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ON THE JUMPS IN THE SERIES OF RAMIFICATIONS GROUPS

by

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1. Notations.

Let k be a finite extension of the field Q_p of rational p-adic numbers, which does not contain the p-th roots of unity. We denote by n the ramification index of k/Q_p and by $q = p^f$ the number of elements in the residue class field of k.

Let K be a finite extension of k with Galois group G. In G we have the descending chain of ramification groups G_j (For definitions and properties, used in the following, lool at CL, IV). We denote by v_i the jumps in this series for $j \geq 1$:

$$G \supseteq G_o \supseteq G_1 = \dots = G_{v_1} \supset G_{v_1+1} = \dots = G_{v_2} \supset \dots \supset G_{v_{r-1}+1} = \dots = G_{v_r} \supset G_{v_r+1} = 1.$$

We put in addition $v_o = 0$, $v_{r+1} = \infty$.

The jumps v_i together with the orders $g_i = (G_{v_i} : 1)$, $i = 0 , - , r$, constitute the numerical ramification invariants of K/k . We propose to study these invariants in dependence of the group structure of G . In order to do this, it proves more convenient, to work with a system of invariants, which is equivalent to the system (v_i, g_i) :

Let $\varphi_{K/k}$ be the Hasse-function of K/k , that is the piecewise linear function starting at 0 , with breaks at the v_i and derivation $\frac{g_{i+1}}{g_o}$ between v_i and v_{i+1} . Then the $t_i = \varphi_{K/k}(v_i)$ are the breaks of the inverse function $\psi_{K/k}$ of $\varphi_{K/k}$. We shall investigate the system (t_i, g_i) , that is essentially the set $T(K/k) = \{t_i , i=1,-,r\}$.

Corresponding to the use of the t_i we use the "upper" numbering of the ramification groups, defined by

$$G^r = G_{\{\psi_{K/k}(r)\}} \qquad \text{for all real } r \geq 0 .$$

Then $T(K/k)$ is the set of jumps in the upper numbering and we shall call the t_i the upper jumps of K/k . Let L be a normal extension of k within K and let N be the corresponding subgroup of L in G. The following properties are fundamental :

$$(G/N)^x = G^x N/N \qquad \text{(Herbrand)}$$

$$\varphi_{K/k} = \varphi_{L/k} \circ \varphi_{K/L} \qquad \text{(Tamagawa)}$$

An immediate consequence is : $T(L/k) \subseteq T(K/k)$.

Remarks : 1) As we are only interested in wild ramification we can and shall restrict ourselves to normal p-extensions, where p is the characteristic of the residue class field.

2) The restriction on regular p-adic number fields k is not essential. Similar results hold for more general types of local fields, including those where class field theory is not available.

2. Abelian Extensions.

In the case of abelian extensions class field theory provides a full description of the ramification groups (CL XV). We recall the fundamental result of Hasse [1]: The upper jumps of an abelian extension are integers.

But there are further restrictions in the abelian case, depending on the structure of G. We give two exemples :

PROPOSITION 1. *Let K/k be an abelian extension of exponent p^n. Then $T(K/k)$ is contained in the set of all natural numbers $t \le n(n + \frac{1}{p-1})$ with*

$$t \not\equiv 0 \mod p^n \quad \text{for} \quad t - \frac{n}{p-1} \le n$$

$$t \not\equiv in \mod p^{n-i} \quad \text{for} \quad in < t - \frac{n}{p-1} \le (i+1)n \ ,$$

$$i = 1, \ldots, n-1 \ .$$

If K is the maximal abelian extension of k with exponent p^n, then $T(K)$ consists of all t with the above properties.

The other exemple gives a full description of $T(K/k)$ in the case of cyclic extensions.

$$\text{Let } P(t) = \begin{cases} p^t & \text{if} \quad t \le \frac{n}{p-1} \\ t+n & \text{if} \quad t > \frac{n}{p-1} \end{cases}$$

PROPOSITION 2. [3] : <u>Let</u> $t_1 \leq t_2 \leq \dots \leq t_n$ <u>be a finite set</u> T <u>of natural numbers. Then there exists a purely ramified cyclic extension</u> K/k <u>with</u> T(K/k)=T <u>if and only if</u>

(1) $t_{i+1} \geq P(t_i)$ <u>for</u> $i = 1,\dots,n-1$ <u>and</u>

(2) $t_i = P^m(t')$ <u>with</u> $0 < t' \leq \frac{np}{p-1}$, $p \nmid t'$ <u>implies</u> $t' \in T$.

