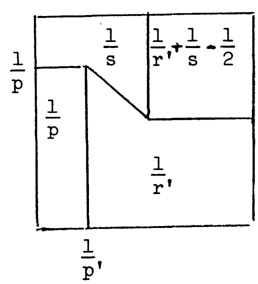




$\underline{\lambda_I(r,s,\pi_1)}$

$1 < p < 2$





<u>The limit orders</u> $\lambda_I(r,s,\pi_p)$ <u>and</u> $\lambda_I(r,s,\nu_{p'})$ with <u>$2 < p < \infty$</u>

Since by theorem 3 and 3* for $1 \leqslant s \leqslant 2$ we have

$$\lambda_I(r,s,\pi_p) = \lambda_I(r,s,\pi_2) \quad \text{and} \quad \lambda_I(s,r,\nu_{p'}) = \lambda_I(s,r,\nu_2)$$

in the following we need only consider the case $2 < s \leqslant \infty$.
Since

$$\lambda_I(r,s,\pi_p) \leqslant \lambda_I(r,s,\nu_p)$$

by (3*) we obtain

$$(4) \qquad \lambda_I(r,s,\pi_p) \leqslant \begin{cases} 1/r' & \text{if} \quad 1 \leqslant r \leqslant p', \quad p \leqslant s \leqslant \infty, \\ 1/p & \text{if} \quad p' \leqslant r \leqslant \infty, \quad p \leqslant s \leqslant \infty, \\ 1/r' & \text{if} \quad 1 \leqslant r \leqslant s', \quad 2 \leqslant s \leqslant p, \\ 1/s & \text{if} \quad s' \leqslant r \leqslant \infty, \quad 2 \leqslant s \leqslant p \end{cases} .$$

It follows from lemma 6 and 10 that

$$\lambda_I(s,r,\nu_{p'}) \leqslant \lambda_I(s,\infty,\|\cdot\|) + \lambda_I(\infty,p',\nu_{p'}) + \lambda_I(p',r,\|\cdot\|)$$

$$\leqslant 0 + 1/p' + \begin{cases} 1/r - 1/p' & \text{if} \quad p' \geqslant r, \\ 0 & \text{if} \quad p' \leqslant r \ . \end{cases}$$

On the other hand lemma 6 and 8 imply that

$$\lambda_I(s,r,\nu_{p'}) \leqslant \lambda_I(s,1,\|\cdot\|) + \lambda_I(1,2,\nu_{p'}) + \lambda_I(2,r,\|\cdot\|)$$

$$\leqslant 1/s' + 0 + 0 \quad \text{if} \quad 2 \leqslant r \ .$$

In each case, choosing the best result we obtain

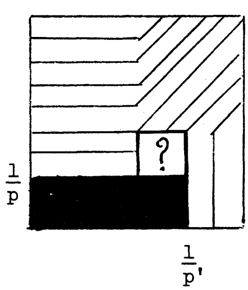
$$(4^*) \qquad \lambda_I(s,r,\nu_{p'}) \leqslant \begin{cases} 1/r & \text{if} \quad 1 \leqslant r \leqslant p', \quad p \leqslant s \leqslant \infty, \\ 1/p' & \text{if} \quad p' \leqslant r \leqslant \infty, \quad p \leqslant s \leqslant \infty, \\ 1/r & \text{if} \quad 1 \leqslant r \leqslant p', \quad 2 \leqslant s \leqslant p, \\ 1/s' & \text{if} \quad 2 \leqslant r \leqslant \infty, \quad 2 \leqslant s \leqslant p \ . \end{cases}$$

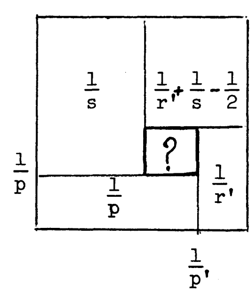
Because the square

$$Q_{I,p} := \{ (1/r, 1/s) : p' < r < 2, \ 2 < s < p \}$$

does not appear in (4*), we have the open problem wether identity holds for all r and s in (4).

$$\frac{\lambda_I(r,s,\pi_p)}{2 < p < \infty}$$





§ 8. LITTELWOOD OPERATORS IN l_r^n-SPACES.

