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SYMMETRY TYPES OF HYPERELLIPTIC RIEMANN SURFACES

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Abstract. — A compact Riemann surface X is symmetric if it admits an antianalytic involution $\tau : X \to X$. Such an involution is called a real structure. Two real structures are isomorphic if they are conjugate in the full group $\mathrm{Aut}^{\pm}X$ of analytic and antianalytic automorphisms of X. In this memoir we classify up to isomorphism the real structures of all symmetric hyperelliptic Riemann surfaces of genus $g \geqslant 2$. The topological invariants of each isomorphism class are also computed. We give the list of groups which act as the full group of analytic and antianalytic automorphisms of such surfaces. Moreover, the complex algebraic curve associated to any such Riemann surface is described in terms of polynomial equations. We also find the explicit formula of a real structure in each isomorphism class.

Résumé **(Types de symétrie des surfaces de Riemann hyperelliptiques)**

Une surface de Riemann compacte X est dite symétrique si elle admet une involution antiholomorphe $\tau : X \to X$. On appelle structure réelle une telle involution. Deux structures réelles sont isomorphes si elles sont conjuguées par le groupe complet $\mathrm{Aut}^{\pm}X$ des automorphismes holomorphes et anti-holomorphes de X. Dans ce mémoire, nous classifions à isomorphisme près les structures réelles de toutes les surfaces de Riemann hyperelliptiques de genre $g \geqslant 2$. Nous calculons aussi les invariants topologiques de chaque classe d'isomorphisme. Nous donnons la liste des groupes qui agissent comme le groupe des automorphismes holomorphes et anti-holomorphes d'une telle surface. De plus, nous décrivons la courbe algébrique complexe associée à une telle surface en terme d'équations polynomiales. Nous donnons enfin une formule explicite pour une structure réelle dans chaque classe d'isomorphisme.

CONTENTS

INTRODUCTION

Let X be a compact Riemann surface. A *real structure* of X is an antianalytic involution $\tau : X \to X$. We will also say that τ is a *symmetry* of X. The surface X is said to be symmetric if it admits some real structure τ. A *real form* of X is the conjugacy class of a real structure with respect to the group $\mathrm{Aut}^{\pm}X$ of all analytic and antianalytic automorphisms of X.

The origin of these names comes from the uniformization theorem of Koebe and Poincaré, since it implies that each compact Riemann surface is conformally equivalent to an irreducible smooth complex algebraic curve. Let $F_1, \ldots, F_m$ be a set of polynomials defining such a curve X. If each F_i turns out to have real coefficients then the complex conjugation determines an antianalytic involution τ on X. Thus, X is symmetric and τ is a real structure of X. A pair (X, τ) consisting of an irreducible smooth complex algebraic curve X and an antianalytic involution τ on it, is called a *real algebraic curve*. The complex curve X is said to be its *complexification*.

Most complex algebraic curves have no real form and others have more than one. For example, let X be the elliptic curve defined by

$$X = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{C}) : x_0 x_2^2 = x_1(x_1^2 + x_0^2)\}.$$

Clearly, the restriction τ to X of the complex conjugation on $\mathbb{P}^2$ is a real structure of X. Also $\varphi \circ \tau$ is a real structure of X, where φ is the birational automorphism of X given by

$$\varphi : [x_0 : x_1 : x_2] \longmapsto [x_0 : -x_1 : ix_2], \quad \text{where} \quad i = \sqrt{-1}.$$

It is easy to see that the fixed point set $\mathrm{Fix}(\tau)$ of τ has one connected component whilst $\mathrm{Fix}(\varphi \circ \tau)$ has two; hence τ and $\varphi \circ \tau$ are non-conjugate real structures of X and so (X, τ) and $(X, \varphi \circ \tau)$ are non-isomorphic real algebraic curves with the same complexification.

Along this memoir the terms "compact Riemann surface" and "complex algebraic curve" will be used indistinctly.

Let $k \geqslant 0$ be the number of connected components of $\mathrm{Fix}(\tau)$ and let ε be the *separability character* of τ defined as $\varepsilon = -1$ if $X - \mathrm{Fix}(\tau)$ is connected and $\varepsilon = 1$ otherwise. Note that we do not exclude the possibility of $\mathrm{Fix}(\tau)$ to be empty. The parameters k and ε classify τ topologically. Clearly a conjugate in $\mathrm{Aut}^{\pm}X$ of τ is also a symmetry with the same topological type; that is, k and ε just depend on the conjugacy class of τ. So we define the *species* $\operatorname{sp}(\tau)$ of the real form represented by τ to be the integer εk. The *symmetry type* of X is the (finite) set of species of all real forms of X.

The computation of the symmetry type of a hyperelliptic Riemann surface is a classic problem, posed by Felix Klein in 1893 and solved by himself in the case $|\mathrm{Aut}^{\pm} X| = 4$. Partial solutions have appeared since, as in the cases of low genus or special families of Riemann surfaces (see below). These solutions are immediate consequence of the results in this memoir since here we completely solve this problem. Namely, *we compute the symmetry types of all compact hyperelliptic Riemann surfaces of genus* $g \geqslant 2$.

We also obtain, for each $g \geqslant 2$, *the list of groups which act as the full group of analytic and antianalytic automorphisms of a genus g symmetric hyperelliptic Riemann surface.* This extends the results of Brandt and Stichtenoth in [**4**] and Bujalance, Gamboa and Gromadzki in [**15**].

The uniformization theorem makes the theory of Fuchsian groups a fruitfull technique to deal with compact Riemann surfaces. However, there is an increasing interest in describing them via defining equations. In this memoir *we compute explicit polynomial equations of each symmetric hyperelliptic Riemann surface. The formula of a representative of each real form is also given.*

The most elementary case for computing symmetry types is that of algebraic curves of genus zero. The Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ admits exactly two real forms, namely those represented by the symmetries

$$\tau_1 : z \longmapsto \overline{z} \quad \text{and} \quad \tau_2 : z \longmapsto \frac{-1}{\overline{z}}.$$

The first fixes the real axis, which disconnects $\widehat{\mathbb{C}}$, while the second is fixed-point free. Therefore, the symmetry type of $\widehat{\mathbb{C}}$ is $\{1, 0\}$. The case of curves of genus one was completely solved by Alling [**1**]. Elliptic curves are tori, and each torus is isomorphic to the quotient $X_\gamma = \mathbb{C}/L_\gamma$ where $L_\gamma = \mathbb{Z} + \gamma\,\mathbb{Z}$ and $\gamma \in P = \{\gamma \in \mathbb{C} : |\gamma| \geqslant 1,\ |\mathrm{Re}(\gamma)| \leqslant 1/2\}$. The symmetric tori correspond to the points of P on the imaginary axis or on the boundary of P. For $\gamma = i$, the symmetry type of X_γ is $\{-1, 0, 2\}$. If $\mathrm{Re}(\gamma) = 0$ and $\mathrm{Im}(\gamma) > 1$ then the symmetry type of X_γ is $\{0, 0, 2, 2\}$, and the symmetry type of the other symmetric tori is $\{-1, -1\}$.

Among the pioneers in the study of real forms of a complex algebraic curve, Harnack [**32**], Weichold [**52**] and Klein [**35**] stand out. The first two determined the admissible values of the species of the real forms of a curve of genus g. As said above, Klein obtained the first result concerning the symmetry types of curves of genus $g \geqslant 2$. More precisely, he proved that the symmetry type of a hyperelliptic curve of genus $g \geqslant 2$ whose group of automorphisms has order 4 is one of the following: $\{-1, -1\}, \{-2, -2\}, \ldots, \{-g, -g\}$, $\{g + 1, g + 1\}$ or $\{0, 1\}$ if g is even, and $\{-1, -1\}, \{-2, -2\}, \ldots, \{-g, -g\}$, $\{g + 1, g + 1\}$, $\{0, 2\}$ or $\{0, 0\}$ if g is odd.

After these pioneer works, interest was lost in studying real forms of algebraic curves until the seventies, with the foundational research of Alling and Greenleaf [**2**], Earle [**24**] and Gross and Harris [**31**]. Moreover, the development of proper techniques of the real algebraic geometry, see for example the book of Bochnak, Coste and Roy [**3**], has propelled ahead this new field of research. Alling and Greenleaf studied systematically *Klein surfaces*, mainly compact ones, which may be seen as quotient spaces $X/\langle\tau\rangle$ where τ is a real structure of the compact Riemann surface X. With the obvious definition, conjugate real structures give rise to isomorphic Klein surfaces. They showed that the categories of compact Klein surfaces and real algebraic curves are equivalent. Earle introduced the moduli of compact Riemann surfaces with symmetries, while Gross and Harris showed, among other things, that the invariants k and ε of a symmetry are determined by the first homology group $H_1(X, \mathbb{Z}_2)$, and conversely. They also described the topology of hyperelliptic real algebraic curves.

Related to the problem of existence of symmetries in a Riemann surface, we mention here the work of Singerman [**48**]. He obtained conditions for a Riemann surface with large automorphism group to be symmetric. For example, he showed that all Riemann surfaces admitting automorphisms of order greater than $2g + 2$ are symmetric. However, he also exhibited an example of a Riemann surface having Hurwitz automorphism group which is not symmetric. (A Riemann surface of genus $g \geqslant 2$ has Hurwitz automorphism group if it admits the maximum number $84(g - 1)$ of automorphisms that a genus g Riemann surface may admit.)

In the same line of Klein's results quoted above, Bujalance and Singerman [**18**] calculated the 18 symmetry types of symmetric Riemann surfaces of genus 2. Since all such surfaces are hyperelliptic, these symmetry types appear naturally in this memoir. They showed, for example, that such a surface always admits a real form with non-zero species. They also characterized, in terms of the full group of automorphisms, the surfaces admitting a unique real form. Explicit polynomial equations for these surfaces and their real forms have been calculated by Cirre in [**21**], where the same description has also been done for the family of curves admitting the maximum number of real forms with non-zero species. More recently, Melekoğlu in [**39**] has calculated the symmetry types of curves of genus 3.

It is worthwhile mentioning other results in the same line. For example, Natanzon obtained in [**41**], [**42**] and [**43**] the symmetry types of those algebraic curves of genus g admitting a real form of species $g+1$ or $-g$. Using combinatorial methods, Bujalance and Costa in [**9**] also studied the symmetries of these curves. In [**12**] Bujalance, Costa and Gamboa calculated the symmetry types of the algebraic curves whose group of analytic automorphisms has prime order. This extends Klein's results quoted above. Bujalance and Costa [**10**] found the symmetry type $\{-2, 0\}$ of the famous Macbeath's curve of genus 7. It must be pointed out that the aid of the symbolic language CAYLEY has been very useful to compute finite generating sets and conjugacy classes of some groups, and also to decide the separability character of some symmetries. More recently, Broughton, Bujalance, Costa, Gamboa and Gromadzki in [**5**] and [**6**] obtained the symmetry types of those curves on which $\mathrm{PSL}(2,q)$ acts as a Hurwitz automorphism group, and of the Accola-Maclachlan and Kulkarni curves, respectively.

All these last results were obtained by the combinatorial methods to be explained below. By using purely algebraic arguments, Turbek [**51**] calculated the symmetry type of the so called Kulkarni curve. It should be remarked that only this last paper, [**6**] and [**21**] provide explicit formulae for the symmetries representing the real forms.

Sometimes it is helpful to know, before computing the symmetry type of an algebraic curve, the number of their real forms. To that end, some upper bounds have been obtained in the last twenty years. Natanzon [**44**], using topological methods, proved that an algebraic curve of genus g has at most $2(\sqrt{g}+1)$ real forms of nonzero species. He also showed that this bound is attained for infinitely many values of g, those of the form $g=(2^n-1)^2$. Later on, Bujalance, Gromadzki and Singerman [**17**] obtained a combinatorial proof of this result and proved that these are the only values of g for which the bound is sharp. This has been considerably improved recently by Bujalance, Gromadzki and Izquierdo [**16**]. If $g=1+2^{r-1}u$ with u odd, then every algebraic curve of genus g has at most 2^{r+1} real forms with nonzero species. In particular, it follows a striking corollary which was first proved by Gromadzki and Izquierdo [**30**]: each algebraic curve of even genus has at most 4 real forms with nonzero species. A bound for the number of real forms with zero species will appear in the paper [**8**] by Bujalance, Conder, Gamboa, Gromadzki and Izquierdo.

Also related with this subject we mention here the papers by Natanzon [**45**], Singerman [**50**] and Gromadzki [**28**], [**29**], where they get upper bounds for the sum of the number of connected components of the real structures of an algebraic curve. In particular the hyperelliptic case is treated.

Other results concerning topological properties of symmetries of Riemann surfaces have been obtained by Bujalance, Costa, Natanzon and Singerman in [**13**], Bujalance and Costa in [**11**] and Izquierdo and Singerman in [**34**].

Closely connected with the study of symmetries of hyperelliptic algebraic curves is that of the so called *pseudo-symmetries* due to Singerman [**49**]. Each symmetry τ of the hyperelliptic curve X induces a symmetry $\widehat{\tau}$ of the Riemann sphere $\widehat{\mathbb{C}}$. However the converse is not always true: some symmetries $\widehat{\tau}$ of $\widehat{\mathbb{C}}$ admit liftings $\tau : X \to X$ of order 4. They are called pseudo-symmetries and will appear in a natural way in our work.

Computational aspects in the theory of Riemann surfaces are an increasing subject of research. One of the main goals is to pass explicitly between defining equations, Fuchsian groups and period matrices. This is the classical uniformization problem. Among the recent results in this direction, we mention here the paper by Gianni, Seppälä, Silhol and Trager [**26**] where they have designed an algorithm to compute a representation of a compact Riemann surface as an algebraic curve and to approximate its period matrix. The family of hyperelliptic Riemann surfaces is the best source of study. They have the good property that an equation for such a curve can be obtained from period matrices via Theta characteristics. In turn, a standard period matrix can be recovered from an equation of the curve, as done by Buser and Silhol in [**20**] for the case of hyperelliptic real curves.

The memoir is organized as follows. In Chapter 1, the combinatorial methods to be employed in the subsequent chapters are described in detail. Special interest have Subsections 1.1.4 and 1.3.2. In the first one we explain how to decide if a finite group acting on a compact Riemann surface as a group of automorphisms coincides with the full automorphism group. In Subsection 1.3.2 it is explained how to compute the species of the liftings of a given symmetry on the Riemann sphere, Theorems 1.3.4 and 1.3.5.

Chapter 2 is devoted to find presentations of the full group of automorphisms $\mathrm{Aut}^{\pm}X$ of a symmetric hyperelliptic Riemann surface X. This allows us to count the number of conjugacy classes of symmetries in $\mathrm{Aut}^{\pm}X$, and so the number of real forms of X.

Chapter 3 is the core of the memoir. We compute the symmetry type of any hyperelliptic Riemann surface X. To that end we use both a combinatorial and a geometric method. The latter also allows us to get polynomial equations of X and the formula of a representative of each of its real forms. The chapter is naturally divided into ten sections, according to the nature of the automorphism group $\mathrm{Aut}^{\pm}X_{\widehat{\mathbb{C}}}$ induced by $\mathrm{Aut}^{\pm}X$ on the Riemann sphere.

The referee proposed us the following problem: find a complete set of invariants that could distinguish conjugacy classes of symmetries on hyperelliptic Riemann surfaces. We thank here him or her for this very interesting question. In case of genus $g = 0$, we already pointed out that the species itself constitutes such a complete set of invariants: the unique two conjugacy classes of symmetries on the sphere have different species, which are $+1$ for those conjugate to complex conjugation and 0 for those conjugate to the antipodal map. The situation is much more involved in higher genus, and as far as we know it remains open except in some particular cases, see *e.g.,* [**11**]. In this paper, it is given a criterion to distinguish between symmetries with the same species in case their fixed point sets have a large number of connected components. However, the answer relies on the knowledge of the full automorphism group $\mathrm{Aut}^{\pm}X$ of X. Once such a complete set of invariants were found, the natural subsequent question would be to exhibit a representative of each conjugacy class. In this memoir we have achieved this last goal avoiding the first one.

CHAPTER 1

PRELIMINARIES

1.1. Klein surfaces and NEC groups

A classical Riemann surface is a topological surface without boundary together with an analytic structure. This analytic structure makes it orientable. However, nonorientable or orientable surfaces with boundary may admit a *dianalytic structure* which behaves in many aspects as the analytic structure of a classical Riemann surface. Roughly speaking, a dianalytic structure is the equivalence class of an atlas whose transition functions are either analytic or antianalytic (a function is antianalytic if its composite with complex conjugation is analytic). The easiest way in which such a non-classical surface K arises is as the quotient of a classical Riemann surface S under the action of an *antianalytic involution.* Such a quotient is usually known as *Klein surface.* For the time being this is the definition of Klein surface we shall use (see [**2**, section 1.2] for a rigorous one). We define the genus of K as that of S. Throughout this memoir, unless otherwise stated, all surfaces considered will be compact of genus $g \geqslant 2$.

1.1.1. NEC groups. — Uniformization theorem for Riemann surfaces also has its counterpart for Klein surfaces. Namely, every Klein surface can be viewed as the quotient of the hyperbolic plane $\mathcal{H}$ under the action of certain subgroup of the group $\mathrm{Aut}^{\pm}\mathcal{H}$ of analytic and antianalytic selfhomeomorphisms of $\mathcal{H}$. As a trivial consequence of the maximum modulus principle it follows that

$$\begin{aligned}\mathrm{Aut}^{\pm}\mathcal{H} = &\left\{ f : z \mapsto \frac{az+b}{cz+d} \ \text{ with } \ \{a,b,c,d\} \subset \mathbb{R} \ \text{ and } \ ad-bc>0 \right\} \\ &\cup \left\{ f : z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d} \ \text{ with } \ \{a,b,c,d\} \subset \mathbb{R} \ \text{ and } \ ad-bc<0 \right\}.\end{aligned}$$

The elements of $\mathrm{Aut}^{\pm}\mathcal{H}$ are called automorphisms of $\mathcal{H}$. We denote by $\mathrm{Aut}\,\mathcal{H}$ the subgroup of its analytic automorphisms.

Let $\mathrm{GL}(2;\mathbb{R})$ be the group of 2×2 non-singular matrices with real entries. It is clear that the mapping

$$\begin{aligned}\mathrm{GL}(2;\mathbb{R}) &\longrightarrow \mathrm{Aut}^{\pm}\mathcal{H}\\ A=\begin{pmatrix} a & b\\ c & d\end{pmatrix} &\longmapsto f_A : z \mapsto \begin{cases} \dfrac{az+b}{cz+d} & \text{if} \quad \det A>0,\\[2ex] \dfrac{a\overline{z}+b}{c\overline{z}+d} & \text{if} \quad \det A<0\end{cases}\end{aligned}$$

is a group epimorphism with kernel $\{\lambda I_2 : \lambda\in\mathbb{R}^*\}$, where I_2 is the identity matrix. Thus we may identify $\mathrm{Aut}^{\pm}\mathcal{H}$ with the quotient group $\mathrm{PGL}(2;\mathbb{R}) := \mathrm{GL}(2;\mathbb{R})/\{\lambda I_2\}$. Thus $\mathrm{Aut}^{\pm}\mathcal{H}$ *is a topological group* and it makes sense to talk about its discrete subgroups.

Definitions 1.1.1

(1) Let Γ be a subgroup of $\mathrm{Aut}^{\pm}\mathcal{H}$. We say that Γ is a *non-euclidean crystallographic group* (shortly NEC group) if it is a discrete subgroup and the quotient space $\mathcal{H}/\Gamma$ is compact.
(2) An NEC group Γ is said to be a *Fuchsian group* if it is contained in $\mathrm{Aut}\,\mathcal{H}$. Otherwise Γ is said to be a *proper* NEC group.
(3) Given a proper NEC group Γ, its *canonical Fuchsian group* is $\Gamma^+ := \Gamma\cap\mathrm{Aut}\,\mathcal{H}$. Obviously, $[\Gamma:\Gamma^+]=2$ and Γ^+ is the unique subgroup of index 2 in Γ contained in $\mathrm{Aut}\,\mathcal{H}$.

If Γ is an NEC group then the quotient space $\mathcal{H}/\Gamma$ can be endowed with a structure of a (compact) Klein surface (see [**2**, 1.8.4]). A fundamental region for Γ can be constructed as a convex bounded hyperbolic polygon with a finite number of sides. A suitable labelling of the sides gives the following *canonical surface symbol:*

$$(+) \qquad \alpha_1\beta_1\alpha_1'\beta_1'\dots\alpha_g\beta_g\alpha_g'\beta_g'\xi_1\xi_1'\dots\xi_r\xi_r'\varepsilon_1\gamma_{10}\dots\gamma_{1s_1}\varepsilon_1'\dots\varepsilon_k\gamma_{k0}\dots\gamma_{ks_k}\varepsilon_k'$$

if $\mathcal{H}/\Gamma$ is orientable, or

$$(-) \qquad \alpha_1\alpha_1^*\dots\alpha_g\alpha_g^*\xi_1\xi_1'\dots\xi_r\xi_r'\varepsilon_1\gamma_{10}\dots\gamma_{1s_1}\varepsilon_1'\dots\varepsilon_k\gamma_{k0}\dots\gamma_{ks_k}\varepsilon_k'$$

otherwise. A primed side is paired to the corresponding unprimed side by means of an orientation-preserving automorphism while a starred side is paired to the corresponding unstarred side by means of an orientation reversing automorphism.

Taking into account the surface symbol it is possible to obtain the following presentation of Γ :

generators:

- $x_1,\dots,x_r$ (elliptic elements);
- $c_{10},\dots,c_{1s_1},\dots,c_{k0},\dots,c_{ks_k}$ (reflections);
- $e_1,\dots,e_k$ (orientation preserving elements);
- $a_1,b_1,\dots,a_g,b_g$ (hyperbolic elements) in case $(+)$;
- $d_1,\dots,d_g$ (glide reflections) in case $(-)$;

and relations:
- $x_i^{m_i} = 1$ for $1 \leqslant i \leqslant r$;
- $c_{is_i} = e_i^{-1} c_{i0} e_i$ for $i = 1, \dots, k$;
- $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1} c_{ij})^{n_{ij}} = 1$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant s_i$;
- $x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1$ in case $(+)$;
- $x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_g^2 = 1$ in case $(-)$.

Along this memoir, a set of generators as the above will be called *a set of canonical generators* of Γ.

The first presentations for NEC groups appeared in [**53**] and their structure was clarified by the introduction of signatures in [**36**].

1.1.2. Signatures

Definitions 1.1.2

(1) An (abstract) *signature* is a collection of symbols and non-negative integers of the form

$$\sigma = (g; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}).$$

If the sign "+" appears then we write $\text{sign}(\sigma) =$ "+"; otherwise $\text{sign}(\sigma) =$ "$-$". The integers $m_1, \dots, m_r$ are called the *proper periods of* σ and the n_{ij} are called the *link periods* of the period cycle $(n_{i1}, \dots, n_{is_i})$. An empty set of proper periods, (*i.e.,* $r = 0$), will be denoted by $[-]$, an empty period-cycle (*i.e.,* $s_i = 0$) by $(-)$, and the fact that σ has no period-cycles (*i.e.,* $k = 0$) by $\{-\}$. The non-negative integer g is called the *orbit genus* of σ.

(2) Now given an NEC group Γ with the above presentation we define its *signature* $\sigma(\Gamma)$ as

$$\sigma(\Gamma) = (g; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$$

the sign of $\sigma(\Gamma)$ being "+" in the case $(+)$ and "$-$" otherwise. Since a Fuchsian group contains no orientation reversing elements, its signature has no period cycles and its sign is always "+". Hence we may drop such data and in the sequel the signature of a Fuchsian group will be represented simply by

$$(g; [m_1, \dots, m_r]).$$

In the obvious manner, a presentation of an NEC group Γ can be read off from its signature. In fact, signatures give a procedure to classify NEC groups up to isomorphism.

Proposition 1.1.3. — *Let* Γ *be an* NEC *group with signature* $\sigma(\Gamma)$ *as above. Let* Γ' *be another* NEC *group with signature*

$$\sigma' = \sigma(\Gamma') = (g'; \pm; [m'_1, \dots, m'_{r'}]; \{(n'_{11}, \dots, n'_{1s'_1}), \dots, (n'_{k'1}, \dots, n'_{k's'_{k'}})\}).$$

Let us write $C_i = (n_{i1}, \dots, n_{is_i})$ *and* $C'_i = (n'_{i1}, \dots, n'_{is'_i})$. *Then* Γ *and* Γ' *are isomorphic as abstract groups if and only if*

(i) *sign*$(\sigma) =$ *sign*(σ');

(ii) $g = g'$; $r = r'$; $k = k'$; $s_i = s'_i$ *for* $i = 1, \dots, k$;

(iii) $\{m_1, \dots, m_r\} = \{m'_1, \dots, m'_r\}$;

(iv) *if sign*$(\sigma) =$ "+" *then there exists a permutation* ϕ *of* $\{1, \dots, k\}$ *such that for each* $i \in \{1, \dots, k\}$ *one of the following conditions holds true:*
 C'_i *is a cyclic permutation of* $C_{\phi(i)}$;
 C'_i *is a cyclic permutation of the inverse of* $C_{\phi(i)}$.

(v) *if sign*$(\sigma) =$ "−" *then there exists a permutation* ϕ *of* $\{1, \dots, k\}$ *such that either* C'_i *is a cyclic permutation of* $C_{\phi(i)}$ *or* C'_i *is a cyclic permutation of the inverse of* $C_{\phi(i)}$.

We will see in Chapter 3 that in addition to this algebraic information, the signature of an NEC group Γ also provides topological information of the canonical projection $\mathcal{H} \to \mathcal{H}/\Gamma$ (see Proposition 3.1.1 for a precise statement).

The signature of the canonical Fuchsian group Γ^+ of an NEC group Γ can be obtained from that of Γ.

Proposition 1.1.4. — *If*

$$\sigma(\Gamma) = (g; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$$

is the signature of the NEC *group* Γ *then the signature of its canonical Fuchsian group* Γ^+ *is*

$$\sigma(\Gamma^+) = (\eta g + k - 1; [m_1, m_1, \dots, m_r, m_r, n_{11}, \dots, n_{1s_1}, \dots, n_{k1}, \dots, n_{ks_k}]),$$

where $\eta = 2$ *if sign*$(\sigma(\Gamma)) =$ "+" *and* $\eta = 1$ *otherwise.*

Definition 1.1.5. — Let

$$\sigma = (g; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$$

be an abstract signature and define η as above. The *area* of σ is defined to be

$$\mu(\sigma) = 2\pi \left(\eta g + k - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right) \right).$$

The following justifies this definition.

Theorem 1.1.6

(1) *Let* Γ *be an* NEC *group with signature* $\sigma(\Gamma)$. *Then the hyperbolic area* $\mu(\Gamma)$ *of an arbitrary fundamental region for* Γ *is* $\mu(\sigma(\Gamma))$.

(2) *The signature* σ *is the signature of some* NEC *group* Γ *if and only if* $\mu(\sigma) > 0$ *and* $\eta + g \geqslant 2$.

(3) *If* Γ' *is a subgroup of finite index of an* NEC *group* Γ *then* Γ' *is also an* NEC *group and the so called* Hurwitz-Riemann formula *holds:*

$$[\Gamma:\Gamma'] = \frac{\mu(\Gamma')}{\mu(\Gamma)}\,.$$

1.1.3. Uniformization and automorphism groups of Klein surfaces. — As said above, if Γ is an NEC group then the quotient $\mathcal{H}/\Gamma$ admits a structure of compact Klein surface. The importance of NEC groups comes from the fact that, under certain restrictions, the converse is also true. To explain this we need the notion of surface NEC group.

Definition 1.1.7. — An NEC group Γ having signature

$$\sigma(\Gamma) = (g;\pm;[-];\{(-),\overset{k}{\ldots},(-)\}),\quad k\geqslant 0$$

is said to be a *surface group.* If $k>0$ then Γ is a *bordered surface group.* Equivalently, an NEC group Γ is a surface group if and only if it has no orientation preserving elements of finite order.

Surface NEC groups uniformize compact Klein surfaces in the following sense.

Theorem 1.1.8 ([**46**]). — *Let* K *be a compact Klein surface of genus* $g\geqslant 2$*. Then there exists a surface* NEC *group* Γ *such that* $K=\mathcal{H}/\Gamma$ *as Klein surfaces.*

Remarks 1.1.9

(1) If K is the quotient of the compact Riemann surface S under the action of the antianalytic involution τ and Γ is as above then $S=\mathcal{H}/\Gamma^+$ as compact Riemann surfaces and $\langle\tau\rangle = \Gamma/\Gamma^+$, where Γ^+ is the canonical Fuchsian group of Γ.

(2) If we write the Klein surfaces K and K' as $\mathcal{H}/\Gamma$ and $\mathcal{H}/\Gamma'$ respectively, then K and K' are isomorphic if and only if Γ and Γ' are conjugate in $\mathrm{Aut}^{\pm}\mathcal{H}$.

This theorem, together with the classification of NEC groups by means of signatures, opens the door to the combinatorial approach to the theory of Klein surfaces and their automorphism groups. We now summarize some general results concerning automorphisms of Klein surfaces. The full group of dianalytic automorphisms of the Klein surface K will be denoted by $\mathrm{Aut}^{\pm}K$.

Theorem 1.1.10 ([**38**]). — *Let* $\mathrm{Aut}^{\pm}K$ *be the full group of automorphisms of the Klein surface* $K=\mathcal{H}/\Gamma$ *and let* $N(\Gamma)$ *be the normalizer of* Γ *in* $\mathrm{Aut}^{\pm}\mathcal{H}$*. Then*

(1) $N(\Gamma)$ *is an* NEC *group;*
(2) $\mathrm{Aut}^{\pm}K \simeq N(\Gamma)/\Gamma$;
(3) *A group* G *is a subgroup of* $\mathrm{Aut}^{\pm}K$ *if and only if it is isomorphic to* Λ/Γ *for some* NEC *group* Λ *containing* Γ *as a normal subgroup.*

1.1.4. Maximal NEC groups. — It is rather difficult to decide if a given group G of automorphisms of the Klein surface K coincides with the full group $\mathrm{Aut}^{\pm}K$. Signatures of NEC groups prove to be a useful tool in this problem. The key point is that almost all NEC signatures are maximal in some sense, and given such a maximal signature σ there exists a maximal NEC group Λ with signature σ. In particular, for every surface NEC group Γ contained in Λ as a normal subgroup, Λ/Γ is the full automorphism group $\mathrm{Aut}^{\pm}(\mathcal{H}/\Gamma)$ of the Klein surface $\mathcal{H}/\Gamma$. The following results concerning maximality may be found in Section 5.1 in [**14**].

Definition 1.1.11. — An NEC group is said to be *maximal* if there does not exist another NEC group containing it properly.

An (abstract) signature σ is an NEC (respectively Fuchsian) signature if it is the signature of an NEC (respectively Fuchsian) group.

Definition 1.1.12. — An NEC signature σ is said to be *maximal* if for every NEC group Λ' containing an NEC group Λ with $\sigma(\Lambda) = \sigma$ the equality $\dim(\Lambda) = \dim(\Lambda')$ (dimensions of Teichmüller spaces) implies $\Lambda = \Lambda'$. The dimension of the Teichmüller space of the NEC group Λ is $\dim(\Lambda) = \dim(\Lambda^+)/2 = 3(g-1) + r$ if the signature of the Fuchsian group Λ^+ is $(g; [m_1, \dots, m_r])$.

Remarks 1.1.13

(1) Let σ be the signature of a proper NEC group. If the signature σ^+ of its canonical Fuchsian group is maximal then so is σ.

(2) Almost all Fuchsian signatures turn out to be maximal. A list of those which fail to be so was obtained by Greenberg in [**27**] and completed by Singerman in [**47**]; this list is given in Table 1 below. Signatures in the first column are the non-maximal ones and for each σ in it, the corresponding signature σ' is the signature of a Fuchsian group Λ'^+ properly containing a group Λ^+ with signature σ and such that $\dim(\Lambda^+) = \dim(\Lambda'^+)$. The group Λ'^+ can be chosen containing Λ^+ as a normal subgroup just in the first eight cases and then we say that the pair (σ, σ') is *normal*. In the third column we write $[\sigma' : \sigma] = [\Lambda'^+ : \Lambda^+]$. The corresponding list of normal pairs of NEC signatures was obtained in [**7**].

It must be pointed out that the maximality of the NEC signature $\sigma(\Lambda)$ does not imply the maximality of the NEC group Λ. However:

Theorem 1.1.14. — *If σ is a maximal* NEC *signature then there exists a maximal* NEC *group Λ with $\sigma(\Lambda) = \sigma$.*

Next corollary plays a key role in this memoir.

Corollary 1.1.15. — *Let Λ be an* NEC *group containing a surface Fuchsian group Γ as a normal subgroup. Assume that $\sigma(\Lambda^+)$ is maximal. Then the topological surface $\mathcal{H}/\Gamma$ can be endowed with a structure of Riemann surface such that* $\mathrm{Aut}^{\pm}(\mathcal{H}/\Gamma) = \Lambda/\Gamma$.

σ	σ'	$[\sigma' : \sigma]$
$(1; [t])$	$(0; [2, 2, 2, 2t])$	2
$(1; [t, t])$	$(0; [2, 2, 2, 2, t])$	2
$(2; [-])$	$(0; [2, 2, 2, 2, 2, 2])$	2
$(0; [t, t, t, t]),\ \ t \geqslant 3$	$(0; [2, 2, 2, t])$	4
$(0; [t, t, u, u]),\ \ t + u \geqslant 5$	$(0; [2, 2, t, u])$	2
$(0; [t, t, t]),\ \ t \geqslant 4$	$(0; [3, 3, t])$	3
$(0; [t, t, t]),\ \ t \geqslant 4$	$(0; [2, 3, 2t])$	6
$(0; [t, t, u]),\ \ t \geqslant 3,\ \ t + u \geqslant 7$	$(0; [2, t, 2u])$	2
$(0; [7, 7, 7])$	$(0; [2, 3, 7])$	24
$(0; [2, 7, 7])$	$(0; [2, 3, 7])$	9
$(0; [3, 3, 7])$	$(0; [2, 3, 7])$	8
$(0; [4, 8, 8])$	$(0; [2, 3, 8])$	12
$(0; [3, 8, 8])$	$(0; [2, 3, 8])$	10
$(0; [9, 9, 9])$	$(0; [2, 3, 9])$	12
$(0; [4, 4, 5])$	$(0; [2, 4, 5])$	6
$(0; [n, 4n, 4n]),\ n \geqslant 2$	$(0; [2, 3, 4n])$	6
$(0; [n, 2n, 2n]),\ n \geqslant 3$	$(0; [2, 4, 2n])$	4
$(0; [3, n, 3n]),\ n \geqslant 3$	$(0; [2, 3, 3n])$	4
$(0; [2, n, 2n]),\ n \geqslant 4$	$(0; [2, 3, 2n]),$	3

TABLE 1. Non-maximal Fuchsian signatures

1.2. Symmetric Riemann surfaces

DEFINITION 1.2.1. — Let X be a compact Riemann surface of genus $g \geqslant 2$. A *symmetry* of X is an antianalytic involution $\tau : X \to X$. We will also say that τ is a *real structure* on X. A Riemann surface admitting a symmetry is called *symmetric*.

The topological nature of a symmetry τ is determined by the properties of its fixed point set $\mathrm{Fix}(\tau)$. Indeed, the following facts are well known.

(1) $\mathrm{Fix}(\tau)$ consists of k disjoint Jordan curves, where $0 \leqslant k \leqslant g + 1$ (Harnack's theorem).
(2) $X - \mathrm{Fix}(\tau)$ consists of one connected component if the Klein surface X/τ is non-orientable, and two otherwise. In the first case, τ is called *non-separating*, and *separating* in the second one.

Definition 1.2.2. — Let τ be a symmetry of X and suppose that $\mathrm{Fix}(\tau)$ consists of k disjoint Jordan curves. Then we define the *species* of τ to be

$$\mathrm{sp}(\tau) = \begin{cases} 0 & \text{if } k = 0, \\ k & \text{if } k > 0 \text{ and } X - \mathrm{Fix}(\tau) \text{ has two components,} \\ -k & \text{if } k > 0 \text{ and } X - \mathrm{Fix}(\tau) \text{ has one component.} \end{cases}$$

Fixed the genus g, the values that the species of a symmetry τ may attain are known since Klein. In fact,

(i) There always exists a genus g Riemann surface admitting a symmetry of species 0;

(ii) There exists a genus g Riemann surface admitting a symmetry of species k if and only if $k \leqslant g + 1$ and $k \equiv g + 1 \pmod 2$;

(ii) There exists a genus g Riemann surface admitting a symmetry of species $-k$ if and only if $k \leqslant g$.

For example, the possible species of a genus 2 Riemann surface are 0, 1, 3, -2 and -1.

Let $\mathrm{Aut}^{\pm} X$ denote the group of analytic and antianalytic selfhomeomorphisms of the Riemann surface X. It is called the *automorphism group of* X, and its elements *automorphisms of* X. Let $\mathrm{Aut}\, X$ be its subgroup consisting of the analytic selfhomeomorphisms. Then a symmetry of X is an involution τ in $\mathrm{Aut}^{\pm} X$ which lies outside $\mathrm{Aut}\, X$. Clearly, a conjugate in $\mathrm{Aut}^{\pm} X$ of a symmetry is another symmetry with the same species. This motivates the following definition.

Definition 1.2.3. — The *symmetry type* of X is the unordered list of species of all conjugacy classes of symmetries of X.

NEC groups prove to be a useful tool for the study of symmetric Riemann surfaces. We first show a characterization in terms of NEC groups of the symmetric nature of a Riemann surface.

Lemma 1.2.4. — *The Riemann surface $X = \mathcal{H}/\Gamma$ is symmetric if and only if there exists a proper* NEC *group Λ such that $\Gamma < \Lambda$ with index* 2.

Proof. — If such an NEC group Λ exists then it is the disjoint union $\Lambda = \Gamma \cup \Gamma t$, where $t \in \mathrm{Aut}^{\pm}\mathcal{H} - \mathrm{Aut}\,\mathcal{H}$. As $t\Gamma = \Gamma t$ we see that, with the obvious notations, $\tau : \mathrm{orbit}(z, \Gamma) \mapsto \mathrm{orbit}(t(z), \Gamma)$ is a symmetry of X. Conversely, if $\tau : X \to X$ is a symmetry then τ lifts to a mapping $t : \mathcal{H} \to \mathcal{H}$ where $t \in \mathrm{Aut}^{\pm}\mathcal{H} - \mathrm{Aut}\,\mathcal{H}$ (as τ is antianalytic) and $t^2 \in \Gamma$ (as $\tau^2 = id$). Now the disjoint union $\Lambda := \Gamma \cup \Gamma t$ is the required proper NEC group. ☐

As a consequence of this lemma, every symmetry τ of the Riemann surface $X = \mathcal{H}/\Gamma$ may be represented as Λ/Γ for certain proper NEC group Λ. Notice that in these conditions, Γ coincides with Λ^+, the canonical Fuchsian group of Λ. Therefore

Λ contains no orientation preserving elements of finite order since Λ^+ is a surface Fuchsian group. In other words, Λ *is a surface* NEC *group.* It turns out that the species of τ can be read off from the signature of Λ. For example, the number of period cycles of $\sigma(\Lambda)$ coincides with the number of boundary components of the Klein surface $\mathcal{H}/\Lambda = (\mathcal{H}/\Gamma)/(\Lambda/\Gamma) = X/\langle\tau\rangle$ (see Proposition 3.1.1), which is the number of disjoint Jordan curves of $\mathrm{Fix}(\tau)$. Moreover, the sign of the signature of Λ is the same as the sign of the species of τ, as the next lemma shows (see *e.g.,* [**18**]).

LEMMA 1.2.5. — *With the above notations, $X - \mathrm{Fix}(\tau)$ has two components if and only if $\mathcal{H}/\Lambda$ is an orientable Klein surface.*

Finally, the formulae for the areas of $\sigma(\Gamma)$ and $\sigma(\Lambda)$ together with Hurwitz-Riemann relation give that the genus of $\mathcal{H}/\Lambda$ equals $g-k+1$ if it is non-orientable or $(g-k+1)/2$ otherwise, where g is the genus of X. Summarizing, we have the following theorem.