COROLLARY. <u>Let</u> $p^{\nu-1} \leq \frac{n}{p-1} < p^{\nu}$ <u>and let</u> K/k <u>be a purely ramified cyclic extension of degree</u> $p^n \geq p^{\nu}$. <u>Then</u>

$$t_{\nu+i} = t_{\nu} + in \qquad i = 0,\dots,n-\nu \ .$$

This Corollary has been obtained also by Wyman [7] and Tate [6] .

Remark : In general the characterisation of the numerical ramification invariants of an abelian extension K/k in terms of the structure invariants of G cannot be reduced to the cyclic case. This can be done if and only if K/k is the composite of arithmetically disjoint cyclic extension in the sense of [3] . The results of [4] imply that in general there is no such decomposition of K/k . For further results on abelian extensions see also Marschall [2].

3. Non-abelian extensions.

It is well known, that there are extensions with broken upper jumps and we shall see in a moment, that "in general" one has to expect broken upper jumps in the non-abelian case. On the other hand, there exist non-abelian extensions with entire upper jumps. More generally : given a normal extension K/k , then there exists an extension $\bar{k}/k$ such that $K\bar{k}/\bar{k}$ has group theoretically the same ramification groups as K/k but with entire upper jumps [3].

I do not know if there is a class of groups, strictly larger than the class of the abelian ones, for which the upper jumps are always integers.

Now let us investigate the upper jumps in dependence of the length of the series of Frattini subgroups and of a central series of G .

Let $K_o = k$ and let k_i be the maximal abelian extension of k_{i-1} of exponent p .

Let $k_{(o)} = k$ and let $k_{(i)}$ be the maximal abelian extension of $k_{(i-1)}$ of exponent p such that $\mathrm{Gal}(k_{(i)} \mid k_{(i-1)})$ is contained in the center of $\mathrm{Gal}(k_{(i)}/k)$.

By class field theory we know that each finite p-extension of k is contained in some k_i resp $k_{(j)}$.

The results on abelian extensions enable us, to compute $T(k_n/k)$, but it would be rather lengthy to give the full result. We content ourselves by giving an approximative result, which concerns the growth and the denominators of the upper jumps.

Let t_{r_n} (resp. $t_{r_{(n)}}$) be the last upper jump of k_n/k (resp. $k_{(n)}/k$) .

PROPOSITION 3. <u>There are positive constants</u> c_1 , c_2 , <u>depending only on</u> k , <u>such that</u>

$$c_1 < n\frac{n}{p-1} - t_{r_n} < c_2 .$$

<u>One may take for instance</u> : $c_1 = \frac{1}{q}$, $c_2 = 1 + \frac{3}{q-1}$ <u>for</u> $n > 1$.

<u>Remark</u> : a somewhat weaker result may be found in [3].

PROPOSITION 4. <u>Let</u> m <u>be determined by</u> $2^m \leq n < 2^{m+1}$. <u>Then there are constants</u> c_1 , c_2 , <u>depending only on</u> k , <u>such that</u>

$$c_1 < t_{r_{(n)}} - n\,n < m\frac{n}{p-1} + c_2 .$$

<u>One may take for instance</u> : $c_1 = \frac{n}{p-1} - 2$, $c_2 = \frac{n}{p-1}$.

Let

$$R_n = \{t \in Q \; ; \; t = \frac{a}{q^{1+p+\ldots+p^{n-3}}} \; , \; a \in \mathbb{N} \; , \; 1 \leq t \leq t_{r_n}\} \text{ for } n \geq 3$$

$$R'_n = \{t \in Q \; ; \; t = \frac{a}{e_{n-1}} \; , \; a \in \mathbb{N} \; , \; 1 \leq t \leq t_{r_n}\}$$

Here e_n is the ramification index of k_n/k , given recursively by

$$e_n = e_{n-1}\, q^{p^{n-1} n\, e_{n-1}} .$$

PROPOSITION 5.

$$R_n \subseteq T(k_n/k) \subseteq R'_n \quad \underline{\text{for}} \quad n \geq 3$$

$$R_{n+1} \not\subseteq T(k_n/k) \not\subseteq R'_{n-1} \quad \underline{\text{for}} \quad n \geq 2 .$$

The precise result may be found in [3] .