In the following n ranges over the set of all natural numbers $n = 2^k$ with $k = 1,2,\ldots$. The symmetric Littlewood operators $A_n = (\alpha_{ik}^{(n)})$ are defined inductively by (cf. [20])

$$A_2 := \begin{pmatrix} 1, & 1 \\ 1, & -1 \end{pmatrix}, \ldots, A_{2n} := \begin{pmatrix} A_n, & A_n \\ A_n, & -A_n \end{pmatrix}, \ldots .$$

Then

$$A_n^2 = n\, I_n \quad \text{and} \quad \alpha_{ik}^{(n)} = \pm 1 .$$

The limit orders $\lambda_A(r,s,\pi_p)$ and $\lambda_A(r,s,\nu_p)$ are introduced in the same way as in the case of identity operators.

Lemma 12 : If $\alpha + \beta \leqslant 2$,

$$\lambda_A(r,s,\pi_p) \leqslant \alpha \quad \text{and} \quad \lambda_A(s,r,\nu_{p'}) \leqslant \beta$$

then

$$\lambda_A(r,s,\pi_p) = \alpha \quad \text{and} \quad \lambda_A(s,r,\nu_{p'}) = \beta .$$

Proof : Since

$$n^2 = \text{trace } (n\, I_n) \leqslant \pi_p(A_n : l_r^n \to l_s^n)\ \nu_{p'}(A_n : l_s^n \to l_r^n)$$

we have

$$2 \leqslant \lambda_A(r,s,\pi_p) + \lambda_A(s,r,\nu_{p'}) \leqslant \alpha + \beta \leqslant 2 .$$

Consequently, identity holds.

Lemma 13 : If $2 \leqslant s \leqslant \infty$ then

$$\lambda_A(r,s, \|\cdot\|) \leqslant 1/s .$$

Proof : Since the operator $n^{-1/2} A_n$ is unitary we have

$$\| A_n : l_2^n \to l_2^n \| \leqslant n^{1/2} .$$

On the other hand, because $|\alpha_{ik}^{(n)}| = 1$, it follows that

$$\|A_n : l_1^n \to l_\infty^n\| \leqslant 1 \quad .$$

Finally, if $2 \leqslant s \leqslant \infty$, the M. Riesz' convexity theorem implies

$$\|A_n : l_{s'}^n \to l_s^n\| \leqslant n^{1/s}$$

Lemma 14 :

$$\lambda_A(1,2,\nu_1) \leqslant 1/2 \ .$$

Proof : The result follows from

$$\begin{aligned} \nu_1(A_n : l_1^n \to l_\infty^n) &= \pi_1(A_n : l_1^n \to l_\infty^n) \\ &= \pi_1(I_n : l_1^n \to l_2^n) \, \|A_n : l_2^n \to l_2^n\| \, \|I_n : l_2^n \to l_\infty^n\| \\ &\leqslant c_L \, n^{1/2} \quad . \end{aligned}$$

The limit orders $\lambda_A(r,s,\|\cdot\|)$ and $\lambda_A(s,r,\nu_1)$

By lemma 6 and 13 we have

$$\lambda_A(r,s,\|\cdot\|) \leqslant \lambda_I(r,2,\|\cdot\|) + \lambda_A(2,2,\|\cdot\|) + \lambda_I(2,s,\|\cdot\|)$$

$$\leqslant \begin{cases} (1/2 - 1/r) + 1/2 + (1/s - 1/2) & \text{if } r \geqslant 2,\ 2 \geqslant s, \\ \quad 0 \quad + 1/2 + (1/s - 1/2) & \text{if } r \leqslant 2,\ 2 \geqslant s, \\ (1/2 - 1/r) + 1/2 + \quad 0 & \text{if } r \geqslant 2,\ 2 \leqslant s. \end{cases}$$

On the other hand we obtain

$$\begin{aligned} \lambda_A(r,s,\|\cdot\|) &\leqslant \lambda_I(r,s',\|\cdot\|) + \lambda_A(s',s,\|\cdot\|) \\ &\leqslant 0 + 1/s \quad \text{if} \quad r \leqslant s' \quad \text{and} \quad 2 \leqslant s, \end{aligned}$$

and

$$\begin{aligned} \lambda_A(r,s,\|\cdot\|) &\leqslant \lambda_A(r,r',\|\cdot\|) + \lambda_I(r',s,\|\cdot\|) \\ &\leqslant 1/r' + 0 \quad \text{if} \quad r \leqslant 2 \quad \text{and} \quad r' \leqslant s \ . \end{aligned}$$

