MÉMOIRES DE LA S. M. F.

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An average form of Artin's conjecture

AN AVERAGE FORM OF ARTIN'S CONJECTURE

by

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A more detailed account of the theorems described in this lecture is published in [1] .

The order, $e_p(a)$, of a modulo p is defined to be the least positive integer n for which $a^r \equiv 1 \pmod{p}$. It is well-known that there are $\phi(d)$ numbers of order d in any interval of length p . In particular, there are $\phi(p-1)$ primitive roots in any interval of length p . We define

$$M_p(N) = \sum_{\substack{a \le N \\ e_p(a)=p-1}} 1 , \qquad (1)$$

and deduce from the above remark that

$$M_p(N) = \phi(p-1) \left\{ \frac{N}{p} + 0(1) \right\} . \qquad (2)$$

Using the notation

$$N_a(x) = \sum_{\substack{p \le x \\ e_p(a)=p-1}} 1 . \qquad (3)$$

Artin conjectured that

$$N_a(x) \sim A(a) \, x / \log x \ .$$

Forty years elapsed before Hooley proved that

$$N_a(x) = \frac{A(a) \, x}{\log x} + 0 \left(\frac{x \log \log x}{\log^2 x} \right) ,$$

is a consequence of the Riemann hypothesis holding for certain algebraic number fields.

A year later Goldfeld showed, without any hypothesis, that

$$N_a(x) = A \, \mathrm{li}(x) + 0 \left(x / (\log x)^D \right) \qquad (4)$$

where $A = \prod_p \left(1 - \frac{1}{p(p-1)} \right)$ is true for almost all $a \le N$ where N satisfied

$$x \ge N > \exp \left\{ c_1 (\log x \log \log x)^\delta \right\} .$$

He also found a bound for the number of exceptions.

In this note we shall outline the proof of the following theorem.

THEOREM A .   If $N > \exp \{4(\log x \log \log x)^{1/2}\}$, then

$$\frac{1}{N} \sum_{a \leq N} N_a(x) = A \operatorname{li}(x) + 0(\frac{x}{(\log x)^D}) \qquad (5)$$

where D is an arbitrary constant greater than 1 .

By proving the following theorem it is possible to deduce a bound for the number of exceptions to (4) . We only state it here.

THEOREM B.   If $N > \exp \{6(\log x \log \log x)^{1/2}\}$, then

$$\frac{1}{N} \sum_{a \leq N} \{N_a(x) - A \operatorname{li}(x) \}^2 \ll x^2 / (\log x)^E ,$$

where E is an arbitrary constant greater than 2 .

The following lemma may be proved using the Siegel-Walfisz theorem.

LEMMA.   For any choice of the constant D we have

$$\sum_{p \leq x} \frac{\phi(p-1)}{p} = A \operatorname{li}(x) + 0 (\frac{x}{(\log x)^D}) . \qquad (6)$$

We define

$$c(\chi) = \frac{1}{(p-1)} \sum_a' \chi(a)$$

where $\sum_a'$ runs over all the primitive roots in an interval of length p . Then

$$\sum_{\chi \pmod p} c(\chi) \chi(a) = \begin{cases} 1 & \text{if } a \text{ is a primitive root mod } p , \\ 0 & \text{else.} \end{cases}$$

It is clear that

$$c(\chi_0) = \phi(p-1) / (p-1) .$$

If $\chi \neq \chi_0$, then we may express the sum form $c(\chi)$ in terms of Ramanujan sums to obtain

$$|c(\chi)| \leq 1 / \operatorname{ord} \chi ,$$

where ord χ is the least integer d for which $\chi^d = \chi_0$.