COROLLARY. <u>Let</u> T <u>be a finite set of positive rational numbers.</u> <u>Then there is a finite normal extension</u> K/k <u>with</u> $T(K/k) \supseteq T$.

For we can get all p-power denominators by Proposition 3 and 5 , taking a sufficiently large n . To succeed in the general case one has to begin with a suitable tamely ramified extension.

4. Infinite Extensions.

Let K be an arbitrary normal extension of k and let as before $G = \mathrm{Gal}(K/k)$. One can still define ramification groups in the upper numbering (CL IV) :

$$G^x = \lim_{\substack{\leftarrow \\ L}} G^x_{L/K} \quad ,$$

L running over all finite normal extension of k, contained in K.

Let $T(K/k) = \{t \in \mathbb{R} \; ; \; G^t \neq G^{t+\varepsilon} \text{ for all } \varepsilon > 0\}$ be the set of jumps in the upper numbering.

As usual things become more simple but less precise in passing to infinite extensions. We give some exemples :

PROPOSITION 6. Let k_{ab} be the maximal abelian extension of k. The $T(k_{ab}) = \mathbb{N}$.

This follows for instance from Proposition 1 .

PROPOSITION 7. Let Z be a totally ramified normal extension of k with $\mathrm{Gal}(Z/k) \simeq Z_p$. Then $T(Z/k)$ is a discrete set t_i of natural numbers $t_1 < t_2 < \dots$ with the following properties :

(1) $\quad t_{i+1} \geq P(t_i)$ for $i = 1,2,\dots$ and

(2) $\quad$ there is a ν with $\frac{n}{p-1} < t_\nu \leq \frac{np}{p-1}$ and $t_{i+1} = t_i + n$ for all $i \geq \nu$.

This follows from Proposition 2.

PROPOSITION 8. Let $k_{2\text{-}ab}$ be the maximal 2-step-abelian p-extension of k. Then

$$T(k_{2\text{-}ab}/k) = \bigcup_{n=1}^{\infty} \{n, n + \frac{1}{q^n}, \dots, n + \frac{q^n-1}{q^n}\} .$$

This follows from Proposition 1 .

PROPOSITION 9. Let k_{p^∞} be the maximal p-extension of k, that is the union of all finite normal p-extensions of k. Then

$$T(k_{p^\infty}/k) = \{x \in \mathbb{R} \; ; \; x \geq 1\} .$$

This follows form Proposition 3 and 5 .

PROPOSITION IO. <u>Let</u> $\bar{k}$ <u>be the algebraic closure of</u> k . <u>Then</u>

$$T(\bar{k}/k) = \{ x \in \mathbb{R} \quad , \quad x \geq 0 \} \quad .$$

This follows form the corollary of Proposition 5 .

5. Generators and defining relations.

Let $G = \mathrm{Gal}(k_{p^\infty}/k)$, let t be a real number ≥ 1 and let k_t be the subfield of k_{p^∞} , corresponding to the closed normal subgroup G^t .

Then k_t is the union of all finite normal p-extensions of k with upper jumps $< t$. One may ask, how to describe $\mathrm{Gal}(k_t/k)$ in terms of generators and defining relations. In our situation G is a free topological p-group of rank $d = [k : \mathbb{Q}_p] + 1$. Therefore $\mathrm{Gal}(k_t/k) = G/G^t$ has at most d generators and the question is, how to generate G^t as closed normal subgroup of G . This seems to be a difficult problem ; an answer must involve an arithmetic normalisation of the free generators of G . One may ask more modestly if there are finitely many defining relations, that is if there exist finitely many elements of G^t , which generate G^t as closed normal subgroups of G . In this direction we have the following result :

Let $r(t)$ be the minimal number of such generators of G^t ; let $\{t\}_n$ be the smallest number in $T(k_n/k)$, which is $\geqq t$; let $m(t)$ be the number of different $\{t\}_n$ for $n = 1,2, \ldots$.

PROPOSITION 11. $\quad r(t) \geq \mathfrak{f}\, m(t)$ <u>for all</u> $t > 1$.

COROLLARY. <u>If</u> t <u>is not a rational number with p-power denominator, then there are infinitely many defining relation for</u> $\mathrm{Gal}(k_t/k)$.

One may guess, that $r(t)$ is finite for t a rational number with p-power denominator.

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