Summarizing the results we have

$$(5)\qquad \lambda_A(r,s,\|\cdot\|) \leqslant \begin{cases} 1/r' + 1/s - 1/2 & \text{if } 2 \leqslant r \leqslant \infty,\ 1 \leqslant s \leqslant 2, \\ 1/s & \text{if } 1 \leqslant r \leqslant 2,\ 1 \leqslant s \leqslant r', \\ 1/r' & \text{if } s' \leqslant r \leqslant \infty,\ 2 \leqslant s \leqslant \infty \ . \end{cases}$$

With the known values of $\lambda_I(s,r,\nu_1)$ we obtain

$$\lambda_A(s,r,\nu_1) \leqslant \lambda_I(s,1,\nu_1) + \lambda_A(1,r,\|\cdot\|) \leqslant 1 + 1/r,$$

and

$$\lambda_A(s,r,\nu_1) \leqslant \lambda_A(s,\infty,\|\cdot\|) + \lambda_I(\infty,r,\nu_1) \leqslant 1/s' + 1 \ .$$

On the other hand it follows from lemma 14 that

$$\lambda_A(s,r,\nu_1) \leqslant \lambda_I(s,1,\|\cdot\|) + \lambda_A(1,\infty,\nu_1) + \lambda_I(\infty,r,\|\cdot\|)$$

$$\leqslant 1/s' + 1/2 + 1/r \ .$$

In each case, choosing the best result we obtain,

$$(5^*)\qquad \lambda_A(s,r,\nu_1) \leqslant \begin{cases} 1/r + 1/s' + 1/2 & \text{if } 2 \leqslant r \leqslant \infty,\ 1 \leqslant s \leqslant 2, \\ 1/s' + 1 & \text{if } 1 \leqslant r \leqslant 2,\ 1 \leqslant s \leqslant r', \\ 1/r + 1 & \text{if } s' \leqslant r \leqslant \infty,\ 2 \leqslant s \leqslant \infty \ . \end{cases}$$

Finally, lemma 12 implies that identity holds in (5) and (5*).

$\underline{\lambda_A(r,s,\|\cdot\|)}$

<u>The limit orders</u> $\lambda_A(r,s,\pi_2)$ <u>and</u> $\lambda_A(s,r,\nu_2)$

Since

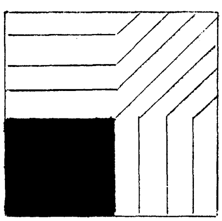
$$\lambda_A(r,s,\pi_2) \leqslant \lambda_I(r,2,\pi_2) + \lambda_A(2,s,\|\cdot\|)$$

we obtain, using the known values of $\lambda_I(r,2,\pi_1)$ and $\lambda_A(2,s,\|\cdot\|)$,

$$(6) \qquad \lambda_A(r,s,\pi_2) \leqslant \begin{cases} 1/r' + 1/s & \text{if } 1 \leqslant r \leqslant 2, \quad 1 \leqslant s \leqslant 2, \\ 1/2 + 1/s & \text{if } 2 \leqslant r \leqslant \infty, \quad 1 \leqslant s \leqslant 2, \\ 1/r' + 1/2 & \text{if } 1 \leqslant r \leqslant 2, \quad 2 \leqslant s \leqslant \infty, \\ 1/2 + 1/2 & \text{if } 2 \leqslant r \leqslant \infty, \quad 2 \leqslant s \leqslant \infty \end{cases} .$$

Finally, it follows from lemma 12 and $\lambda_A(s,r,\nu_2) = \lambda_A(s,r,\pi_2)$ that identity holds in (6)

<u>$\lambda_A(r,s,\pi_2)$</u>



$\frac{1}{2}+\frac{1}{s}$	$\frac{1}{r'}+\frac{1}{s}$
1	$\frac{1}{r'}+\frac{1}{2}$

<u>The limit orders</u> $\lambda_A(r,s,\pi_p)$ <u>and</u> $\lambda_A(s,r,\nu_{p'})$ <u>with $1 \leqslant p < 2$</u>

Since by theorem 2 and 2* for $1 \leqslant r \leqslant 2$ we have

$$\lambda_A(r,s,\pi_p) = \lambda_A(r,s,\pi_2) \quad \text{and} \quad \lambda_A(s,r,\nu_{p'}) = \lambda_A(s,r,\nu_2)$$

in the following we need only consider the case $2 < r \leqslant \infty$.