THEOREM 1.2.6. — *Let $X = \mathcal{H}/\Gamma$ be a compact Riemann surface of genus $g \geqslant 2$ (where Γ is a Fuchsian surface group) which admits a symmetry τ. Write $\langle\tau\rangle = \Lambda/\Gamma$ where Λ is a proper* NEC *group such that $\Gamma < \Lambda$ with index 2. Then*

$$\mathrm{sp}(\tau) = \begin{cases} 0 & \text{if } \sigma(\Lambda) = (g+1; -; [-]; \{-\}); \\ k & \text{if } \sigma(\Lambda) = ((g+1-k)/2; +; [-]; \{(-), \overset{k}{\ldots}, (-)\}); \\ -k & \text{if } \sigma(\Lambda) = (g+1-k; -; [-]; \{(-), \overset{k}{\ldots}, (-)\}). \end{cases}$$

1.3. Hyperelliptic real algebraic curves

We now consider the algebraic counterpart of compact Riemann surfaces, that is, we view them as complex algebraic curves. This is the point of view we adopt in this section. In it we obtain the necessary tools to deal with real forms of hyperelliptic complex curves. We first recall the notion of hyperellipticity.

DEFINITION 1.3.1. — A complex algebraic curve X of genus $g \geqslant 2$ is *hyperelliptic* if any of the following equivalent conditions holds

(1) there exists a meromorphic function $\pi_X : X \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ of degree 2;
(2) there exists an automorphism $\rho_X : X \to X$ of order 2 with $2g+2$ fixed points;
(3) there exists an automorphism $\rho_X : X \to X$ of order 2 such that the quotient X/ρ_X has genus 0.

The automorphism ρ_X is called the *hyperelliptic involution.* In the sequel we will simply denote it by ρ. It is *central* in the full group $\mathrm{Aut}^{\pm}X$ of analytic and antianalytic automorphisms of X, that is, it commutes with all elements in $\mathrm{Aut}^{\pm}X$.

Hyperelliptic complex algebraic curves enjoy the property of admitting an easy representation by polynomials. In fact, it is well known that any such curve can be obtained from an equation of the form $y^2 = P(x)$ where P is a monic polynomial with distinct roots (see, for example, Sections 1 and 4, Chapter III in [**40**]). Let us fix some notations which will be used along this memoir.

Notations 1.3.2. — A hyperelliptic complex algebraic curve X of genus g will be represented by the affine plane model

$$X = \{(x, y) \in \mathbb{C}^2 : y^2 = P_X(x)\}$$

where

$$P_X(x) = (x - e_1) \cdots (x - e_{2g+1+\delta}) \quad \text{with} \quad e_i \neq e_j \quad \text{if} \quad i \neq j \quad \text{and} \quad \delta = 0 \text{ or } 1.$$

In this model we identify the characteristic elements of a hyperelliptic curve. First, the projection π_X on the first coordinate

$$\begin{aligned} \pi_X : \quad X &\longrightarrow \widehat{\mathbb{C}} \\ (x, y) &\longmapsto x \end{aligned}$$

is a meromorphic function of degree 2. Its branch points are thus the roots of P_X and possibly ∞. They constitute what we call (by abuse of language) the *branch point set of* X,

$$B_X = \begin{cases} \{e_1, \dots, e_{2g+2}\} & \text{if } \delta = 1, \\ \{e_1, \dots, e_{2g+1}, \infty\} & \text{if } \delta = 0. \end{cases}$$

The automorphism interchanging the two sheets of π_X, *i.e.,* the hyperelliptic involution, has the following formula

$$\begin{aligned} \rho : \quad X &\longrightarrow X \\ (x, y) &\longmapsto (x, -y). \end{aligned}$$

Isomorphisms between hyperelliptic complex curves are closely related to automorphisms of the Riemann sphere, *i.e.,* to Möbius transformations. We now describe this relation in terms of equations of such curves.

Let Y be another hyperelliptic curve and B_Y its branch point set. Every birational isomorphism $f : X \to Y$ induces a Möbius transformation $\widehat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which maps B_X onto B_Y. In fact, $\widehat{f}$ is defined by $\widehat{f} : \pi_X(p) \mapsto \pi_Y(f(p))$ for any $p \in X$.

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \widehat{\mathbb{C}} & \xrightarrow{\ \widehat{f}\ } & \widehat{\mathbb{C}} \end{array}$$

Conversely, every Möbius transformation $m : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which maps B_X onto B_Y induces exactly two birational isomorphisms $f_1,\ f_2 : X \to Y$ such that $\widehat{f_i} = m$, $i = 1, 2$ (in fact, $f_2 = f_1 \circ \rho = \rho \circ f_1$). We call these isomorphisms *liftings of* m. Their formulae can be calculated explicitly.

1.3.1. Liftings of Möbius transformations. — Let B_X and B_Y be the branch point sets of the genus g hyperelliptic complex curves $X = \{y^2 = P_X(x)\}$ and $Y = \{w^2 = P_Y(z)\}$. Let

$$m(x) = \frac{ax+b}{cx+d} \quad (\{a,b,c,d\} \subset \mathbb{C}) \quad \text{with} \quad \det m := ad - bc \neq 0$$

be a Möbius transformation such that $m(B_X) = B_Y$. Then the formulae of its liftings, say f and $f \circ \rho$, depend on whether $\infty \in B_X$ or not, and whether m fixes ∞ or not. Explicitly (see [**22**]),

- if $\infty \in B_X$ and $m(\infty) = \infty$ then
$$f(x,y) = \left(\frac{a}{d}x + \frac{b}{d}, y \cdot \left(\frac{a}{d}\right)^g \frac{\sqrt{\det m}}{d}\right).$$
- If $\infty \in B_X$ and $m(\infty) \neq \infty$ then
$$f(x,y) = \left(\frac{ax+b}{cx+d}, \frac{y \cdot c^g}{(cx+d)^{g+1}} \sqrt{-\det m \cdot P_Y'(a/c)}\right).$$
- If $\infty \notin B_X$ and $m(\infty) = \infty$ then
$$f(x,y) = \left(\frac{a}{d}x + \frac{b}{d}, y \cdot \left(\frac{a}{d}\right)^{g+1}\right).$$
- If $\infty \notin B_X$ and $m(\infty) \neq \infty$ then
$$f(x,y) = \left(\frac{ax+b}{cx+d}, \frac{y \cdot c^{g+1}}{(cx+d)^{g+1}} \sqrt{P_Y(a/c)}\right).$$

If the Möbius transformation mapping B_X onto B_Y is not analytic but antianalytic, *i.e.,* of the form $(a\overline{x}+b)/(c\overline{x}+d)$ then the formulae of its liftings are exactly the same than the above just changing x and y by their complex conjugates $\overline{x}$ and $\overline{y}$.

In this memoir we are specially interested in automorphisms of X, whose formulae are obtained by making $Y = X$ in the above. In this case, the relation between isomorphisms of hyperelliptic curves and Möbius transformations states that the group $\mathrm{Aut}^{\pm} X$ of automorphisms of X consists exactly of the liftings of those Möbius transformations (analytic and antianalytic) which preserve the branch point set B_X. For notational convenience, we denote such group by $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$:

$$\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}} := \{m : m(B_X) = B_X\}.$$

Observe that algebraically, $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ is nothing but the quotient $(\mathrm{Aut}^{\pm} X)/\langle\rho\rangle$.

According to the classification of the finite automorphism groups of the sphere, there are ten different classes of such groups containing an antianalytic involution. They will appear in a natural way as $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ in the combinatorial study of $\mathrm{Aut}^{\pm} X$ that we make in the next chapter.

1.3.2. Species of real structures. — Assume now that the hyperelliptic complex curve X admits a real structure τ. The pair (X,τ) is called a *hyperelliptic real algebraic curve* and our purpose here is to determine its topology. To be more precise, we compute the species of τ in terms of both its formula and the equation of X. This is summarized in Theorems 1.3.4 and 1.3.5, which are based on results in [**31**], Section 6, adapted to our point of view.

As said above, the antianalytic automorphism $\tau : X \to X$ induces an antianalytic automorphism $\widehat{\tau} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which also has order 2. If $\widehat{\tau}$ fixes points then it is conjugate in $\mathrm{Aut}^{\pm}\mathcal{H}$ to complex conjugation $x \mapsto \overline{x}$; otherwise it is conjugate in $\mathrm{Aut}^{\pm}\mathcal{H}$ to the antipodal map $x \mapsto -1/\overline{x}$, [**2**, Theorem 1.9.4].

If $\widehat{\tau}$ is *complex conjugation* then one of its liftings is $\tau : (x,y) \mapsto (\overline{x},\overline{y})$ and so its fixed point set consists exactly of the real points of the complex curve $X = \{y^2 = P_X(x)\}$. Note that here P_X is a real polynomial since its roots are permuted by $\widehat{\tau}$. It is then easy to see that if P_X has $2k > 0$ real roots then $\mathrm{Fix}(\tau)$ consists exactly of k ovals. Moreover, the separability character of $\mathrm{Fix}(\tau)$ depends on the number of real roots of P_X, or more precisely (since ∞ may be a branch point), on the number of branch points of X fixed by $\widehat{\tau}$. Indeed, if $2k$ is the number of branch points fixed by $\widehat{\tau}$ then the species of τ is the following ([**31**, Proposition 6.3]):

$$\mathrm{sp}(\tau) = \begin{cases} g+1 & \text{if } k = g+1; \\ -k & \text{if } 0 < k < g+1; \\ 1 & \text{if } k = 0 \text{ and } g \text{ is even;} \\ 2 & \text{if } k = 0 \text{ and } g \text{ is odd.} \end{cases}$$

The other lifting of $\widehat{\tau}$ is $\tau \circ \rho : (x,y) \mapsto (\overline{x}, -\overline{y})$. If $\widehat{\tau}$ fixes no branch point then P_X is always positive on $\mathbb{R}$. So $\mathrm{Fix}(\tau \circ \rho)$ is empty, that is, $\mathrm{sp}(\tau \circ \rho) = 0$. If $\widehat{\tau}$ fixes some branch point we claim that the species of $\tau \circ \rho$ coincides with the species of τ. To show this we use the obvious fact that if α and α' are real structures of two different complex curves X and X' respectively, and $f \circ \alpha = \alpha' \circ f$ for some isomorphism $f : X \to X'$ then the species of α and α' coincide.

Lemma 1.3.3. — *If $\widehat{\tau}$ fixes some branch point then* $\mathrm{sp}\,(\tau \circ \rho) = \mathrm{sp}\,(\tau)$.

Proof. — Let r be the greatest real root of P_X and let m be the real Möbius transformation $m : x \mapsto -1/(x-r)$. Consider the hyperelliptic complex curve X' whose branch point set is $B_{X'} = m(B_X)$. Clearly, $\tau' : (x,y) \mapsto (\overline{x},\overline{y})$ is a real structure of X'. Denoting by f a lifting of m it is easy to check that $f \circ \tau \circ \rho = \tau' \circ f$ and so $\mathrm{sp}(\tau \circ \rho) = \mathrm{sp}(\tau')$. It remains to see that $\mathrm{sp}(\tau') = \mathrm{sp}(\tau)$. Since both real structures τ' and τ have the same formulae $(x,y) \mapsto (\overline{x},\overline{y})$, it follows from the above that $\mathrm{sp}(\tau') = \mathrm{sp}(\tau)$ if and only if the number of branch points of X' fixed by $\widehat{\tau}'$ coincides with the number of branch points of X fixed by $\widehat{\tau}$. But this is clear because $m \circ \widehat{\tau} \circ m^{-1} = \widehat{\tau}'$ and so the branch points of X' fixed by $\widehat{\tau}'$ *are* the images by m of the branch points of X fixed by $\widehat{\tau}$. ☐

We have just calculated the species of each lifting of complex conjugation.

THEOREM 1.3.4. — *Suppose that the branch point set of the hyperelliptic curve X of genus g is preserved by complex conjugation $\widehat{\tau}$. Then X admits the real structures given by $\tau : (x, y) \mapsto (\overline{x}, \overline{y})$ and $\tau \circ \rho : (x, y) \mapsto (\overline{x}, -\overline{y})$. Let $2k$ be the number of branch points of X fixed by $\widehat{\tau}$.*

(1) *If $k > 0$ then*

$$\mathrm{sp}\,(\tau) = \mathrm{sp}\,(\tau \circ \rho) = \begin{cases} g+1 & \text{if } k = g+1; \\ -k & \text{if } k < g+1. \end{cases}$$

(2) *If $k = 0$ then*

$$\mathrm{sp}\,(\tau) = \begin{cases} 1 & \text{if } g \text{ is even,} \\ 2 & \text{if } g \text{ is odd;} \end{cases}$$

$$\mathrm{sp}\,(\tau \circ \rho) = 0.$$

If the branch point set of X is preserved by an antianalytic involution $\widehat{\tau}'$ which is not complex conjugation but conjugate in $\mathrm{Aut}^{\pm}\mathcal{H}$ to it, then we claim that the species of its liftings τ' and $\tau' \circ \rho$ also depend on the number of branch points fixed by $\widehat{\tau}'$. This can be proved in a similar way than the above lemma. If $m \circ \widehat{\tau}' \circ m^{-1} = \widehat{\tau}$ then $\tau : (x, y) \mapsto (\overline{x}, \overline{y})$ and $\tau \circ \rho$ are real structures of the hyperelliptic curve Y whose branch point set is $m(B_X)$. In fact, if f is a lifting of m then either $f \circ \tau' \circ f^{-1} = \tau$ (and so $f \circ \tau' \circ \rho \circ f^{-1} = \tau \circ \rho$) or $f \circ \tau' \circ f^{-1} = \tau \circ \rho$ (and so $f \circ \tau' \circ \rho \circ f^{-1} = \tau$). In both cases $\{\mathrm{sp}(\tau'), \mathrm{sp}(\tau' \circ \rho)\} = \{\mathrm{sp}(\tau), \mathrm{sp}(\tau \circ \rho)\}$ which means that the species of τ' and $\tau' \circ \rho$ depend on the number of branch points of Y fixed by $\widehat{\tau}$. But the branch points of Y fixed by $\widehat{\tau}$ are the images by m of the branch points of X fixed by $\widehat{\tau}'$. This proves our claim.

Suppose now that $\widehat{\tau}$ is the *antipodal map* $\widehat{\tau} : x \mapsto -1/\overline{x}$. We claim that in this case the genus g must be odd. Otherwise its liftings would not be involutions. Indeed, using the formulae of its liftings we see immediately that their square is $\tau^2 = (\tau \circ \rho)^2 = (x, y \cdot (-1)^{g+1})$. So for g even, $\tau^2 = \rho$, *i.e.,* τ is a pseudo-symmetry, see [**49**]. As to their species, since $\widehat{\tau}$ fixes no point the same happens to both its liftings. Therefore,

$$\mathrm{sp}(\tau) = \mathrm{sp}(\tau \circ \rho) = 0.$$

Summarizing, we have shown the following theorem, which gives the species of liftings of an antianalytic involution $\widehat{\tau}$. If $\widehat{\tau}$ is conjugate to complex conjugation we call it *reflection.*

THEOREM 1.3.5. — *Let $\widehat{\tau}$ be an antianalytic involution preserving the branch point set of the hyperelliptic curve X of genus g. Let τ and $\tau \circ \rho$ be its liftings and assume that both have order* 2.

(1) *If* $\widehat{\tau}$ *is conjugate to the antipodal map then* g *is odd and*

$$\mathrm{sp}\,(\tau) = \mathrm{sp}\,(\tau \circ \rho) = 0.$$

(2) *If* $\widehat{\tau}$ *is a reflection which fixes* $2k > 0$ *branch points of* X *then*

$$\mathrm{sp}\,(\tau) = \mathrm{sp}\,(\tau \circ \rho) = \begin{cases} g+1 & \textit{if } k = g+1; \\ -k & \textit{if } k < g+1. \end{cases}$$

(3) *If* $\widehat{\tau}$ *is a reflection which fixes no branch point of* X *then*

$$\{\mathrm{sp}\,(\tau), \mathrm{sp}\,(\tau \circ \rho)\} = \begin{cases} \{1,0\} & \textit{if } g \textit{ is even;} \\ \{2,0\} & \textit{if } g \textit{ is odd.} \end{cases}$$

Remark 1.3.6. — In case (3) we do *not* know which is the lifting with nonzero species except if $\widehat{\tau}$ is complex conjugation (see Theorem 1.3.4). When dealing with equations of curves we will exhibit concrete formulae of both liftings and so we will have to decide which one fixes points.

CHAPTER 2

AUTOMORPHISM GROUPS OF SYMMETRIC HYPERELLIPTIC RIEMANN SURFACES

Let X be a hyperelliptic Riemann surface X admitting a symmetry τ. Then, as the hyperelliptic involution ρ commutes with it, the composite $\tau \circ \rho$ is also a symmetry. Both generate a Klein 4-group $\{id, \tau, \tau \circ \rho, \rho\}$ of automorphisms of X. If X admits no analytic automorphisms other than ρ then $\mathrm{Aut}^{\pm} X$ coincides with $\langle \tau, \rho \rangle$ and so it has exactly two conjugacy classes of symmetries. Their species were already calculated by Klein [**35**] using geometric methods. The same result was obtained in [**18**, Thm 5.2] using NEC groups. *The symmetry types of symmetric hyperelliptic Riemann surfaces of genus g which admit no analytic automorphisms other than the hyperelliptic involution are*

$$\{-k, -k\} \quad \textit{where} \quad 1 \leqslant k \leqslant g, \quad \{g+1, g+1\} \quad \textit{and} \quad \begin{cases} \{1,0\} & \textit{if } g \textit{ is even} \\ \{2,0\}, \ \{0,0\} & \textit{if } g \textit{ is odd.} \end{cases}$$

From now on we will assume therefore that X admits more analytic automorphisms than ρ, that is $\mathrm{order}(\mathrm{Aut}^{\pm} X) > 4$. The aim of this chapter is to find a presentation of $\mathrm{Aut}^{\pm} X$. This will be achieved in Section 2.3, where we give the presentation of all groups which may act as $\mathrm{Aut}^{\pm} X$. This extends the results of Brandt and Stichtenoth in [**4**] and Bujalance, Gamboa and Gromadzki in [**15**]. The knowledge of $\mathrm{Aut}^{\pm} X$ enables us to compute the number of real forms of X and a representative of each of them. The calculation of the species of each real form is postponed to the next chapter.

2.1. On signatures of the automorphism groups

In this section we calculate the signatures of NEC groups which may realize groups of automorphisms of symmetric hyperelliptic surfaces. We begin by fixing some notations which will be used throughout.

Notation 2.1.1. — For short, the full automorphism group $\mathrm{Aut}^{\pm} X$ of X will be denoted by G. Since we are dealing only with the case $\mathrm{order}(G) > 4$ and G contains a Klein 4-group, we write

$$\mathrm{order}(G) = 4N \quad \text{for some } N \geqslant 2.$$

If we write $X = \mathcal{H}/\Gamma$ for some Fuchsian surface group Γ then G is identified with Λ/Γ for some NEC group Λ containing Γ as a normal subgroup. We denote by

$$\theta : \Lambda \longrightarrow G$$

the canonical epimorphism. Also $\langle\rho\rangle$ can be represented as Γ_ρ/Γ for some Fuchsian group Γ_ρ. Set

$$\widehat{G} := G/\rho = \Lambda/\Gamma_\rho$$

and let

$$\widehat{\theta} : \Lambda \longrightarrow \widehat{G}$$

be the composite of θ and the canonical projection $G \to \widehat{G}$. Note that

$$\Gamma_\rho = \ker \widehat{\theta}.$$

The existence of the Fuchsian group Γ_ρ also characterizes the hyperellipticity of X. In fact, *X is hyperelliptic if and only if there exists a unique Fuchsian group Γ_ρ with signature* $(0; [2, \overset{2g+2}{\ldots}, 2])$ *containing* Γ *as a subgroup of index* 2, ([**37**, Lemma 2]). Note that in particular, any non-trivial element of finite order of $\ker \widehat{\theta}$ is an involution.

The rest of this section is devoted to obtain a presentation of the group G. More precisely, we look for the signatures of Λ and the corresponding epimorphism θ. A detailed study of each case will be the subject of the next section.

Assume that Λ has signature

$$\sigma(\Lambda) = (g'; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

Recall that $m_i = \operatorname{order}(x_i)$ and $n_{ij} = \operatorname{order}(c_{i,j-1}c_{ij})$ where x_i and c_{ij} are canonical generators of Λ (elliptic transformations and reflections, respectively). Clearly $c_{ij} \notin \ker \widehat{\theta} = \Gamma_\rho$ for arbitrary i, j. Let $\widehat{m}_i$ and $\widehat{n}_{i,j}$ be the orders of the images by $\widehat{\theta}$ of x_i and $c_{i,j-1}c_{ij}$ in $\widehat{G}$ respectively. Since any non-trivial element of finite order of $\ker \widehat{\theta}$ has order 2, the integers $m_i/\widehat{m}_i$ and $n_{ij}/\widehat{n}_{ij}$ are equal to 1 or 2. Now, Theorem 2.2.4 of [**14**] gives a relation between the integers m_i, $\widehat{m}_i$, n_{ij}, $\widehat{n}_{ij}$, the index $[\Lambda : \Gamma_\rho] = 2N$ and the number $2g + 2$ of proper periods of Γ_ρ. In fact, denoting $I := \{i \mid m_i = 2\widehat{m}_i\}$ and $K := \{(i, j) \mid n_{ij} = 2\widehat{n}_{ij}\}$ we obtain the following equality:

$$2g + 2 = \sum_{i \in I} 2N/\widehat{m}_i + \sum_{(i,j) \in K} N/\widehat{n}_{ij}. \tag{1}$$

Combining this with the Hurwitz-Riemann formula for Λ and Γ_ρ gives

$$-2 = 2N(\alpha g' + k - 2 + \sum_{i=1}^{r}(1 - 1/\widehat{m}_i) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - 1/\widehat{n}_{ij})).$$

Hence $\alpha g' + k < 2$, and in fact $0 < \alpha g' + k$ since otherwise Λ would contain no orientation reversing element. Therefore, either $g' = 0$ and $k = 1$ or $\alpha = g' = 1$ and $k = 0$. In the first case Λ has signature

$$\sigma(\Lambda) = (0; +; [m_1, \ldots, m_r]; \{(n_1, \ldots, n_s)\})$$

and the above equation reads

$$\sum_{i=1}^{r}\left(1-\frac{1}{\widehat{m}_i}\right)+\frac{1}{2}\sum_{j=1}^{s}\left(1-\frac{1}{\widehat{n}_j}\right)=1-\frac{1}{N}.$$

In the second case the signature of Λ is

$$\sigma(\Lambda)=(1;-;[m_1,\dots,m_r];\{-\})$$

and the corresponding relation between the integers $\widehat{m}_i$ and N is

$$\sum_{i=1}^{r}\left(1-\frac{1}{\widehat{m}_i}\right)=1-\frac{1}{N}.$$

Our task now is to find all the integers $\widehat{m}_i$ and $\widehat{n}_j$ satisfying each formula. Note that $\widehat{m}_i$ and $\widehat{n}_j$ are divisors of $2N$ since they are orders of elements of the group $\widehat{G}$ of order $2N$. Furthermore, proof of Theorem 2.2.4 of [**14**] shows that $\widehat{n}_j$ is a divisor of N. It turns out that there are eight different solutions for the first formula and only one for the second. They are the following.

Case	$\{\widehat{m}_1,\dots,\widehat{m}_r\}$	$\{\widehat{n}_1,\dots,\widehat{n}_s\}$
I	$\{1,\dots,1\}$	$\{1,\dots,1,N,N\}$
II	$\{1,\dots,1,N\}$	$\{1,\dots,1\}$
III	$\{1,\dots,1,2\}$	$\{1,\dots,1,N/2\}$
IV	$\{1,\dots,1\}$	$\{1,\dots,1,2,2,N/2\}$
V	$\{1,\dots,1,3\}$	$\{1,\dots,1,2\}$
VI	$\{1,\dots,1\}$	$\{1,\dots,1,2,3,3\}$
VII	$\{1,\dots,1\}$	$\{1,\dots,1,2,3,4\}$
VIII	$\{1,\dots,1\}$	$\{1,\dots,1,2,3,5\}$
IX	$\{1,\dots,1,N\}$	$-$

Obviously, cases III and IV occur only for N even. Moreover, $N\geqslant 4$ since for $N=2$ cases III and IV coincide with cases II and I respectively.

Each of these solutions gives rise to different signatures for Λ according to the number of coincidences of the integers m_i and n_j with the integers $\widehat{m}_i$ and $\widehat{n}_j$ respectively. Recall that $m_i/\widehat{m}_i=1$ or 2 and the same happens to $n_j/\widehat{n}_j$. Further, a relation between g, N and the number of coincidences is obtained from formula (1). As an example, we find the signatures that Λ may have in case I. In this case $\{\widehat{m}_1,\dots,\widehat{m}_r\}=\{1,\dots,1\}$ and so all the proper periods of Λ are equal to 2. Thus

$$\sum_{i\in I}2N/\widehat{m}_i=2Nr.$$

As to the link periods, $s-2$ of them are equal to 2, and the other two are equal to N or $2N$. Since cyclic permutations of link periods preserve the isomorphism class of

Λ (see Proposition 1.1.3), three possibilities must be considered:

$$(n_1,\dots,n_s)=\begin{cases}(2,\overset{p_1}{\dots},2,N,2,\overset{p_2}{\dots},2,N) & \text{and } \sum_{j\in K}N/\widehat{n}_j=N(p_1+p_2),\\ (2,\overset{p_1}{\dots},2,2N,2,\overset{p_2}{\dots},2,2N) & \text{and } \sum_{j\in K}N/\widehat{n}_j=N(p_1+p_2)+2,\\ (2,\overset{p_1}{\dots},2,N,2,\overset{p_2}{\dots},2,2N) & \text{and } \sum_{j\in K}N/\widehat{n}_j=N(p_1+p_2)+1.\end{cases}$$

So case I splits into three subcases. Proceeding likewise with the others we obtain all signatures that Λ may have. We list them in the following tables, where for shortness we have denoted $2,\overset{s}{\dots},2$ by 2^s. We also write the relation between the integers p_i, r, g and N.

Case I		
Ia	$(0;+;[2^r];\{(2^{p_1},N,2^{p_2},N)\})$	$2r+p_1+p_2=(2g+2)/N$
Ib	$(0;+;[2^r];\{(2^{p_1},2N,2^{p_2},2N)\})$	$2r+p_1+p_2=2g/N$
Ic	$(0;+;[2^r];\{(2^{p_1},N,2^{p_2},2N)\})$	$2r+p_1+p_2=(2g+1)/N$

Case II		
IIa	$(0;+;[2^r,N];\{(2^p)\})$	$2r+p=(2g+2)/N$
IIb	$(0;+;[2^r,2N];\{(2^p)\})$	$2r+p=2g/N$

Case III		
IIIa	$(0;+;[2^r,2];\{(2^p,N/2)\})$	$2r+p=(2g+2)/N$
IIIb	$(0;+;[2^r,2];\{(2^p,N)\})$	$2r+p=2g/N$
IIIc	$(0;+;[2^r,4];\{(2^p,N/2)\})$	$2r+p=(2g+2)/N-1$
IIId	$(0;+;[2^r,4];\{(2^p,N)\})$	$2r+p=2g/N-1$

Case IV		
IVa	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},2,2^{p_3},N/2)\})$	$2r+p_1+p_2+p_3=(2g+2)/N;$
IVb	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},4,2^{p_3},N/2)\})$	$2r+p_1+p_2+p_3=(2g+2)/N-1/2$
IVc	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},4,2^{p_3},N/2)\})$	$2r+p_1+p_2+p_3=(2g+2)/N-1$
IVd	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},2,2^{p_3},N)\})$	$2r+p_1+p_2+p_3=2g/N$
IVe	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},4,2^{p_3},N)\})$	$2r+p_1+p_2+p_3=2g/N-1/2$
IVf	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},4,2^{p_3},N)\})$	$2r+p_1+p_2+p_3=2g/N-1;$

Case V		
Va	$(0;+;[2^r,3];\{(2^p,2)\})$	$2r+p=(g+1)/6$
Vb	$(0;+;[2^r,6];\{(2^p,2)\})$	$2r+p=(g-3)/6$
Vc	$(0;+;[2^r,3];\{(2^p,4)\})$	$2r+p=(g-2)/6$
Vd	$(0;+;[2^r,6];\{(2^p,4)\})$	$2r+p=(g-6)/6$

Case VI		
VIa	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},3)\})$	$2r+p_1+p_2+p_3=(g+1)/6$
VIb	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},6)\})$	$2r+p_1+p_2+p_3=(g-1)/6$
VIc	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},6)\})$	$2r+p_1+p_2+p_3=(g-3)/6$
VId	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},3)\})$	$2r+p_1+p_2+p_3=(g-2)/6$
VIe	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},6)\})$	$2r+p_1+p_2+p_3=(g-4)/6$
VIf	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},6)\})$	$2r+p_1+p_2+p_3=(g-6)/6$

Case VII		
VIIa	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},4)\})$	$2r+p_1+p_2+p_3=(g+1)/12$
VIIb	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},4)\})$	$2r+p_1+p_2+p_3=(g-3)/12$
VIIc	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},8)\})$	$2r+p_1+p_2+p_3=(g-2)/12$
VIId	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},8)\})$	$2r+p_1+p_2+p_3=(g-6)/12$
VIIe	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},4)\})$	$2r+p_1+p_2+p_3=(g-5)/12$
VIIf	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},4)\})$	$2r+p_1+p_2+p_3=(g-9)/12$
VIIg	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},8)\})$	$2r+p_1+p_2+p_3=(g-8)/12$
VIIh	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},8)\})$	$2r+p_1+p_2+p_3=(g-12)/12$

Case VIII		
VIIIa	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},5)\})$	$2r+p_1+p_2+p_3=(g+1)/30$
VIIIb	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},10)\})$	$2r+p_1+p_2+p_3=(g-5)/30$
VIIIc	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},10)\})$	$2r+p_1+p_2+p_3=(g-15)/30$
VIIId	$(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},5)\})$	$2r+p_1+p_2+p_3=(g-9)/30$
VIIIe	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},5)\})$	$2r+p_1+p_2+p_3=(g-14)/30$
VIIIf	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},10)\})$	$2r+p_1+p_2+p_3=(g-20)/30$
VIIIg	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},5)\})$	$2r+p_1+p_2+p_3=(g-24)/30$
VIIIh	$(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},10)\})$	$2r+p_1+p_2+p_3=(g-30)/30$

Case IX		
IXa	$(1; -; [2^r, N]; \{-\})$	$r = (g+1)/N$
IXb	$(1; -; [2^r, 2N]; \{-\})$	$r = g/N$

REMARKS 2.1.2

(1) In principle many more signatures of Λ are involved. Namely those obtained from the above ones by permutations of proper periods and link periods. Observe however that, changing the roles of p_1, p_2 and p_3, the effect of any permutation is actually the same as a direct or inverse cyclic one.

(2) We observe that the orders of the groups $G = \Lambda/\Gamma$ occurring in cases V to VIII do not depend on N; in these cases N equals 12, 12, 24 or 60 respectively, and so the order of G is 48, 48, 96 or 240 respectively. Therefore any group theory package can be used to deal with properties of such groups. We shall use the GAP Programm [**25**].

2.2. Number of real forms

This section is devoted to compute the number of conjugacy classes of involutions of G which represent symmetries. The computation splits naturally into nine different cases according to the signatures obtained in the previous section. We recall that a dihedral group of order $2M$ generated by two antianalytic involutions τ_1 and τ_2 has exactly two conjugacy classes of symmetries, with representatives τ_1 and τ_2, if M is even, and only one if M is odd.

Let us explain the strategy. For each of the above signatures σ we look for an epimorphism $\theta : \Lambda \to G$ from an NEC group Λ with signature σ such that

i) $\operatorname{im}\theta = G$ is a group of order $4N$ having a central involution ρ,

ii) $\ker\theta$ is a Fuchsian surface group Γ,

iii) $\theta^{-1}(\langle\rho\rangle)$ is a (normal) subgroup Γ_ρ with signature $(0; [2, \overset{2g+2}{\ldots}, 2])$,

iv) there exists some symmetry in G, that is, an involution which is the image of an orientation reversing element of Λ.

We first analize how the epimorphisms θ and $\widehat{\theta} : \Lambda \to \widehat{G} = G/\langle\,\rho\rangle$ transform the canonical generators of Λ. This provides a presentation of the image of θ and we have to check whether this group satisfies the above conditions. Note that we have already found some necessary conditions for the existence of θ, namely, those relations appearing in the third columns of the above tables. We will see that *except in case* IX *such conditions are also sufficient.*

For the sake of completeness we state precisely the relation between the signature of Λ and the orders of the images by θ of its canonical generators (see remark 2.3.7 in [**14**]). It will be used without mention in the sequel. We will also use without mention that ρ is a central involution of G. We begin with signatures occurring in cases I to VIII.

Proposition 2.2.1. — *Let* Λ *be an* NEC *group with signature*

$$\sigma(\Lambda) = (0; +; [m_1, \ldots, m_s]; \{(n_1, \ldots, n_p)\})$$

and let $\{x_1, \ldots, x_s, c_0, \ldots, c_p, e\}$ *be a set of canonical generators of* Λ. *If* $\theta : \Lambda \to G$ *is a group epimorphism such that* $\ker\theta$ *is a surface Fuchsian group then*

(1) $\theta(e) = \left(\prod_{i=1}^{s} \theta(x_i)\right)^{-1}$; $\theta(c_p) = \theta(e)^{-1}\theta(c_0)\theta(e)$,
(2) $\theta(x_i)$ *has order* m_i *for* $1 \leqslant i \leqslant s$,
(3) $\theta(c_{j-1}c_j)$ *has order* n_j *for* $1 \leqslant j \leqslant p$,
(4) $\theta(c_j)$ *has order* 2 *for* $0 \leqslant j \leqslant p$.

Note that those elements of the form $\theta(c_j)$ are the unique images of the canonical generators of Λ which represent orientation reversing involutions in G. (In particular, condition *iv*) above is always satisfied in cases I to VIII.) So the symmetries of G are words on the symbols $\theta(x_i)$ and $\theta(c_j)$ with an odd exponent sum of symbols $\theta(c_j)$.

More information about the presentation of G can be obtained in our case by comparing the orders of its generators with the orders of the images under $\widehat{\theta} : \Lambda \to \widehat{G} = G/\langle\rho\rangle$ of the generators of Λ. For example, if $\widehat{m}_i = 1$ then $m_i = 2$ and this means that $\theta(x_i)$ has order 2 (see the Proposition above) whilst $\widehat{\theta}(x_i) = 1$. So $\theta(x_i) = \rho$ for such i. Similarly if $\widehat{n}_j = 1$ then $n_j = 2$ and so $\theta(c_{j-1}c_j) = \rho$. This gives $\theta(c_j) = \theta(c_{j-1})\rho$ for such j.

2.2.1. Case I. — Here Λ has one of the following signatures:

Ia: $(0; +; [2^r]; \{(2^{p_1}, N, 2^{p_2}, N)\})$, with $2r + p_1 + p_2 = (2g + 2)/N$,
Ib: $(0; +; [2^r]; \{(2^{p_1}, 2N, 2^{p_2}, 2N)\})$, with $2r + p_1 + p_2 = 2g/N$,
Ic: $(0; +; [2^r]; \{(2^{p_1}, N, 2^{p_2}, 2N)\})$, with $2r + p_1 + p_2 = (2g + 1)/N$.

First consider **case Ia**. As we already observed, $\theta(x_i) = \rho$ for $1 \leqslant i \leqslant r$. Let us denote $\tau_1 := \theta(c_0)$ and $\tau_2 := \theta(c_{p_1+1})$. Then

$$\theta(c_j) = \begin{cases} \tau_1\rho^j & \text{for } 1 \leqslant j \leqslant p_1, \\ \tau_2\rho^{j-(p_1+1)} & \text{for } p_1 + 2 \leqslant j \leqslant p_1 + p_2 + 1. \end{cases}$$

Furthermore, from formulae (1) of Proposition 2.2.1, $\theta(e) = \rho^r$ and thus $\theta(c_{p_1+p_2+2}) = \theta(e)^{-1}\theta(c_0)\theta(e) = \tau_1$. So G is generated by τ_1, τ_2 and ρ. Now $\theta(c_{p_1}c_{p_1+1}) = \tau_1\tau_2\rho^{p_1}$ and $\theta(c_{p_1+p_2+1}c_{p_1+p_2+2}) = \tau_1\tau_2\rho^{p_2}$. Both elements have order N and so $1 = (\tau_1\tau_2\rho^{p_1})^N = (\tau_1\tau_2)^N\rho^{p_1N}$, $1 = (\tau_1\tau_2\rho^{p_2})^N = (\tau_1\tau_2)^N\rho^{p_2N}$.

If N is even then $\tau_1\tau_2$ has order N and thus $G = \mathrm{D}_N \oplus \mathrm{Z}_2 = \langle\tau_1, \tau_2\rangle \oplus \langle\rho\rangle$. Therefore G has four conjugacy classes of symmetries represented by $\tau_1, \tau_2, \tau_1\rho$ and $\tau_2\rho$.

Note that here we can also deduce whether each representative of a conjugacy class of symmetries fixes points or not, *i.e.,* whether its species is 0 or not. For example, $\mathrm{sp}(\tau_1) \neq 0$ for any value of p_1 and p_2 since it is the image of a canonical reflection of Λ, whilst $\mathrm{sp}(\tau_1\rho) \neq 0$ if and only if $p_1 \neq 0$. The same happens replacing τ_1 by τ_2.

However, we will not discuss this aspect in the present chapter since in the next one we will compute the precise value of the species of each symmetry.

If N is odd then p_1+p_2 is even because $p_1+p_2=(2g+2)/N-2r$. If both p_1 and p_2 are even then $\tau_1\tau_2$ has order N and so $G=\mathrm{D}_N\oplus\mathrm{Z}_2=\langle\tau_1,\tau_2\rangle\oplus\langle\rho\rangle$, which is isomorphic to $\langle\tau_1,\tau_2\rho\rangle=\mathrm{D}_{2N}$. Therefore G has two conjugacy classes of symmetries represented by τ_1 and $\tau_2\rho\sim\tau_1\rho$. If both p_1 and p_2 are odd then $(\tau_1\tau_2)^N=\rho$ and so $G=\mathrm{D}_{2N}=\langle\tau_1,\tau_2\rangle$, which has two conjugacy classes of symmetries, represented by τ_1 and $\tau_2\sim\tau_1\rho$.

In **cases Ib and Ic** it is more convenient to work with $\tau_1=\theta(c_{p_1+p_2+2})$ and $\tau_2:=\theta(c_{p_1+p_1+1})$. Both are elements of order 2 whose product has order $2N$. Thus $G=\mathrm{D}_{2N}=\langle\tau_1,\tau_2\rangle$ and $\rho=(\tau_1\tau_2)^N$ since this is the only central element of G. Hence τ_1 and τ_2 are representatives of the unique conjugacy classes of symmetries of G. Note that for N odd, which is always so in case Ic, we have $\tau_2\sim\tau_1\rho$. We claim that the rest of relations in Proposition 2.2.1 also hold here. Indeed, $\theta(c_0)=\theta(e)\theta(c_{p_1+p_2+2})\theta(e)^{-1}=\tau_1$ and so

$$\theta(c_j)=\begin{cases}\tau_1\rho^j & \text{for } 1\leqslant j\leqslant p_1,\\ \tau_2\rho^{p_1+p_2+1-j} & \text{for } p_1+1\leqslant j\leqslant p_1+p_2.\end{cases}$$

But then $\theta(c_{p_1}c_{p_1+1})^N=(\tau_1\tau_2)^N\rho^{N(p_1+p_2)}=\rho\rho^{N(p_1+p_2)}$. So $\theta(c_{p_1}c_{p_1+1})$ has order $2N$ in case Ib because $N(p_1+p_2)$ is even, and order N in case Ic because $N(p_1+p_2)$ is odd.