Then we have

$$\frac{1}{N} \sum_{a \le N} N_a(x) = \frac{1}{N} \sum_{p \le x} M_p(N)$$

$$= \frac{1}{N} \sum_{p \le x} \sum_{a \le N} \sum_{\chi (\bmod\ p)} c(\chi)\, \chi(a)$$

$$= \frac{1}{N} \sum_{p \le x} \sum_{\chi (\bmod\ p)} c(\chi) \sum_{a \le N} \chi(a)$$

$$= \frac{1}{N} \sum_{p \le x} \frac{\phi(p-1)}{(p-1)} \{[N] - [N/p]\}$$

$$+ O\left(\frac{1}{N} \sum_{\substack{p \le x}} \sum_{\substack{\chi (\bmod\ p) \\ \chi \ne \chi_0}} \frac{1}{\operatorname{ord}\chi} \left| \sum_{a \le N} \chi(a)\right|\right)$$

$$= \sum_{p \le x} \frac{\phi(p-1)}{p} + O\left(\frac{x}{N \log x}\right) + O\left(\frac{S_1}{N}\right) , \quad (7)$$

where

$$S_1 = \sum_{\substack{p \le x}} \sum_{\substack{\chi (\bmod\ p) \\ \chi \ne \chi_0}} \frac{1}{\operatorname{ord}\chi} \left| \sum_{a \le N} \chi(a)\right| .$$

Since any non-principal character mod p is primitive, we have

$$S_1 = \sum_{p \le x} \sum_{\chi (\bmod\ p)}^{*} \frac{1}{\operatorname{ord}\chi} \left| \sum_{a \le N} \chi(a)\right|$$

where $\sum^{*}$ denotes summation over primitive characters. We now use Holder's inequality to obtain

$$S_1^{2r} \le \left\{ \sum_{p \le x} \sum_{\chi (\bmod\ p)}^{*} \left(\frac{1}{\operatorname{ord}\chi}\right)^{2r/(2r-1)}\right\}^{2r-1} \sum_{p \le x} \sum_{\chi (\bmod\ p)}^{*} \left| \sum_{a \le N} \chi(a)\right|^{2r} . \quad (8)$$

The first sum on the right hand side of this inequality is easily estimated to be

$$\ll (c_2 x)^{2r-1} . \qquad\qquad (9)$$

The second sum is

$$S_2 = \sum_{p \le x} \sum_{\chi (\bmod\ p)}^{*} \left| \sum_{a \le N} \chi(a)\right|^{2r}$$

$$= \sum_{p \le x} \sum_{\chi (\bmod\ p)} \left| \sum_{a \le N^r} \tau'_r(a)\, \chi(a)\right|^2$$

where $\tau'_r(a)$ is the number of ways of writing a as the product of r factors each of which is less than N . If we define $\tau_r(a)$ to be the number of ways of

writing a as the product of r factors, then $\tau'_r(a) \leq \tau_r(a)$. Then large sieve inequality (see Gallagher [2]) gives

$$S_2 \ll (x^2 + N^r) \sum_{a \leq N^r} \{\tau'_r(a)\}^2$$

$$\ll (x^2 + N^r) \sum_{a \leq N^r} \tau_r^2(a) . \qquad (10)$$

Mardjanichvili [3] estimated the sum on the right hand side of this inequality to be

$$\ll N^r \{\log(e \ N^r)\}^{r^2-1} . \qquad (11)$$

   We thus have from (8) , (9) , (10) and (11) that

$$S_1^{2r} \ll (c_2 x)^{2r-1} (x^2 + N^r) \ N^r \{\log (e \ N^r)\}^{r^2-1} .$$

By choosing $r = \left[\dfrac{2 \log x}{\log N} \right] + 1$, we deduce

$$S_1 \ll x^{1-1/2r} \ N \{\log(e \ x^2)\}^{(r^2-1)/2r}$$

or,

$$S_1/N \ll x / (\log x)^D .$$

This together with (6) and (7) yields (5).

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REFERENCES

[1] P.J. STEPHENS. - "An average result for Artin's conjecture", Mathematika, 16
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[2] P.X. GALLAGHER. - "The large sieve", Mathematika 14, (1967), 14-20.

[3] C. MARDJANICHVILI. - Doklady Akad. Nauk SSSR, 22 (1939), n° 7.

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