Since

$$\lambda_A(r,s,\pi_p) \leqslant \lambda_I(r,p,\pi_p) + \lambda_A(p,s,\|\cdot\|)$$

we obtain, using the known values of $\lambda_I(r,p,\pi_p)$ and $\lambda_A(p,s,\|\cdot\|)$,

$$\lambda_A(r,s,\pi_p) \leqslant 1/p + \begin{cases} 1/p' & \text{if} \quad s \geqslant p', \\ 1/s & \text{if} \quad s \leqslant p' \end{cases}.$$

On the other hand it follows from

$$\lambda_A(r,s,\pi_p) \leqslant \lambda_I(r,r',\pi_p) + \lambda_A(r',s,\|\cdot\|)$$

that

$$\lambda_A(r,s,\pi_p) \leqslant 1/r' + \begin{cases} 1/r & \text{if} \quad s \geqslant r, \\ 1/s & \text{if} \quad s \leqslant r \end{cases}.$$

In each case, choosing the best result we obtain

$$(7) \qquad \lambda_A(r,s,\pi_p) \leqslant \begin{cases} 1 & \text{if} \quad p' \leqslant r \leqslant \infty, \quad p' \leqslant s \leqslant \infty, \\ 1/p + 1/s & \text{if} \quad p' \leqslant r \leqslant \infty, \quad 1 \leqslant s \leqslant p', \\ 1 & \text{if} \quad 2 \leqslant r \leqslant p', \quad r \leqslant s \leqslant \infty, \\ 1/r' + 1/s & \text{if} \quad 2 \leqslant r \leqslant p', \quad 1 \leqslant s \leqslant r. \end{cases}$$

Moreover,

$$\lambda_A(s,r,\nu_{p'}) \leqslant \lambda_A(s,\infty,\|\cdot\|) + \lambda_I(\infty,r,\nu_{p'})$$
$$\leqslant 1/s + \begin{cases} 1/r & \text{if} \quad r \leqslant p', \\ 1/p' & \text{if} \quad r \geqslant p', \end{cases}$$

and

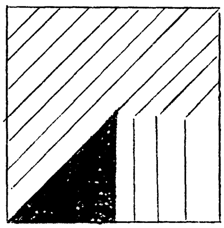
$$\lambda_A(s,r,\nu_{p'}) \leqslant \lambda_A(s,2,\nu_{p'}) + \lambda_I(2,r,\|\cdot\|)$$
$$\leqslant \lambda_A(s,2,\nu_2) \leqslant 1 \quad \text{if} \quad 2 \leqslant s \leqslant \infty.$$

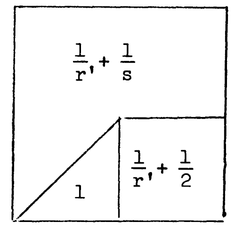
Consequently,

$$(7^*) \qquad \lambda_A(s,r,\nu_{p'}) \leqslant \begin{cases} 1 & \text{if} \quad p' \leqslant r \leqslant \infty, \quad p' \leqslant s \leqslant \infty, \\ 1/p' + 1/s' & \text{if} \quad p' \leqslant r \leqslant \infty, \quad 1 \leqslant s \leqslant p', \\ 1 & \text{if} \quad 2 \leqslant r \leqslant p', \quad r \leqslant s \leqslant \infty, \\ 1/r + 1/s' & \text{if} \quad 2 \leqslant r \leqslant p', \quad 1 \leqslant s \leqslant r. \end{cases}$$

Finally, lemma 12 implies that identity holds in (7) and (7*) .

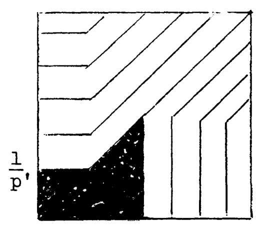
$\underline{\lambda_A(r,s,\pi_1)}$

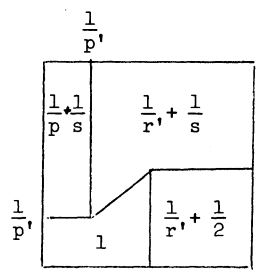




$\underline{\lambda_A(r,s,\pi_p)}$

$1 < p < 2$





<u>The limit orders $\lambda_A(r,s,\pi_p)$ and $\lambda_A(s,r,\nu_{p'})$ with $2 < p < \infty$</u>

Since by theorem 3 and 3* for $1 \leqslant s \leqslant 2$ we have

$$\lambda_A(r,s,\pi_p) = \lambda_A(r,s,\pi_2) \quad \text{and} \quad \lambda_A(s,r,\nu_{p'}) = \lambda_A(s,r,\nu_2)$$

in the following we need only consider the case $2 < s \leqslant \infty$.