In all three cases the quotient group $\widehat{G}=G/\langle\rho\rangle$ is $\langle\widehat{\tau}_1,\widehat{\tau}_2\,|\,(\widehat{\tau}_1)^2,(\widehat{\tau}_2)^2,(\widehat{\tau}_1\widehat{\tau}_2)^N\rangle=\mathrm{D}_N$, and the epimorphism $\widehat{\theta}$ is defined by $\widehat{\theta}(x_i)=1$ for $1\leqslant i\leqslant r$, $\widehat{\theta}(c_i)=\widehat{\tau}_1$ for $0\leqslant i\leqslant p_1$ and $i=p_1+p_2+2$, $\widehat{\theta}(c_j)=\widehat{\tau}_2$ for $p_1+1\leqslant j\leqslant p_1+p_2+1$ and $\widehat{\theta}(e)=1$. This common presentation of $\widehat{G}$ characterizes case I and, in fact, we will see that $\widehat{G}$ is different in each case. It will be used in the next chapter to define the different classes of hyperelliptic symmetric curves.

2.2.2. Case II. — In this case Λ has one of the following signatures:

IIa: $(0;+;[2^r,N];\{(2^p)\})$, with $2r+p=(2g+2)/N$,
IIb: $(0;+;[2^r,2N];\{(2^p)\})$, with $2r+p=2g/N$.

Here $\theta(x_i)=\theta(c_{j-1}c_j)=\rho$ for $1\leqslant i\leqslant r$ and $1\leqslant j\leqslant p$. Thus, if we denote $\tau:=\theta(c_0)$, which is a symmetry, then $\theta(c_j)=\tau\rho^j$. Now, $\widehat{m}_{r+1}=N$ means that $a:=\theta(x_{r+1})$ is an element of order N in case IIa and $2N$ in case IIb. Also $\theta(e)=a^{-1}\rho^r$ and $\tau\rho^p=\theta(c_p)=\theta(e^{-1})\theta(c_0)\theta(e)=\rho^r a\tau a^{-1}\rho^r=a\tau a^{-1}$. So

$$\rho^p=\tau a\tau a^{-1}.$$

Suppose first that ***p* is even**. Then $\tau a=a\tau$ and therefore G is Abelian. In case IIa, the element a has order N and so $G=\langle\rho\rangle\oplus\langle\tau\rangle\oplus\langle a\rangle=\mathrm{Z}_2\oplus\mathrm{Z}_2\oplus\mathrm{Z}_N$. Hence G has two conjugacy classes of symmetries, namely τ and $\tau\rho$ if N is odd, and two more classes represented by $\tau a^{N/2}$ and $\tau\rho a^{N/2}$ if N is even. In case IIb, $a^N=\rho$ and so $G = \langle\tau\rangle \oplus \langle a\rangle = \mathrm{Z}_2 \oplus \mathrm{Z}_{2N}$. Hence in this case G has two conjugacy classes of symmetries represented by τ and $\tau\rho$.

If ***p* is odd**, then $\rho = \tau a\tau a^{-1}$ and by the relations between r, p and N we see that N is even. So in case IIa G has presentation

$$G = \langle a, \tau, \rho \,|\, a^N, \tau^2, \rho^2, \rho\tau a\tau a^{-1}, \rho a\rho a^{-1}\,\rangle.$$

It is convenient to observe that the above group is isomorphic to

$$\langle a, \tau \,|\, a^N, \tau^2, (a\tau)^2(\tau a)^{-2}, \tau a^2\tau a^{-2}\,\rangle,$$

where $\rho = \tau a\tau a^{-1}$. We claim that for N even this group has order $4N$. Indeed for N even the Abelian group $H = \mathrm{Z}_N \oplus \mathrm{Z}_2$ admits the presentation $\langle a_1, a_2 \mid a_1^N, a_2^N, a_1^2a_2^{-2}, a_1a_2a_1^{-1}a_2^{-1}\rangle$. Then the semidirect product of H by $\mathrm{Z}_2 = \langle\tau\rangle$ subject to the action $\tau a_1\tau = a_2$, has presentation

$$\langle a_1, a_2, \tau \,|\, a_1^N, a_1^2a_2^{-2}, \tau^2, \tau a_1\tau a_2^{-1}, a_1a_2a_1^{-1}a_2^{-1}\rangle.$$

This group is isomorphic to $\langle a_1, \tau \,|\, a_1^N, \tau^2, (a_1\tau)^2(\tau a_1)^{-2}, \tau a_1^2\tau a_1^{-2}\,\rangle$, which is nothing but G.

Now each element x of G can be written in a unique form as $x = a^\alpha\tau^\delta\rho^\varepsilon$, for some $0 \leqslant \alpha < N$ and $0 \leqslant \varepsilon, \delta \leqslant 1$. But if x represents an orientation reversing automorphism of G, then $\delta = 1$ and therefore either $x = a^\alpha\tau$ or $x = a^\alpha\tau\rho$. Observe however that these two elements are conjugate in G because $a^\alpha\tau\rho = a^{\alpha+1}\tau a^{-1}$. So we see that each element representing an orientation reversing automorphism is conjugate to $a^\alpha\tau$. But if α is odd then $(a^\alpha\tau)^2 = a^{2(\alpha+1)}(\tau a^{-1})^2$. So such element cannot be an involution since otherwise $\tau a\tau = a^{2\alpha+1}$ and therefore G would have order $\leqslant 2N$. So we see that α is even and then $(a^\alpha\tau)^2 = a^{2\alpha}$. Therefore either $\alpha = 0$ or $\alpha = N/2$. So if 4 divides N then G has two conjugacy classes of symmetries: τ and $\tau a^{N/2}$. If 4 does not divide N then G has one conjugacy class of symmetries, represented by τ.

In case IIb, $G = \langle a, \tau \,|\, a^{2N}, \tau^2, \tau a\tau a^{N-1}\,\rangle = \mathrm{Z}_{2N} \rtimes \mathrm{Z}_2$ because the map $a \mapsto a^{N+1}$ induces an automorphism of a cyclic group of order $2N$ for N even. But now if 4 divides N then τ is the only one, up to conjugacy, symmetry of G while in the other case G has two conjugacy classes of symmetries: τ (which is conjugate to $\tau\rho$) and $\tau a^{N/2}$.

Note that in both cases IIa and IIb we have $\widehat{G} = \langle\widehat{a}\rangle \oplus \langle\widehat{\tau}\rangle = \mathrm{Z}_N \oplus \mathrm{Z}_2$, with $\widehat{\theta}(x_i) = 1$ for $1 \leqslant i \leqslant r$, $\widehat{\theta}(x_{r+1}) = \widehat{a}$, $\widehat{\theta}(c_j) = \widehat{\tau}$ for $0 \leqslant j \leqslant p$ and $\widehat{\theta}(e) = \widehat{a}$.

2.2.3. Case III. — Here Λ has one of the following signatures

IIIa: $(0; +; [2^r, 2]; \{(2^p, N/2)\})$, with $2r + p = (2g+2)/N$,
IIIb: $(0; +; [2^r, 2]; \{(2^p, N)\})$, with $2r + p = 2g/N$,
IIIc: $(0; +; [2^r, 4]; \{(2^p, N/2)\})$, with $2r + p = (2g+2)/N - 1$ or
IIId: $(0; +; [2^r, 4]; \{(2^p, N)\})$, with $2r + p = 2g/N - 1$.

In all cases $\theta(x_i) = \rho$ for $1 \leqslant i \leqslant r$ and $\tau_1 := \theta(c_0)$ and $\tau_2 := \theta(c_{p+1})$ are elements in G of order 2. However, $a := \theta(x_{r+1})$ has order 2 in cases IIIa, IIIb and order 4 in cases IIIc and IIId. Clearly $\theta(c_j) = \tau_1\rho^j$ for $1 \leqslant j \leqslant p$ and $\theta(c_{p+1}) = \theta(e)^{-1}\theta(c_0)\theta(e) = a\tau_1 a^{-1}$ since $\theta(e) = a^{-1}\rho^r$. Recall that $N \geqslant 4$ and N is even along case III.

Case IIIa. Here $\tau_2 = \theta(c_{p+1}) = a\tau_1 a$ and the element $\tau_1\tau_2\rho^p = \theta(c_p)\theta(c_{p+1})$ has order $N/2$. Thus if $pN/2$ is even then $\tau_1\tau_2$ has also order $N/2$ and so $G = \langle \tau_1, a \mid a^2, \tau_1^2, (\tau_1 a)^N\rangle \oplus \langle\rho\rangle = \mathrm{D}_N \oplus \mathrm{Z}_2$. If $pN/2$ is odd then $\rho = (\tau_1\tau_2)^{N/2} = (\tau_1 a)^N$ and therefore $G = \langle a, \tau_1\rangle = \mathrm{D}_{2N}$. Consequently, G has two conjugacy classes of symmetries, represented by τ_1 and $\tau_1\rho$ if $N/2$ is even, four conjugacy classes of symmetries, represented by $\tau_1, \tau_1\rho$, $(\tau_1 a)^{N/2}$ and $(\tau_1 a)^{N/2}\rho$ if $N/2$ is odd and p even and finally one conjugacy class of symmetries represented by τ_1 if $N/2$ and p are odd.

Case IIIb. Also here $\tau_2 = \theta(c_{p+1}) = a\tau_1 a$. Now $\rho = (\tau_1\tau_2\rho^p)^{N/2} = (\tau_1\tau_2)^{N/2}\rho^{pN/2}$. So if $pN/2$ is even then $\rho = (\tau_1\tau_2)^{N/2}$ and thus $G = \langle a, \tau_1\rangle = \mathrm{D}_{2N}$, which has one conjugacy class of symmetries, represented by τ_1. If $pN/2$ is odd then $(\tau_1\tau_2)^{N/2} = 1$ and now $G = \langle\tau_1, a\rangle \oplus \langle\rho\rangle = \mathrm{D}_N \oplus \mathrm{Z}_2$, which has four conjugacy classes of symmetries, represented by $\tau_1, (\tau_1 a)^{N/2}, \tau_1\rho$ and $(\tau_1 a)^{N/2}\rho$.

Case IIIc. Now $a^2 = \rho$ and $\tau_2 = \theta(c_{p+1}) = a\tau_1 a^{-1}$. But then $1 = (\theta(c_p c_{p+1}))^{N/2} = (\tau_1\tau_2)^{N/2}\rho^{pN/2} = (\tau_1 a\tau_1 a\rho)^{N/2}\rho^{pN/2} = (\tau_1 a)^N\rho^{(p+1)N/2}$. So if $(p+1)N/2$ is even then $(\tau_1 a)^N = 1$ and thus

$$G = \langle a, \tau_1 \mid a^4, \tau_1^2, (\tau_1 a^2)^2, (a\tau_1)^N\rangle$$

since clearly all relations hold in G and the right hand side group also has order $4N$. To see this, consider the group

$$H_{N/2} = \langle a_1, a_2 \mid a_1^4, a_1^2 a_2^2, (a_1 a_2)^{N/2}\rangle,$$

which has order $2N$, see [**4**]. Now the semidirect product of $H_{N/2}$ by $\mathrm{Z}_2 = \langle\tau\rangle$ subject to the action $\tau a_1\tau = a_2$, has presentation

$$\langle a_1, a_2, \tau \mid a_1^4, (a_1 a_2)^{N/2}, a_1^2 a_2^{-2}, \tau^2, \tau a_1\tau a_2^{-1}, \tau a_2\tau a_1^{-1}\rangle,$$

which is nothing but $\langle a_1, \tau \mid a_1^4, \tau^2, (\tau a_1^2)^2, (a_1\tau)^N\rangle$.

Each element x of this group can be written in a unique way as $x = (a\tau_1)^\alpha a^\delta$ for some $0 \leqslant \alpha < N$ and $0 \leqslant \delta \leqslant 3$. We now look for all symmetries of G. If x is one of them then α is odd. For $\delta = 1$, x is conjugate to $\tau_1\rho$ or τ_1 according to $\alpha \equiv 1$ or 3 (mod 4). So, for $\delta = 3$, x is conjugate to τ_1 or $\tau_1\rho$ according to $\alpha \equiv 1$ or 3 (mod 4). For δ even it is $x^2 = (a\tau_1)^{2\alpha}$ and so $\alpha = N/2$, which must be odd. Therefore if $N/2$ is even then τ_1 and $\tau_1\rho$ are the unique conjugacy classes of symmetries of G. If $N/2$ is odd then $(a\tau_1)^{N/2}$ and $(a\tau_1)^{N/2}\rho$ are also symmetries of G; however, both are conjugate via a, and the same happens to τ_1 and $\tau_1\rho$ via $\tau_1(\tau_1 a)^{N/2}$. Consequently, if $N/2$ and p are odd then G has two conjugacy classes of symmetries, with representatives τ_1 and $(\tau_1 a)^{N/2}$.

If $(p+1)N/2$ is odd then $\rho = (\tau_1 a)^N$. So

$$G = \langle \rho, \tau_1, a \mid a^4, \tau_1^2, (\tau_1 a^2)^2, \rho(\tau_1 a)^N, \rho a^2 \rangle$$

and similarly we show that now G has two conjugacy classes of symmetries represented by τ_1 and $\tau_1\rho$.

Case IIId. Also here $a^2 = \rho$ and $\tau_2 = \theta(c_{p+1}) = a\tau_1 a^{-1}$. So $\rho = (\theta(c_p c_{p+1}))^{N/2} = (\tau_1 a)^N \rho^{(p+1)N/2}$. Therefore if $(p+1)N/2$ is even then $\rho = (\tau_1 a)^N$ and thus

$$G = \langle a, \tau_1 \,|\, a^4, \tau_1^2, a^2(a\tau_1)^N, (a^2\tau_1)^2 \rangle$$

since the right hand side group also has order $4N$. This can be proved similarly as in case IIIc but using the group $G_{N/2} = \langle a_1, a_2 \,|\, a_1^2 a_2^{N/2}, a_2^N, a_1^{-1} a_2 a_1 a_2 \rangle$ (which has order $2N$, see [**4**]), instead of $H_{N/2}$. Again each element x of such group can be written in a unique way as $x = (a\tau_1)^\alpha a^\delta$ for some $0 \leqslant \alpha < N$ and $0 \leqslant \delta \leqslant 3$. If x represents a symmetry then α is odd. Moreover, $\delta = 1$ or 3 since otherwise $1 = x^2 = (a\tau_1)^{2\alpha}$ and so $\alpha = 0$ or N, which is impossible. As in case IIIc, for $\delta = 1$, $x \sim \tau_1\rho$ if $\alpha \equiv 1 \pmod 4$ and $x \sim \tau_1$ otherwise; for $\delta = 3$, $x \sim \tau_1$ if $\alpha \equiv 1 \pmod 4$ and $x \sim \tau_1\rho$ otherwise. We claim that $\tau_1 \sim \tau_1\rho$ if and only if $N/2$ is even. Indeed if $N/2$ is even then $(a\tau_1)^{N/2}\tau_1(a\tau_1)^{-N/2} = (a\tau_1)^{N/2}\tau_1(\tau_1 a)^{N/2} = (a\tau_1)^N\tau_1 = a^2\tau_1 = \rho\tau_1$. Conversely assume that $N/2$ is odd. Observe that each element of G acts by conjugation on τ_1 in the same way as $(a\tau_1)^\alpha$ or $(a\tau_1)^\alpha a$. For the element of the first form $(a\tau_1)^\alpha\tau_1(a\tau_1)^{-\alpha} = (a\tau_1)^{2\alpha-1}a^{2\alpha+1}$. The last equals $\tau_1 a^2$ if and only if $(\tau_1 a)^{2\alpha} = \rho^{\alpha-1}$. This implies for α odd that $(\tau_1 a)^{2\alpha} = 1$ and so $\alpha = 0$, contrary to our assumption. For α even this implies $(\tau_1 a)^{4\alpha} = 1$ and so $\alpha = N/2$, which is odd by assumption; a contradiction again. Similarly one can show that the conjugation of τ_1 by an arbitrary element of the second form is not equal to $\tau_1\rho$. Therefore here G has one conjugacy class of symmetries, represented by τ_1, if $N/2$ is even and two conjugacy classes, represented by τ_1 and $\tau_1\rho$, if $N/2$ and p are odd.

If $(p+1)N/2$ is odd then $(\tau_1 a)^N = 1$. So

$$G = \langle\ \tau_1, a \,|\, a^4, \tau_1^2, (a^2\tau_1)^2, (a\tau_1)^N \rangle$$

since the right hand side group has order $4N$. As in the previous subcases one can show that now G has two conjugacy classes of symmetries represented by τ_1 and $(\tau_1 a)^{N/2}$.

Observe that in all cases IIIa-IIId we have $\widehat{G} = \langle \widehat{a}, \widehat{\tau}_1 \rangle \simeq \mathrm{D}_N$ since $(\widehat{a})^2 = (\widehat{\tau}_1)^2 = (\widehat{a}\widehat{\tau}_1)^N = 1$. The epimorphism $\widehat{\theta}$ is given by $\widehat{\theta}(x_i) = 1$ for $1 \leqslant i \leqslant r$, $\widehat{\theta}(x_{r+1}) = \widehat{a}$, $\widehat{\theta}(c_i) = \widehat{\tau}_1$ for $0 \leqslant i \leqslant p$, $\widehat{\theta}(c_{p+1}) = \widehat{a}\widehat{\tau}_1\widehat{a}$ and $\widehat{\theta}(e) = \widehat{a}$.

2.2.4. Case IV. — Here Λ has one of the following signatures:

IVa: $(0; +; [2^r]; \{(2^{p_1}, 2, 2^{p_2}, 2, 2^{p_3}, N/2)\})$, with $2r + p_1 + p_2 + p_3 = (2g+2)/N$,

IVb: $(0; +; [2^r]; \{(2^{p_1}, 2, 2^{p_2}, 4, 2^{p_3}, N/2)\})$, with $2r + p_1 + p_2 + p_3 = (2g+2)/N - 1/2$,

IVc: $(0; +; [2^r]; \{(2^{p_1}, 4, 2^{p_2}, 4, 2^{p_3}, N/2)\})$, with $2r + p_1 + p_2 + p_3 = (2g+2)/N - 1$,

IVd: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},2,2^{p_3},N)\})$, with $2r+p_1+p_2+p_3=2g/N$,
IVe: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},4,2^{p_3},N)\})$, with $2r+p_1+p_2+p_3=2g/N-1/2$, or
IVf: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},4,2^{p_3},N)\})$, with $2r+p_1+p_2+p_3=2g/N-1$.

Recall that N is even $\geqslant 4$ along this case. If $N=4$ then signatures of cases IVb and IVe give rise to the same configuration of symmetries than cases IVd and IVc for $N=4$ respectively; moreover, in the former cases the relation between r, p_i, g and N forces N to be a multiple of 4. So, in cases IVb and IVe we will suppose $N\equiv 0 \pmod 4$ and $N\geqslant 8$.

In all cases $\theta(x_i)=\rho$ for $1\leqslant i\leqslant r$ and $\tau_1:=\theta(c_0)$, $\tau_2:=\theta(c_{p_1+1})$ and $\tau_3:=\theta(c_{p_1+p_2+2})$ are elements of order 2. Then

$$\theta(c_j)=\begin{cases}\tau_1\rho^j & \text{for } 1\leqslant j\leqslant p_1,\\ \tau_2\rho^{j-p_1-1} & \text{for } p_1+2\leqslant j\leqslant p_1+p_2+1,\\ \tau_3\rho^{j-p_1-p_2-2} & \text{for } p_1+p_2+3\leqslant j\leqslant p_1+p_2+p_3+2.\end{cases}$$

Note that $\theta(c_{p_1+p_2+p_3+3})=\theta(e)^{-1}\theta(c_0)\theta(e)=\rho^r\tau_1\rho^{-r}=\tau_1$.

Case IVa. Here $1=\theta(c_{p_1}c_{p_1+1})^2=(\tau_1\tau_2)^2$ and analogously $(\tau_2\tau_3)^2=1$. Also $(\tau_3\tau_1)^{N/2}\rho^{p_3N/2}=1$ because $\tau_3\rho^{p_3}\tau_1=\theta(c_{p_1+p_2+p_3+2}c_{p_1+p_2+p_3+3})$ has order $N/2$. So if $p_3N/2$ is even then $(\tau_1\tau_3)^{N/2}=1$ and thus

$$G=\langle\tau_1,\tau_3\rangle\oplus\langle\tau_2\rangle\oplus\langle\rho\rangle\simeq \mathrm{D}_{N/2}\oplus\mathrm{Z}_2\oplus\mathrm{Z}_2.$$

If $p_3N/2$ is odd then $\rho=(\tau_1\tau_3)^{N/2}$ and therefore $G=\langle\tau_1,\tau_3\rangle\oplus\langle\tau_2\rangle\simeq\mathrm{D}_N\oplus\mathrm{Z}_2$. This group also has presentation

$$G=\langle\tau_1,\tau_3\rho\rangle\oplus\langle\tau_2\rangle\oplus\langle\rho\rangle\simeq \mathrm{D}_{N/2}\oplus\mathrm{Z}_2\oplus\mathrm{Z}_2.$$

Consequently, only the parity of $N/2$ affects the number of conjugacy classes of symmetries of G. If $N/2$ is even then G has eight, with representatives τ_i, $\tau_i\rho$ for $i=1,2,3$, $(\tau_1\tau_3)^{N/4}\tau_2$ and $(\tau_1\tau_3)^{N/4}\tau_2\rho$. If $N/2$ is odd then it has four, represented by τ_i, $\tau_i\rho$ for $i=1,2$.

Case IVb. Here $(\tau_1\tau_2)^2=1$, $(\tau_2\tau_3)^2=\rho$ and so $\tau_1(\tau_2\tau_3)^2=(\tau_2\tau_3)^2\tau_1$. Furthermore, $1=(\tau_1\tau_3\rho^{p_3})^{N/2}=(\tau_1\tau_3)^{N/2}\rho^{p_3N/2}=(\tau_1\tau_3)^{N/2}$ because $N\equiv 0\pmod 4$ in case IVb. Hence

$$G=\langle\tau_1,\tau_2,\tau_3\,|\,\tau_1^2,\tau_2^2,\tau_3^2,(\tau_1\tau_2)^2,(\tau_2\tau_3)^4,(\tau_1\tau_3)^{N/2},(\tau_1(\tau_2\tau_3)^2)^2\,\rangle$$

since these relations clearly hold in G and the right hand side group has order $4N$. To see this consider the group $V_{N/2}$ with presentation $\langle x,y\,|\,x^4,y^{N/2},(xy)^2,(x^{-1}y)^2\rangle$, which has order $2N$, [**4**]. Thus the semidirect product of $V_{N/2}$ by $\mathrm{Z}_2=\langle\tau\rangle$ subject to the action $\tau y\tau=y^{-1}$, $\tau x\tau=x^{-1}$, has order $4N$ and presentation

$$\langle x,y,\tau\,|\,x^4,y^{N/2},(xy)^2,(x^{-1}y)^2,\tau^2,\tau^2,(\tau y)^2,(\tau x)^2\rangle.$$

But this group is isomorphic to the above one via the mapping $\tau_1 \mapsto \tau y$, $\tau_2 \mapsto x\tau$, $\tau_3 \mapsto \tau$. This proves our assertion. Such group can also be represented as

$$G = (\langle \tau_1, \tau_3 \rangle \oplus \langle \rho \rangle) \rtimes \langle \tau_2 \rangle \simeq (\mathrm{D}_{N/2} \oplus \mathrm{Z}_2) \rtimes \mathrm{Z}_2,$$

where τ_2 acts on $\mathrm{D}_{N/2} \oplus \mathrm{Z}_2$ by $\tau_1^{\tau_2} = \tau_1$, $\tau_3^{\tau_2} = \rho\tau_3$, $\rho^{\tau_2} = \rho$. Clearly, τ_1, τ_3, $\tau_1\rho$ and $\tau_3\rho$ are representatives of all conjugacy classes of symmetries in the factor $\mathrm{D}_{N/2} \oplus \mathrm{Z}_2$. However $\tau_2\tau_3\tau_2^{-1} = \tau_3\rho$, while the remaining elements τ_1, τ_3 and $\tau_1\rho$ are still nonconjugate in G. Let us look for representatives of all conjugacy classes of symmetries of G. Each $x \in G$ can be written in a unique way as $x = (\tau_1\tau_3)^\alpha \tau_1^\beta \tau_2^\delta \rho^\gamma$ with $0 \leqslant \alpha < N/2$, $0 \leqslant \beta, \delta, \gamma \leqslant 1$. If x is a symmetry then $\beta + \delta$ is odd. Thus, if $\beta = 1$ then $\delta = 0$, *i.e.,* $x \in \mathrm{D}_{N/2} \oplus \mathrm{Z}_2$ and so x is conjugate to τ_1, τ_3 or $\tau_1\rho$. If $\beta = 0$ then $\delta = 1$, *i.e.,* $x = (\tau_1\tau_3)^\alpha \tau_2 \rho^\gamma$ and $x^2 = (\tau_1\tau_3)^{2\alpha}\rho^\alpha$, as is easy to check. Since $\rho \notin \langle \tau_1, \tau_3 \rangle$ it follows that α is even and so $(\tau_1\tau_3)^{2\alpha} = 1$. Thus, either $\alpha = 0$ and so $x = \tau_2\rho^\gamma$, or $\alpha = N/4$ (if $N/4$ is even) and so $x = (\tau_1\tau_3)^{N/4}\tau_2\rho^\gamma$. But $\tau_2\rho = \tau_3\tau_2\tau_3$, *i.e.,* τ_2 and $\tau_2\rho$ are conjugate in G, and the same holds for $(\tau_1\tau_3)^{N/4}\tau_2$ and $(\tau_1\tau_3)^{N/4}\tau_2\rho$.

Summing up, G admits exactly five nonconjugate symmetries represented by τ_i, for $i = 1, 2, 3$, $\tau_1\rho$ and $(\tau_1\tau_3)^{N/4}\tau_2$ if $N/4$ is even and four conjugacy classes of symmetries represented by τ_1, τ_2, τ_3 and $\tau_1\rho$ if $N/4$ is odd.

Case IVc. Here $(\tau_1\tau_2)^2 = \rho = (\tau_2\tau_3)^2$ and

$$G = \langle \tau_1, \tau_2, \tau_3 \mid \tau_1^2, \tau_2^2, \tau_3^2, (\tau_1\tau_2)^4, (\tau_2\tau_3)^4, (\tau_1\tau_3)^{N/2}, (\tau_1\tau_2)^2(\tau_2\tau_3)^2 \rangle$$

since clearly all relations hold in G and in addition the right hand side group has order $4N$. This can be proved similarly as in case IVb but using the group $H_{N/2}$ appearing in case IIIc instead of $V_{N/2}$. So our group can be represented as

$$G = (\langle \tau_1, \tau_3 \rangle \oplus \langle \rho \rangle) \rtimes \langle \tau_2 \rangle \simeq (\mathrm{D}_{N/2} \oplus \mathrm{Z}_2) \rtimes \mathrm{Z}_2,$$

where now τ_2 acts on $\mathrm{D}_{N/2} \oplus \mathrm{Z}_2$ by $\tau_1^{\tau_2} = \tau_1\rho$, $\tau_3^{\tau_2} = \tau_3\rho$, $\rho^{\tau_2} = \rho$.

Again τ_1, τ_3, $\tau_1\rho, \tau_3\rho$ if $N/2$ is even and $\tau_1, \tau_1\rho$ if $N/2$ is odd are representatives of all conjugacy classes of symmetries in the factor $\mathrm{D}_{N/2} \oplus \mathrm{Z}_2$. But now τ_1 and τ_3 are conjugate, via τ_2, to $\tau_1\rho$ and $\tau_3\rho$ respectively. Let us look for all the conjugacy classes of symmetries in G. Each element of G can be represented in a unique way as $x = (\tau_1\tau_3)^\alpha \tau_1^\beta \tau_2^\delta \rho^\gamma$ with $0 \leqslant \alpha < N/2$, $0 \leqslant \beta, \delta, \gamma \leqslant 1$. If $x \in G$ is a symmetry not in $\mathrm{D}_{N/2} \oplus \mathrm{Z}_2$ then $\delta = 1$ and so $\beta = 0$. Hence $1 = x^2 = ((\tau_1\tau_3)^\alpha\tau_2)^2\rho^{2\gamma} = (\tau_1\tau_3)^{2\alpha}$ and so either $\alpha = 0$, and in such a case $x = \tau_2$ or $x = \tau_2\rho$, or else $\alpha = N/4$ (if $N/2$ is even) and in such a case $x = (\tau_1\tau_3)^{N/4}\tau_2$ or $(\tau_1\tau_3)^{N/4}\tau_2\rho$. However, τ_2 and $(\tau_1\tau_3)^{N/4}\tau_2$ are conjugate, via τ_1, to $\tau_2\rho$ and $(\tau_1\tau_3)^{N/4}\tau_2\rho$ respectively. So we conclude that G has four conjugacy classes of symmetries, represented by τ_i for $i = 1, 2, 3$ and $(\tau_1\tau_3)^{N/4}\tau_2$, if $N/2$ is even and two conjugacy classes of symmetries, represented by τ_i for $i = 1, 2$, if $N/2$ is odd.

Case IVd. Here we have

$$G = \langle \tau_1, \tau_2, \tau_3 \mid \tau_1^2, \tau_2^2, \tau_3^2, (\tau_1\tau_2)^2, (\tau_2\tau_3)^2, (\tau_1\tau_3)^N \rangle$$

and $\rho = (\tau_1\tau_3)^{N/2}$, i.e,

$$G = \mathrm{D}_N \oplus \mathrm{Z}_2 = \langle \tau_1, \tau_3 \rangle \oplus \langle \tau_2 \rangle.$$

Thus, τ_i for $i = 1, 2, 3$ and $\rho\tau_2$ are representatives of all conjugacy classes of symmetries of G.

Case IVe. Here we have

$$G = \langle \tau_1, \tau_2, \tau_3 \mid \tau_1^2, \tau_2^2, \tau_3^2, (\tau_1\tau_2)^2, (\tau_2\tau_3)^4, (\tau_1\tau_3)^{N/2}(\tau_2\tau_3)^2 \rangle$$

and $\rho = (\tau_2\tau_3)^2$. Recall that $N \equiv 0 \pmod 4$ and $N \geqslant 8$. Now

$$G = \mathrm{D}_N \rtimes \mathrm{Z}_2 = \langle \tau_1, \tau_3 \rangle \rtimes \langle \tau_2 \rangle,$$

where τ_2 acts on D_N by $\tau_1^{\tau_2} = \tau_1$, $\tau_3^{\tau_2} = \rho\tau_3$. The last equality implies that the symmetries τ_3 and $\rho\tau_3$ are conjugate in G and the same happens to τ_2 and $\tau_2\rho$. The factor D_N admits exactly two conjugacy classes of symmetries, represented by τ_1 and τ_3. Now, each symmetry of G which is not in $\langle \tau_1, \tau_3 \rangle$ has the form $x = (\tau_1\tau_3)^\alpha\tau_2$ with $0 \leqslant \alpha < N$. But $x^2 = (\tau_1\tau_3)^\alpha\tau_2(\tau_1\tau_3)^\alpha\tau_2 = (\tau_1\tau_3)^{2\alpha}\rho^\alpha$. For α even, x has order 2 if and only if $(\tau_1\tau_3)^{2\alpha} = 1$; thus either $\alpha = 0$ and so $x = \tau_2$, or $\alpha = N/2$ and so $x = \tau_2\rho$. For α odd, x has order 2 if and only if $(\tau_1\tau_3)^{4\alpha} = 1$; thus either $\alpha = N/4$ and so $x = (\tau_1\tau_3)^{N/4}\tau_2$, or $\alpha = 3N/4$ and so $x = (\tau_1\tau_3)^{N/4}\tau_2\rho$ (note that this happens only for $N/4$ odd and that $(\tau_1\tau_3)^{N/4}\tau_2\rho \sim (\tau_1\tau_3)^{N/4}\tau_2$ via τ_2).

In particular, G has three conjugacy classes of symmetries represented by τ_1, τ_2, τ_3 if $N/4$ is even and one more class represented by $(\tau_1\tau_3)^{N/4}\tau_2$ if $N/4$ is odd.

Case IVf. Here we have

$$G = \langle \tau_1, \tau_2, \tau_3 \mid \tau_1^2, \tau_2^2, \tau_3^2, (\tau_1\tau_2)^4, (\tau_1\tau_2)^2(\tau_2\tau_3)^2, (\tau_1\tau_2)^2(\tau_1\tau_3)^{N/2} \rangle,$$

with $\rho = (\tau_1\tau_2)^2$. Therefore

$$G = \mathrm{D}_N \rtimes \mathrm{Z}_2 = \langle \tau_1, \tau_3 \rangle \rtimes \langle \tau_2 \rangle,$$

where the action of τ_2 is given by $\tau_1^{\tau_2} = \rho\tau_1$ and $\tau_3^{\tau_2} = \rho\tau_3$. So $\tau_1 \sim \tau_1\rho$, $\tau_2 \sim \tau_2\rho$ and $\tau_3 \sim \tau_3\rho$. We claim that τ_1 and τ_3 are conjugate in G if and only if $N/2$ is odd. Indeed, if they are conjugate then their images $\widehat{\tau}_1$ and $\widehat{\tau}_3$ in $\widehat{G} = G/\langle \rho \rangle = \mathrm{D}_{N/2} \oplus \mathrm{Z}_2 = \langle \widehat{\tau}_1, \widehat{\tau}_3 \rangle \oplus \langle \widehat{\tau}_2 \rangle$ remain conjugate and so $N/2$ is odd. Conversely, if $N/2$ is odd then $\tau_1\tau_3\rho$ has order $N/2$ and so $\tau_1 \sim \tau_3\rho$.

Furthermore each symmetry of G which is not in $\langle \tau_1, \tau_3 \rangle$ has the form $x = (\tau_1\tau_3)^\alpha\tau_2$ with $0 \leqslant \alpha < N$. Then $x^2 = (\tau_1\tau_3)^{2\alpha}$ and so x has order 2 if and only if either $x = \tau_2$ or $x = \rho\tau_2$. But $\rho\tau_2 = (\tau_2\tau_3)^2\tau_2$ and so $\rho\tau_2$ is conjugate to τ_2. Thus G has two conjugacy classes of symmetries, represented by τ_1 and τ_2, if $N/2$ is odd, and one more, represented by τ_3, if $N/2$ is even.

Observe that $\widehat{G} = \mathrm{D}_{N/2} \oplus \mathrm{Z}_2 = \langle \widehat{\tau}_1, \widehat{\tau}_3 \rangle \oplus \langle \widehat{\tau}_2 \rangle$ along case IV, and for the epimorphism $\widehat{\theta}$ we have $\widehat{\theta}(x_i) = 1$ for $i \leqslant r$, $\widehat{\theta}(e) = 1$ and

$$\widehat{\theta}(c_i) = \begin{cases} \widehat{\tau}_1 \text{ for } 0 \leqslant i \leqslant p_1, \\ \widehat{\tau}_2 \text{ for } p_1 + 1 \leqslant i \leqslant p_1 + p_2 + 1, \\ \widehat{\tau}_3 \text{ for } p_1 + p_2 + 2 \leqslant i \leqslant p_1 + p_2 + p_3 + 2, \\ \widehat{\tau}_1 \text{ for } i = p_1 + p_2 + p_3 + 3. \end{cases}$$

In cases V to VIII the order of the group G does not depend on N. Indeed, in these cases the order is 48, 48, 96 or 240. This allows us to use GAP Programm to deal with properties of such groups. We explain just some cases in detail, restricting ourselves to give the results in the others.

2.2.5. Case V. — Recall that here the signature of Λ is one of the following:

$$\mathrm{Va} : (0; +; [2^r, 3]; \{(2^p, 2)\}), \ \mathrm{Vb} : (0; +; [2^r, 6]; \{(2^p, 2)\}),$$
$$\mathrm{Vc} : (0; +; [2^r, 3]; \{(2^p, 4)\}), \ \mathrm{Vd} : (0; +; [2^r, 6]; \{(2^p, 4)\}).$$

In all cases we have $\theta(x_i) = \rho$ for $1 \leqslant i \leqslant r$, whilst $a := \theta(x_{r+1})$ has order 3 in cases Va, Vc and order 6 in cases Vb and Vd, in which $a^3 = \rho$. Let us write $\tau_1 := \theta(c_0)$, which represents a symmetry in G. Then $\theta(c_j) = \tau_1 \rho^j$ for $1 \leqslant j \leqslant p$ and $\theta(c_{p+1}) = \theta(e)^{-1}\theta(c_0)\theta(e) = a\tau_1 a^{-1}$ since $\theta(e) = a^{-1}\rho^r$. In cases Va and Vb, $\theta(c_p c_{p+1})^2 = 1$ whilst $\theta(c_p c_{p+1})^2 = \rho$ in Vc and Vd.

Let us consider for example **case Vd**. Here $\rho = a^3$ and also $\rho = \theta(c_p c_{p+1})^2 = (\tau_1 a \tau_1 a^{-1})^2$. So G has presentation

$$\langle a, \tau_1, \rho \mid \tau_1^2, \rho^2, \rho a^3, (\tau_1 \rho)^2, \rho(\tau_1 a \tau_1 a^{-1})^2 \rangle$$

since this group has order 48. Now using GAP we find that τ_1 is, up to conjugacy, the only symmetry of G.

In the remaining cases Va, Vb and Vc a presentation of the group G and representatives of its conjugacy classes of symmetries are those given in the corresponding table at the end of this section. In all cases, $\widehat{G} = \langle \widehat{\tau}_1, \widehat{a} \mid \widehat{\tau}_1^2, \widehat{a}^3, (\widehat{\tau}_1 \widehat{a} \widehat{\tau}_1 \widehat{a}^{-1})^2 \rangle$ and for the epimorphism $\widehat{\theta}$ we have $\widehat{\theta}(x_i) = 1$ for $i \leqslant r$, $\widehat{\theta}(x_{r+1}) = \widehat{a}$, $\widehat{\theta}(e) = \widehat{a}^{-1}$, and

$$\widehat{\theta}(c_i) = \begin{cases} \widehat{\tau}_1 & \text{for } 0 \leqslant i \leqslant p, \\ \widehat{a}\widehat{\tau}_1\widehat{a}^{-1} & \text{for } i = p + 1. \end{cases}$$

In cases VI, VII and VIII we have $\theta(x_i) = \rho$ for $1 \leqslant i \leqslant r$ and $G = \langle \rho, \tau_1, \tau_2, \tau_3 \rangle$ with $\tau_1 := \theta(c_0)$, $\tau_2 := \theta(c_{p_1+1})$ and $\tau_3 := \theta(c_{p_1+p_2+2})$. Then

$$\theta(c_j) = \begin{cases} \tau_1 \rho^j & \text{for } 1 \leqslant j \leqslant p_1, \\ \tau_2 \rho^{j-p_1-1} & \text{for } p_1 + 2 \leqslant j \leqslant p_1 + p_2 + 1, \\ \tau_3 \rho^{j-p_1-p_2-2} & \text{for } p_1 + p_2 + 3 \leqslant j \leqslant p_1 + p_2 + p_3 + 2. \end{cases}$$

Necessarily $\theta(c_{p_1+p_2+p_3+3}) = \tau_1$.

2.2.6. Case VI. — Recall that here the signature of Λ is one of the following:

$$\begin{aligned}&\text{VIa} : (0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},3)\}),\ \text{VIb} : (0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},6)\}),\\ &\text{VIc} : (0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},6)\}),\ \text{VId} : (0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},3)\}),\\ &\text{VIe} : (0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},6)\}),\ \text{VIf} : (0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},6)\}).\end{aligned}$$

In **case VIa**, $\theta(c_{p_1}c_{p_1+1})$, $\theta(c_{p_1+p_2+1}c_{p_1+p_2+2})$ and $\theta(c_{p_1+p_2+p_3+2}c_{p_1+p_2+p_3+3})$ have orders $2,3$ and 3 respectively. Further, $\theta(c_{p_1}c_{p_1+1})^2 = (\tau_1\tau_2)^2$, $\theta(c_{p_1+p_2+1}c_{p_1+p_2+2})^3 = (\tau_2\tau_3)^3\rho^{3p_2} = (\tau_2\tau_3)^3\rho^{p_2}$ and $\theta(c_{p_1+p_2+p_3+2}c_{p_1+p_2+p_3+3})^3 = (\tau_3\tau_1)^3\rho^{3p_3} = (\tau_3\tau_1)^3\rho^{p_3}$. So G has presentation

$$\langle \rho,\tau_1,\tau_2,\tau_3 \mid \tau_1^2,\tau_2^2,\tau_3^2,(\tau_1\tau_2)^2,\rho^{p_2}(\tau_2\tau_3)^3,\rho^{p_3}(\tau_1\tau_3)^3\rangle$$

since for any parity of p_2 and p_3 this group has order 48. With the aid of the computer we obtain that G has two conjugacy classes of symmetries represented by τ_1 and $\tau_1\rho$ for any value of p_2 and p_3.

The same happens in **cases VIb and VIc** because a presentation of the corresponding group G differs from the above only in the parity of p_2 and p_3. Indeed, in case VIb we have the relation $\rho^{p_3+1}(\tau_1\tau_3)^3 = 1$, whilst in case VIc, in addition to that, we also have $\rho^{p_2+1}(\tau_2\tau_3)^3 = 1$.

In **case VId**, $\rho = \theta(c_{p_1}c_{p_1+1})^2 = (\tau_1\tau_2)^2$. Furthermore both $\theta(c_{p_1+p_2+1}c_{p_1+p_2+2})$ and $\theta(c_{p_1+p_2+p_3+2}c_{p_1+p_2+p_3+3})$ have order 3 and on the other hand their cubes are $(\tau_2\tau_3)^3\rho^{p_2}$ and $(\tau_3\tau_1)^3\rho^{p_3}$ respectively. So G has presentation

$$\langle \tau_1,\tau_2,\tau_3,\rho \mid \tau_1^2,\tau_2^2,\tau_3^2,\rho(\tau_1\tau_2)^2,\rho^{p_2}(\tau_2\tau_3)^3,\rho^{p_3}(\tau_1\tau_3)^3\rangle$$

since for any parity of p_2 and p_3 this group has order 48. With the aid of the computer we obtain that G has one conjugacy class of symmetries represented by τ_1 for any value of p_2 and p_3.

The same happens in **cases VIe and VIf** because a presentation of the corresponding group G differs from the above only in the parity of p_2 and p_3. Indeed, in case VIe we have the relation $\rho^{p_3+1}(\tau_1\tau_3)^3 = 1$, whilst in case VIf, in addition to that, we also have $\rho^{p_2+1}(\tau_2\tau_3)^3 = 1$.

In all cases,

$$\widehat{G} = \langle \widehat{\tau}_1,\widehat{\tau}_2,\widehat{\tau}_3 \mid \widehat{\tau}_1^2,\widehat{\tau}_2^2,\widehat{\tau}_3^2,(\widehat{\tau}_1\widehat{\tau}_2)^2,(\widehat{\tau}_2\widehat{\tau}_3)^3,(\widehat{\tau}_1\widehat{\tau}_3)^3\rangle$$

and for the epimorphism $\widehat{\theta}$ we have $\widehat{\theta}(x_i) = 1$ for $i \leqslant r$, $\widehat{\theta}(e) = 1$, and

$$\widehat{\theta}(c_j) = \begin{cases} \widehat{\tau}_1 \text{ for } 0 \leqslant j \leqslant p_1 \text{ and } j = p_1+p_2+p_3+3,\\ \widehat{\tau}_2 \text{ for } p_1+1 \leqslant j \leqslant p_1+p_2+1,\\ \widehat{\tau}_3 \text{ for } p_1+p_2+2 \leqslant j \leqslant p_1+p_2+p_3+2.\end{cases}$$

2.2.7. Case VII. — Recall that here the signature of Λ is one of the following:

VIIa: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},4)\})$, VIIb: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},4)\})$,
VIIc: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},8)\})$, VIId: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},8)\})$,
VIIe: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},4)\})$, VIIf: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},4)\})$,
VIIg: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},8)\})$, VIIh: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},8)\})$.

In **case VIIa**, $\tau_1\tau_2\rho^{p_1}=\theta(c_{p_1}c_{p_1+1})$ has order 2, $\tau_2\tau_3\rho^{p_2}=\theta(c_{p_1+p_2+1}c_{p_1+p_2+2})$ has order 3 and $\tau_3\tau_1\rho^{p_3}=\theta(c_{p_1+p_2+p_3+2}c_{p_1+p_2+p_3+3})$ has order 4. In particular $\mathrm{order}(\tau_1\tau_2)=2$, $\mathrm{order}(\tau_1\tau_3)=4$ and $(\tau_2\tau_3)^3=\rho^{p_2}$. Thus G has presentation

$$\langle \tau_1,\tau_2,\tau_3,\rho \mid \tau_1^2,\tau_2^2,\tau_3^2,(\tau_1\tau_2)^2,\rho^{p_2}(\tau_2\tau_3)^3,(\tau_1\tau_3)^4\rangle$$

since for any value of p_2 this group has order 96. With the aid of the computer we obtain that, independently of the parity of p_2, the group G has 6 conjugacy classes of symmetries, represented by τ_1, τ_3, $\tau_1\rho$, $\tau_3\rho$, $(\tau_1\tau_2\tau_3)^3$ and $(\tau_1\tau_2\tau_3)^3\rho$.

The same happens in **case VIIb** because a presentation of the corresponding group G is the same as the above but with p_2+1 instead of p_2.

In **case VIIc**, $\tau_1\tau_2\rho^{p_1}$ has order 2, $\tau_2\tau_3\rho^{p_2}$ has order 3 and $(\tau_3\tau_1\rho^{p_3})^4=\rho$. Thus G has presentation

$$\langle \tau_1,\tau_2,\tau_3,\rho \mid \tau_1^2,\tau_2^2,\tau_3^2,(\tau_1\tau_2)^2,\rho^{p_2}(\tau_2\tau_3)^3,\rho(\tau_1\tau_3)^4\rangle$$

since for p_2 even or odd this group has order 96. With the aid of the computer we obtain that G has 2 conjugacy classes of symmetries, represented by τ_1 and τ_3, for any parity of p_2. The same happens in case **case VIId** because a presentation of the corresponding group G is the same as the above but with p_2+1 instead of p_2.

Case VIIe. As before $(\tau_1\tau_2)^2=\rho,(\tau_1\tau_3)^4=1$ and $(\tau_2\tau_3)^3\rho^{3p_2}=1$. Thus G has presentation

$$\langle \tau_1,\tau_2,\tau_3,\rho \mid \tau_1^2,\tau_2^2,\tau_3^2,\rho(\tau_1\tau_2)^2,\rho^{p_2}(\tau_2\tau_3)^3,(\tau_1\tau_3)^4\rangle$$

since for p_2 even or odd this group has order 96. With the aid of the computer we obtain that here for p_2 even or odd G has 3 conjugacy classes of symmetries, represented by τ_1,τ_3 and $(\tau_1\tau_2\tau_3)^3$.

The same happens in **case VIIf** because the corresponding G has the same presentation but with p_2+1 instead of p_2.

Finally, **case VIIg** gives rise to a group G with presentation

$$\langle \rho,\tau_1,\tau_2,\tau_3 \mid \tau_1^2,\tau_2^2,\tau_3^2,\rho(\tau_1\tau_2)^2,\rho^{p_2}(\tau_2\tau_3)^3,\rho(\tau_1\tau_3)^4\rangle.$$

It has order 96 and two conjugacy classes of symmetries, represented by τ_1 and τ_3, for any value of p_2. The same happens in **case VIIh**.

Observe that

$$\widehat{G}=\langle \widehat{\tau}_1,\widehat{\tau}_2,\widehat{\tau}_3 \mid \widehat{\tau}_1^2,\widehat{\tau}_2^2,\widehat{\tau}_3^2,(\widehat{\tau}_1\widehat{\tau}_2)^2,(\widehat{\tau}_2\widehat{\tau}_3)^3,(\widehat{\tau}_1\widehat{\tau}_3)^4\rangle$$

along case VII, and for the epimorphism $\widehat{\theta}$ we have $\widehat{\theta}(x_i) = 1$ for $i \leqslant r$, $\widehat{\theta}(e) = 1$, and

$$\widehat{\theta}(c_j) = \begin{cases} \widehat{\tau}_1 \text{ for } 0 \leqslant j \leqslant p_1 \text{ and } j = p_1 + p_2 + p_3 + 3, \\ \widehat{\tau}_2 \text{ for } p_1 + 1 \leqslant j \leqslant p_1 + p_2 + 1, \\ \widehat{\tau}_3 \text{ for } p_1 + p_2 + 2 \leqslant j \leqslant p_1 + p_2 + p_3 + 2. \end{cases}$$

2.2.8. Case VIII. — Recall that here the signature of Λ is one of the following:

$$\begin{array}{ll} \text{VIIIa} : ([2^r]; \{(2^{p_1}, 2, 2^{p_2}, 3, 2^{p_3}, 5)\}), & \text{VIIIb} : ([2^r]; \{(2^{p_1}, 2, 2^{p_2}, 3, 2^{p_3}, 10)\}), \\ \text{VIIIc} : ([2^r]; \{(2^{p_1}, 2, 2^{p_2}, 6, 2^{p_3}, 10)\}), & \text{VIIId} : ([2^r]; \{(2^{p_1}, 2, 2^{p_2}, 6, 2^{p_3}, 5)\}), \\ \text{VIIIe} : ([2^r]; \{(2^{p_1}, 4, 2^{p_2}, 3, 2^{p_3}, 5)\}), & \text{VIIIf} : ([2^r]; \{(2^{p_1}, 4, 2^{p_2}, 3, 2^{p_3}, 10)\}), \\ \text{VIIIg} : ([2^r]; \{(2^{p_1}, 4, 2^{p_2}, 6, 2^{p_3}, 5)\}), & \text{VIIIh} : ([2^r]; \{(2^{p_1}, 4, 2^{p_2}, 6, 2^{p_3}, 10\}), \end{array}$$

where in order to save space we have omitted the orbit genus and the sign of $\sigma(\Lambda)$, which are 0 and "+" respectively.

In **case VIIIa**, $\tau_1\tau_2\rho^{p_1} = \theta(c_{p_1}c_{p_1+1})$ has order 2, $\tau_2\tau_3\rho^{p_2} = \theta(c_{p_1+p_2+1}c_{p_1+p_2+2})$ has order 3 and $\tau_3\tau_1\rho^{p_3} = \theta(c_{p_1+p_2+p_3+2}c_{p_1+p_2+p_3+3})$ has order 5. Thus G has presentation

$$\langle \rho, \tau_1, \tau_2, \tau_3 \mid \tau_1^2, \tau_2^2, \tau_3^2, (\tau_1\tau_2)^2, \rho^{p_2}(\tau_2\tau_3)^3, \rho^{p_3}(\tau_1\tau_3)^5 \rangle$$

since for any parity of p_2 and p_3 this group has order 240. With the aid of the computer we obtain that also for any parity of p_2 and p_3 the group G has 4 conjugacy classes of symmetries, represented by τ_1, $\tau_1\rho$, $(\tau_1\tau_2\tau_3)^5$ and $(\tau_1\tau_2\tau_3)^5\rho$.

The same happens in **cases VIIIb-VIIId** because a presentation of the corresponding group G differs from the above only in the parity of p_2 and p_3. Indeed, $\rho^{p_2+1}(\tau_2\tau_3)^3 = 1$ in cases VIIIc and VIIId, and $\rho^{p_3+1}(\tau_1\tau_3)^5 = 1$ in cases VIIIb and VIIIc.

In **case VIIIe**, $\tau_2\tau_3\rho^{p_2}$ and $\tau_3\tau_1\rho^{p_3}$ have orders 3 and 5 respectively, but now $(\tau_1\tau_2)^2 = \rho$. Thus G has presentation

$$\langle \rho, \tau_1, \tau_2, \tau_3 \mid \tau_1^2, \tau_2^2, \tau_3^2, \rho(\tau_1\tau_2)^2, \rho^{p_2}(\tau_2\tau_3)^3, \rho^{p_3}(\tau_1\tau_3)^5 \rangle$$

since for any parity of p_2 and p_3 this group has order 240. With the aid of the computer we obtain that also for any parity of p_2 and p_3 the group G has one conjugacy class of symmetries represented by τ_1.

The same happens in **cases VIIIf-VIIIh** because a presentation of the corresponding group G differs from the above only in the parity of p_2 and p_3. Indeed, $\rho^{p_2+1}(\tau_2\tau_3)^3 = 1$ in cases VIIIg and VIIIh, and $\rho^{p_3+1}(\tau_1\tau_3)^5 = 1$ in cases VIIIf and VIIIh.

Observe that

$$\widehat{G} = \langle \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3 \mid \widehat{\tau}_1^2, \widehat{\tau}_2^2, \widehat{\tau}_3^2, (\widehat{\tau}_1\widehat{\tau}_2)^2, (\widehat{\tau}_2\widehat{\tau}_3)^3, (\widehat{\tau}_1\widehat{\tau}_3)^5 \rangle$$

along case VIII, and for the epimorphism $\widehat{\theta}$ we have $\widehat{\theta}(x_i) = 1$ for $i \leqslant r$, $\widehat{\theta}(e) = 1$, and

$$\theta(c_j) = \begin{cases} \widehat{\tau}_1 \text{ for } 0 \leqslant j \leqslant p_1 \text{ and } j = p_1 + p_2 + p_3 + 3, \\ \widehat{\tau}_2 \text{ for } p_1 + 1 \leqslant j \leqslant p_1 + p_2 + 1, \\ \widehat{\tau}_3 \text{ for } p_1 + p_2 + 2 \leqslant j \leqslant p_1 + p_2 + p_3 + 2. \end{cases}$$

2.2.9. Case IX. — Here Λ has one of the following signatures:
IXa: $(1; -; [2^r, N]; \{-\})$, with $r = (g+1)/N$,
IXb: $(1; -; [2^r, 2N]; \{-\})$, with $r = g/N$.
Proposition 2.2.1 is no longer valid for these signatures. Instead, we have the following:

Proposition 2.2.2. — *Let $x_1, \ldots, x_s, d$ be a set of canonical generators of an* NEC *group Λ with signature*

$$\sigma(\Lambda) = (1; -; [m_1, \ldots, m_s]; \{-\}).$$

If $\theta : \Lambda \to G$ is a group epimorphism such that $\ker\theta$ *is a surface Fuchsian group then*
(1) *$\theta(x_i)$ has order m_i for $i = 1, \ldots, s$;*
(2) $\prod_{i=1}^{s} \theta(x_i)\theta(d)^2 = 1$.

In our case, the relation between each m_i and the order $\widehat{m}_i$ of $\widehat{\theta}(x_i)$ gives that $\theta(x_i) = \rho$ for $1 \leqslant i \leqslant r$ whilst $a := \theta(x_{r+1})$ is an element of order N in case IXa and order $2N$ in case IXb, in which $a^N = \rho$. Note also that $\gamma := \theta(d)$ is the unique image of an orientation reversing canonical generator of Λ. Hence, symmetries of G are words of order 2 on γ, ρ and a with an odd exponent sum of letters γ. From condition (2) in the above proposition we have

$$a = \gamma^{-2}\rho^r.$$

Thus G is generated by γ and ρ.

In **case IXa**, a has order N and so $\gamma^{2N} = \rho^{rN}$. If rN is odd then γ has order $4N$, which gives $G = \langle\gamma\rangle$. But the unique involution in this group, γ^{2N}, does not represent an orientation reversing element. Therefore $rN = g + 1$ must be even. So γ has order $2N$ and, in addition, $\rho \notin \langle\gamma\rangle$ since otherwise G would have order $2N$. Hence $G = \langle\gamma\rangle \oplus \langle\rho\rangle = \mathrm{Z}_{2N} \oplus \mathrm{Z}_2$. Observe that for N even this group has no orientation reversing involution. We conclude that in case IXa both N and g must be odd and G contains two conjugacy classes of symmetries, with representatives γ^N and $\gamma^N\rho$.

In **case IXb**, $a^N = \rho$ and so $\gamma^{2N} = \rho^{rN+1}$. For rN even γ would have order $4N$ and G would contain no symmetry. Therefore $rN = g$ must be odd and so $\mathrm{order}(\gamma) = 2N$. We conclude as before that $G = \langle\gamma\rangle \oplus \langle\rho\rangle = \mathrm{Z}_{2N} \oplus \mathrm{Z}_2$, which contains two conjugacy classes of symmetries, with representatives γ^N and $\gamma^N\rho$.

Observe that in both cases

$$\widehat{G} = \langle\widehat{\gamma}\rangle = \mathrm{Z}_{2N}$$

and the epimorphism $\widehat{\theta}$ satisfies $\widehat{\theta}(x_i) = 1$ for $1 \leqslant i \leqslant r$, $\widehat{\theta}(d) = \widehat{\gamma}$ and $\widehat{\theta}(x_{r+1}) = \widehat{\gamma}^2$.

We summarize the results obtained in this section in the following tables. In each case we give a presentation of G, the values of p and N for which such a presentation occurs, and a set of representatives of the conjugacy classes of symmetries of G (last column). The letters τ and τ_i stand for symmetries of G whilst ρ stands for a central involution. For simplicity we omit their corresponding relations in the presentation of G.

Case I			
Ia	$2\mid N$	$\langle \rho, \tau_1, \tau_2 \mid (\tau_1\tau_2)^N \rangle = \mathrm{D}_N \oplus \mathrm{Z}_2$	$\{\tau_1, \tau_1\rho, \tau_2, \tau_2\rho\}$
	$2\nmid N$	$\langle \rho, \tau_1, \tau_2 \mid \rho(\tau_1\tau_2)^N \rangle = \mathrm{D}_{2N}$	$\{\tau_1, \tau_1\rho\}$
Ib		$\langle \rho, \tau_1, \tau_2 \mid \rho(\tau_1\tau_2)^N \rangle = \mathrm{D}_{2N}$	$\{\tau_1, \tau_2\}$
Ic	$2\nmid N$	$\langle \rho, \tau_1, \tau_2 \mid \rho(\tau_1\tau_2)^N \rangle = \mathrm{D}_{2N}$	$\{\tau_1, \tau_1\rho\}$

Case II			
IIa	$2\mid p$	$\langle \rho, \tau, a \mid a^N, \tau a \tau a^{-1} \rangle = \mathrm{Z}_2 \oplus \mathrm{Z}_2 \oplus \mathrm{Z}_N$	$\{\tau, \tau\rho, \tau a^{N/2}, \tau\rho a^{N/2}\}$ if $2\mid N$ $\{\tau, \tau\rho\}$ if $2\nmid N$
	$2\nmid p$	$\langle \tau, a \mid a^N, (a\tau)^2(\tau a)^{-2} \rangle$	$\{\tau, \tau a^{N/2}\}$ if $4\mid N$ $\{\tau\}$ if $4\nmid N$
IIb	$2\mid p$	$\langle \tau, a \mid a^{2N}, \tau a \tau a^{-1} \rangle = \mathrm{Z}_{2N} \oplus \mathrm{Z}_2$	$\{\tau, \tau\rho\}$
	$2\nmid p$	$\langle \tau, a \mid a^{2N}, \tau a \tau a^{N-1} \rangle$	$\{\tau, \tau a^{N/2}\}$ if $4\nmid N$ $\{\tau\}$ if $4\mid N$

Case III			
IIIa	$4\mid pN$	$a^2 = (a\tau_1)^N = 1$	$\{\tau_1, \tau_1\rho\}$ if $4\mid N$ $\{\tau_1, \tau_1\rho, (\tau_1 a)^{N/2}, (\tau_1 a)^{N/2}\rho\}$ if $4\nmid N$
	$4\nmid pN$	$a^2 = \rho(a\tau_1)^N = 1$	$\{\tau_1\}$
IIIb	$4\mid pN$	$a^2 = (a\tau_1)^{2N} = 1$	$\{\tau_1\}$
	$4\nmid pN$	$a^2 = (a\tau_1)^N = 1$	$\{\tau_1, \tau_1\rho, (a\tau_1)^{N/2}, (a\tau_1)^{N/2}\rho\}$
IIIc	$4\mid(p+1)N$	$\rho a^2 = (a\tau_1)^N = 1$	$\{\tau_1, \tau_1\rho\}$ if $4\mid N$ $\{\tau_1, (\tau_1 a)^{N/2}\}$ if $4\nmid N$
	$4\nmid(p+1)N$	$\rho a^2 = \rho(a\tau_1)^N = 1$	$\{\tau_1, \tau_1\rho\}$
IIId	$4\mid(p+1)N$	$\rho a^2 = \rho(a\tau_1)^N = 1$	$\{\tau_1\}$ if $4\mid N$ $\{\tau_1, \tau_1\rho\}$ if $4\nmid N$
	$4\nmid(p+1)N$	$\rho a^2 = (a\tau_1)^N = 1$	$\{\tau_1, (\tau_1 a)^{N/2}\}$

Recall that N is even $\geqslant 4$ in case III. For simplicity we omit this condition in the previous table. We also omit the generators of G, which are a, ρ and τ, and just write the non-obvious relations.

Also N is even $\geqslant 4$ in case IV. For simplicity we omit this condition in the following table. We also omit the generators of G, which are ρ, τ_1, τ_2 and τ_3, and write just the non-obvious relations.

Case IV			
IVa	$4\mid N$	$(\tau_1\tau_2)^2=(\tau_2\tau_3)^2=(\tau_1\tau_3)^{N/2}=1$	$\{\tau_i\rho^j, (\tau_1\tau_3)^{N/4}\tau_2\rho^j \,:\, i\leqslant 3, j\leqslant 2\}$
	$4\nmid N$	$(\tau_1\tau_2)^2=(\tau_2\tau_3)^2=(\tau_1\tau_3\rho)^{N/2}=1$	$\{\tau_1,\tau_2,\tau_1\rho,\tau_2\rho\}$
IVb	$4\mid N\geqslant 8$	$(\tau_1\tau_2)^2=\rho(\tau_2\tau_3)^2=(\tau_1\tau_3)^{N/2}=1$	$\{\tau_1,\tau_2,\tau_3,\tau_1\rho,\tau_2(\tau_1\tau_3)^{N/4}\}$ if $8\mid N$ $\{\tau_1,\tau_2,\tau_3,\tau_1\rho\}$ if $8\nmid N$
IVc		$\rho(\tau_1\tau_2)^2=\rho(\tau_2\tau_3)^2=(\tau_1\tau_3)^{N/2}=1$	$\{\tau_1,\tau_2,\tau_3,\tau_2(\tau_1\tau_3)^{N/4}\}$ if $4\mid N$ $\{\tau_1,\tau_2\}$ if $4\nmid N$
IVd		$(\tau_1\tau_2)^2=(\tau_2\tau_3)^2=\rho(\tau_1\tau_3)^{N/2}=1$	$\{\tau_1,\tau_2,\tau_2\rho,\tau_3\}$
IVe	$4\mid N\geqslant 8$	$(\tau_1\tau_2)^2=\rho(\tau_2\tau_3)^2=\rho(\tau_1\tau_3)^{N/2}=1$	$\{\tau_1,\tau_2,\tau_3\}$ if $8\mid N$ $\{\tau_1,\tau_2,\tau_3,\tau_2(\tau_1\tau_3)^{N/4}\}$ if $8\nmid N$
IVf		$\rho(\tau_1\tau_2)^2=\rho(\tau_2\tau_3)^2=\rho(\tau_1\tau_3)^{N/2}=1$	$\{\tau_1,\tau_2,\tau_3\}$ if $4\mid N$ $\{\tau_1,\tau_2\}$ if $4\nmid N$

The condition for the existence of the group G in case V is $N = 12$. The generators of G are a, τ and ρ.

Case V		
Va	$a^3 = (\tau a\tau a^{-1})^2 = 1$	$\{\tau,\tau\rho,(\tau a)^3,(\tau a)^3\rho\}$
Vb	$\rho a^3 = (\tau a\tau a^{-1})^2 = 1$	$\{\tau,\tau\rho,(\tau a)^3,(\tau a)^3\rho\}$
Vc	$a^3 = \rho(\tau a\tau a^{-1})^2 = 1$	$\{\tau\}$
Vd	$\rho a^3 = \rho(\tau a\tau a^{-1})^2 = 1$	$\{\tau\}$

Observe that the assignment $a \mapsto a\rho$, $\tau \mapsto \tau$ and $\rho \mapsto \rho$ induces an isomorphism between the groups appearing in cases Va and Vb, and the same happens to those appearing in Vc and Vd.

In tables corresponding to cases VI to VIII we have changed the presentation of the groups G in such a way that now the parameters p_2 and p_3 do not appear. The new presentation preserves, of course, the representatives of the conjugacy classes of symmetries chosen in the text. For example, in case VIa, the assignment $\tau_1 \mapsto \tau_1\rho^{p_3}$, $\tau_2 \mapsto \tau_2\rho^{p_2}$, $\tau_3 \mapsto \tau_3$ induces an isomorphism between

$\langle \tau_1, \tau_2, \tau_3, \rho \,|\, (\tau_1\tau_2)^2, \rho^{p_2}(\tau_1\tau_3)^3, \rho^{p_3}(\tau_2\tau_3)^3 \rangle$ and $\langle \tau_1, \tau_2, \tau_3, \rho \,|\, (\tau_1\tau_2)^2, (\tau_1\tau_3)^3, (\tau_2\tau_3)^3 \rangle$. In particular, this shows that the groups appearing in the table in cases VIa, VIb and VIc are isomorphic. The same assignment works in cases VId, VIIIa and VIIId. In cases VIIa, VIIc, VIIe and VIIg we may take $\tau_1 \mapsto \tau_1$, $\tau_2 \mapsto \tau_2\rho^{p_2}$ and $\tau_3 \mapsto \tau_3$. Therefore, this gives the following isomorphisms between the corresponding groups: VId $\simeq$ VIe $\simeq$ VIf, VIIa $\simeq$ VIIb, VIIc $\simeq$ VIId, VIIe $\simeq$ VIIf, VIIg $\simeq$ VIIh, VIIIa $\simeq$ VIIIb $\simeq$ VIIIc $\simeq$ VIIId and VIIIe $\simeq$ VIIIf $\simeq$ VIIIg $\simeq$ VIIIh.

In all cases the group G is generated by ρ, τ_1, τ_2 and τ_3. Also $N = 12$ in case VI, $N = 24$ in case VII and $N = 60$ in case VIII.

Case VI		
VIa	$(\tau_1\tau_2)^2 = (\tau_1\tau_3)^3 = (\tau_2\tau_3)^3 = 1$	$\{\tau_1, \tau_1\rho\}$
VIb	$(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^3 = 1$	$\{\tau_1, \tau_1\rho\}$
VIc	$(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^3 = 1$	$\{\tau_1, \tau_1\rho\}$
VId	$\rho(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = (\tau_1\tau_3)^3 = 1$	$\{\tau_1\}$
VIe	$\rho(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^3 = 1$	$\{\tau_1\}$
VIf	$\rho(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^3 = 1$	$\{\tau_1\}$

Case VII		
VIIa	$(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = (\tau_1\tau_3)^4 = 1$	$\{\tau_1, \tau_3, \tau_1\rho, \tau_3\rho, (\tau_1\tau_2\tau_3)^3, (\tau_1\tau_2\tau_3)^3\rho\}$
VIIb	$(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = (\tau_1\tau_3)^4 = 1$	$\{\tau_1, \tau_3, \tau_1\rho, \tau_3\rho, (\tau_1\tau_2\tau_3)^3, (\tau_1\tau_2\tau_3)^3\rho\}$
VIIc	$(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^4 = 1$	$\{\tau_1, \tau_3\}$
VIId	$(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^4 = 1$	$\{\tau_1, \tau_3\}$
VIIe	$\rho(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = (\tau_1\tau_3)^4 = 1$	$\{\tau_1, \tau_3, (\tau_1\tau_2\tau_3)^3\}$
VIIf	$\rho(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = (\tau_1\tau_3)^4 = 1$	$\{\tau_1, \tau_3, (\tau_1\tau_2\tau_3)^3\}$
VIIg	$\rho(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^4 = 1$	$\{\tau_1, \tau_3\}$
VIIh	$\rho(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^4 = 1$	$\{\tau_1, \tau_3\}$

Case VIII		
VIIIa	$(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = (\tau_1\tau_3)^5 = 1$	$\{\tau_1, \tau_1\rho, (\tau_1\tau_2\tau_3)^5, (\tau_1\tau_2\tau_3)^5\rho\}$
VIIIb	$(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^5 = 1$	$\{\tau_1, \tau_1\rho, (\tau_1\tau_2\tau_3)^5, (\tau_1\tau_2\tau_3)^5\rho\}$
VIIIc	$(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^5 = 1$	$\{\tau_1, \tau_1\rho, (\tau_1\tau_2\tau_3)^5, (\tau_1\tau_2\tau_3)^5\rho\}$
VIIId	$(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = (\tau_1\tau_3)^5 = 1$	$\{\tau_1, \tau_1\rho, (\tau_1\tau_2\tau_3)^5, (\tau_1\tau_2\tau_3)^5\rho\}$
VIIIe	$\rho(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = (\tau_1\tau_3)^5 = 1$	$\{\tau_1\}$
VIIIf	$\rho(\tau_1\tau_2)^2 = (\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^5 = 1$	$\{\tau_1\}$
VIIIg	$\rho(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = (\tau_1\tau_3)^5 = 1$	$\{\tau_1\}$
VIIIh	$\rho(\tau_1\tau_2)^2 = \rho(\tau_2\tau_3)^3 = \rho(\tau_1\tau_3)^5 = 1$	$\{\tau_1\}$

Recall that case IX occurs only for N odd. In both cases IXa and IXb the group G has the same presentation and so the same conjugacy classes of symmetries. Here γ is an orientation reversing element of G which does not represent a symmetry.

Case IX			
IXa	$N\|(g+1)$	$\langle\gamma\rangle \oplus \langle\rho\rangle = \mathrm{Z}_{2N} \oplus \mathrm{Z}_2$	$\{\gamma^N, \rho\gamma^N\}$
IXb	$N\|g$	$\langle\gamma\rangle \oplus \langle\rho\rangle = \mathrm{Z}_{2N} \oplus \mathrm{Z}_2$	$\{\gamma^N, \rho\gamma^N\}$

2.3. Automorphism groups of symmetric hyperelliptic Riemann surfaces

So far we have computed the number of conjugacy classes of symmetries of G. This gives the number of real forms of any hyperelliptic curve X such that its full automorphism group $\mathrm{Aut}^{\pm}X$ coincides with G. Our goal now is to study the existence of such curves. The idea is, of course, to take an NEC group Λ with signature appearing in the first section of this chapter and consider the curve $X = \mathcal{H}/\Gamma$, where $\Gamma = \ker\theta$ and $\theta : \Lambda \to G$ is the corresponding epimorphism defined above. The point is that the full automorphism group $\mathrm{Aut}^{\pm}X$ of such surface may be bigger than G. However, the equality can always be assured if the associated Fuchsian signature $\sigma(\Lambda)^+$ is maximal (see Corollary 1.1.15). But non-maximal Fuchsian signatures may also provide full automorphism groups, as we shall see.

Definition 2.3.1. — Let σ be one of the NEC signatures obtained in the first section and let θ be the corresponding epimorphism. Note that θ is defined for any NEC group with signature σ and that $\sigma(\ker\theta) = (g; [-])$. We say that σ is *g-hyperelliptic maximal* if there exists an NEC group Λ with signature σ such that $\theta : \Lambda \to G$ cannot be extended to an epimorphism $\theta' : \Lambda' \to G'$ such that $1 \neq [\Lambda' : \Lambda] = [G' : G]$. Notice that for such an epimorphism θ', it is $\ker\theta = \ker\theta'$.

We also say that a *Fuchsian signature* σ_1 *is g-hyperelliptic maximal* if there exist a Fuchsian group Δ with signature σ_1 and an epimorphism $\varphi : \Delta \to H$ such that $\mathcal{H}/\ker\varphi$ is a genus g hyperelliptic Riemann surface and φ satisfies the same property as the above θ.

If σ is an NEC signature appearing in the first section, then for any NEC group Λ with signature σ the quotient $\Lambda/\ker\theta$ is a group of automorphisms of the genus g hyperelliptic Riemann surface $\mathcal{H}/\ker\theta$. Now, such an NEC group Λ can be chosen so that $\Lambda/\ker\theta$ is the full group of automorphisms if and only if σ is g-hyperelliptic maximal. Our goal then is to find which signatures are g-hyperelliptic maximal. Recall that if σ^+ is a maximal Fuchsian signature then σ is a maximal NEC signature; in such a case, it is obvious that σ is g-hyperelliptic maximal (see Theorem 1.1.14).

We also observe that if σ is a g-hyperelliptic maximal NEC signature then σ^+ is a g-hyperelliptic maximal Fuchsian signature. Indeed if Λ is an NEC group with signature σ such that θ cannot be extended as above then Λ^+ is a Fuchsian group with signature σ^+ and we claim that the restriction $\theta^+ : \Lambda^+ \to G^+$ of θ cannot be extended. Otherwise $G^+ = \Lambda^+/\ker\theta$ would not be the full group of analytic automorphisms of $X = \mathcal{H}/\ker\theta$, which contradicts the fact that $G = \Lambda/\ker\theta$ is the full group of analytic and antianalytic automorphisms of X.

Therefore, in order to determine which NEC signatures are g-hyperelliptic maximal, it remains to study those whose associated Fuchsian signatures are non-maximal and g-hyperelliptic maximal. Such Fuchsian signatures have been calculated in [**15**]. They appear in the following table, together with a presentation of the group G^+ they give rise. Of course G^+ is the subgroup of analytic automorphisms of G and so its presentation may be given in terms of generators of G.

Non-maximal Fuchsian signatures which are g-hyperelliptic maximal			
Case	g, N	σ^+	G^+
Ic	$2g+1=N$	$(0;[2,N,2N])$	$\langle \rho\tau_1\tau_2\rangle = \mathrm{Z}_{2N}$
IIIc	$g=3,\ N=4$	$(0;[2,2,4,4])$	$\langle (a\tau_1)^2, a\rangle = \mathrm{H}_2 = \mathrm{Z}_2\oplus\mathrm{Z}_4$
IVb	$g=3,\ N=16$	$(0;[2,4,8])$	$\langle \tau_3\tau_2, \tau_1\tau_3\rangle = \mathrm{V}_8$
IVb	$g=5,\ N=8$	$(0;[2,2,4,4])$	$\langle \tau_3\tau_2, \tau_1\tau_3\rangle = \mathrm{V}_4$
IVc	$g=3,\ N=4$	$(0;[2,2,4,4])$	$\langle \tau_1\tau_2, \tau_2\tau_3\rangle = \mathrm{H}_2 = \mathrm{Z}_2\oplus\mathrm{Z}_4$
Va	$g=5,\ N=12$	$(0;[2,2,3,3])$	$\langle\rho\rangle \oplus \langle \tau a\tau a^{-1}, a\rangle = \mathrm{Z}_2\oplus\mathrm{A}_4$
Vb	$g=9,\ N=12$	$(0;[2,2,6,6])$	$\langle\rho\rangle \oplus \langle \tau a\tau a^{-1}, \rho a\rangle = \mathrm{Z}_2\oplus\mathrm{A}_4$
VIa	$g=5,\ N=12$	$(0;[2,2,3,3])$	$\langle\rho\rangle \oplus \langle \tau_1\tau_2, \tau_2\tau_3\rangle = \mathrm{Z}_2\oplus\mathrm{A}_4$
VIc	$g=9,\ N=12$	$(0;[2,2,6,6])$	$\langle\rho\rangle \oplus \langle \tau_1\tau_2, \rho\tau_2\tau_3\rangle = \mathrm{Z}_2\oplus\mathrm{A}_4$
VIIe	$g=5,\ N=24$	$(0;[3,4,4])$	$\mathrm{W}_2 = \langle a, b \mid a^4, b^3, ba^2b^{-1}a^2, (ab)^4\rangle$ with $a=\tau_1\tau_2,\ b=\tau_2\tau_3$
VIIf	$g=9,\ N=24$	$(0;[4,4,6])$	$\mathrm{W}_2 = \langle \tau_1\tau_2, \rho\tau_2\tau_3\rangle$

We point out that, in the original table in [**15**], the group $\mathrm{Z}_2\oplus\mathrm{Z}_4$ is excluded as the full group of analytic automorphisms of a hyperelliptic Riemann surface of genus 3. However, this is not so, as the next example shows (see [**23**]). For any complex number ω such that $(\omega^9-\omega)(\omega^8-194\omega^4+1)\neq 0$, the automorphism group of the hyperelliptic Riemann surface of genus 3 given by $y^2=(x^4-1)(x^2-\omega^2)(x^2-1/\omega^2)$ is $\mathrm{Z}_2\oplus\mathrm{Z}_4 = \langle f\rangle\oplus\langle g\rangle$, where $f(x,y)=(-x,y)$ and $g(x,y)=(1/x, yi/x^4)$.

Given then an NEC signature σ such that σ^+ appears in the above table, our task now is to determine whether σ is g-hyperelliptic maximal or not. The next proposition shows that the answer is always affirmative.

PROPOSITION 2.3.2. — *Let σ be an* NEC *signature appearing in the first section. Then σ is g-hyperelliptic maximal if and only if σ^+ is g-hyperelliptic maximal.*

Proof. — The "only if" part has already been proved. So let Δ be a Fuchsian group with signature σ^+ for which there exists an epimorphism $\varphi : \Delta \to H$ such that $X = \mathcal{H}/\ker\varphi$ is a genus g hyperelliptic Riemann surface and φ cannot be extended in the sense of Definition 2.3.1. We divide the proof into two cases according to the dimension $\dim(\sigma^+)$ of the Teichmüller space of Δ.

If $\mathbf{dim(\boldsymbol{\sigma}^+) = 0}$ then all Fuchsian groups with signature σ^+ are conjugate in $\mathrm{Aut}^{\pm}\mathcal{H}$. In particular, if Λ is a g-hyperelliptic NEC group with signature σ then $\Delta = h\Lambda^+h^{-1}$ for some $h \in \mathrm{Aut}^{\pm}\mathcal{H}$. Let $\Lambda' = h\Lambda h^{-1}$. Clearly, $\Lambda'^+ = \Delta$, and since $\Delta/\ker\varphi$ is the full group $\mathrm{Aut}\,X$ of analytic automorphisms of X, $\Lambda'/\ker\varphi$ is the full group $\mathrm{Aut}^{\pm}X$. This shows that $\sigma(\Lambda') = \sigma$ is g-hyperelliptic maximal.

The case $\mathbf{dim(\boldsymbol{\sigma}^+) = 1}$ is more laborious. For each of the remainder NEC signatures σ (namely, those in which σ^+ has four proper periods) we have to show that there exists an NEC group Λ such that the corresponding epimorphism $\theta : \Lambda \to G$ cannot be extended.

Assume, by the way of contradiction, that for any NEC group Λ with signature σ there exists another NEC group Λ' containing Λ properly such that $\theta : \Lambda \to G$ extends to $\theta' : \Lambda' \to G'$ with $[\Lambda' : \Lambda] = [G' : G]$. The signature σ' of Λ' must be one of those calculated in the first section, and our goal is to find it out. First note that as σ is not a maximal NEC signature (otherwise we are done) Λ can be chosen so that $\dim(\Lambda') = \dim(\Lambda)$ (see the arguments in the proof of Theorem 5.1.2 in [**14**]). Thus also $\dim(\Lambda^+) = \dim(\Lambda'^+)$ and therefore the pair (σ^+, σ'^+) appears in the table of non-maximal Fuchsian signatures given in Chapter 1. It is in fact, a normal pair with $[\sigma'^+ : \sigma^+] = 2$ and so (σ, σ') is a normal pair of NEC signatures. Applying then the list of such pairs obtained in [**7**] we get the signature σ' we are looking for.

Such signature σ' provides a group of automorphisms of a hyperelliptic Riemann surface. However, in some cases the Riemann surface does not have genus g. For example, in case Va with $g = 5$ we have $\sigma = (0; +; [3]; \{(2, 2)\})$ and the list in [**7**] gives $\sigma' = (0; +; [-]; \{(2, 2, 2, 3)\})$ as the unique possibility; but whenever σ' occurs in the first section, it is $g \neq 5$. Therefore σ is a 5-hyperelliptic maximal NEC signature. It is easy to check that the same happens in cases IVb with $g = 5$ and $N = 8$, Vb with $g = 9$, VIa with $g = 5$ and VIc with $g = 9$.