From (7*) and

$$\lambda_A(r,s,\pi_p) \leqslant \lambda_A(r,s,\nu_p)$$

we obtain

$$(8) \qquad \lambda_A(r,s,\pi_p) \leqslant \begin{cases} 1/r' + 1/s & \text{if} \quad 1 \leqslant r \leqslant s, \quad 2 \leqslant s \leqslant p, \\ 1 & \text{if} \quad s \leqslant r \leqslant \infty, \quad 2 \leqslant s \leqslant p, \\ 1/r' + 1/p & \text{if} \quad 1 \leqslant r \leqslant p, \quad p \leqslant s \leqslant \infty, \\ 1 & \text{if} \quad p \leqslant r \leqslant \infty, \quad p \leqslant s \leqslant \infty \end{cases}.$$

On the other hand, we have

$$\lambda_A(s,r,\nu_{p'}) \leq \lambda_I(s,p',\nu_{p'}) + \lambda_A(p',r,\|\cdot\|)$$

$$\leq 1/p' + \begin{cases} 1/p & \text{if} \quad p \leq r, \\ 1/r & \text{if} \quad p \geq r, \end{cases}$$

and

$$\lambda_A(s,r,\nu_{p'}) \leq \lambda_I(s,2,\|\cdot\|) + \lambda_A(2,2,\nu_{p'}) + \lambda_I(2,r,\|\cdot\|)$$

$$\leq (1/2-1/s) + 1 + (1/r-1/2) \quad \text{if} \quad 1 \leq r \leq 2 \quad \text{and} \quad 2 \leq s \leq \infty .$$

In each case, choosing the best result we obtain

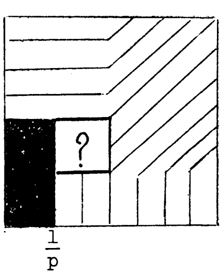
$$(8^*) \qquad \lambda_A(s,r,\nu_{p'}) \leq \begin{cases} 1/r + 1/s' & \text{if} \quad 1 \leq r \leq 2, \quad 2 \leq s \leq p, \\ 1 & \text{if} \quad p \leq r \leq \infty, \quad 2 \leq s \leq p, \\ 1/r + 1/p' & \text{if} \quad 1 \leq r \leq p, \quad p \leq s \leq \infty, \\ 1 & \text{if} \quad p \leq r \leq \infty, \quad p \leq s \leq \infty . \end{cases}$$

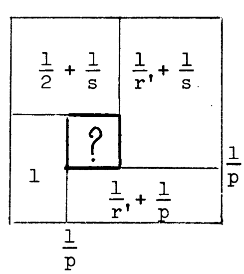
Because the square

$$Q_{A,p} := \{ (1/r', 1/s) : 2 < r < p, \quad 2 < s < p \}$$

does not appear in (8*), we have the open problem wether identity holds for all r and s in (8).

$$\frac{\lambda_A(r,s,\pi_p)}{2 < p < \infty}$$





Final remark (Cf. end of § 5)

Let L_r and L_s be infinite dimensional. Then $\rho_p(L_r,L_s)$ is strictly increasing

1) if $2 \leqslant r \leqslant \infty$, $1 \leqslant s \leqslant 2$, and $r' \leqslant p \leqslant 2$ since $\lambda_A(r,s,\pi_p) = 1/p + 1/s$,
2) if $1 \leqslant r \leqslant 2$, $2 \leqslant s \leqslant \infty$, and $2 \leqslant p \leqslant s$ since $\lambda_A(r,s,\pi_p) = 1/p + 1/r'$,
3) if $2 \leqslant r \leqslant \infty$, $2 \leqslant s \leqslant \infty$, and $r' \leqslant p \leqslant s$ since $\lambda_I(r,s,\pi_p) = 1/p$.

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