It remains to show that signatures in cases IIIc and IVc with $g = 3$ and $N = 4$ are 3-hyperelliptic maximal. We develop in detail the case IIIc, the other being similar.

Suppose then that for every NEC group Λ with signature $\sigma = (0; +; [4]; \{(2, 2)\})$ there exists an NEC group Λ' containing it properly such that $\theta : \Lambda \to G$ extends to $\theta' : \Lambda' \to G'$. Recall that in case IIIc with $N = 4$, we have $G = \langle a, \tau_1 \mid a^4, (\tau_1)^2, (\tau_1 a^2)^2, (a\tau_1)^4 \rangle$. As said above, $[\Lambda' : \Lambda] = [G' : G] = 2$ and in this case $\sigma(\Lambda') = (0; +; [-]; \{(2, 2, 2, 4)\})$. Let $\{c'_0, \ldots, c'_4\}$ and $\{x_1, e_1, c_0, c_1, c_2\}$ be canonical sets of generators of Λ' and Λ respectively. Using the (unique) epimorphism from Λ' onto Z_2 with kernel Λ we find the expression of the canonical generators of Λ in terms of those of Λ'. Explicitly, $x_1 = c'_3c'_4$, $e_1 = c'_4c'_3$, $c_0 = c'_4c'_2c'_4$, $c_1 = c'_1$ and $c_2 = c'_2$. Since θ' extends θ it follows that $\theta'(c'_1) = \theta(c_1) = \tau_1 a^2$ and $\theta'(c'_2) = \theta(c_2) = a\tau_1 a^{-1}$. Writing $\tau' = \theta'(c'_0 = c'_4)$ (which has order 2) we obtain $\theta'(c'_3) = \tau' a^{-1}$. So G' is the group generated by a, τ_1 and τ'. Now, the relations in G' induced via θ' by those in Λ' yields in particular that $a\tau_1 = \tau_1 a$, as is easy to check. However, this relation does not hold in G and so the extension θ' cannot exist. Therefore $\sigma = (0; +; [4]; \{(2, 2)\})$ is a 3-hyperelliptic maximal NEC signature. □

This way we find all g-hyperelliptic maximal NEC signatures; this provides the complete list of full automorphism groups of genus g symmetric hyperelliptic Riemann surfaces. For example, the associated Fuchsian signature in case Ia is $\sigma^+ = (0; [2^{2r+p_1+p_2}, N, N])$, where $2r + p_1 + p_2 = (2g + 2)/N$. If $(2g + 2)/N > 2$ then σ^+ is maximal since it has more than four proper periods. If $(2g + 2)/N = 1$ then $\sigma^+ = (0; [2, N, N])$ with $N = 2g+2 \geqslant 6$ and so it appears in the table of non-maximal Fuchsian signatures; evenmore, it does not appear in the above table of g-hyperelliptic maximal signatures. The same happens if $(2g+2)/N = 2$, as is easy to check. Therefore, we conclude that the corresponding group $G = \langle \tau_1, \tau_2 \rangle \oplus \langle \rho \rangle = D_N \oplus Z_2$ acts as the full group of automorphisms of a genus g hyperelliptic Riemann surface if and only if N is a divisor of $2g + 2$ with $(2g + 2)/N > 2$.

The same analysis in the other cases proves the following theorem.

Theorem 2.3.3. — *For a fixed $g \geqslant 2$ a finite group G of order $4N > 4$ acts as the full group of analytic and antianalytic automorphisms of a genus g symmetric hyperelliptic Riemann surface if and only if it appears in the following tables with N and g satisfying the conditions in the last columns. A presentation of G is given in the second columns; the letters τ and τ_i stand for symmetries whilst ρ stands for the hyperelliptic involution, which is central. For simplicity we omit their corresponding relations in the presentation.*

Ia	$\langle \rho, \tau_1, \tau_2 \mid (\tau_1\tau_2)^N \rangle = D_N \oplus Z_2$	$N \mid (2g + 2),\ (2g + 2)/N > 2$
Ib	$\langle \rho, \tau_1, \tau_2 \mid \rho(\tau_1\tau_2)^N \rangle = D_{2N}$	$N \mid 2g,\ 2g/N > 2$
Ic	$\langle \rho, \tau_1, \tau_2 \mid \rho(\tau_1\tau_2)^N \rangle = D_{2N}$	$N \mid (2g + 1)$

IIa	$\langle \rho, \tau, a \mid a^N, \tau a \tau a^{-1} \rangle = \mathrm{Z}_2 \oplus \mathrm{Z}_2 \oplus \mathrm{Z}_N$ $\langle \tau, a \mid a^N, (a\tau)^2(\tau a)^{-2} \rangle$	$(2g+2)/N$ *even* > 2 $(2g+2)/N$ *odd* > 2
IIb	$\langle \tau, a \mid a^{2N}, \tau a \tau a^{-1} \rangle = \mathrm{Z}_{2N} \oplus \mathrm{Z}_2$ $\langle \tau, a \mid a^{2N}, \tau a \tau a^{N-1} \rangle$	$2g/N$ *even* > 2 $2g/N$ *odd* > 2

In cases III *and* IV, *N is even $\geqslant 4$. We omit this condition in the following tables.*

IIIa	$\langle \rho, \tau, a \mid a^2, (a\tau)^N \rangle = \mathrm{D}_N \oplus \mathrm{Z}_2$ $\langle \rho, \tau, a \mid a^2, \rho(a\tau)^N \rangle = \mathrm{D}_{2N}$	$(N \mid (2g+2),\ 4 \mid N)$ or $(4 \nmid N,\ N \mid (g+1))$ $N \nmid (g+1),\ N \mid (2g+2),\ 4 \nmid N$
IIIb	$\langle \tau, a \mid a^2, (a\tau)^{2N} \rangle = \mathrm{D}_{2N}$ $\langle \tau, a \rangle \oplus \langle \rho \rangle = \mathrm{D}_N \oplus \mathrm{Z}_2$	$(N \mid 2g,\ 4 \mid N)$ or $(4 \nmid N, N \mid g)$ $N \nmid g,\ N \mid 2g,\ 4 \nmid N$
IIIc	$\langle \rho, \tau, a \mid \rho a^2, (\tau a)^N \rangle$ $\langle \rho, \tau, a \mid \rho a^2, \rho(\tau a)^N \rangle$	$(N \mid (2g+2),\ 4 \mid N)$ or $(4 \nmid N,\ N \mid (g+1))$, $(2g+2)/N > 1$ $N \nmid (g+1),\ N \mid (2g+2),\ 4 \nmid N$, $(2g+2)/N > 1$
IIId	$\langle \rho, \tau, a \mid \rho a^2, \rho(\tau_1 a)^N \rangle$ $\langle \rho, \tau, a \mid \rho a^2, (\tau_1 a)^N \rangle$	$(N \mid 2g,\ 4 \mid N)$ or $(4 \nmid N,\ N \mid g)$, $2g/N > 1$ $N \nmid g,\ N \mid 2g,\ 4 \nmid N,\ 2g/N > 1$

IVa	$\langle \tau_1, \tau_2, \tau_3, \rho \mid (\tau_1\tau_2)^2, (\tau_2\tau_3)^2, (\tau_1\tau_3)^{N/2} \rangle$	$N \mid (2g+2)$
IVb	$\langle \tau_1, \tau_2, \tau_3, \rho \mid (\tau_1\tau_2)^2, \rho(\tau_2\tau_3)^2, (\tau_1\tau_3)^{N/2} \rangle$	$(4g+4)/N$ odd, $N \geqslant 8$
IVc	$\langle \tau_1, \tau_2, \tau_3, \rho \mid \rho(\tau_1\tau_2)^2, \rho(\tau_2\tau_3)^2, (\tau_1\tau_3)^{N/2} \rangle$	$N \mid (2g+2),\ (2g+2)/N > 1$
IVd	$\langle \tau_1, \tau_2, \tau_3, \rho \mid (\tau_1\tau_2)^2, (\tau_2\tau_3)^2, \rho(\tau_1\tau_3)^{N/2} \rangle$	$N \mid 2g$
IVe	$\langle \tau_1, \tau_2, \tau_3, \rho \mid (\tau_1\tau_2)^2, \rho(\tau_2\tau_3)^2, \rho(\tau_1\tau_3)^{N/2} \rangle$	$4g/N$ odd, $N \geqslant 8,\ g > 2$
IVf	$\langle \tau_1, \tau_2, \tau_3, \rho \mid \rho(\tau_1\tau_2)^2, \rho(\tau_2\tau_3)^2, \rho(\tau_1\tau_3)^{N/2} \rangle$	$N \mid 2g,\ 2g/N > 1$

Va, Vb	$\langle a, \tau, \rho \mid a^3, (\tau a \tau a^{-1})^2 \rangle$	$g \equiv 5$ or $3 \pmod 6$, $g \neq 3$
Vc, Vd	$\langle a, \tau, \rho \mid a^3, \rho(\tau a \tau a^{-1})^2 \rangle$	$g \equiv 2$ or $0 \pmod 6$, $g \neq 2, 6$

VIa-VIc	$\langle \tau_1, \tau_2, \tau_3, \rho \mid (\tau_1\tau_2)^2, (\tau_1\tau_3)^3, (\tau_2\tau_3)^3 \rangle$	$g \equiv 5, 1$ or $3 \pmod 6$, $g \neq 3$
VId-VIf	$\langle \tau_1, \tau_2, \tau_3, \rho \mid \rho(\tau_1\tau_2)^2, (\tau_2\tau_3)^3, (\tau_1\tau_3)^3 \rangle$	$g \equiv 2, 4$ or $0 \pmod 6$, $g \neq 2, 6$

VIIa, VIIb	$\langle \tau_1, \tau_2, \tau_3, \rho \mid (\tau_1\tau_2)^2, (\tau_2\tau_3)^3, (\tau_1\tau_3)^4 \rangle$	$g \equiv 11$ or $3 \pmod{12}$
VIIc, VIId	$\langle \tau_1, \tau_2, \tau_3, \rho \mid (\tau_1\tau_2)^2, (\tau_2\tau_3)^3, \rho(\tau_1\tau_3)^4 \rangle$	$g \equiv 2$ or $6 \pmod{12}$
VIIe, VIIf	$\langle \tau_1, \tau_2, \tau_3, \rho \mid \rho(\tau_1\tau_2)^2, (\tau_2\tau_3)^3, (\tau_1\tau_3)^4 \rangle$	$g \equiv 5$ or $9 \pmod{12}$
VIIg, VIIh	$\langle \tau_1, \tau_2, \tau_3, \rho \mid \rho(\tau_1\tau_2)^2, (\tau_2\tau_3)^3, \rho(\tau_1\tau_3)^4 \rangle$	$g \equiv 8$ or $0 \pmod{12}$

VIIIa-VIIId	$\langle \tau_1, \tau_2, \tau_3, \rho \mid (\tau_1\tau_2)^2, (\tau_2\tau_3)^3, (\tau_1\tau_3)^5 \rangle$	$g \equiv 29, 5, 19$ or $9 \pmod{30}$
VIIIe-VIIIh	$\langle \tau_1, \tau_2, \tau_3, \rho \mid \rho(\tau_1\tau_2)^2, (\tau_2\tau_3)^3, (\tau_1\tau_3)^5 \rangle$	$g \equiv 14, 20, 24$ or $0 \pmod{30}$

IXa, IXb	$\langle \tau \rangle \oplus \langle a \rangle = \mathrm{Z}_{2N} \oplus \mathrm{Z}_2$	N, g odd and $(N \mid (g+1)$ or $g/N > 1)$

Note that there are groups occurring in different cases which are isomorphic as abstract groups. However, we may distinguish them by the geometric action of its elements. This is reflected in the presentation of each G. Indeed, the presentation distinguishes the analytic automorphisms from the antianalytic ones and it also points out the central involution which plays the role of the hyperelliptic involution.

Definition 2.3.4. — Let X be a symmetric hyperelliptic Riemann surface. We define the *class* of X according to the case in which $\mathrm{Aut}^{\pm} X$ occurs. So, for example, we say that X is of class Ia or IIIb and so on. Sometimes it will be necessary to stress the order of the group $\mathrm{Aut}^{\pm} X$ (which is $4N$) and we will say that X is of class $(\mathrm{Ia})_N$ or $(\mathrm{IIIb})_N$ and so on.

Observe that all groups occurring in the same case have the same quotient group $\widehat{G} = G/\rho$. Moreover, $\widehat{G}$ characterizes each case. Thus we may characterize the class of a symmetric hyperelliptic Riemann surface X in terms of its induced group of Möbius transformations $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$. So, for example, X is of class I if and only if $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ has presentation $\langle \tau_1, \tau_2 \mid (\tau_1)^2, (\tau_2)^2, (\tau_1\tau_2)^N \rangle = \mathrm{D}_N$, where τ_1 and τ_2 are antianalytic. Conversely, for notational convenience we say that a finite group of Möbius transformations is of class I if it has a presentation $\langle \alpha, \beta \rangle = \mathrm{D}_N$ with α and β antianalytic. Analogous definitions will be used for the ten different classes of finite groups of Möbius transformations occurring as $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$.

CHAPTER 3

SYMMETRY TYPES OF HYPERELLIPTIC RIEMANN SURFACES

Throughout this chapter we keep the notations of the preceding one. A hyperelliptic Riemann surface of genus $g \geqslant 2$ will be represented as $X = \mathcal{H}/\Gamma$, where Γ is a Fuchsian surface group; its hyperelliptic involution will be denoted by ρ. Let G, $\widehat{G}$, Λ, Γ_ρ, θ and $\widehat{\theta}$ be the groups and epimorphisms defined in 2.1.1. As before, the letters r, p, p_1, p_2 and p_3 will denote non-negative integers. In this chapter we compute the species of each of the real forms calculated in the previous one. Thus the symmetry type of any hyperelliptic Riemann surface is obtained. The goal is to compute the signatures of certain NEC groups associated to symmetries. We do so in two ways, a combinatorial and a geometrical one. The latter essentially consists in studying the ramification data of the projection $S^2 \to S^2/\widehat{G}$, where we view the Riemann sphere S^2 as the quotient $\mathcal{H}/\Gamma_\rho$. Moreover, this geometrical method allows us to get polynomial equations of X and the formulae of its real forms. Then, once each real form is explicitly calculated, Theorems 1.3.4 and 1.3.5 are applied to find out which real form realizes each species of the symmetry type of X.

Let τ be an antianalytic involution of the hyperelliptic symmetric Riemann surface $X = \mathcal{H}/\Gamma$. Together with the hyperelliptic involution ρ it generates a Klein 4-group $\{id, \tau, \rho, \tau \circ \rho\}$ of automorphisms of X. Then such a group may be written as Λ_1/Γ for certain NEC group Λ_1 containing Γ as a normal subgroup. It turns out that the species of τ and $\tau \circ \rho$ are determined by the signature of Λ_1. More precisely, they depend on the number of link periods of Λ_1, if any, as the next theorem shows. It is a reformulation of Theorems 6.1.3 in [**14**] and 2.2 in [**19**] and it will be used throughout this chapter.

Theorem 3.0.1. — *With the above notations, the signature of Λ_1 is one of the following:*

(1) $(0; +; [-]; \{(2, \overset{2g+2}{\ldots}, 2)\})$; *in such a case*

$$\operatorname{sp}(\tau) = \operatorname{sp}(\tau \circ \rho) = g + 1.$$

(2) $(0;+;[2,\overset{p}{\ldots},2];\{(2,\overset{2k}{\ldots},2)\})$ *with* $p\cdot k\neq 0$*; in such a case*

$$\mathrm{sp}\,(\tau)=\mathrm{sp}\,(\tau\circ\rho)=-k.$$

(3) $(0;+;[2,\overset{g+1}{\ldots},2];\{(-)\})$*; in such a case*

$$\{\mathrm{sp}\,(\tau),\mathrm{sp}\,(\tau\circ\rho)\}=\begin{cases}\{1,0\} & \textit{if } g \textit{ is even;}\\ \{2,0\} & \textit{if } g \textit{ is odd.}\end{cases}$$

(4) $(1;-;[2,\overset{g+1}{\ldots},2];\{-\})$*; in such a case*

$$\mathrm{sp}\,(\tau)=\mathrm{sp}\,(\tau\circ\rho)=0.$$

Notice that on a hyperelliptic curve the species of a separating symmetry only may attain the values 1, 2 and $g+1$; however, the species of a non-separating one may attain any of the values $0,\,-1,-2,\ldots,-g$.

3.1. Symmetry types of hyperelliptic algebraic curves of class I

Recall that these curves are those whose induced group $\widehat{G}$ of Möbius transformations is dihedral generated by two elements $\widehat{\tau}_1$ and $\widehat{\tau}_2$ which represent orientation reversing involutions. The epimorphism $\widehat{\theta}:\Lambda\to\widehat{G}$ is given by

- $\widehat{\theta}(x_i)=1$ for $1\leqslant i\leqslant r$,
- $\widehat{\theta}(c_i)=\begin{cases}\widehat{\tau}_1 \text{ for } 0\leqslant i\leqslant p_1 \text{ and } i=p_1+p_2+2,\\ \widehat{\tau}_2 \text{ for } p_1+1\leqslant i\leqslant p_1+p_2+1,\end{cases}$
- $\widehat{\theta}(e)=1$.

The analysis splits naturally into 3 cases according to Λ having signature Ia, Ib or Ic. This algebraic distinction will be translated into a geometric one in terms of the branch points of the hyperelliptic curve.

3.1.1. Class Ia. — In this case Λ has signature

$$\sigma=(0;+;[2,\overset{r}{\ldots},2];\{(2,\overset{p_1}{\ldots},2,N,2,\overset{p_2}{\ldots},2,N)\})\ \text{ with }\ 2r+p_1+p_2=\frac{2g+2}{N}>2.$$

The inequality is given by Theorem 2.3.3. Recall that for N even $G=\mathrm{D}_N\oplus\mathrm{Z}_2=\langle\tau_1,\tau_2\rangle\oplus\langle\rho\mid\rho^2\rangle$ which has exactly four conjugacy classes of symmetries, namely, those represented by τ_1, τ_2, $\tau_1\rho$ and $\tau_2\rho$. For N odd, $G=\mathrm{D}_{2N}=\langle\tau_1,\tau_2\rangle$ and τ_1 and $\tau_1\rho$ are representatives of its only two conjugacy classes of symmetries. Suppose we know the signature of the NEC group Λ_1 associated to the group generated by τ_1 and $\tau_1\rho$, *i.e.,* $\langle\tau_1,\tau_1\rho\rangle=\Lambda_1/\Gamma$. In such a case the species of the symmetries τ_1 and $\tau_1\rho$ are completely determined by Theorem 3.0.1. Analogously, we can compute the species of the symmetries τ_2 and $\tau_2\rho$ if we know the signature of the NEC group Λ_2 associated to $\langle\tau_2,\tau_2\rho\rangle$.

Observe that Λ_1 and Λ_2 are the inverse images under $\widehat{\theta}:\Lambda\to\widehat{G}$ of the subgroups of $\widehat{G}$ generated by $\widehat{\tau}_1$ and $\widehat{\tau}_2$ respectively. In [**33**] Hoare gives a combinatorial method to compute the signatures of Λ_1 and Λ_2. The only we need is to know how the canonical generators of Λ act on the cosets of $\widehat{G}/\langle\widehat{\tau}_1\rangle$ and $\widehat{G}/\langle\widehat{\tau}_2\rangle$ respectively. For brevity, we explain only the case N even since the case N odd is similar.

Note that $\widehat{G}/\langle\widehat{\tau}_1\rangle = \{\langle\widehat{\tau}_1\rangle, \langle\widehat{\tau}_1\rangle\widehat{\tau}_1\widehat{\tau}_2, \dots \langle\widehat{\tau}_1\rangle(\widehat{\tau}_1\widehat{\tau}_2)^{N-1}\}$. To shorten notation we represent these cosets by $0, 1, \dots, N-1$. The action of the canonical generators of Λ is

$$c_i \longmapsto \begin{cases} (0)\,(\frac{N}{2})\,(1, N-1)\cdots(\frac{N}{2}-1, \frac{N}{2}+1), & \text{for } i = 0, \dots, p_1,\ p_1{+}p_2{+}2, \\ (0,1)\,(2, N-1)\,(3, N-2)\cdots(\frac{N}{2}, \frac{N}{2}+1), & \text{for } i = p_1+1, \dots, p_1{+}p_2{+}1, \end{cases}$$

$$x_i \longmapsto (0)(1)\dots(N-1), \qquad \text{for } \ i = 1, \dots, r,$$

$$e \longmapsto (0)(1)\dots(N-1).$$

It is now easy to calculate the orbit of each coset under the elements of the group $\langle c_i, c_{i+1}\rangle$, associated to the periods of the period cycle of Λ:

$$\langle c_i, c_{i+1}\rangle \to \begin{cases} \{0\}, \{\frac{N}{2}\}, \{1, N-1\}, \dots, \{\frac{N}{2}-1, \frac{N}{2}+1\}, & i = 0, \dots, p_1-1, \\ \{0,1\}, \{2, N-1\}, \dots, \{\frac{N}{2}, \frac{N}{2}+1\}, & i = p_1+1, \dots, p_1+p_2, \\ \{0, 1, 2, \dots, N-1\}, & i = p_1,\ p_1+p_2+1. \end{cases}$$

Then Theorem 1 of [**33**] gives that Λ_1 has $rN + (N-2)p_1/2 + Np_2/2$ proper periods, all of them equal to 2, and one period cycle of length $2p_1$ in which all periods are also equal to 2. So using the Hurwitz-Riemann formula we obtain that the genus of Λ_1 is 0 and therefore its signature is the following

$$\sigma(\Lambda_1) = (0; +; [2, \overset{rN+(N-2)p_1/2+Np_2/2}{\dots\dots}, 2]; \{(2, \overset{2p_1}{\dots}, 2)\}).$$

The action of Λ on the cosets $\widehat{G}/\langle\widehat{\tau}_2\rangle$ is similar and in the same way we obtain the following signature for Λ_2 :

$$\sigma(\Lambda_2) = (0; +; [2, \overset{rN+(N-2)p_2/2+Np_1/2}{\dots\dots}, 2]; \{(2, \overset{2p_2}{\dots}, 2)\}).$$

In case N is odd it turns out that the signatures of Λ_1 and Λ_2 coincide:

$$\sigma(\Lambda_1) = \sigma(\Lambda_2) = (0; +; [2, \overset{rN+(N-1)(p_1+p_2)/2}{\dots\dots}, 2]; \{(2, \overset{p_1+p_2}{\dots}, 2)\}).$$

Before proceeding with cases Ib and Ic, we again obtain $\sigma(\Lambda_1)$ and $\sigma(\Lambda_2)$ in a different way.

A geometric method for computing the signatures of Λ_1 and Λ_2. — This method is mainly based on the topological interpretation of the symbols and integers occurring in the signature of an NEC group. The method is new and somewhat more general than needed here. It allows us to calculate polynomial equations of X and the formulae of its real forms. So it will be the strategy employed in other sections.

Let Δ be an NEC group and consider the canonical projection $f : \mathcal{H} \to \mathcal{H}/\Delta$. At each $P \in \mathcal{H}$, f behaves locally as $z \mapsto z^k$ (see [**2**, section 5] for the rigorous statement). The integer k is the *ramification index of f at P;* we say that f is *ramified at P* if $k > 1$. The ramified points of f are precisely those fixed by orientation preserving elements of Δ. The elliptic elements of Δ fixing such a point P constitute a cyclic group, and its order is the ramification index at such a point. Moreover, it turns out that all the points in the same fiber as P have the same ramification index. We say that $f(P)$ is a *branch point of f with branching order k.*

Proposition 3.1.1 ([**53**]). — *Let Δ be an* NEC *group with signature*

$$\sigma = (g; \pm; [m_1, \ldots, m_t]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})$$

and let $S = \mathcal{H}/\Delta$. Then:

(1) *g is the topological genus of S.*
(2) *sign$(\sigma(\Delta))$="+" if and only if S is orientable.*
(3) *$m_1, \ldots, m_t$ are the branching orders with respect to the canonical projection $\mathcal{H} \to \mathcal{H}/\Delta$ of the t conic points lying in the interior of S.*
(4) *k is the number of connected components of the boundary of S.*
(5) *$n_{i1}, \ldots, n_{is_i}$ are the branching orders with respect to the canonical projection $\mathcal{H} \to \mathcal{H}/\Delta$ of the s_i corner points lying on the i-th connected component of the boundary of S.*

For simplicity, g, k and " $\pm$ " will be called the *topological data* of the projection whilst the integers m_i and n_{ij} will be its *branching data.* This proposition shows that the knowledge of the topological and branching data of the projection $\mathcal{H} \to \mathcal{H}/\Delta$ is equivalent to that of the signature of Δ.

Recall we are given the epimorphism $\widehat{\theta} : \Lambda \to \widehat{G}$. Its kernel Γ_ρ is a Fuchsian group with signature $(0; [2, \overset{2g+2}{\ldots}, 2])$. Then, Proposition 3.1.1 shows that the quotient $\mathcal{H}/\Gamma_\rho$ is the Riemann sphere S^2 and that the canonical projection $\mathcal{H} \to S^2$ has $2g+2$ branch points. Clearly, $\widehat{G} = \Lambda/\Gamma_\rho$ acts on $\mathcal{H}/\Gamma_\rho$ permuting these points. In class I, $\widehat{G}$ is dihedral of order $2N$ generated by $\widehat{\tau}_1 = \widehat{\theta}(c_0)$ and $\widehat{\tau}_2 = \widehat{\theta}(c_{p_1+1})$. Since c_0 and c_{p_1+1} are both canonical reflections of Λ it follows that $\widehat{\tau}_1$ and $\widehat{\tau}_2$ represent anticonformal involutions acting with fixed points on the sphere (those coming from fixed points of the reflections c_i). Therefore we may suppose that $\widehat{\tau}_1$ and $\widehat{\tau}_2$ are two reflections with respect to planes through the center of the sphere with angle π/N. Consequently, a fundamental set for the action of $\widehat{G}$ is the two sided spherical polygon with vertices P_0 and P_∞ illustrated in Figure 1.

Consider now $S^2/\widehat{G}$ as the quotient $(\mathcal{H}/\Gamma_\rho)/(\Lambda/\Gamma_\rho) = \mathcal{H}/\Lambda$. Since Λ has signature $(0;+;[2,\overset{r}{\ldots},2];\{(2,\overset{p_1}{\ldots},2,N,2,\overset{p_2}{\ldots},2,N)\})$, the quotient $\mathcal{H}/\Lambda$ is a genus 0 surface with one boundary component, as the figure illustrates. Its boundary consists of those points of the fundamental set fixed by either of the reflections $\widehat{\tau}_1$, $\widehat{\tau}_2$. The signature of Λ also shows that the canonical projection $\mathcal{H} \to \mathcal{H}/\Lambda$ branches over r points in the interior of the fundamental set and $p_1 + p_2 + 2$ points on its boundary.

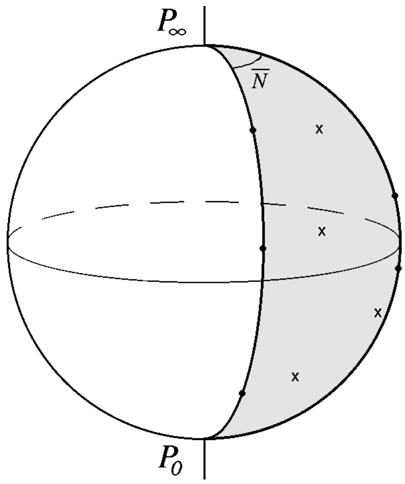
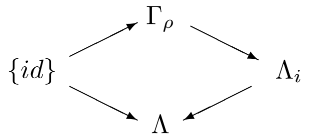


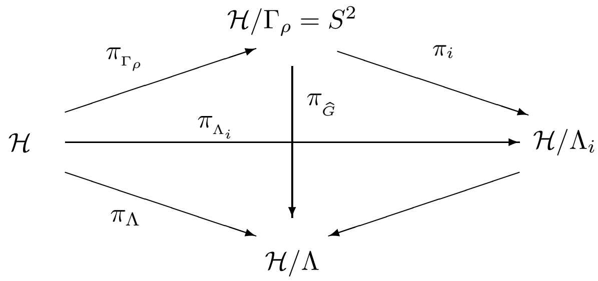
FIGURE 1

Among the boundary branch points, those two whose branching order is N lie in the intersection $\mathrm{Fix}(\widehat{\tau}_1) \cap \mathrm{Fix}(\widehat{\tau}_2)$ of the fixed point set of $\widehat{\tau}_1$ with that of $\widehat{\tau}_2$, p_1 of them (with branching order 2) lie on $\mathrm{Fix}(\widehat{\tau}_1) - \mathrm{Fix}(\widehat{\tau}_2)$, and the other p_2 (with branching order 2) lie on $\mathrm{Fix}(\widehat{\tau}_2) - \mathrm{Fix}(\widehat{\tau}_1)$. This follows from the way in which the epimorphism $\widehat{\theta} : \Lambda \to \widehat{G}$ maps the $p_1 + p_2 + 3$ canonical reflections of Λ. Indeed, the image under the canonical projection $\pi_\Lambda : \mathcal{H} \to \mathcal{H}/\Lambda$ of the fixed point of the orientation preserving element $c_i c_{i+1} \in \Lambda$ is a boundary branch point of π_Λ. Moreover, it lies in (the projection under the action of $\widehat{G}$ of) $\mathrm{Fix}(\widehat{\tau}_1)$ for $i = 0, \ldots, p_1 - 1$ or in (the projection under the action of $\widehat{G}$ of) $\mathrm{Fix}(\widehat{\tau}_2)$ for $i = p_1 + 1, \ldots, p_1 + p_2$ because $\widehat{\theta}(c_i) = \widehat{\tau}_1$ for $i = 0, \ldots, p_1 - 1$ and $\widehat{\theta}(c_i) = \widehat{\tau}_2$ for $i = p_1 + 1, \ldots, p_1 + p_2$. For $i = p_1$ and $i = p_1 + p_2 + 1$, it lies in $\mathrm{Fix}(\widehat{\tau}_1) \cap \mathrm{Fix}(\widehat{\tau}_2)$, and has branching order $N = \mathrm{order}(c_i c_{i+1})$.

Let us begin with the computation of the signatures of Λ_1 and Λ_2. For $i = 1, 2$ consider the following commutative diagram of NEC group monomorphisms



It induces the corresponding one of projections between quotient spaces:



We begin by calculating the genus, the orientability and the number of boundary components of $\mathcal{H}/\Lambda_i$. To that end note that $\mathcal{H}/\Lambda_i$ is the quotient of the Riemann sphere $\mathcal{H}/\Gamma_\rho$ under the action of the group Λ_i/Γ_ρ. But

$$\Lambda_i/\Gamma_\rho = \frac{\Lambda_i/\Gamma}{\Gamma_\rho/\Gamma} = \frac{\langle \tau_i, \tau_i \circ \rho \rangle}{\langle \rho \rangle} = \langle \widehat{\tau}_i \rangle.$$

Since $\widehat{\tau}_i$ is the reflection with respect to a plane, the quotient $\mathcal{H}/\Lambda_i$ is, topologically, a closed disc, *i.e.,* it is an orientable surface of topological genus 0 with one boundary component. Therefore Proposition 3.1.1 gives that the signature of Λ_i has the following form:

$$\sigma(\Lambda_i) = (0; +; [\text{proper periods}]; \{(\text{periods})\}).$$

To calculate the branching data of the projection $\pi_{\Lambda_i} : \mathcal{H} \to \mathcal{H}/\Lambda_i$ we write it as the composite $\pi_{\Lambda_i} = \pi_i \circ \pi_{\Gamma_\rho}$. Note that π_i is the quotient map for the action of the group generated just by an orientation reversing involution, namely $\widehat{\tau}_i$. It follows that the branch points of π_{Λ_i} are exactly the images under π_i of the branch points of π_{Γ_ρ} and, moreover, their branching orders coincide with those of the branch points of π_{Γ_ρ}, *i.e.,* they are equal to 2. In terms of signatures this means that all the proper periods and the periods of the non-empty period cycle of $\sigma(\Lambda_i)$ are equal to 2:

$$\sigma(\Lambda_i) = (0; +; [2, \ldots, 2]; \{(2, \ldots, 2)\}).$$

It is clear that the above also applies to any symmetry τ such that the induced antianalytic involution $\widehat{\tau}$ in the Riemann sphere is the reflection with respect to a plane. We may also obtain the signatures of the NEC groups corresponding to those antianalytic involutions in the Riemann sphere which are conjugate to the antipodal map. Although they do not occur in class I, it is convenient to do it now. We include both types of signatures in the following lemma for further reference.

LEMMA 3.1.2. — *With the above notations, the signature of the* NEC *group* Λ_i *associated to a symmetry* τ_i *is*

(1) $(0;+;[2,\overset{u}{\dots},2];\{(2,\overset{v}{\dots},2)\})$ *with* $2u+v=2g+2$ *if* $\widehat{\tau}_i$ *is a reflection with respect to a plane, or*

(2) $(1;-;[2,\overset{g+1}{\dots},2];\{-\})$ *if* $\widehat{\tau}_i$ *is the antipodal map.*

Recall that a reflection with respect to a plane and the antipodal map are representatives of the only two conjugacy classes of antianalytic involutions of the Riemann sphere.

Proof. — All the arguments done above with a reflection are also valid for the antipodal map, except that the quotient of the Riemann sphere $\mathcal{H}/\Gamma_\rho$ under the action of the antipodal map Λ_i/Γ_ρ is, topologically, a real projective plane. So in case (2), the signature $\sigma(\Lambda_i)$ has the form

$$(1;-;[\text{proper periods}=2];\{-\}).$$

It only remains to prove the equalities concerning u, v and g. Recall that the projection $\pi_{\Gamma_\rho}:\mathcal{H}\to S^2$ ramifies over $2g+2$ points, say $e_1,\dots,e_{2g+2}$. They are permuted by the involution $\widehat{\tau}_i$ and so the $\widehat{\tau}_i$-orbit of a branch point e_j consists of two points if it is not fixed by $\widehat{\tau}_i$ and one point otherwise. Those e_j fixed by $\widehat{\tau}_i$ lie in the boundary of $\mathcal{H}/\Lambda_i$ and correspond to periods of the period cycle of $\sigma(\Lambda_i)$, if any. The rest lie in the interior of $\mathcal{H}/\Lambda_i$ and correspond to proper periods of $\sigma(\Lambda_i)$. Summing up cardinals of $\widehat{\tau}_i$-orbits yields the desired equalities. □

REMARK 3.1.3. — It follows from the proof that in case (1) v equals the number of branch points e_j lying on $\mathrm{Fix}(\widehat{\tau}_i)$.

We now return to the particular case of curves of class Ia. Since both $\widehat{\tau}_1$ and $\widehat{\tau}_2$ are reflections with respect to planes, part (1) of the above lemma gives that for $i=1,2$,

$$\sigma(\Lambda_i)=(0;+;[2,\overset{u_i}{\dots},2],(\{(2,\overset{v_i}{\dots},2)\})\quad\text{with}\quad 2u_i+v_i=2g+2,$$

where v_i equals the number of branch points e_j lying on $\mathrm{Fix}(\widehat{\tau}_i)$. Let us calculate v_i.

The points e_j are permuted by the elements of the dihedral group $\widehat{G}=\langle\widehat{\tau}_1,\widehat{\tau}_2\rangle$ of order $2N$. The branching order with respect to π_{Γ_ρ} of each e_j is 2. The unique points of S^2 fixed by some non-trivial orientation preserving element of $\widehat{G}$ are the fixed points P_0 and P_∞ of the rotation $\widehat{a}:=\widehat{\tau}_2\circ\widehat{\tau}_1$. So their images under the canonical projection $\pi_{\widehat{G}}:S^2\to\mathcal{H}/\Lambda$ are the unique branch points of $\pi_{\widehat{G}}$. Moreover, their branching order with respect to $\pi_{\widehat{G}}$ coincides with the order of $\widehat{a}$, *i.e.*, N. Therefore, P_0 and P_∞ are not among the e_j because otherwise π_Λ would have branch points of branching order $2N$, contrary to the signature of Λ.

Now, P_0 and P_∞ divide $\mathrm{Fix}(\widehat{\tau}_i)$ into two semicircles, one of them being a part of the boundary of the fundamental set illustrated in Figure 1. This semicircle contains exactly p_i branch points e_j. Both semicircles are paired by an element of $\widehat{G}$ if and only if N is even. So the number of e_j lying on the other semicircle is also p_i if and only if N is even. Since P_0 and P_∞ are not among them, we conclude that the total number v_i of branch points e_j lying on $\mathrm{Fix}(\widehat{\tau}_i)$ is $2p_i$ if N is even and p_1+p_2 otherwise. So,

$$\sigma(\Lambda_i) = (0;+;[2,\overset{g+1-p_i}{\ldots},2];\{(2,\overset{2p_i}{\ldots},2)\}) \text{ for } i=1,2 \qquad \text{if } N \text{ is even,}$$

$$\sigma(\Lambda_1)=\sigma(\Lambda_2) = (0;+;[2,\overset{g+1-(p_1+p_2)/2}{\ldots},2];\{(2,\overset{p_1+p_2}{\ldots},2)\}) \text{ if } N \text{ is odd.}$$

Note that this fits in with the signatures obtained by combinatorial methods since $g+1 = N(2r+p_1+p_2)/2$.

Computation of species. — Recall that for N even τ_1, $\tau_1\rho$, τ_2 and $\tau_2\rho$ are representatives of the unique four conjugacy classes of symmetries of G. In view of the signature of Λ_i for $i=1,2$, it follows from Theorem 3.0.1 that the species of τ_i and $\tau_i\rho$ are the following.

Ia, N even	$p_i = 0$	$0 < p_i < g+1$	$p_i = g+1$
$\{\mathrm{sp}(\tau_i), \mathrm{sp}(\tau_i\rho)\}$	$\{1,0\}$ if g even $\{2,0\}$ if g odd	$\{-p_i,-p_i\}$	$\{g+1,g+1\}$

Note that p_i ranges from 0 to $g+1$. However, if $p_i = g+1$ then $p_j = 0$ and $N=2$, as is easy to check from the equality $2r+p_1+p_2 = (2g+2)/N$.

For N odd G has only two conjugacy classes of symmetries, represented by τ_1 and $\tau_1\rho$. Their species are given in the following table.

Ia, N odd	$p_1+p_2 = 0$	$0 < p_1+p_2 \leqslant (2g+2)/N$
$\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\}$	$\{1,0\}$ if g even $\{2,0\}$ if g odd	$\{-(p_1+p_2)/2, -(p_1+p_2)/2\}$

Note that p_1+p_2 cannot attain the value $2g+2$.

We are now in a position to give the list of symmetry types of hyperelliptic curves of class $(\mathrm{Ia})_N$. We know from Theorem 2.3.3 that there exists such a curve X if and only if N is a divisor of $2g+2$ with $(2g+2)/N \neq 1,2$. As said before, X has 4 real forms if N is even and 2 otherwise. In either case, the integers p_1 and p_2 satisfy the equality $p_1+p_2 = (2g+2)/N - 2r$; so $p_1+p_2 \leqslant (2g+2)/N$ and both sides of the inequality have the same parity. We also have to take into account the parity of g in case $\mathrm{sp}(\tau_i) \neq \mathrm{sp}(\tau_i\rho)$. It is then straightforward to show the following.

THEOREM 3.1.4. — *The symmetry type of a hyperelliptic algebraic genus g curve of class* $(\mathrm{Ia})_N$ *is one of the listed below. The integers p_1 and p_2 appearing in the list satisfy*

$$0 < p_i \leqslant \frac{2g+2}{N} \quad \textit{and} \quad p_1 + p_2 \leqslant \frac{2g+2}{N}.$$

(1) *If g is even:*
  (1.1) *If N is even:*
    - $\{1, 0, g+1, g+1\}$, *only if $N = 2$;*
    - $\{1, 0, -p_2, -p_2\}$ *where p_2 is odd;*
    - $\{-p_1, -p_1, -p_2, -p_2\}$ *where $p_1 + p_2$ is odd.*
  (1.2) *If N is odd:*
    - $\{1, 0\}$;
    - $\{-(p_1+p_2)/2, -(p_1+p_2)/2\}$ *where $p_1 + p_2$ is even.*

(2) *If g is odd:*
  (2.1) *If N and $(2g+2)/N$ are even:*
    - $\{2, 0, 2, 0\}$;
    - $\{2, 0, g+1, g+1\}$, *only if $N = 2$;*
    - $\{2, 0, -p_2, -p_2\}$ *where p_2 is even;*
    - $\{-p_1, -p_1, -p_2, -p_2\}$ *where $p_1 + p_2$ is even.*
  (2.2) *If N is even and $(2g+2)/N$ is odd:*
    - $\{2, 0, -p_2, -p_2\}$ *where p_2 is odd;*
    - $\{-p_1, -p_1, -p_2, -p_2\}$ *where $p_1 + p_2$ is odd.*
  (2.3) *If N is odd:*
    - $\{2, 0\}$;
    - $\{-(p_1+p_2)/2, -(p_1+p_2)/2\}$ *where $p_1 + p_2$ is even.*

Conversely, each of these is the symmetry type of a genus g curve of class $(\mathrm{Ia})_N$.

Proof. — Only the converse needs some explanation. Let $g \geqslant 2$ and N be given such that $(2g+2)/N > 2$, and let T be any of the above uplas, say given by parameters p_1 and p_2. Consider the NEC signature $\sigma = (0; +; [2, \overset{r}{\dots}, 2]; \{(2, \overset{p_1}{\dots}, 2, N, 2, \overset{p_2}{\dots}, 2, N)\})$, where $2r = (2g+2)/N - p_1 - p_2$. Since $(2g+2)/N > 2$, the proof of Theorem 2.3.3 assures then the existence of an NEC group Λ with signature σ such that $\Lambda/\ker\theta$ is the full group of automorphisms of $X = \mathcal{H}/\ker\theta$, where $\theta : \Lambda \to \Lambda/\ker\theta$ is the epimorphism studied in case Ia in the preceding chapter. Therefore X is of class $(\mathrm{Ia})_N$ and, according to the above computations, its symmetry type is T. □

We may obtain the complete list of symmetry types of hyperelliptic curves of class Ia. For that we have to collect the above symmetry types as N runs over all the divisors of $2g+2$ with $(2g+2)/N > 2$. If we write $2g+2 = 2^\alpha q$ with q odd then $N = 2^\beta q'$ with $\beta \leqslant \alpha$ and $q'|q$. With this notation, N is even if and only if $\beta > 0$, $(2g+2)/N$ is even if and only if $\beta < \alpha$, and g is even if and only if $\alpha = 1$.

COROLLARY 3.1.5. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* Ia *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integers p_1 and p_2 appearing in the list satisfy*

$$0 < p_i < g+1 \quad and \quad p_1 + p_2 \leqslant g+1.$$

We write $2g+2 = 2^\alpha q$ with q odd, and if $q \neq 1$ let $q_1 \neq 1$ be its smallest divisor.

(1) *If g is even:*
  - $\{1, 0, g+1, g+1\}$*;*
  - $\{1, 0, -p_2, -p_2\}$ *where p_2 is odd;*
  - $\{-p_1, -p_1, -p_2, -p_2\}$ *where $p_1 + p_2$ is odd;*
  - $\{1, 0\}$*, only if $g+1 \neq$ prime number;*
  - $\{-(p_1+p_2)/2, -(p_1+p_2)/2\}$ *where p_1+p_2 is even $\leqslant (2g+2)/q_1$, only if $g+1 \neq$ prime number.*

(2) *If g is odd:*
  - $\{2, 0, 2, 0\}$;
  - $\{2, 0, g+1, g+1\}$*;*
  - $\{2, 0, -p_2, -p_2\}$ *where p_2 is* $\begin{cases} \textit{even} & \textit{if } q = 1; \\ \textit{even or} \leqslant q & \textit{if } q \neq 1; \end{cases}$
  - $\{-p_1, -p_1, -p_2, -p_2\}$ *where $p_1 + p_2$ is* $\begin{cases} \textit{even} & \textit{if } q = 1; \\ \textit{even or} \leqslant q & \textit{if } q \neq 1; \end{cases}$
  - $\{2, 0\}$ *if $q \neq 1$;*
  - $\{-(p_1+p_2)/2, -(p_1+p_2)/2\}$ *where p_1+p_2 is even $\leqslant (2g+2)/q_1$ if $q \neq 1$.*

The same kind of corollaries may be obtained in other classes. However, it is more accurate to take into account the order of $\mathrm{Aut}^{\pm} X$ when listing symmetry types. So, in order to avoid unnecessary complications, we will omit them.

3.1.2. Class Ib. — In this case Λ has signature

$$\sigma = (0; +; [2, \overset{r}{\ldots}, 2]; \{(2, \overset{p_1}{\ldots}, 2, 2N, 2, \overset{p_2}{\ldots}, 2, 2N)\}) \text{ with } 2r + p_1 + p_2 = \frac{2g}{N} > 2.$$

Recall that here $G = \langle \tau_1, \tau_2 \rangle = \mathrm{D}_{2N}$ which has exactly 2 conjugacy classes of symmetries, represented by τ_1 and τ_2. The signatures of the NEC groups Λ_1 and Λ_2 are easily computed with the geometric method. The unique difference with the preceding case is that now the projection $\mathcal{H} \to \mathcal{H}/\Lambda$ has two (boundary) branch points of branching order $2N$ instead of N. With the same notations, this forces the two fixed points P_0 and P_∞ of the rotation $\widehat{a} := \widehat{\tau}_1 \circ \widehat{\tau}_2$ to be two of the $2g+2$ branch points e_j of the projection $\mathcal{H} \to \mathcal{H}/\Gamma_\rho = S^2$.

As a consequence, the number of e_j lying on $\mathrm{Fix}(\widehat{\tau}_i)$ is 2 greater than in the preceding case. So,

$$\sigma(\Lambda_i) = (0; +; [2, \overset{g-p_i}{\ldots}, 2]; \{(2, \overset{2p_i+2}{\ldots}, 2)\}) \text{ for } i = 1, 2 \qquad \text{if } N \text{ is even,}$$

$$\sigma(\Lambda_1) = \sigma(\Lambda_2) = (0; +; [2, \overset{g-(p_1+p_2)/2}{\ldots}, 2]; \{(2, \overset{p_1+p_2+2}{\ldots}, 2)\}) \text{ if } N \text{ is odd.}$$

Computation of species. — According to Theorem 3.0.1 the species of the unique two conjugacy classes of symmetries of G, represented by τ_1 and τ_2, are those given in the tables below. Either if N is even or odd, $p_1 + p_2$ ranges from 0 to $2g/N$. So, for N even, p_i can attain the value g although if $p_i = g$ then $p_j = 0$ and $N = 2$. For N odd $p_1 + p_2$ cannot attain the value $2g$.

Ib, N even	$p_i < g$	$p_i = g$
$\mathrm{sp}(\tau_i)$	$-(p_i + 1)$	$g + 1$

Ib, N odd	$p_1 + p_2 \leqslant 2g/N$
$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_2)$	$-(p_1 + p_2 + 2)/2$

The next theorem lists all the symmetry types of genus g curves of class $(\mathrm{Ib})_N$. We know from Theorem 2.3.3 that such curves exist if and only if N is a divisor of $2g$ and $2g/N \neq 1, 2$ (in particular, $g > 2$). According to the above tables the species of the real forms depend on the parity of N. Moreover, the range of the parameters p_i depends on the parity of $2g/N$ since $p_1 + p_2 = 2g/N - 2r$.

THEOREM 3.1.6. — *The symmetry type of a genus g hyperelliptic algebraic curve of class $(\mathrm{Ib})_N$ is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integers p_1 and p_2 satisfy*

$$0 \leqslant p_i \leqslant \frac{2g}{N} \quad \text{and} \quad p_1 + p_2 \leqslant \frac{2g}{N}.$$

(1) *If N is even:*
- $\{-1, g+1\}$, *only if* $N = 2$;
- $\{-(p_1 + 1), -(p_2 + 1)\}$ *where* $p_1 + p_2$ *is* $\begin{cases} \text{even} & \text{if } 2g/N \text{ is even,} \\ \text{odd} & \text{if } 2g/N \text{ is odd.} \end{cases}$

(2) *If N is odd:*
- $\{-(p_1 + p_2 + 2)/2, -(p_1 + p_2 + 2)/2\}$ *where* $p_1 + p_2$ *is even.*

3.1.3. Class Ic. — In this case Λ has signature

$$\sigma = (0; +; [2, \overset{r}{\ldots}, 2]; \{(2, \overset{p_1}{\ldots}, 2, N, 2, \overset{p_2}{\ldots}, 2, 2N)\}) \quad \text{with} \quad 2r + p_1 + p_2 = \frac{2g+1}{N}.$$

Note that N must be odd and the same happens to $p_1 + p_2$. Here $G = \langle \tau_1, \tau_2 \rangle = \mathrm{D}_{2N}$ which has exactly 2 conjugacy classes of symmetries, represented by τ_1 and $\tau_1 \rho \sim \tau_2$. The unique difference with case Ib is that now the projection $\mathcal{H} \to \mathcal{H}/\Lambda$ has one (boundary) branch point of branching order $2N$ and another of branching order N. With the same notations, this forces just one of the two fixed points P_0 and P_∞ of the rotation $\widehat{a} := \widehat{\tau}_1 \circ \widehat{\tau}_2$ to be one of the $2g + 2$ branch points e_j of the projection $\mathcal{H} \to \mathcal{H}/\Gamma_\rho = S^2$.

As a consequence, the number of e_j lying on $\mathrm{Fix}(\widehat{\tau}_i)$ is 1 less than in the preceding case. So,

$$\sigma(\Lambda_1) = (0; +; [2, \overset{g-(p_1+p_2-1)/2}{\ldots\ldots\ldots}, 2]; \{(2, \overset{p_1+p_2+1}{\ldots\ldots}, 2)\}).$$

The species of τ_1 and $\tau_1 \rho$ are therefore the following.

Ic	$p_1 + p_2 \leqslant (2g+1)/N$
$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho)$	$-(p_1 + p_2 + 1)/2$

Note that the equality $p_1 + p_2 = (2g+1)/N - 2r$ prevents $p_1 + p_2 + 1$ to attain the value $2g + 2$.

THEOREM 3.1.7. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{Ic})_N$ *is*

$$\{-(p_1 + p_2 + 1)/2, -(p_1 + p_2 + 1)/2\}$$

for some non-negative integers p_1, p_2 such that $p_1 + p_2$ is odd $\leqslant (2g+1)/N$. Conversely, for any p_1, p_2 in these conditions, the above is the symmetry type of a genus g curve of class $(\mathrm{Ic})_N$.

3.1.4. Equations of curves of class I and their real forms. — The last part of this section is devoted to find explicit polynomial equations defining hyperelliptic curves of classes Ia, Ib and Ic. We also obtain the formula of a real structure realizing each of their real forms.

So far the distinction between these classes has been algebraic: they are distinguished by the corresponding NEC group Λ and the epimorphism θ. However, the geometric method used for computing the signature of Λ_i provides another way of characterizing each class. This alternative characterization is specially useful to obtain equations since it is based on the distribution of the branch points of the projection π_{Γ_ρ}. Indeed we have seen that

a) in class Ia none of the two fixed points P_0 and P_∞ of the rotation $\widehat{\tau}_2 \circ \widehat{\tau}_1$ is a branch point;
b) in class Ib both are branch points;
c) in class Ic just one of them is a branch point.

Recall that a hyperelliptic complex algebraic curve X of genus g is represented by its affine plane model

$$X = \{y^2 = P_X(x) := (x - e_1) \cdots (x - e_{2g+1+\delta})\}$$

with $e_i \neq e_j$ if $i \neq j$ and $\delta = 0$ or 1. The *branch point set* of the meromorphic function of degree 2 given by $\pi_X : X \to \widehat{\mathbb{C}} : (x, y) \mapsto x$ is

$$B_X = \begin{cases} \{e_1, \ldots, e_{2g+2}\} & \text{if } \delta = 1, \\ \{e_1, \ldots, e_{2g+1}, \infty\} & \text{if } \delta = 0, \end{cases}$$

and the *hyperelliptic involution* ρ is given by

$$\begin{aligned} \rho : \quad X &\longrightarrow X \\ (x, y) &\longmapsto (x, -y). \end{aligned}$$

With the notations of the preceding sections, B_X coincides with the branch point set of the canonical projection $\pi_{\Gamma_\rho} : \mathcal{H} \to \mathcal{H}/\Gamma_\rho = S^2$. This follows from the fact that π_{Γ_ρ} is the composite of the unramified cover $\mathcal{H} \to \mathcal{H}/\Gamma = X$ with the meromorphic function π_X. Its branch points are permuted by the elements of the group $\widehat{G}$ generated by two reflections $\widehat{\tau}_1$ and $\widehat{\tau}_2$ with respect to planes through the center of the sphere with angle π/N. It is now more convenient to think of the Riemann sphere as the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ instead of the subset S^2 of the 3-euclidean space. We change from one model to the other by means of the stereographic projection $\Phi : S^2 \to \widehat{\mathbb{C}}$ given by

$$\Phi(x_1, x_2, x_3) = \begin{cases} \dfrac{x_2}{x_1+1} + i\dfrac{x_3}{x_1+1} & \text{if } x_1 \neq -1 \\ \infty & \text{if } x_1 = -1. \end{cases}$$

This projection maps the equator $\{x_3 = 0\}$ onto the real axis $\mathbb{R} \cup \{\infty\}$; the images of two antipodal points P and $-P$ are of the form α and $-1/\overline{\alpha}$ (we adopt throughout the conventions that $z/\infty = 0$ and $z/0 = \infty$ for any $z \in \mathbb{C}$). Viewing $\widehat{\tau}_1$ and $\widehat{\tau}_2$ as Möbius transformations in $\widehat{\mathbb{C}}$ via Φ we may choose as generators for $\widehat{G}$ the following:

$$\widehat{\tau}_1 : x \longmapsto \overline{x} \quad \text{and} \quad \widehat{\tau}_2 : x \longmapsto \overline{x}e^{2\pi i/N},$$

where we have denoted transformations of $\widehat{\mathbb{C}}$ in the same way than the corresponding transformations of S^2. Thus the two fixed points of the rotation $\widehat{a} := \widehat{\tau}_2 \circ \widehat{\tau}_1 : x \mapsto xe^{2\pi i/N}$ are 0 and ∞. It is then easy to check that the orbit under the action of $\widehat{G}$ of a point $\alpha \in \widehat{\mathbb{C}}$ is the following:

$$\operatorname{orbit}(\alpha, \widehat{G}) = \begin{cases} \text{roots of } (x^N - \alpha^N)(x^N - \overline{\alpha}^N) & \text{if } \alpha \text{ is fixed by no antianalytic involution on } \widehat{G}; \\ \text{roots of } (x^N - \alpha^N) & \text{if } \alpha \neq 0, \infty \text{ is fixed by an antianalytic involution on } \widehat{G}; \\ \{\alpha\} & \text{if } \alpha = 0 \text{ or } \infty. \end{cases}$$

If α is fixed by no antianalytic involution then α^N is a complex non-real number. If $\alpha \neq 0, \infty$ is fixed by an antianalytic involution of G then α^N is a real number. Its sign depends on $\arg(\alpha)$ and the parity of N. This justifies the choice of parameters λ_i, μ_i and w_i appearing in the following theorem. In it we assume that N is an integer such that the corresponding value of g is an integer $\geqslant 2$.

THEOREM 3.1.8. — *For almost every choice of*

- $\{w_1, \dots, w_r\} \subset \mathbb{C}^+ := \{z \in \mathbb{C} : \Im(z) > 0\}$ *with* $w_i \neq w_j$ *if* $i \neq j$,
- $0 < \lambda_1 < \cdots < \lambda_{p_1} \subset \mathbb{R}$,
- $0 < \mu_1 < \cdots < \mu_{p_2} \subset \mathbb{R}$,

the equation

$$y^2 = x^\varepsilon \cdot \prod_{j=1}^{r}(x^N - w_j)(x^N - \overline{w}_j) \cdot \prod_{j=1}^{p_1}(x^N - \lambda_j) \cdot \prod_{j=1}^{p_2}(x^N + \mu_j)$$

defines a hyperelliptic complex curve X of class

a) $(\mathrm{Ia})_N$ *and genus* $g = N(2r + p_1 + p_2)/2 - 1$ *if* $\varepsilon = 0$ *and* $2r + p_1 + p_2 > 2$;

b) $(\mathrm{Ib})_N$ *and genus* $g = N(2r + p_1 + p_2)/2$ *if* $\varepsilon = 1$ *and* $2r + p_1 + p_2 > 2$;

c) $(\mathrm{Ic})_N$ *and genus* $g = N(2r + p_1 + p_2)/2 - 1/2$ *if* $\varepsilon = 0$.

Conversely, each hyperelliptic genus g curve of class I *is (isomorphic to another) of the above form for some N, r, p_1, p_2 and ε satisfying the above conditions.*

Representatives of all the real forms of X are the following:

Class $(\mathrm{Ia})_N$	
N *even*	τ_1, $\tau_1\rho$, τ_2, $\tau_2\rho$
N *odd*	τ_1, $\tau_1\rho$

Class $(\mathrm{Ib})_N$
τ_1, τ_2

Class $(\mathrm{Ic})_N$
τ_1, $\tau_1\rho$

where

$$\tau_1 : (x, y) \longmapsto (\overline{x}, \overline{y}), \quad \text{and} \quad \tau_2 : (x, y) \longmapsto \left(\overline{x} \cdot e^{2\pi i/N},\ \overline{y} \cdot (e^{\pi i/N})^{\varepsilon}\right).$$

In all cases, the species of each real form is also given.

Proof. — The above equation defines a hyperelliptic curve of the prescribed genus g since it is of the form $y^2 = P(x)$ with $P(x)$ a polynomial with simple roots, whose degree $(2r + p_1 + p_2)N + \varepsilon$ equals $2g + 2$ in case *a*) and $2g + 1$ in cases *b*) and *c*).

We begin by showing that the curve X is of class I. Its branch points are permuted by the elements of the dihedral group $\langle \widehat{\tau}_1, \widehat{\tau}_2 \rangle$, where $\widehat{\tau}_1 : x \mapsto \overline{x}$ and $\widehat{\tau}_2 : x \mapsto \overline{x}e^{2\pi i/N}$ are reflections with respect to planes (viewed in S^2). In other words, $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ contains the subgroup $\langle \widehat{\tau}_1, \widehat{\tau}_2 \rangle$ of class I. Now, for almost every choice of the parameters $w_1, \ldots, w_r$, $\lambda_1, \ldots, \lambda_{p_1}$, $\mu_1, \ldots, \mu_{p_2}$ the branch points of the curve X are "in general position" and so they are permuted by no other Möbius transformation than those in $\langle \widehat{\tau}_1, \widehat{\tau}_2 \rangle$. Therefore $\langle \widehat{\tau}_1, \widehat{\tau}_2 \rangle$ coincides with $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ and this proves that X is a class I curve. We note that the condition $2r + p_1 + p_2 \geqslant 3$ gives us a "large enough" number of parameters to prevent X from having more automorphisms. In fact, if $2r + p_1 + p_2 = 1$ or 2 then there exist Möbius transformations in $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}} - \langle \widehat{\tau}_1, \widehat{\tau}_2 \rangle$. Explicitly, if $p_1 = 1$ and $r = p_2 = 0$, then the Möbius transformation $x \mapsto \alpha^2/x$ with $\alpha^N = \lambda_1$ belongs to $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}} - \langle \widehat{\tau}_1, \widehat{\tau}_2 \rangle$; the case $p_2 = 1$ and $r = p_1 = 0$ is analogous. In case $r = 1$ and $p_1 = p_2 = 0$, consider the transformation $x \mapsto \beta^2/x$ with $\beta^N = \omega_1$; in case $r = 0$, $p_1 = p_2 = 1$ the transformation to be considered is $x \mapsto \alpha\beta e^{i\pi/N}/x$, with $\alpha^N = \lambda_1$ and $\beta^N = -\mu_1$; finally, in case $p_1 = 2$ and $r = p_2 = 0$ (case $p_2 = 2$ and $r = p_1 = 0$ is analogous) the transformation $x \mapsto \alpha_1\alpha_2/x$ with $\alpha_i^N = \lambda_i$ works.

In case *a*) the curve X is of class $(\mathrm{Ia})_N$ since none of the two fixed points 0 and ∞ of the rotation $\widehat{\tau}_1 \circ \widehat{\tau}_2$ is a branch point. In case *b*) X is of class $(\mathrm{Ib})_N$ because both are branch points, and in case *c*) X is of class $(\mathrm{Ic})_N$ because only one of them, namely ∞, is a branch point.

Conversely, let X be a hyperelliptic complex curve of class I, say with $|\mathrm{Aut}^{\pm}X_{\widehat{\mathbb{C}}}| = 2N$. Then we may suppose that this group is generated by $\widehat{\tau}_1 : x \mapsto \overline{x}$ and $\widehat{\tau}_2 : x \mapsto \overline{x}e^{2\pi i/N}$. In particular the branch point set B_X of X consists of orbits of points $\alpha \in \widehat{\mathbb{C}}$ under the action of $\langle \widehat{\tau}_1, \widehat{\tau}_2 \rangle$. It follows from the computations preceding the theorem that:

a) if X is of class $(\mathrm{Ia})_N$ then B_X consists of the roots of a polynomial $P(x)$ as in the statement of the theorem with $\varepsilon = 0$;
b) if X is of class $(\mathrm{Ib})_N$ then the same holds true with $\varepsilon = 1$.
c) if X is of class $(\mathrm{Ic})_N$ then either 0 or ∞ is a branch point. In case $0 \in B_X$ (otherwise we are done) we change X by the curve X' whose branch point set is the image of B_X under the transformation $x \mapsto 1/\overline{x}$. Note that $\mathrm{Aut}^{\pm}X'_{\widehat{\mathbb{C}}}$ is still generated by $\widehat{\tau}_1$ and $\widehat{\tau}_2$ because $x \mapsto 1/\overline{x}$ commutes with both.

We now calculate real forms and species. In all cases, $\mathrm{Aut}^{\pm}X_{\widehat{\mathbb{C}}}$ is the group generated by $\widehat{\tau}_1$ and $\widehat{\tau}_2$ and so $\mathrm{Aut}^{\pm}X$ is generated by their liftings and the hyperelliptic involution. The formulae of these liftings coincide with those given in the theorem, as is easy to check.

In **case *a*)** the composite $\tau_2 \circ \tau_1 : (x,y) \mapsto (xe^{2\pi i/N}, y)$ has order N and so $\mathrm{Aut}^{\pm}X = \langle \tau_1, \tau_2 \rangle \oplus \langle \rho \rangle = \mathrm{D}_N \oplus \mathrm{Z}_2$. Hence, as we know from Chapter 2, there are exactly four conjugacy classes of symmetries with representatives τ_1, τ_2, $\tau_1\rho$ and $\tau_2\rho$ if N is even, and two with representatives τ_1 and $\tau_1\rho$ if N is odd. Since $\widehat{\tau}_i$ is a reflection for $i = 1, 2$ the species of each symmetry depends on the number of branch points fixed by $\widehat{\tau}_i$ (see Theorems 1.3.4 and 1.3.5).

We begin with the case N odd. The species of the liftings τ_1 and $\tau_1\rho$ of complex conjugation $\widehat{\tau}_1$ depends on the number of real roots of the polynomial $P(x)$. Since such number is $p_1 + p_2$ it follows from Theorem 1.3.4 that

$$\mathrm{sp}(\tau_1) = \begin{cases} -(p_1+p_2)/2 & \text{if } 0 < p_1 + p_2 < 2g+2, \\ 1 & \text{if } p_1 + p_2 = 0 \text{ and } g \text{ is even}, \\ 2 & \text{if } p_1 + p_2 = 0 \text{ and } g \text{ is odd}; \end{cases}$$

$$\mathrm{sp}(\tau_1\rho) = \begin{cases} -(p_1+p_2)/2 & \text{if } 0 < p_1 + p_2 < 2g+2, \\ 0 & \text{if } p_1 + p_2 = 0. \end{cases}$$

Note that in this case it cannot happen that all the roots of $P(x)$ are real.

In case N is even, $P(x)$ has $2p_1$ real roots and so

$$\mathrm{sp}(\tau_1) = \begin{cases} g+1 & \text{if } p_1 = g+1, \\ -p_1 & \text{if } 0 < p_1 < g+1, \\ 1 & \text{if } p_1 = 0 \text{ and } g \text{ is even}, \\ 2 & \text{if } p_1 = 0 \text{ and } g \text{ is odd}. \end{cases} \qquad \mathrm{sp}(\tau_1\rho) = \begin{cases} g+1 & \text{if } p_1 = g+1, \\ -p_1 & \text{if } 0 < p_1 < g+1, \\ 0 & \text{if } p_1 = 0. \end{cases}$$

As to the species of τ_2 and $\tau_2\rho$ they may be computed by a direct application of Theorem 1.3.5 only if $p_2 \neq 0$ because only in this case $\widehat{\tau}_2$ fixes branch points. Indeed, the roots of $P(x)$ which are fixed by $\widehat{\tau}_2 : x \mapsto \overline{x} \cdot e^{2\pi i/N}$ are exactly the $2p_2$ roots of the form $\pm\alpha_j e^{\pi i/N}$ for $j = 1, \dots, p_2$, where $\alpha_j^N = \mu_j$ ($\alpha_j > 0$). So

- if $p_2 > 0$ then $\mathrm{sp}(\tau_2) = \mathrm{sp}(\tau_2\rho) = \begin{cases} g+1 & \text{if } p_2 = g+1, \\ -p_2 & \text{if } p_2 < g+1. \end{cases}$

In case $p_2 = 0$ Theorem 1.3.5 only gives that $\{\mathrm{sp}(\tau_2), \mathrm{sp}(\tau_2\rho)\} = \{1, 0\}$ if g is even and $\{2, 0\}$ otherwise. We have to find which real structure realizes each species. We will see that the species of each real form depends on the parity of p_1. Let m be the Möbius transformation $m : x \mapsto x \cdot e^{-\pi i/N}$ and consider the hyperelliptic curve X' which ramifies over $m(B_X)$. Its equation is

$$X' = \{y^2 = \prod_{j=1}^{r}(x^N + w_j)(x^N + \overline{w}_j) \cdot \prod_{j=1}^{p_1}(x^N + \lambda_j)\}.$$

Clearly, $\tau' : (x, y) \mapsto (\overline{x}, \overline{y})$ and $\tau'\rho$ are real structures on X' and so we may apply Theorem 1.3.4 to calculate their species: $\mathrm{sp}(\tau' \circ \rho) = 0$, and $\mathrm{sp}(\tau') = 1$ if g is even and 2 otherwise. But

$$f \circ \tau_2 \circ f^{-1} = \begin{cases} \tau' & \text{if } p_1 \text{ is even,} \\ \tau'\rho & \text{if } p_1 \text{ is odd,} \end{cases}$$

where $f : X \to X' : (x, y) \mapsto (x \cdot e^{-\pi i/N}, \sqrt{(-1)^{p_1}} \cdot y)$ is a lifting of m. So, taking into account that now p_1 even implies g odd, we get:

- if $p_2 = 0$ and p_1 even then $\mathrm{sp}(\tau_2) = 2$ and $\mathrm{sp}(\tau_2\rho) = 0$.
- if $p_2 = 0$ and p_1 odd then $\mathrm{sp}(\tau_2) = 0$ and $\mathrm{sp}(\tau_2\rho) = \begin{cases} 1 & \text{if } g \text{ is even,} \\ 2 & \text{if } g \text{ is odd.} \end{cases}$

In **case *b*)** the composite $\tau_2 \circ \tau_1 : (x, y) \mapsto (xe^{2\pi i/N}, ye^{\pi i/N})$ has order $2N$ and so $\mathrm{Aut}^{\pm} X = \langle \tau_1, \tau_2 \rangle = \mathrm{D}_{2N}$. Hence there are exactly two conjugacy classes of symmetries with representatives τ_1 and τ_2.

The number of branch points of X fixed by complex conjugation $\widehat{\tau}_1$ is $2p_1 + 2 > 0$ if N is even and $p_1 + p_2 + 2$ if N is odd (0 and ∞ are among them). So Theorem 1.3.4 gives

$$\mathrm{sp}(\tau_1) = \begin{cases} g+1 & \text{if } N \text{ is even and } p_1 = g, \\ -(p_1+1) & \text{if } N \text{ is even and } p_1 < g, \\ -(p_1+p_2+2)/2 & \text{if } N \text{ is odd.} \end{cases}$$

Note that in case N odd it cannot happen that all the roots of $P(x)$ are real.

In addition to 0 and ∞, the branch points of X fixed by $\widehat{\tau}_2$ are those roots of $P(x)$ of the form $\alpha = \pm|\alpha|e^{\pi i/N}$, where α^N equals either λ_j or $-\mu_j$. For N even, α^N is always negative and so $\alpha^N = -\mu_j$ for some $j \in \{1, \dots, p_2\}$; hence there are $2p_2 + 2$ branch points fixed by $\widehat{\tau}_2$. For N odd, α^N is negative $= -\mu_j$ for some $j \in \{1, \dots, p_2\}$ if $\alpha = |\alpha|e^{\pi i/N}$, and positive $= \lambda_j$ for some $j \in \{1, \dots, p_1\}$ if $\alpha = -|\alpha|e^{\pi i/N}$; hence there are $p_1 + p_2 + 2$ branch points fixed by $\widehat{\tau}_2$. In both cases it fixes some branch point and so Theorem 1.3.4 gives

$$\mathrm{sp}(\tau_2) = \begin{cases} g+1 & \text{if } N \text{ is even and } p_2 = g, \\ -(p_2+1) & \text{if } N \text{ is even and } p_2 < g, \\ -(p_1+p_2+2)/2 & \text{if } N \text{ is odd.} \end{cases}$$

In **case c)** the composite $(\tau_2\rho)\circ\tau_1 : (x,y) \mapsto (xe^{2\pi i/N}, -y)$ has order $2N$ because N is odd. So $\mathrm{Aut}^{\pm} X = \langle \tau_1, \tau_2\rho\rangle = \mathrm{D}_{2N}$ which has two conjugacy classes of symmetries, with representatives τ_1 and $\tau_2\rho$. But $\mathrm{order}(\tau_2\rho\circ\tau_1\rho) = N$ odd and so $\tau_2\rho \sim \tau_1\rho$. Here the number of real roots of $P(x)$ is p_1+p_2. Since ∞ is also a branch point fixed by $\widehat{\tau}_1$ it follows that

$$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho) = -(p_1+p_2+1)/2.$$

□

3.2. Symmetry types of hyperelliptic algebraic curves of class II

These are the curves whose induced group $\widehat{G}$ of Möbius transformations is $\widehat{G} = \mathrm{Z}_N \oplus \mathrm{Z}_2 = \langle \widehat{a}\rangle \oplus \langle \widehat{\tau}_1\rangle$ where $\widehat{a}$ represents an orientation preserving transformation and $\widehat{\tau}_1$ an orientation reversing one. We distinguish 2 cases according to Λ having signature IIa or IIb.

3.2.1. Class IIa.

— In this case Λ has signature

$$(0;+;[2,\overset{r}{\ldots},2,N];\{(2,\overset{p}{\ldots},2)\}) \quad \text{with} \quad 2r+p = \frac{2g+2}{N} > 2.$$

Recall that for p even, $G = \mathrm{Z}_2 \oplus \mathrm{Z}_2 \oplus \mathrm{Z}_N = \langle\rho\rangle \oplus \langle\tau_1\rangle \oplus \langle a\rangle$, where $a = \theta(x_{r+1})$ preserves orientation. Hence, for N odd G has exactly two conjugacy classes of symmetries, represented by τ_1 and $\tau_1\rho$, whilst for N even it has four, represented by τ_1, $\tau_1\rho$, $\tau_2 := \tau_1 a^{N/2}$ and $\tau_2\rho$. For p odd N must be even and G has presentation $\langle a, \tau_1 \mid a^N, \tau_1^2, (a\tau_1)^2(\tau_1 a)^{-2}, \tau_1 a^2 \tau_1 a^{-2}\rangle$. If $4|N$ then this group has two conjugacy classes of symmetries, represented by τ_1 and $\tau_2 := \tau_1 a^{N/2}$; if $4\nmid N$ then τ_1 represents the unique conjugacy class of symmetries of G.

In order to obtain species, we compute the signatures of the NEC groups Λ_1 and Λ_2 associated to $\langle\tau_1, \tau_1\rho\rangle$ and $\langle\tau_2, \tau_2\rho\rangle$ respectively. Here we do it by using only the geometric method. To that end we must study the action of $\widehat{G}$ on the sphere S^2.

Recall that $\widehat{G} = \Lambda/\ker\widehat{\theta}$ where $\widehat{\theta} : \Lambda \to \widehat{G}$ is the epimorphism given by $\widehat{\theta}(x_i) = 1$ for $1 \leqslant i \leqslant r$, $\widehat{\theta}(x_{r+1}) = \widehat{a}$, $\widehat{\theta}(c_i) = \widehat{\tau}_1$ for $0 \leqslant i \leqslant p$ and $\widehat{\theta}(e) = \widehat{a}$. So, as said above, $\widehat{a}$ represents a conformal transformation on the Riemann sphere S^2 and $\widehat{\tau}_1$ an anticonformal one. Since $\widehat{\tau}_1$ has fixed points (those coming from fixed points of c_0) we may assume that it is the reflection with respect to a plane through the center of the sphere. We claim that in this case $\widehat{a}$ has to be a rotation around an axis orthogonal to the fixed point set $\mathrm{Fix}(\widehat{\tau}_1)$ of $\widehat{\tau}_1$. Indeed, the fixed point of the canonical elliptic generator x_{r+1} projects onto an interior branch point of the canonical projection $\mathcal{H} \to \mathcal{H}/\Lambda$ whilst the fixed points of the canonical reflection c_0 project into the boundary of $\mathcal{H}/\Lambda = S^2/\widehat{G}$. So, the axis of $\widehat{a}$ cannot be contained in $\mathrm{Fix}(\widehat{\tau}_1)$. Moreover, the angle between the axis of $\widehat{a}$ and $\mathrm{Fix}(\widehat{\tau}_1)$ has to be right because $\widehat{a}$ and $\widehat{\tau}_1$ commute.

Therefore a fundamental set for the action of $\widehat{G} = \langle \widehat{a}, \widehat{\tau}_1 \rangle$ on the sphere is a spherical triangle with angles π/N, $\pi/2$, $\pi/2$ at the vertices P_0, P_1, and P_2 respectively, as illustrated in Figure 2. The action of $\widehat{a}$ identifies the arcs P_0P_1 and P_0P_2, where P_0 is a fixed point of $\widehat{a}$. It follows that $S^2/\widehat{G}$ is a topological closed disc whose boundary is contained in the fixed point set of $\widehat{\tau}_1$. Considering now $S^2/\widehat{G}$ as the quotient $(\mathcal{H}/\Gamma_\rho)/(\Lambda/\Gamma_\rho) = \mathcal{H}/\Lambda$ and taking into account the topological interpretation of the signature of Λ (see Proposition 3.1.1) we observe that the canonical projection $\mathcal{H} \to \mathcal{H}/\Lambda$ branches over p points on the boundary of the disc (represented by dots in the figure) and $r+1$ points in its interior (represented by x's).

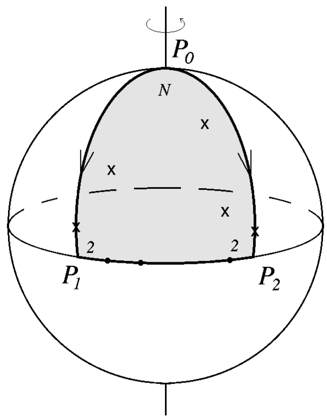


FIGURE 2

Note that the p boundary branch points have branching order 2, r of the interior branch points have branching order 2 and one has branching order N. Obviously the last point is the fixed point P_0 of the orientation preserving transformation $\widehat{a}$.

Let us now compute $\sigma(\Lambda_1)$ and $\sigma(\Lambda_2)$. Since $\widehat{\tau}_1$ is the reflection with respect to a plane, part (1) of Lemma 3.1.2 gives that $\sigma(\Lambda_1)$ is of the form

$$(0; +; [2, \overset{u}{\ldots}, 2]; \{(2, \overset{v}{\ldots}, 2)\}) \ \text{ with } \ 2u + v = 2g + 2,$$

where v equals the number of points e_j lying on $\mathrm{Fix}(\widehat{\tau}_1)$. Here $e_1, \ldots, e_{2g+2}$ are the branch points of π_{Γ_ρ}. Let us compute v. The points e_j are permuted by the elements of $\widehat{G}$ but none of those lying on $\mathrm{Fix}(\widehat{\tau}_1)$ is fixed by an orientation preserving element of $\widehat{G}$. Indeed, the unique points of S^2 fixed by such an element are the fixed points P_0 and P_∞ of the rotation $\widehat{a}$. So the p boundary branch points of $\pi_\Lambda = \pi_{\widehat{G}} \circ \pi_{\Gamma_\rho}$ are exactly the images under $\pi_{\widehat{G}}$ of the v branch points of π_{Γ_ρ} lying on $\mathrm{Fix}(\widehat{\tau}_1)$. From the geometric action of $\widehat{G}$ we deduce that $p = v/N$. Therefore

$$\sigma(\Lambda_1) = (0; +; [2, \overset{g+1-pN/2}{\ldots\ldots}, 2]; \{(2, \overset{pN}{\ldots}, 2)\}).$$

For N even, $\widehat{\tau}_2 := \widehat{\tau}_1 \widehat{a}^{N/2}$ is also an antianalytic involution. In fact, $\widehat{\tau}_2$ is the antipodal map since $\widehat{a}^{N/2}$ is a rotation of order 2 whose axis is orthogonal to $\mathrm{Fix}(\widehat{\tau}_1)$. Therefore, part (2) of Lemma 3.1.2 gives

$$\sigma(\Lambda_2) = (1; -; [2, \overset{g+1}{\ldots}, 2]; \{-\}).$$

In view of signatures $\sigma(\Lambda_1)$ and $\sigma(\Lambda_2)$ it follows from Theorem 3.0.1 that the species of the candidates to be representatives of the real forms of a curve of class $(\mathrm{IIa})_N$ are the following.

IIa	$p = 0$	$0 < p < (2g+2)/N$	$p = (2g+2)/N$
$\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\}$	$\{1, 0\}$ if g even $\{2, 0\}$ if g odd	$\{-pN/2, -pN/2\}$	$\{g+1, g+1\}$
$\mathrm{sp}(\tau_2) = \mathrm{sp}(\tau_2\rho) = 0$			

We can now give the list of symmetry types of hyperelliptic genus g curves of class $(\mathrm{IIa})_N$. Recall that such a curve X exists if and only if N is a divisor of $2g+2$ and $(2g+2)/N \neq 1, 2$. As said at the beginning, representatives of the real forms of X are $\tau_1, \tau_1\rho$ if p is even and N is odd; $\tau_1, \tau_1\rho, \tau_2, \tau_2\rho$ if p and N are even; τ_1, τ_2 if p is odd and 4 divides N, and τ_1 if p is odd and 4 does not divide N. Using the above table and the fact that *p is even if and only if N is a divisor of $g+1$* (which is immediate from equality $p = (2g+2)/N - 2r$), we get the following.

THEOREM 3.2.1. — *The symmetry type of a genus g hyperelliptic algebraic curve of class $(\mathrm{IIa})_N$ is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integer p appearing in the list is positive and has the same parity as $(2g+2)/N$.*

(1) *If $N|(g+1)$:*
- (1.1) *If N is odd:*
  - $\{1, 0\}$, *only if g is even;*
  - $\{2, 0\}$, *only if g is odd;*
  - $\{-pN/2, -pN/2\}$;
  - $\{g+1, g+1\}$.
- (1.2) *If N is even:*
  - $\{2, 0, 0, 0\}$;
  - $\{-pN/2, -pN/2, 0, 0\}$;
  - $\{g+1, g+1, 0, 0\}$.

(2) *If* $N \not| (g+1)$ *(in such a case* N *is even):*
  (2.1) *If* $4|N$*:*
    - $\{g+1, 0\}$;
    - $\{-pN/2, 0\}$.
  (2.2) *If* $4 \not| N$*:*
    - $\{g+1\}$;
    - $\{-pN/2\}$.

3.2.2. Class IIb. — In this case Λ has signature

$$(0; +; [2, \overset{r}{\ldots}, 2, 2N]; \{(2, \overset{p}{\ldots}, 2)\}) \ \text{ with } \ 2r + p = \frac{2g}{N} > 2.$$

For p even, $G = \mathrm{Z}_2 \oplus \mathrm{Z}_{2N} = \langle \tau_1 \rangle \oplus \langle a \rangle$, which has two conjugacy classes of symmetries, with representatives τ_1 and $\tau_1\rho$. For p odd, $G = \langle a, \tau_1 \mid a^{2N}, \tau_1^2, \tau_1 a \tau_1 a^{N-1} \rangle = \mathrm{Z}_{2N} \rtimes \mathrm{Z}_2$, which has one conjugacy class of symmetries, represented by τ_1, if $4|N$ and two, represented by τ_1 and $\tau_2 := \tau_1 a^{N/2}$, if $4 \not| N$.

In order to compute their species we proceed as in the preceding case, namely, we analyze geometrically the above signature and the action of $\widehat{G}$ on S^2. The unique diference is that now the projection $\mathcal{H} \to \mathcal{H}/\Lambda$ has an interior branch point of branching order 2N. With the same notations, this forces the two fixed points P_0 and P_∞ of the rotation $\widehat{a}$ to be two of the branch points $e_1, \ldots, e_{2g+2}$ of the projection $\mathcal{H} \to \mathcal{H}/\Gamma_\rho = S^2$.

This is the geometric distinction between cases IIa and IIb in terms of the distribution of the branch points e_j which will be reflected in the equations of curves of class II.

Although now P_0 and P_∞ are two of the e_j they do not lie on $\mathrm{Fix}(\widehat{\tau}_1)$ and so the signature of $\sigma(\Lambda_1)$ does not change:

$$\sigma(\Lambda_1) = (0; +; [2, \overset{g+1-pN/2}{\ldots\ldots\ldots}, 2]; \{(2, \overset{pN}{\ldots}, 2)\}).$$

Obviously, the same happens to the signature of the NEC group Λ_2 associated to the antipodal map $\widehat{\tau}_2$:

$$\sigma(\Lambda_2) = (1; -; [2, \overset{g+1}{\ldots}, 2]; \{-\}).$$

Therefore the species of the candidates to be representatives of the real forms of a genus g curve of class $(\mathrm{IIb})_N$ are those given in the next table. Note that in this case p cannot attain the value $(2g+2)/N$ since here $p = 2g/N - 2r$.

IIb	$p = 0$	$0 < p \leqslant 2g/N$
$\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\}$	$\{1, 0\}$ if g even $\{2, 0\}$ if g odd	$\{-pN/2, -pN/2\}$
$\mathrm{sp}(\tau_2) = 0$		

The conjugacy classes of symmetries of G depend on the parity of p (see above). Here N is a divisor of $2g$ with $2g/N \neq 1$ or 2. Since $p = 2g/N - 2r$ it follows that p *is even if and only if* N *is a divisor of* g. It is then straighforward to prove the next theorem.

Theorem 3.2.2. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{IIb})_N$ *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integer p appearing in the list satisfies*

$$0 < p \leqslant \frac{2g}{N}.$$

(1) *If* $N|g$:
  - $\{1, 0\}$, *only if g is even;*
  - $\{2, 0\}$, *only if g is odd;*
  - $\{-pN/2, -pN/2\}$ *where p is even.*

(2) *If* $N \nmid g$ *(in such a case N is even):*
  (2.1) *If* $4|N$*:*
    - $\{-pN/2\}$ *where p is odd.*

  (2.2) *If* $4 \nmid N$*:*
    - $\{-pN/2, 0\}$ *where p is odd.*

3.2.3. Equations of curves of class II and their real forms. — Using the extended complex plane $\widehat{\mathbb{C}}$ as a model of the Riemann sphere we find explicit polynomial equations of hyperelliptic curves of class II. For that we have to describe in $\widehat{\mathbb{C}}$ the branch point set of such curves. This set is preserved by the elements of the Abelian group $\widehat{G} = \mathrm{Z}_N \oplus \mathrm{Z}_2$ generated by a rotation $\widehat{a}$ of order N and a reflection $\widehat{\tau}_1$ with respect to a plane orthogonal to the axis of $\widehat{a}$. Viewing these transformations in $\widehat{\mathbb{C}}$ by means of a suitable stereographic projection, we may choose the following as generators of $\widehat{G}$:

$$\widehat{a} : x \longmapsto xe^{2\pi i/N} \quad \text{and} \quad \widehat{\tau}_1 : x \longmapsto \frac{1}{\overline{x}}.$$

It follows easily that the orbit under the action of $\widehat{G}$ of a point $\alpha \in \widehat{\mathbb{C}}$ is the following

$$\mathrm{orbit}(\alpha, \widehat{G}) = \begin{cases} \text{roots of } (x^N - \alpha^N)(x^N - 1/\overline{\alpha}^N) & \text{if } \alpha \neq 0, \infty \text{ is not fixed by } \widehat{\tau}_1; \\ \text{roots of } (x^N - \alpha^N) & \text{if } \alpha \neq 0, \infty \text{ is fixed by } \widehat{\tau}_1; \\ \{0, \infty\} & \text{if } \alpha = 0 \text{ or } \infty. \end{cases}$$

The fixed point set of $\widehat{\tau}_1$ is the unit circle $\{|\alpha| = 1\}$ and $\widehat{\tau}_1$ interchanges its interior $\{|\alpha| < 1\}$ with its exterior $\{|\alpha| > 1\}$. Note that the rotation $\widehat{a}$ preserves each of these sets. This justifies the choice of the parameters ω_i and λ_i appearing in the following theorem. In it we assume that N is an integer such that the corresponding value of g is an integer $\geqslant 2$.

Theorem 3.2.3. — *For almost every choice of pairwise distinct complex numbers*
- $\{\omega_1, \ldots, \omega_r\} \subset \{0 < |z| < 1\}$,
- $\{\lambda_1, \ldots, \lambda_p\} \subset \{|z| = 1\}$,

with

$$\prod_{j=1}^{r} \frac{\omega_j}{\overline{\omega}_j} \cdot \prod_{j=1}^{p} (-\lambda_j) = 1,$$

the equation

$$y^2 = x^{\varepsilon} \cdot \prod_{j=1}^{r} (x^N - \omega_j)(x^N - 1/\overline{\omega}_j) \cdot \prod_{j=1}^{p} (x^N - \lambda_j)$$

defines a hyperelliptic complex curve X of class

a) $(\mathrm{IIa})_N$ *and genus* $g = N(2r+p)/2 - 1$ *if* $\varepsilon = 0$ *and* $2r+p > 2$;

b) $(\mathrm{IIb})_N$ *and genus* $g = N(2r+p)/2$ *if* $\varepsilon = 1$ *and* $2r+p > 2$.

Conversely, each hyperelliptic genus g curve of class II *is (isomorphic to another) of the above form for some N, r, p and ε satisfying the above conditions.*

Representatives of all the real forms of X are the following.

Class $(\mathrm{IIa})_N$		
$N \mid (g+1)$	N *odd*	$\tau_1,\ \tau_1\rho$
	N *even*	$\tau_1,\ \tau_1\rho,\ \tau_2,\ \tau_2\rho$
$N \nmid (g+1)$	$4 \mid N$	$\tau_1,\ \tau_2$
	$4 \nmid N$	τ_1

Class $(\mathrm{IIb})_N$		
$N \mid g$		$\tau_1,\ \tau_1\rho$
$N \nmid g$	$4 \mid N$	τ_1
	$4 \nmid N$	$\tau_1,\ \tau_2$

where

$$\tau_1 : (x,y) \longmapsto \left(\frac{1}{\overline{x}}, \frac{\overline{y}}{\overline{x}^{g+1}}\right) \quad \textit{and} \quad \tau_2 : (x,y) \longmapsto \left(\frac{-1}{\overline{x}}, \frac{\overline{y} i^{\varepsilon}}{\overline{x}^{g+1}}\right).$$

In all cases, the species of each real form is given.

Proof. — The above equation defines a hyperelliptic curve X of the prescribed genus g since it is of the form $y^2 = P(x)$ where $P(x)$ is a polynomial with simple roots, whose degree $(2r+p)N + \varepsilon$ equals $2g+2$ in case *a*) and $2g+1$ in case *b*).

We begin by showing that X is of class II. First, its branch points are permuted by the elements of the Abelian group $\langle \widehat{a} \rangle \oplus \langle \widehat{\tau}_1 \rangle$ where $\widehat{a}$ is the rotation $x \mapsto xe^{2\pi i/N}$ and $\widehat{\tau}_1$ is the reflection $x \mapsto 1/\overline{x}$. For almost every choice of the parameters $\omega_1, \ldots, \omega_r$, $\lambda_1, \ldots, \lambda_p$ these are the unique Möbius transformations permuting the branch points. So $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ coincides with $\langle \widehat{a} \rangle \oplus \langle \widehat{\tau}_1 \rangle$, which is a class II group. We observe that if $2r+p = 1$ or 2 then X is not of class II. Indeed, it is easy to check that for such values, $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ contains the Möbius transformation $x \mapsto 1/x$, which is not in $\langle \widehat{a} \rangle \oplus \langle \widehat{\tau}_1 \rangle$.

We may distinguish curves of class IIa from curves of class IIb by the distribution of their branch points; namely, none of the two fixed points of the rotation $\widehat{a}$ is a branch point of X if and only if X is of class IIa. In our case, the two fixed points of $\widehat{a}$ are 0 and ∞. Therefore in case *a*) the curve X is of class $(\mathrm{IIa})_N$ whilst in case *b*) X is of class $(\mathrm{IIb})_N$.

Conversely, if X is a curve of class II then we may suppose that $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ is generated by the transformations $\widehat{a}$ and $\widehat{\tau}_1$ described above. In particular, the finite branch points of X constitute the roots of a polynomial $P(x)$ as in the statement of the theorem. The condition $\prod \omega_j/\overline{\omega}_j \prod \lambda_j = (-1)^p$ can be achieved by rotating the branch points an appropriate angle. Indeed, the product in the left hand side of the equality is an unimodular number whose argument is nothing but the sum of the arguments of the N-th powers of the finite branch points. Therefore, by means of a suitable rotation of the form $x \mapsto xe^{i\theta}$ we may assume that the sum of such arguments is either 0 or π according to the parity of p. Note that the presentation of $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ is still the same because such a rotation commutes with both $\widehat{a}$ and $\widehat{\tau}_1$.

The statement concerning the representatives of the real forms of X according to the values of p and N was already shown in the previous chapter. In fact, we saw that liftings of $\widehat{\tau}_1$ and $\widehat{\tau}_2 := \widehat{\tau}_1 \widehat{a}^{N/2}$ (if N is even) may be chosen as representatives of real forms. Using the condition $\prod \omega_j/\overline{\omega}_j \prod \lambda_i = (-1)^p$ it is easy to check that in both cases a) and b) the formula of a lifting τ_1 of $\widehat{\tau}_1$ is that appearing in the statement of the theorem. For N even, the automorphism τ_2 is a lifting of $\widehat{\tau}_2$, with $\varepsilon = 0$ in case a) and $\varepsilon = 1$ in case b). We now calculate the species of each representative.

Since $\widehat{\tau}_2$ is the antipodal map $x \mapsto -1/\overline{x}$ the species of its liftings (if symmetries) are always zero (Theorem 1.3.5):

$$\mathrm{sp}(\tau_2) = \mathrm{sp}(\tau_2\rho) = 0.$$

Since $\widehat{\tau}_1$ is a reflection, the species of its liftings depend on the number of branch points fixed by $\widehat{\tau}_1$. Since there are pN such points, Theorem 1.3.5 gives

$$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho) = \begin{cases} -pN/2 & \text{if } 0 < pN/2 < g+1, \\ \ \ g+1 & \text{if } pN/2 = g+1, \text{ (only in case } a). \end{cases}$$

The case $p = 0$ must be treated separately since Theorem 1.3.5 only gives that $\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\} = \{1, 0\}$ if g is even and $\{2, 0\}$ if g is odd. We now proceed to find which symmetry fixes points of the curve $X = \{y^2 = P_X\}$.

A point (α, β) is fixed by τ_1 if and only if $|\alpha| = 1$ and $\beta/\overline{\beta} = \alpha^{g+1}$. Using the equality $\prod \omega_j/\overline{\omega}_j = 1$ it is easy to check that for any unimodular number α the following equality holds in both cases a) and b):

$$\frac{P_X(\alpha)}{\overline{P_X(\alpha)}} = \alpha^{2g+2}.$$

In particular, $P_X(1)$ is a real number. So if $P_X(1)$ is positive then $\sqrt{P_X(1)}/\overline{\sqrt{P_X(1)}} = 1$ which gives that $(1, \sqrt{P_X(1)})$ is a point fixed by τ_1. If $P_X(1)$ is negative then $(1, \sqrt{P_X(1)})$ is fixed by $\tau_1\rho$, where $\sqrt{P_X(1)}$ is any of the (non-real) square roots of

$P_X(1)$. Summarizing, for $p = 0$,

$$\mathrm{sp}(\tau_1) = \begin{cases} 0 & \text{if } P_X(1) < 0, \\ 1 & \text{if } P_X(1) > 0 \text{ and } g \text{ is even}, \\ 2 & \text{if } P_X(1) > 0 \text{ and } g \text{ is odd}. \end{cases}$$

$$\mathrm{sp}(\tau_1\rho) = \begin{cases} 0 & \text{if } P_X(1) > 0, \\ 1 & \text{if } P_X(1) < 0 \text{ and } g \text{ is even}, \\ 2 & \text{if } P_X(1) < 0 \text{ and } g \text{ is odd}. \end{cases}$$

□

3.3. Symmetry types of hyperelliptic algebraic curves of class III

These are the curves whose induced group $\widehat{G}$ of Möbius transformations is $\widehat{G} = \langle \widehat{\tau}_1, \widehat{a} \,|\, \widehat{\tau}_1^2, \widehat{a}^2, (\widehat{\tau}_1\widehat{a})^N \rangle = \mathrm{D}_N$, where $\widehat{a}$ represents an orientation preserving transformation and $\widehat{\tau}_1$ an orientation reversing one. In fact, the epimorphism $\widehat{\theta} : \Lambda \to \widehat{G}$ maps the canonical elliptic generator x_{r+1} onto $\widehat{a}$ and the canonical reflections $c_0, \dots, c_p$ onto $\widehat{\tau}_1$. It also maps c_{p+1} onto $\widehat{a}\widehat{\tau}_1\widehat{a}$. The signature of Λ is one of the following

IIIa: $(0; +; [2^r, 2]; \{(2^p, N/2)\})$, with $2r + p = (2g+2)/N$,
IIIb: $(0; +; [2^r, 2]; \{(2^p, N)\})$, with $2r + p = 2g/N$,
IIIc: $(0; +; [2^r, 4]; \{(2^p, N/2)\})$, with $2r + p = (2g+2)/N - 1 > 0$ or
IIId: $(0; +; [2^r, 4]; \{(2^p, N)\})$, with $2r + p = 2g/N - 1 > 0$.

Recall that along this class, N is even $\geqslant 4$.

We begin by interpreting geometrically the action of $\widehat{G}$ on the sphere S^2. We may assume that $\widehat{\tau}_1$ is the reflection with respect to a plane through the origin because it has fixed points. Since the composite $\widehat{a} \circ \widehat{\tau}_1$ has order $N > 2$ the axis of the rotation $\widehat{a}$ (of order 2) neither is contained in nor is orthogonal to the fixed point set $\mathrm{Fix}(\widehat{\tau}_1)$ of $\widehat{\tau}_1$. In fact, the angle between the axis and $\mathrm{Fix}(\widehat{\tau}_1)$ is π/N.

Therefore a fundamental set for the action of $\widehat{G}$ is the spherical triangle with vertices P_0, P_1 and P_2 as illustrated in Figure 3. The action of $\widehat{a}$ identifies the half side P_1A with the half side AP_2, where A is a fixed point of $\widehat{a}$. It follows that the quotient $S^2/\widehat{G}$ is a topological closed disc whose boundary consists of the sides P_0P_1 and P_2P_0 (with vertices P_1 and P_2 identified). Now the signature of Λ shows that the canonical projection $\mathcal{H} \to \mathcal{H}/\Lambda = S^2/\widehat{G}$ has $p+1$ boundary branch points (represented by dots in the figure) and $r+1$ interior branch points (represented by x's).

Note that A and P_0 are branch points of $\mathcal{H} \to \mathcal{H}/\Lambda$ since they are projections of points fixed by non-trivial orientation preserving elements of Λ. Indeed, A corresponds to the fixed point of the elliptic generator x_{r+1} and so it is an interior branch point, whilst P_0 correspond to the fixed point of the composite c_pc_{p+1} (P_0 is fixed by the rotation $(\widehat{\tau}_1\widehat{a})^2 = \widehat{\theta}(c_pc_{p+1})$) and so it is a boundary branch point.

We now proceed to calculate the species of the real forms of X.

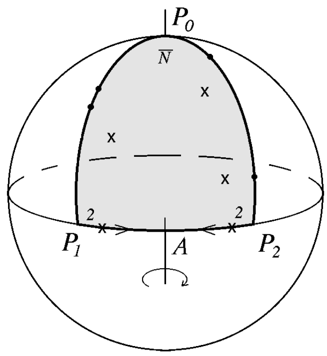


FIGURE 3

The representatives of the real forms chosen in the preceding chapter are always liftings of $\widehat{\tau}_1$ and $\widehat{\tau}_2 := (\widehat{\tau}_1\widehat{a})^{N/2}$ (if $N/2$ is odd). Thus our goal is to compute the signatures of the NEC groups Λ_1 and Λ_2 associated to these antianalytic involutions. In fact, it is enough to compute $\sigma(\Lambda_1)$ since $\widehat{\tau}_2$ is the antipodal map, as is easy to check. So the species of $\tau_2 = (\tau_1 a)^{N/2}$ and $\tau_2\rho$ (if symmetries) are

Class III
$\mathrm{sp}(\tau_2) = \mathrm{sp}(\tau_2\rho) = 0$

As to $\widehat{\tau}_1$, it is the reflection with respect to a plane and so part (1) of Lemma 3.1.2 gives that the signature of Λ_1 is of the form

$$\sigma(\Lambda_1) = (0; +; [2, \overset{u}{\ldots}, 2]; \{(2, \overset{v}{\ldots}, 2)\}) \text{ with } 2u + v = 2g + 2,$$

where v equals the number of points e_i lying on $\mathrm{Fix}(\widehat{\tau}_1)$ ($e_1, \ldots, e_{2g+2}$ are the branch points of the canonical projection $\pi_{\Gamma_\rho} : \mathcal{H} \to \mathcal{H}/\Gamma_\rho = S^2$). Let us calculate v.

The image of $\mathrm{Fix}(\widehat{\tau}_1)$ under the canonical projection $\pi_{\widehat{G}} : S^2 \to \mathcal{H}/\Lambda$ coincides with the boundary of $\mathcal{H}/\Lambda$. Indeed, the arc P_2P_0 is the image of the arc $P_1P_\infty \subset \mathrm{Fix}(\widehat{\tau}_1)$ under $\widehat{a}$, where P_∞ is the antipodal image of P_0. Therefore the images under $\pi_{\widehat{G}}$ of the v branch points e_i lying on $\mathrm{Fix}(\widehat{\tau}_1)$ are among the $p+1$ boundary branch points of π_Λ. We now investigate whether these two sets coincide. The element $(\widehat{\tau}_1\widehat{a})^2$ is a non-trivial orientation preserving element of order $N/2$ of $\widehat{G}$. So the image under $\pi_{\widehat{G}}$ of its two fixed points P_0 and P_∞, say z_0, is a branch point of $\pi_{\widehat{G}}$ with branching order $N/2$. Thus, the branching order of z_0 as branch point of $\pi_\Lambda = \pi_{\widehat{G}} \circ \pi_{\Gamma_\rho}$ is $N/2$ if P_0 and P_∞ are not two of the e_i, and N otherwise. The first case corresponds to Λ having signature IIIa or IIIc and the second to signature IIIb or IIId. Note that P_0 and P_∞ are the unique points on $\mathrm{Fix}(\widehat{\tau}_1)$ fixed by some non-trivial rotation of $\widehat{G}$. In other words, z_0 is the unique boundary branch point of π_Λ which is also a branch point of $\pi_{\widehat{G}}$. The remainder boundary branch points of π_Λ are the images of those e_i different from P_0 and P_∞ lying on $\mathrm{Fix}(\widehat{\tau}_1)$.

From the geometrical action of $\widehat{G}$ on S^2 we see that the restriction of $\pi_{\widehat{G}}$ to $\mathrm{Fix}(\widehat{\tau}_1)$ is two-to-one. So, if P_0 and P_∞ are two of the e_i then the number of e_i lying on $\mathrm{Fix}(\widehat{\tau}_1)$ is twice the number v of boundary branch points of π_Λ, that is, $v = 2p + 2$. If P_0 and P_∞ are not among the e_i then $v = 2p$. Thus,

$$\sigma(\Lambda_1) = \begin{cases} (0; +; [2, \overset{g+1-p}{\ldots}, 2]; \{(2, \overset{2p}{\ldots}, 2)\}) & \text{in cases IIIa and IIIc,} \\ (0; +; [2, \overset{g-p}{\ldots}, 2]; \{(2, \overset{2p+2}{\ldots}, 2)\}) & \text{in cases IIIb and IIId.} \end{cases}$$

The conditions between p, g and N guarantee the existence of proper periods in $\sigma(\Lambda_1)$, that is, p cannot attain the value neither $g + 1$ in cases IIIa and IIIc nor g in cases IIIb and IIId. So none of the symmetries associated to Λ_1 has species $g + 1$. In fact, the species are those given in the following table.

IIIa	$p = 0$	$0 < p \leqslant (2g + 2)/N$
$\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\}$	$\{1, 0\}$ if g even $\{2, 0\}$ if g odd	$\{-p, -p\}$

IIIb	$p \leqslant 2g/N$
$\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\}$	$\{-(p + 1), -(p + 1)\}$

IIIc	$p = 0$	$0 < p \leqslant (2g + 2)/N - 1$
$\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\}$	$\{1, 0\}$ if g even $\{2, 0\}$ if g odd	$\{-p, -p\}$

IIId	$p \leqslant 2g/N - 1$
$\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\}$	$\{-(p + 1), -(p + 1)\}$

Not all the values of p in the above ranges are attained. For example, in class IIIa g even implies p odd, and so $\{1, 0\}$ does not occur.

As seen above, curves of classes IIIa and IIIc have the fixed points P_0 and P_∞ of the rotation $(\widehat{\tau}_1\widehat{a})^2$ as branch points, whilst curves of classes IIIb and IIId do not. It is convenient to give also here a geometric distinction between classes IIIa and IIIc and between classes IIIb and IIId. With the same notations as above, the projection $\pi_{\widehat{G}}$ has exactly one more branch point other than z_0 (z_0 is the projection of P_0 and P_∞), namely, the projection z_1 of the point A fixed by the rotation $\widehat{a}$. Note that z_1 is an interior point of $\mathcal{H}/\Lambda$. Since $\widehat{a}$ has order 2 we see that the branching order of z_1 with respect to $\pi_\Lambda = \pi_{\widehat{G}} \circ \pi_{\Gamma_\rho}$ is 2 if A is not among the branch points $e_1, \ldots, e_{2g+2}$ of π_{Γ_ρ}, and 4 otherwise. The first case corresponds to Λ having signature IIIa or IIIb and the second to signature IIIc or IIId. We will take into account these distinct distributions of branch points when giving explicit polynomial equations of curves of class III.

3.3.1. Class IIIa. — Recall that in this case the number of real forms of X depends on the parity of p and $N/2$. Indeed, if $pN/2$ is odd then X has a unique real form, represented by τ_1; if p is even and $N/2$ is odd then it has four, represented by τ_1, $\tau_1\rho$, τ_2 and $\tau_2\rho$; finally, if $N/2$ is even then τ_1 and $\tau_1\rho$ are representatives of the unique two real forms of X. Now, from equality $2r + p = (2g + 2)/N$, it follows that $pN/2$ *is even if and only if g is odd.* So we may list the symmetry types of curves of class IIIa in terms of the parity of $N/2$ and g.

THEOREM 3.3.1. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\text{IIIa})_N$ *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integer p appearing in the list satisfies*

$$0 < p \leqslant \frac{2g+2}{N} \quad \textit{and} \quad p \equiv \frac{2g+2}{N} \pmod 2.$$

(1) *If g is even:*
  - $\{-p\}$;

(2) *If g is odd:*
  (2.1) *If $N/2$ is even:*
    - $\{2, 0\}$, *only if* $(2g+2)/N$ *is even,*
    - $\{-p, -p\}$;

  (2.2) *If $N/2$ is odd:*
    - $\{2, 0, 0, 0\}$,
    - $\{-p, -p, 0, 0\}$.

3.3.2. Class IIIb. — The number of real forms of X depends on the parity of $pN/2$. Indeed, if $pN/2$ is odd then X has four real forms, represented by τ_1, $\tau_1\rho$, τ_2 and $\tau_2\rho$; if $pN/2$ is even then τ_1 is a representative of the unique real form of X. But $2r + p = 2g/N$ and so $pN/2$ *is even if and only if g is even.* Thus we may list the symmetry types of curves of class IIIb in terms of the parity of g.

THEOREM 3.3.2. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\text{IIIb})_N$ *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integer p appearing in the list satisfies*

$$p \leqslant \frac{2g}{N} \quad \textit{and} \quad p \equiv \frac{2g}{N} \pmod 2.$$

(1) *If g is even:*
  - $\{-(p+1)\}$.

(2) *If g is odd:*
  - $\{-(p+1), -(p+1), 0, 0\}$.

3.3.3. Class IIIc. — In this case X has always two real forms. As representatives we may choose the following: τ_1 and τ_2 if $pN/2$ is odd; τ_1 and $\tau_1\rho$ if $N/2$ is even; τ_1 and $\tau_1\rho$ if $(p+1)N/2$ is odd. Now $(p+1)N/2$ *is odd if and only if* g *is even* because here $2r+p=(2g+2)/N-1>0$. Note that the inequality gives that *the genus of a curve of class* IIIc *cannot be of the form* $g =$ *prime number* -1. We distinguish cases according to the parity of g and $N/2$.

Theorem 3.3.3. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{IIIc})_N$ *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integer p appearing in the list satisfies*

$$0 < p \leqslant \frac{2g+2}{N} - 1 \quad \textit{and} \quad p \equiv \frac{2g+2}{N} - 1 \pmod 2.$$

(1) *If g is even:*
- $\{1,0\}$,
- $\{-p,-p\}$.

(2) *If g is odd:*
  (2.1) *If $N/2$ is even:*
  - $\{2,0\}$, *only if* $(2g+2)/N$ *is odd,*
  - $\{-p,-p\}$.

  (2.2) *If $N/2$ is odd:*
  - $\{-p,0\}$.

3.3.4. Class IIId. — Representatives of the real forms of X are the following: τ_1 if $N/2$ is even; τ_1 and $\tau_1\rho$ if $pN/2$ is odd; τ_1 and τ_2 if $(p+1)N/2$ is odd. Now $(p+1)N/2$ *is odd if and only if* g *is odd* because here $2r+p=2g/N-1>0$. Note that the inequality gives that *the genus of a curve of class* IIId *cannot be a prime number.* We distinguish cases according to the parity of g and $N/2$.

Theorem 3.3.4. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{IIId})_N$ *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integer p appearing in the list satisfies*

$$p \leqslant \frac{2g}{N} - 1 \quad \textit{and} \quad p \equiv \frac{2g}{N} - 1 \pmod 2.$$

(1) *If g is odd:*
- $\{-(p+1),0\}$.

(2) *If g is even:*
  (2.1) *If $N/2$ is even:*
  - $\{-(p+1)\}$.

  (2.2) *If $N/2$ is odd:*
  - $\{-(p+1),-(p+1)\}$.

3.3.5. Equations of curves of class III and their real forms. — The branch point set of such curves is preserved by the elements of the dihedral group of order $2N$ generated by a reflection $\widehat{\tau}_1$ with respect to a plane through the origin and a rotation $\widehat{a}$ of order 2 whose axis makes angle π/N with the plane of $\widehat{\tau}_1$. Viewing these transformations in $\widehat{\mathbb{C}}$ we may choose $\widehat{\tau}_1$ as complex conjugation and $\widehat{a}$ as the rotation of order 2 fixing $\pm e^{\pi i/N}$, that is,

$$\widehat{\tau}_1 : x \longmapsto \overline{x} \quad \text{and} \quad \widehat{a} : x \longmapsto \frac{e^{2\pi i/N}}{x}.$$

Thus the two fixed points of the rotation $(\widehat{\tau}_1\widehat{a})^2 : x \mapsto x\,e^{-4\pi i/N}$ are 0 and ∞. It is then easy to check that the orbit under the action of $\widehat{G}$ of a point $\alpha \in \widehat{\mathbb{C}}$ is the following:

$$\text{orbit}(\alpha, \widehat{G}) = \begin{cases} \text{roots of } (x^{N/2}-\alpha^{N/2})(x^{N/2}-\overline{\alpha}^{N/2})(x^{N/2}+1/\alpha^{N/2})(x^{N/2}+1/\overline{\alpha}^{N/2}) \\ \qquad\text{if } \alpha \text{ is fixed by no element of } \widehat{G}; \\ \text{roots of } (x^{N/2}-\alpha^{N/2})(x^{N/2}+1/\alpha^{N/2}) \\ \qquad\text{if } \alpha \neq 0, \infty \text{ is fixed by an antianalytic involution of } \widehat{G}; \\ \text{roots of } \ x^N+1 \ \text{ if } \alpha \text{ is fixed by a rotation} \\ \qquad\text{of the form } \widehat{a}(\widehat{\tau}_1\widehat{a})^{2j} \ (j=1,\dots,N/2); \\ \{0,\infty\} \ \text{ if } \alpha = 0 \text{ or } \infty. \end{cases}$$

Note that if α is fixed by no element of $\widehat{G}$ then $\alpha^{N/2}$ is a complex non-real number and different to $\pm i$. Therefore $w := \alpha^{N/2} - 1/\alpha^{N/2}$ is also a complex non-real number different to $\pm 2i$. If $\alpha \neq 0, \infty$ is fixed by an antianalytic involution of $\widehat{G}$ then $\alpha^{N/2}$ is a real number. Therefore so is $\lambda := \alpha^{N/2} - 1/\alpha^{N/2}$.

For each $z \in \mathbb{C}$ let P_z denote the following polynomial

$$P_z(T) = T^2 - zT - 1.$$

Products of polynomials of this form serve to describe the above orbits since

- if z is not real and $z \neq \pm 2i$ then the product $P_z \cdot P_{\overline{z}}$ has four different roots which are of the form β, $\overline{\beta}$, $-1/\beta$ and $-1/\overline{\beta}$.
- if z is real then P_z has two different real roots which are of the form β and $-1/\beta$;

This justifies the choice of parameters λ_i and w_i appearing in the following theorem. Recall that $\mathbb{C}^+$ stands for the open upper semiplane $\{x + iy : y > 0\}$.

THEOREM 3.3.5. — *For $z \in \mathbb{C}$ let P_z denote the polynomial $P_z(T) := T^2 - zT - 1$. For almost every choice of*

- $\{w_1, \dots, w_r\} \subset \mathbb{C}^+ - \{2i\}$ *with* $w_i \neq w_j$ *if* $i \neq j$ *and*
- $\lambda_1 < \cdots < \lambda_p \subset \mathbb{R}$,

the equation

$$y^2 = x^{\varepsilon_1} \cdot (x^N + 1)^{\varepsilon_2} \prod_{j=1}^{r} P_{w_j}(x^{N/2}) \cdot P_{\overline{w}_j}(x^{N/2}) \cdot \prod_{j=1}^{p} P_{\lambda_j}(x^{N/2}),$$

where $\varepsilon_i \in \{0, 1\}$ *and* N *is even* $\geqslant 4$*, defines a hyperelliptic complex curve* X *of class*

a) $(\mathrm{IIIa})_N$ *and genus* $g = (2r + p)N/2 - 1$ *if* $\varepsilon_1 = \varepsilon_2 = 0$;
b) $(\mathrm{IIIb})_N$ *and genus* $g = (2r + p)N/2$ *if* $\varepsilon_1 = 1$ *and* $\varepsilon_2 = 0$;
c) $(\mathrm{IIIc})_N$ *and genus* $g = (2r + p + 1)N/2 - 1$ *if* $\varepsilon_1 = 0$, $\varepsilon_2 = 1$ *and* $2r + p > 0$;
d) $(\mathrm{IIId})_N$ *and genus* $g = (2r + p + 1)N/2$ *if* $\varepsilon_1 = \varepsilon_2 = 1$ *and* $2r + p > 0$.

Conversely, each genus g *hyperelliptic curve of class* III *is (isomorphic to another) of the above form for some* N*,* r*,* p*,* ε_1 *and* ε_2 *satisfying the above conditions.*

Write

$$\tau_1 : (x, y) \longmapsto (\overline{x}, \overline{y}) \quad \textit{and} \quad \tau_2 : (x, y) \longmapsto \left(\frac{-1}{\overline{x}}, \frac{\overline{y}\, i^{\varepsilon_2}}{\overline{x}^{g+1}} \right).$$

Then representatives of all real forms of X *are given in the following tables.*

Class $(\mathrm{IIIa})_N$		
g *even*	g *odd*	
	$N/2$ *even*	$N/2$ *odd*
τ_1	τ_1, $\tau_1\rho$	τ_1, $\tau_1\rho$, τ_2, $\tau_2\rho$

Class $(\mathrm{IIIb})_N$	
g *even*	g *odd*
τ_1	τ_1, $\tau_1\rho$, τ_2, $\tau_2\rho$

Class $(\mathrm{IIIc})_N$		
g *even*	g *odd*	
	$N/2$ *even*	$N/2$ *odd*
τ_1, $\tau_1\rho$	τ_1, $\tau_1\rho$	τ_1, τ_2

Class $(\mathrm{IIId})_N$		
g *even*		g *odd*
$N/2$ *even*	$N/2$ *odd*	
τ_1	τ_1, $\tau_1\rho$	τ_1, τ_2

In all cases the species of each real form is given.

Proof. — We only deal with the computation of the species of each real form. The other statements follow easily from the considerations preceding the theorem.

First, $\mathrm{sp}(\tau_2) = \mathrm{sp}(\tau_2\rho) = 0$ (if symmetries) because both are liftings of the antipodal map $\widehat{\tau}_2 : x \mapsto -1/\overline{x}$.

The species of τ_1 and $\tau_1\rho$ depend on the number of branch points fixed by $\widehat{\tau}_1$. It is easy to check that for any parity of $N/2$ the curve X has $2(p+\varepsilon_1)$ branch points lying on $\mathrm{Fix}(\widehat{\tau}_1) = \mathbb{R} \cup \{\infty\}$. In addition, it also has branch points lying outside $\mathbb{R} \cup \{\infty\}$

because $N/2 \geqslant 2$. Hence, Theorem 1.3.4 gives

$$\operatorname{sp}(\tau_1) = \begin{cases} -(p+\varepsilon_1) & \text{if } p+\varepsilon_1 \neq 0, \\ 1 & \text{if } p = \varepsilon_1 = 0 \text{ and } g \text{ even}, \\ 2 & \text{if } p = \varepsilon_1 = 0 \text{ and } g \text{ odd}. \end{cases}$$

$$\operatorname{sp}(\tau_1\rho) = \begin{cases} -(p+\varepsilon_1) & \text{if } p+\varepsilon_1 \neq 0, \\ 0 & \text{if } p+\varepsilon_1 = 0. \end{cases}$$

□

Observe that in cases IIIc and IIId the condition $2r + p > 0$ is necessary since otherwise X would not be of class III. Indeed, if $r = p = 0$ then X is given by $y^2 = x^{\varepsilon_1}(x^N + 1)$ and it is clear that, for example, the rotation $x \mapsto 1/x$ belongs to $\operatorname{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ but not to $\langle \widehat{a}, \widehat{\tau}_1 \rangle$.

3.4. Symmetry types of hyperelliptic algebraic curves of class IV

These are the curves whose induced group $\widehat{G}$ of Möbius transformations is $\widehat{G} = \langle \widehat{\tau}_1, \widehat{\tau}_3 \rangle \oplus \langle \widehat{\tau}_2 \rangle = \mathrm{D}_{N/2} \oplus \mathrm{Z}_2$, where $\widehat{\tau}_1$, $\widehat{\tau}_2$ and $\widehat{\tau}_3$ represent orientation reversing involutions. In fact, each $\widehat{\tau}_i$ is the image of a canonical reflection $c_j \in \Lambda$ under the epimorphism $\widehat{\theta} : \Lambda \to \widehat{G}$. The signature of Λ is one of the following:

IVa: $(0; +; [2^r]; \{(2^{p_1}, 2, 2^{p_2}, 2, 2^{p_3}, N/2)\})$, with $2r + p_1 + p_2 + p_3 = (2g+2)/N$,
IVb: $(0; +; [2^r]; \{(2^{p_1}, 2, 2^{p_2}, 4, 2^{p_3}, N/2)\})$, with $2r + p_1 + p_2 + p_3 = (2g+2)/N - 1/2$,
IVc: $(0; +; [2^r]; \{(2^{p_1}, 4, 2^{p_2}, 4, 2^{p_3}, N/2)\})$, with $2r + p_1 + p_2 + p_3 = (2g+2)/N - 1 > 0$,
IVd: $(0; +; [2^r]; \{(2^{p_1}, 2, 2^{p_2}, 2, 2^{p_3}, N)\})$, with $2r + p_1 + p_2 + p_3 = 2g/N$,
IVe: $(0; +; [2^r]; \{(2^{p_1}, 2, 2^{p_2}, 4, 2^{p_3}, N)\})$, with $2r + p_1 + p_2 + p_3 = 2g/N - 1/2$ and $g > 2$,
IVf: $(0; +; [2^r]; \{(2^{p_1}, 4, 2^{p_2}, 4, 2^{p_3}, N)\})$, with $2r + p_1 + p_2 + p_3 = 2g/N - 1 > 0$.

Recall that along this class, N is even $\geqslant 4$ and in cases IVb and IVe, $N \equiv 0 \pmod 4$ with $N \geqslant 8$.

Let us interpret geometrically the action of $\widehat{G}$ on the sphere S^2. This group is generated by three anticonformal involutions with fixed points and so they are reflections with respect to planes. Since $\widehat{G}$ is finite the planes have a common point, which we may assume it to be the origin. Since $\widehat{\tau}_1$ and $\widehat{\tau}_3$ generate a dihedral group of order N their planes make angle $2\pi/N$. Moreover, each of them is orthogonal to that of $\widehat{\tau}_2$ because $\widehat{\tau}_1$ and $\widehat{\tau}_3$ commute with $\widehat{\tau}_2$.

Therefore a fundamental set for the action of $\widehat{G}$ is a spherical triangle with vertices V_0, V_1 and V_2 as illustrated in Figure 4, where $V_0 \in \operatorname{Fix}(\widehat{\tau}_1\widehat{\tau}_3)$, $V_1 \in \operatorname{Fix}(\widehat{\tau}_1\widehat{\tau}_2)$ and $V_2 \in \operatorname{Fix}(\widehat{\tau}_3\widehat{\tau}_2)$. Each side is contained in the fixed point set of some reflection $\widehat{\tau}_j$ and so the quotient $S^2/\widehat{G}$ is a topological closed disc whose boundary consists of the sides V_0V_1, V_1V_2 and V_2V_0. If we consider now $S^2/\widehat{G}$ as the quotient $\mathcal{H}/\Lambda$ then the signature of Λ shows that the canonical projection $\mathcal{H} \to \mathcal{H}/\Lambda$ has $p_1 + p_2 + p_3 + 3$ boundary branch points and r interior branch points.

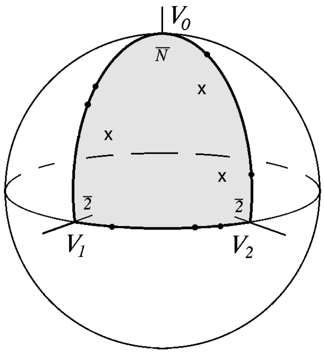


FIGURE 4

Note that p_1 of the boundary branch points lie in $V_0V_1 \subset \mathrm{Fix}(\widehat{\tau}_1)$, p_2 in $V_1V_2 \subset \mathrm{Fix}(\widehat{\tau}_2)$ and p_3 in $V_2V_0 \subset \mathrm{Fix}(\widehat{\tau}_3)$. They all have branching order 2. The other 3 boundary branch points are V_0, V_1 and V_2. This follows from the way in which the epimorphism $\widehat{\theta} : \Lambda \to \widehat{G}$ maps the $p_1 + p_2 + p_3 + 4$ canonical reflections of Λ. The branching order of each V_i depends on whether it is a branch point of the canonical projection $\pi_{\Gamma_\rho} : \mathcal{H} \to \mathcal{H}/\Gamma_\rho = S^2$ or not. This branching order can be read off from the signature of Λ : that of V_0 is $N/2$ in cases IVa, IVb and IVc, and N otherwise; that of V_1 is 2 in cases IVa, IVb, IVd and IVe, and 4 otherwise; that of V_2 is 2 in cases IVa and IVd, and 4 otherwise.

We now compute the signature of the NEC group associated to each $\widehat{\tau}_i$. Before proceeding note that $\widehat{\tau}_4 := \widehat{\tau}_2(\widehat{\tau}_1\widehat{\tau}_3)^{N/4}$ (if $4|N$) is also a symmetry of $\widehat{G}$. It is the antipodal map and so the species of $\tau_4 := \tau_2(\tau_1\tau_3)^{N/4}$ and $\tau_4\rho$ (if symmetries) are

Class IV
$\mathrm{sp}(\tau_4) = \mathrm{sp}(\tau_4\rho) = 0$

As to the others, each $\widehat{\tau}_i$ is a reflection with respect to a plane and so part (1) of Lemma 3.1.2 gives that the signature of its associated NEC group Λ_i is

$$\sigma(\Lambda_i) = (0; +; [2, \overset{u_i}{\ldots}, 2]; \{(2, \overset{v_i}{\ldots}, 2)\}) \quad \text{with} \quad 2u_i + v_i = 2g + 2,$$

where v_i is the number of points e_j lying in $\mathrm{Fix}(\widehat{\tau}_i)$ ($e_1, \ldots, e_{2g+2}$ are the branch points of the projection $\pi_{\Gamma_\rho} : \mathcal{H} \to \mathcal{H}/\Gamma_\rho = S^2$). We only give the details for computing the signature of Λ_2; the same ideas apply to Λ_1 and Λ_3 and so we restrict ourselves to give their signatures.

The unique branch points of $\pi_{\widehat{G}}$ on the side V_1V_2 are precisely the projections of the vertices V_1 and V_2. Hence, the p_2 branch points of $\pi_\Lambda = \pi_{\widehat{G}} \circ \pi_{\Gamma_\rho}$ lying in the side V_1V_2 and different to V_1 and V_2 are the projections under $\pi_{\widehat{G}}$ of Np_2 branch points e_j lying in $\mathrm{Fix}(\widehat{\tau}_2)$ (N is the number of preimages under $\pi_{\widehat{G}}$ of a point $e \neq V_1, V_2$ lying in V_1V_2). So if neither V_1 nor V_2 is an e_j (*i.e.,* if their branching orders with respect to π_Λ are 2) then there are exactly Np_2 branch points e_j lying in $\mathrm{Fix}(\widehat{\tau}_2)$. Therefore

$$\sigma(\Lambda_2) = (0; +; [2, \overset{g+1-p_2N/2}{\ldots\ldots}, 2]; \{(2, \overset{Np_2}{\ldots}, 2)\}).$$

This case corresponds to either signature IVa and IVd.

If either V_1 or V_2 is an e_j then there are exactly $Np_2 + N/2$ branch points e_j lying in $\mathrm{Fix}(\widehat{\tau}_2)$ ($N/2$ is the number of preimages under $\pi_{\widehat{G}}$ of either V_1 and V_2). Therefore

$$\sigma(\Lambda_2) = (0; +; [2, \overset{g+1-(2p_2+1)N/4}{\ldots\ldots\ldots}, 2]; \{(2, \overset{Np_2+N/2}{\ldots\ldots}, 2)\}).$$

This case corresponds to either signature IVb and IVe.

If both V_1 and V_2 are among the e_j then there are exactly $N(p_2+1)$ branch points e_j lying in $\mathrm{Fix}(\widehat{\tau}_1)$. Therefore

$$\sigma(\Lambda_2) = (0; +; [2, \overset{g+1-(p_2+1)N/2}{\ldots\ldots\ldots}, 2]; \{(2, \overset{N(p_2+1)}{\ldots\ldots}, 2)\}).$$

This case corresponds to either signature IVc and IVf.

To compute the signature of Λ_1 we must distinguish whether V_0 and V_1 are among the e_j and whether $N/2$ is even or odd. Either of V_0 and V_1 is the projection of 2 different points of $\mathrm{Fix}(\widehat{\tau}_1)$. Now, V_0 and its antipodal image V_∞ divide $\mathrm{Fix}(\widehat{\tau}_1)$ into two semicircles. The semicircle containing V_1 has $2p_1$ branch points e_j whilst the other has $2p_1$ if $N/2$ is even or $2p_3$ if $N/2$ is odd. It is then easy to check that the signature of Λ_1 is the following, where in order to save space we just write the proper and the link periods.

$$\sigma(\Lambda_1) = \begin{cases} ([2, \overset{g+1-2p_1}{\ldots\ldots}, 2]; \{(2, \overset{4p_1}{\ldots}, 2)\}) & \text{in IVa if } N/2 \text{ even, and IVb;} \\ ([2, \overset{g+1-p_1-p_3}{\ldots\ldots}, 2]; \{(2, \overset{2(p_1+p_3)}{\ldots\ldots}, 2)\}) & \text{in IVa if } N/2 \text{ odd;} \\ ([2, \overset{g-2p_1}{\ldots\ldots}, 2]; \{(2, \overset{4p_1+2}{\ldots\ldots}, 2)\}) & \text{in IVc, IVd, both if } N/2 \text{ even,} \\ & \text{and IVe;} \\ ([2, \overset{g-p_1-p_3}{\ldots\ldots}, 2]; \{(2, \overset{2(p_1+p_3+1)}{\ldots\ldots}, 2)\}) & \text{in IVc, IVd, both if } N/2 \text{ odd;} \\ ([2, \overset{g-1-2p_1}{\ldots\ldots}, 2]; \{(2, \overset{4p_1+4}{\ldots\ldots}, 2)\}) & \text{in IVf if } N/2 \text{ even;} \\ ([2, \overset{g-1-p_1-p_3}{\ldots\ldots}, 2]; \{(2, \overset{2(p_1+p_3+2)}{\ldots\ldots}, 2)\}) & \text{in IVf if } N/2 \text{ odd.} \end{cases}$$

Finally, the signature of Λ_3 may be computed in a similar way as that of Λ_1. The number of e_i lying on $\mathrm{Fix}(\widehat{\tau}_3)$ depends on whether V_0 and V_2 are among them and on the parity of $N/2$. We get the following:

$$\sigma(\Lambda_3) = \begin{cases} ([2, \overset{g+1-2p_3}{\ldots\ldots}, 2]; \{(2, \overset{4p_3}{\ldots}, 2)\}) & \text{in IVa if } N/2 \text{ even;} \\ ([2, \overset{g+1-p_1-p_3}{\ldots\ldots}, 2]; \{(2, \overset{2(p_1+p_3)}{\ldots}, 2)\}) & \text{in IVa if } N/2 \text{ odd;} \\ ([2, \overset{g-2p_3}{\ldots\ldots}, 2]; \{(2, \overset{4p_3+2}{\ldots\ldots}, 2)\}) & \text{in IVc, IVd, both if } N/2 \text{ even,} \\ & \text{and IVb;} \\ ([2, \overset{g-p_1-p_3}{\ldots\ldots}, 2]; \{(2, \overset{2(p_1+p_3+1)}{\ldots\ldots}, 2)\}) & \text{in IVc, IVd, both if } N/2 \text{ odd;} \\ ([2, \overset{g-1-2p_3}{\ldots\ldots}, 2]; \{(2, \overset{4p_3+4}{\ldots\ldots}, 2)\}) & \text{in IVf if } N/2 \text{ even, and IVe;} \\ ([2, \overset{g-1-p_1-p_3}{\ldots\ldots}, 2]; \{(2, \overset{2(p_1+p_3+2)}{\ldots\ldots}, 2)\}) & \text{in IVf if } N/2 \text{ odd.} \end{cases}$$

From the conditions among r, p_i, N and g we see that only in some cases the number of link periods of $\sigma(\Lambda_i)$ may attain the maximal value $2g+2$. The range of the parameters p_i and the species of the symmetries τ_i and $\tau_i\rho$ for $i = 1, 2, 3$ are examined separately in each case and the results displayed in the corresponding table.

3.4.1. Class IVa. — The number of real forms of X depends on the parity of $N/2$. If $N/2$ is even then X has eight real forms, with representatives τ_i, $\tau_i\rho$ for $i = 1, 2, 3, 4$. If $N/2$ is odd then τ_1, τ_2, $\tau_1\rho$ and $\tau_2\rho$ are representatives of the unique four real forms of X. The species of each representative depends on the value of the parameters p_i, $i = 1, 2, 3$. According to the signature of Λ_i the species are the following.

IVa			
$i=2$	$p_2 = 0$	$0 < p_2 < (2g+2)/N$	$p_2 = (2g+2)/N$
$\{\mathrm{sp}(\tau_2), \mathrm{sp}(\tau_2\rho)\}$	$\{1,0\}$ if g is even $\{2,0\}$ if g is odd	$\{-p_2N/2, -p_2N/2\}$	$\{g+1, g+1\}$
$N/2$ even, $i = 1,3$	$p_i = 0$	$0 < p_i < (g+1)/2$	$p_i = (g+1)/2$
$\{\mathrm{sp}(\tau_i), \mathrm{sp}(\tau_i\rho)\}$	$\{1,0\}$ if g is even $\{2,0\}$ if g is odd	$\{-2p_i, -2p_i\}$	$\{g+1, g+1\}$
$N/2$ odd, $i = 1$	$p_1 + p_3 = 0$	$0 < p_1 + p_3 \leqslant (2g+2)/N$	
$\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\}$	$\{1,0\}$ if g is even $\{2,0\}$ if g is odd	$\{-(p_1+p_3), -(p_1+p_3)\}$	

Letting the parameters vary under the restrictions $p_1 + p_2 + p_3 \leqslant (2g+2)/N$ and $\equiv (2g+2)/N \pmod 2$ we obtain all the symmetry types of curves of class $(\mathrm{IVa})_N$.

THEOREM 3.4.1. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{IVa})_N$ *is one of the listed below. Conversely, each of them is the symmetry type* *of such a curve. The parameters p_i are positive and the sum of all the distinct p_i appearing in a symmetry type is* $\leqslant (2g+2)/N$ *and* $\equiv (2g+2)/N \pmod 2$.

(1) *If $N/2$ is even (then g is odd):*
   - $\{2,0,2,0,2,0,0,0\}$, *only if $(2g+2)/N$ is even;*
   - $\{2,0,2,0,-2p_3,-2p_3,0,0\}$;
   - $\{2,0,2,0,g+1,g+1,0,0\}$;
   - $\{2,0,-p_2N/2,-p_2N/2,0,0\}$;
   - $\{2,0,-p_2N/2,-p_2N/2,-2p_3,-2p_3,0,0\}$;
   - $\{-2p_1,-2p_1,2,0,-2p_3,-2p_3,0,0\}$;
   - $\{-2p_1,-2p_1,-p_2N/2,-p_2N/2,-2p_3,-2p_3,0,0\}$.

(2) *If $N/2$ is odd; let δ be 1 if g is even and 2 otherwise:*
   - $\{2,0,2,0\}$, *only if g is odd;*
   - $\{-p_3,-p_3,\delta,0\}$;
   - $\{\delta,0,-p_2N/2,-p_2N/2\}$;
   - $\{-p_3,-p_3,-p_2N/2,-p_2N/2\}$;
   - $\{\delta,0,g+1,g+1\}$.

3.4.2. Class IVb. — Here X has five real forms, with representatives τ_1, $\tau_1\rho$, τ_2, τ_3 and τ_4 if $N/4$ is even, and four, with representatives τ_1, $\tau_1\rho$, τ_2 and τ_3, if $N/4$ is odd. (Recall that here $N \equiv 0 \pmod 4$ and $N \geqslant 8$.) According to the signatures of each Λ_i their species are the following.

IVb		
$i=1$	$p_1=0$	$0<p_1$
$\{\mathrm{sp}(\tau_1),\, \mathrm{sp}(\tau_1\rho)\}$	$\{1,0\}$ if g is even $\{2,0\}$ if g is odd	$\{-2p_1,-2p_1\}$
$i=2$	$p_2<(2g+2)/N-1/2$	$p_2=(2g+2)/N-1/2$
$\mathrm{sp}(\tau_2)$	$-(2p_2+1)N/4$	$g+1$
$i=3$	$p_3 \leqslant (2g+2)/N-1/2$	
$\mathrm{sp}(\tau_3)$	$-(2p_3+1)$	

Since in this case $(4g+4)/N$ must be odd it follows that $N/4$ *is odd if and only if g is even.*

Theorem 3.4.2. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{IVb})_N$ *is one of the listed below. Conversely, each of them is the symmetry type* *of such a curve. The sum of all the distinct integers p_i appearing in a symmetry type is* $\leqslant (2g+2)/N - 1/2$ *and* $\equiv (2g+2)/N - 1/2 \pmod 2$. *Also,* p_1 *is positive.*

(1) *If g is even:*
   - $\{1, 0, g+1, -1\}$;
   - $\{1, 0, -(2p_2+1)N/4, -(2p_3+1)\}$;
   - $\{-2p_1, -2p_1, -(2p_2+1)N/4, -(2p_3+1)\}$.

(2) *If g is odd:*
   - $\{2, 0, g+1, -1, 0\}$;
   - $\{2, 0, -(2p_2+1)N/4, -(2p_3+1), 0\}$;
   - $\{-2p_1, -2p_1, -(2p_2+1)N/4, -(2p_3+1), 0\}$.

3.4.3. Class IVc. — For $N/2$ even X has four real forms, with representatives τ_i for $i = 1, 2, 3, 4$, whilst for $N/2$ odd it has two, with representatives τ_1 and τ_2. Their species are the following.

IVc		
$i = 2$	$p_2 < (2g+2)/N - 1$	$p_2 = (2g+2)/N - 1$
$\mathrm{sp}(\tau_2)$	$-(p_2+1)N/2$	$g+1$
$N/2$ even, $i = 1, 3$	$p_i \leqslant (2g+2)/N - 1$	
$\mathrm{sp}(\tau_i)$	$-(2p_i+1)$	
$N/2$ odd, $i = 1$	$p_1 + p_3 \leqslant (2g+2)/N - 1$	
$\mathrm{sp}(\tau_1)$	$-(p_1+p_3+1)$	

Recall that genus g curves of class $(\mathrm{IVc})_N$ exist if and only if $N/2$ is a proper divisor of $g+1$. (In particular $g+1$ cannot be a prime number.)

Theorem 3.4.3. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{IVc})_N$ *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The sum of all the distinct integers p_i appearing in a symmetry type is* $\leqslant (2g+2)/N - 1$ *and* $\equiv (2g+2)/N - 1 \pmod 2$.

(1) *If $N/2$ is even:*
   - $\{-1, g+1, -1, 0\}$;
   - $\{-(2p_1+1), -(p_2+1)N/2, -(2p_3+1), 0\}$.

(2) *If $N/2$ is odd:*
   - $\{-1, g+1\}$;
   - $\{-(p_1+p_3+1), -(p_2+1)N/2\}$.

3.4.4. Class IVd. — Here X has four real forms, with representatives τ_1, τ_2, $\tau_2\rho$ and τ_3, for any value of N, which is a divisor of $2g$. Their species are the following.

IVd		
$i = 2$	$p_2 = 0$	$0 < p_2$
$\{\mathrm{sp}(\tau_2), \mathrm{sp}(\tau_2\rho)\}$	$\{1,0\}$ if g is even $\{2,0\}$ if g is odd	$\{-p_2N/2, -p_2N/2\}$
$i = 1,3$; $N/2$ even	$p_i < g/2$	$p_i = g/2$
$\mathrm{sp}(\tau_i)$	$-(2p_i + 1)$	$g + 1$
$N/2$ odd	$p_1 + p_3 \leqslant 2g/N$	
$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_3)$	$-(p_1 + p_3 + 1)$	

THEOREM 3.4.4. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{IVd})_N$ *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The sum of the distinct integers* p_i *which appear in a symmetry type is* $\leqslant 2g/N$ *and* $\equiv 2g/N \pmod 2$. *Also,* p_2 *is positive.*

(1) *If g is even:*
  (1.1) *If* $N/2$ *is odd:*
   - $\{-(p_1+p_3+1), 1, 0, -(p_1+p_3+1)\}$;
   - $\{-(p_1+p_3+1), -p_2N/2, -p_2N/2, -(p_1+p_3+1)\}$.
  (1.2) *If* $N/2$ *is even:*
   - $\{-(2p_1+1), 1, 0, -(2p_3+1)\}$;
   - $\{-(2p_1+1), -p_2N/2, -p_2N/2, -(2p_3+1)\}$;
   - $\{-1, 1, 0, g+1\}$ *if* $N = 4$.
(2) *If g is odd:*
  - $\{-(p_1+p_3+1), 2, 0, -(p_1+p_3+1)\}$;
  - $\{-(p_1+p_3+1), -p_2N/2, -p_2N/2, -(p_1+p_3+1)\}$.

3.4.5. Class IVe. — Here X has four real forms, with representatives τ_i for $i = 1,2,3,4$ if $N/4$ is odd and three, with representatives τ_i for $i = 1,2,3$ if $N/4$ is even. According to the signatures of each Λ_i their species are the following. In each case, $p_i \leqslant 2g/N - 1/2$.

IVe	
$\mathrm{sp}(\tau_1)$	$-(2p_1+1)$
$\mathrm{sp}(\tau_2)$	$-(2p_2+1)N/4$
$\mathrm{sp}(\tau_3)$	$-(2p_3+2)$

Since in this case $4g/N$ must be odd it follows that $N/4$ *is even if and only if g is even.* Moreover, $g>2$ in this class.

THEOREM 3.4.5. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{IVe})_N$ *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The sum of all the distinct integers p_i appearing in a symmetry type is* $\leqslant 2g/N-1/2$ *and* $\equiv 2g/N-1/2 \pmod 2$.

(1) *If g is even:*
 - $\{-(2p_1+1), -(2p_2+1)N/4, -(2p_3+2)\}$.

(2) *If g is odd:*
 - $\{-(2p_1+1), -(2p_2+1)N/4, -(2p_3+2), 0\}$.

3.4.6. Class IVf. — For $N/2$ even X has three real forms, with representatives τ_1, τ_2 and τ_3, whilst for $N/2$ odd it has two, with representatives τ_1 and τ_2. Their species are the following.

IVf	
$i=2$	$p_2 \leqslant 2g/N-1$
$\mathrm{sp}(\tau_2)$	$-(p_2+1)N/2$
$N/2$ even, $i=1,3$	$p_i \leqslant 2g/N-1$
$\mathrm{sp}(\tau_i)$	$-(2p_i+2)$
$N/2$ odd, $i=1$	$p_1+p_3 \leqslant 2g/N-1$
$\mathrm{sp}(\tau_1)$	$-(p_1+p_3+2)$

Recall that genus g curves of class $(\mathrm{IVf})_N$ exist if and only if $N/2$ is a proper divisor of g. (In particular g cannot be a prime number.)

THEOREM 3.4.6. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* $(\mathrm{IVf})_N$ *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The sum of all the distinct integers p_i appearing in a symmetry type is* $\leqslant 2g/N-1$ *and* $\equiv 2g/N-1 \pmod 2$.

(1) *If $N/2$ is even:*
- $\{-(2p_1+2), -(p_2+1)N/2, -(2p_3+2)\}$.

(2) *If $N/2$ is odd:*
- $\{-(p_1+p_3+2), -(p_2+1)N/2\}$.

3.4.7. Equations of curves of class IV and their real forms. — The branch point set of a curve of class IV is preserved by the elements of the group $\widehat{G} = \mathrm{D}_{N/2} \oplus \mathrm{Z}_2$ of order $2N$ generated by three reflections $\widehat{\tau}_1$, $\widehat{\tau}_2$ and $\widehat{\tau}_3$. The factor $\mathrm{D}_{N/2}$ is generated by $\widehat{\tau}_1$ and $\widehat{\tau}_3$ and so they are reflections with respect to planes making angle $2\pi/N$. Since $\widehat{\tau}_1$ and $\widehat{\tau}_3$ commute with $\widehat{\tau}_2$ the plane of this latter reflection is orthogonal to those of the formers. Viewed in $\widehat{\mathbb{C}}$, we choose the following formulae for these reflections:

$$\widehat{\tau}_1 : x \longmapsto \overline{x}, \quad \widehat{\tau}_2 : x \longmapsto \frac{1}{\overline{x}} \quad \text{and} \quad \widehat{\tau}_3 : x \longmapsto \overline{x} \cdot e^{4\pi i/N}.$$

Thus the two fixed points of the rotation $\widehat{\tau}_3\widehat{\tau}_1 : x \mapsto x \cdot e^{4\pi i/N}$ are 0 and ∞; those of $\widehat{\tau}_2\widehat{\tau}_1 : x \mapsto 1/x$ are ± 1 and those of $\widehat{\tau}_3\widehat{\tau}_2 : x \mapsto e^{4\pi i/N}/x$ are $\pm e^{2\pi i/N}$. It is then easy to check that the orbit under the action of $\widehat{G}$ of a point $\alpha \in \widehat{\mathbb{C}}$ is the following:

$$\operatorname{orbit}(\alpha, \widehat{G}) = \begin{cases} \text{roots of } (x^{N/2}-\alpha^{N/2})(x^{N/2}-\overline{\alpha}^{N/2})(x^{N/2}-1/\alpha^{N/2})(x^{N/2}-1/\overline{\alpha}^{N/2}) \\ \qquad \text{if } \alpha \text{ is fixed by no element of } \widehat{G}; \\ \text{roots of } (x^{N/2}-\alpha^{N/2})(x^{N/2}-1/\alpha^{N/2}) \\ \qquad \text{if } \alpha \neq 0, \infty \text{ is fixed by a symmetry different to } \widehat{\tau}_2; \\ \text{roots of } (x^{N/2}-\alpha^{N/2})(x^{N/2}-\overline{\alpha}^{N/2}) \quad \text{if } \alpha \text{ is fixed only by } \widehat{\tau}_2; \\ \text{roots of } (x^{N/2}-\alpha^{N/2}) \quad \text{if } \alpha \text{ is fixed by } \widehat{\tau}_2 \text{ and a symmetry} \\ \qquad \text{of the form } (\widehat{\tau}_3\widehat{\tau}_1)^j\,\widehat{\tau}_1 \quad (j = 1, \dots, N/2); \\ \{0, \infty\} \quad \text{if } \alpha = 0, \infty. \end{cases}$$

Note that:

- if α is fixed by no element of $\widehat{G}$ then $\alpha^{N/2}$ is a complex non-real number with modulus different from 1. Hence the polynomial $(x^{N/2}-\alpha^{N/2})(x^{N/2}-\overline{\alpha}^{N/2})(x^{N/2}-1/\alpha^{N/2})(x^{N/2}-1/\overline{\alpha}^{N/2})$ may be written as $(x^N - wx^{N/2}+1)(x^N - \overline{w}x^{N/2}+1)$ where w is a complex number with positive imaginary part.
- If $\alpha \neq 0, \infty$ is fixed by a symmetry different to $\widehat{\tau}_2$ then $\alpha^{N/2}$ is a real number. Hence the polynomial $(x^{N/2}-\alpha^{N/2})(x^{N/2}-1/\alpha^{N/2})$ may be written as $x^N - \lambda x^{N/2}+1$ where λ is real with $|\lambda| > 2$.
- If α is fixed only by $\widehat{\tau}_2$ then $\alpha^{N/2} \neq \pm 1$ is a unimodular complex number. Hence the polynomial $(x^{N/2}-\alpha^{N/2})(x^{N/2}-\overline{\alpha}^{N/2})$ may be written as $x^N - \nu x^{N/2}+1$ where ν is a real number with $|\nu| < 2$.
- If α is fixed by $\widehat{\tau}_2$ and a symmetry of the form $(\widehat{\tau}_3\widehat{\tau}_1)^j\widehat{\tau}_1$ $(j = 1, \dots, N/2)$ then $\alpha^{N/2} = \pm 1$.

This leads us to consider polynomials of the following form: for each $z \in \mathbb{C}$ let Q_z be

$$Q_z(T) = T^2 - zT + 1.$$

It is easy to check that

- if $z \in \mathbb{C}^+$ then the product $Q_z \cdot Q_{\overline{z}}$ has four different roots which are of the form β, $\overline{\beta}$, $1/\beta$ and $1/\overline{\beta}$.
- If z is real and > 2 then Q_z has two positive roots.
- If z is real and < -2 then Q_z has two negative roots.
- If z is real and $|z| < 2$ then Q_z has two unimodular complex non-real roots.

Now, given

- $\{w_1, \dots, w_r\} \subset \mathbb{C}^+$ with $w_i \neq w_j$ if $i \neq j$,
- $2 < \lambda_1 < \cdots < \lambda_{p_1}$,
- $-2 < \nu_1 < \cdots < \nu_{p_2} < 2$ and
- $2 < \mu_1 < \cdots < \mu_{p_3}$,

we define the polynomial $R = R^{r,p_1,p_2,p_3}_{w_j,\lambda_j,\nu_j,\mu_j}$ as

$$R(x) = \prod_{j=1}^{r} Q_{w_j}(x^{N/2}) \cdot Q_{\overline{w}_j}(x^{N/2}) \cdot \prod_{j=1}^{p_1} Q_{\lambda_j}(x^{N/2}) \cdot \prod_{j=1}^{p_2} Q_{\nu_j}(x^{N/2}) \cdot \prod_{j=1}^{p_3} Q_{-\mu_j}(x^{N/2}),$$

where N is even. Note that R is a polynomial of degree $N(2r+p_1+p_2+p_3)$. Clearly, it serves to describe orbits of branch points of curves of class IV.

With the same notations as in the beginning of this section, the vertices V_0, V_1 and V_2 of a fundamental set for the $\widehat{G}$-action are fixed points of the rotations $\widehat{\tau}_1\widehat{\tau}_3$, $\widehat{\tau}_1\widehat{\tau}_2$ and $\widehat{\tau}_2\widehat{\tau}_3$ respectively. For our choice of the generators $\widehat{\tau}_1$, $\widehat{\tau}_2$ and $\widehat{\tau}_3$ we may take $V_0 = 0$, $V_1 = 1$ and $V_2 = e^{2\pi i/N}$. Therefore, the orbit of V_0 consists of 0 and ∞, that of V_1 consists of the roots of $x^{N/2} - 1$ and that of V_2 consists of the roots of $x^{N/2} + 1$. Recall that the presence of some of these orbits among the branch points gives a distinction between curves of different classes. Indeed, we have the following table, in which a "yes" means that the vertex V_i is a branch point of a curve of the corresponding class.

	Class IVa	Class IVb	Class IVc	Class IVd	Class IVe	Class IVf
V_0	no	no	no	yes	yes	yes
V_1	no	no	yes	no	no	yes
V_2	no	yes	yes	no	yes	yes

We have shown most of the following theorem. In it we assume that N is an even number $\geqslant 4$ which makes the corresponding value of g an integer $\geqslant 2$.

Theorem 3.4.7. — *With the above notation, for almost every choice of*

- $\{w_1, \dots, w_r\} \subset \mathbb{C}^+$ *with* $w_i \neq w_j$ *if* $i \neq j$,
- $2 < \lambda_1 < \cdots < \lambda_{p_1}$,

- $-2 < \nu_1 < \cdots < \nu_{p_2} < 2$ *and*
- $2 < \mu_1 < \cdots < \mu_{p_3}$,

the equation

$$y^2 = x^{\varepsilon_0} \cdot (x^{N/2} - 1)^{\varepsilon_1} \cdot (x^{N/2} + 1)^{\varepsilon_2} \cdot R(x),$$

where $\varepsilon_i \in \{0,1\}$, *defines a hyperelliptic complex curve* X *of class*

a) $(\mathrm{IVa})_N$ *and genus* $g = (2r + p_1 + p_2 + p_3)N/2 - 1$ *if* $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$;
b) $(\mathrm{IVb})_N$ *and genus* $g = (2r + p_1 + p_2 + p_3 + 1/2)N/2 - 1$ *if* $N \geqslant 8$, $\varepsilon_0 = \varepsilon_1 = 0$ *and* $\varepsilon_2 = 1$;
c) $(\mathrm{IVc})_N$ *and genus* $g = (2r + p_1 + p_2 + p_3 + 1)N/2 - 1$ *if* $2r + p_1 + p_2 + p_3 > 0$, $\varepsilon_0 = 0$ *and* $\varepsilon_1 = \varepsilon_2 = 1$;
d) $(\mathrm{IVd})_N$ *and genus* $g = (2r + p_1 + p_2 + p_3)N/2$ *if* $\varepsilon_0 = 1$ *and* $\varepsilon_1 = \varepsilon_2 = 0$;
e) $(\mathrm{IVe})_N$ *and genus* $g = (2r + p_1 + p_2 + p_3 + 1/2)N/2$ *if* $\varepsilon_0 = \varepsilon_2 = 1$, $\varepsilon_1 = 0$ *and* $g > 2$;
f) $(\mathrm{IVf})_N$ *and genus* $g = (2r + p_1 + p_2 + p_3 + 1)N/2$ *if* $2r + p_1 + p_2 + p_3 > 0$, $N \geqslant 8$ *and* $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$.

Conversely, each genus g hyperelliptic curve of class IV *is (isomorphic to another) of the above form for some* N, r, p_i *and* ε_j *in the above conditions.*

Representatives of all the real forms of X *are the following:*

Class $(\mathrm{IVa})_N$	
$N/2$ *even*	τ_i, $\tau_i\rho$ *for* $i = 1,2,3,4$
$N/2$ *odd*	τ_1, $\tau_1\rho$, τ_2, $\tau_2\rho$

Class $(\mathrm{IVb})_N$	
$N/4$ *even*	τ_1, $\tau_1\rho$, τ_2, τ_3, τ_4
$N/4$ *odd*	τ_1, $\tau_1\rho$, τ_2, τ_3

Class $(\mathrm{IVc})_N$	
$N/2$ *even*	τ_1, τ_2, τ_3, τ_4
$N/2$ *odd*	τ_1, τ_2

Class $(\mathrm{IVd})_N$
τ_1, τ_2, $\tau_2\rho$, τ_3

Class $(\mathrm{IVe})_N$	
$N/4$ *even*	τ_1, τ_2, τ_3
$N/4$ *odd*	τ_1, τ_2, τ_3, τ_4

Class $(\mathrm{IVf})_N$	
$N/2$ *even*	τ_1, τ_2, τ_3
$N/2$ *odd*	τ_1, τ_2

where

$$\tau_1 : (x,y) \longmapsto (\overline{x}, \overline{y}), \qquad \tau_2 : (x,y) \longmapsto \left(\frac{1}{\overline{x}}, \frac{\overline{y} \cdot i^{\varepsilon_1}}{\overline{x}^{g+1}}\right),$$

$$\tau_3 : (x,y) \longmapsto (\overline{x} \cdot e^{4\pi i/N}, \overline{y} \cdot (e^{2\pi i/N})^{\varepsilon_0}), \quad \tau_4 : (x,y) \longmapsto \left(\frac{-1}{\overline{x}}, \frac{\overline{y} \cdot i^{\varepsilon_0+\varepsilon_1}}{\overline{x}^{g+1}}\right).$$

In all cases the species of each real form is given.

Proof. — We restrict ourselves to compute species. First,

$$\mathrm{sp}(\tau_4) = \mathrm{sp}(\tau_4\rho) = 0$$

because both are liftings of the antipodal map. The other symmetries are liftings of the reflections $\widehat{\tau}_i$ and so their species depend on the number of branch points fixed by $\widehat{\tau}_i$.

It is easy to check that the number of branch points of X lying on $\mathrm{Fix}(\widehat{\tau}_1) = \mathbb{R}\cup\{\infty\}$ is $2\varepsilon_0+2\varepsilon_1+4p_1$ if $N/2$ is even and $2\varepsilon_0+\varepsilon_1+\varepsilon_2+2p_1+2p_3$ if $N/2$ is odd. In addition, all the branch points of X lie on $\mathbb{R}\cup\{\infty\}$ if and only if $N = 4$ and $\varepsilon_2 = r = p_2 = p_3 = 0$. Hence, from Theorem 1.3.4, if $N/2$ is odd then

$$\mathrm{sp}(\tau_1) = \begin{cases} -(2\varepsilon_0+\varepsilon_1+\varepsilon_2+2p_1+2p_3)/2 & \text{if } 2\varepsilon_0+\varepsilon_1+\varepsilon_2+2p_1+2p_3 > 0,\\ 1 & \text{if } \varepsilon_0=\varepsilon_1=\varepsilon_2=p_1=p_3=0 \\ & \text{and } g \text{ even},\\ 2 & \text{if } \varepsilon_0=\varepsilon_1=\varepsilon_2=p_1=p_3=0 \\ & \text{and } g \text{ odd},\end{cases}$$

$$\mathrm{sp}(\tau_1\rho) = \begin{cases} -(2\varepsilon_0+\varepsilon_1+\varepsilon_2+2p_1+2p_3)/2 & \text{if } 2\varepsilon_0+\varepsilon_1+\varepsilon_2+2p_1+2p_3 > 0,\\ 0 & \text{if } \varepsilon_0=\varepsilon_1=\varepsilon_2=p_1=p_3=0.\end{cases}$$

If $N/2$ is even then

$$\mathrm{sp}(\tau_1) = \begin{cases} -(\varepsilon_0+\varepsilon_1+2p_1) & \text{if } 0<\varepsilon_0+\varepsilon_1+2p_1<g+1,\\ 1 & \text{if } \varepsilon_0=\varepsilon_1=p_1=0 \text{ and } g \text{ even},\\ 2 & \text{if } \varepsilon_0=\varepsilon_1=p_1=0 \text{ and } g \text{ odd},\\ g+1 & \text{if } N=4 \text{ and } \varepsilon_2=r=p_2=p_3=0,\end{cases}$$

$$\mathrm{sp}(\tau_1\rho) = \begin{cases} -(\varepsilon_0+\varepsilon_1+2p_1) & \text{if } 0<\varepsilon_0+\varepsilon_1+2p_1<g+1,\\ 0 & \text{if } \varepsilon_0=\varepsilon_1=p_1=0,\\ g+1 & \text{if } N=4 \text{ and } \varepsilon_2=r=p_2=p_3=0.\end{cases}$$

In order to compute the species of τ_3 and $\tau_3\rho$ we consider the curve X' whose branch point set $B_{X'}$ is the image of B_X under the rotation $m : x \mapsto xe^{-2\pi i/N}$. This rotation conjugates $\widehat{\tau}_3$ to $\widehat{\tau}_1$. It turns out that the equation of X' has the same aspect than that of X. Indeed, it is easy to check that it is defined by the following parameters:

$$\varepsilon_0' = \varepsilon_0,\ \varepsilon_1' = \varepsilon_2,\ \varepsilon_2' = \varepsilon_1,\ r' = r,\ \omega_j' = -\omega_j,$$
$$p_1' = p_3,\ \lambda_j' = \mu_j,\ p_2' = p_2,\ \nu_j' = -\nu_j,\ p_3' = p_1,\ \mu_j' = \lambda_j.$$

In particular, $\tau_1' : (x,y) \mapsto (\overline{x},\overline{y})$ is a symmetry of X' and it turns out that

$$f\circ\tau_3\circ f^{-1} = \tau_1'\circ\rho,$$

where $f : X \to X'$ is a lifting of m. Thus the species of τ_3 and $\tau_3\rho$ coincide with those of $\tau_1'\rho$ and τ_1' respectively. The latter have already been calculated in the previous case and so changing the primed parameters into non-primed ones we get

- if $N/2$ is odd,

$$\mathrm{sp}(\tau_3) = \mathrm{sp}(\tau_1\rho) \quad \text{and} \quad \mathrm{sp}(\tau_3\rho) = \mathrm{sp}(\tau_1).$$

- If $N/2$ is even,

$$\mathrm{sp}(\tau_3) \;=\; \begin{cases} -(\varepsilon_0+\varepsilon_2+2p_3) & \text{if } 0<\varepsilon_0+\varepsilon_2+2p_3<g+1,\\ 0 & \text{if } \varepsilon_0=\varepsilon_2=p_3=0,\\ g+1 & \text{if } N=4 \text{ and } \varepsilon_1=r=p_1=p_2=0.\end{cases}$$

$$\mathrm{sp}(\tau_3\rho) = \begin{cases} -(\varepsilon_0+\varepsilon_2+2p_3) & \text{if } 0<\varepsilon_0+\varepsilon_2+2p_3<g+1,\\ 1 & \text{if } \varepsilon_0=\varepsilon_2=p_3=0 \text{ and } g \text{ even},\\ 2 & \text{if } \varepsilon_0=\varepsilon_2=p_3=0 \text{ and } g \text{ odd},\\ g+1 & \text{if } N=4 \text{ and } \varepsilon_1=r=p_1=p_2=0.\end{cases}$$

The number of branch points of X fixed by $\widehat{\tau}_2$ is $(\varepsilon_1+\varepsilon_2+2p_2)N/2$, which is maximal (*i.e.*, $2g+2$) if and only if $\varepsilon_0=r=p_1=p_3=0$. Hence,

$$\mathrm{sp}(\tau_2)=\mathrm{sp}(\tau_2\rho)=\begin{cases} -(\varepsilon_1+\varepsilon_2+2p_2)N/4 & \text{if } 0<\varepsilon_1+\varepsilon_2+p_2<g+1,\\ g+1 & \text{if } \varepsilon_0=r=p_1=p_3=0.\end{cases}$$

In case $\varepsilon_1=\varepsilon_2=p_2=0$ we have $\{\mathrm{sp}(\tau_2),\mathrm{sp}(\tau_2\rho)\}=\{1,0\}$ or $\{2,0\}$. We decide which species corresponds to each symmetry by finding which one fixes points of X. A point (α,β) is fixed by τ_2 if and only if $|\alpha|=1$ and $\beta/\overline{\beta}=\alpha^{g+1}$. Now, for any unimodular number α we have the following equality:

$$\frac{P_X(\alpha)}{\overline{P_X(\alpha)}}=\alpha^{2g+2},$$

where P_X is the polynomial defining X. In particular, $P_X(1)$ is a real number. But

$$P_X(1)=(-1)^{p_1}\prod_{j=1}^{r}|2-\omega_j|^2\cdot\prod_{j=1}^{p_1}(\lambda_j-2)\cdot\prod_{j=1}^{p_3}(2+\mu_j),$$

which is positive if and only if p_1 is even. So, if p_1 is even then $\sqrt{P_X(1)}/\overline{\sqrt{P_X(1)}}=1$ and it follows that $(1,\sqrt{P_X(1)})$ is a point fixed by τ_2. If p_1 is odd then $(1,\sqrt{P_X(1)})$ is a point fixed by $\tau_2\rho$. Summarizing, if $\varepsilon_1=\varepsilon_2=p_2=0$ then

$$\mathrm{sp}(\tau_2)=\begin{cases} 0 & \text{if } p_1 \text{ is odd},\\ 1 & \text{if } p_1 \text{ and } g \text{ are even},\\ 2 & \text{if } p_1 \text{ is even and } g \text{ is odd}.\end{cases}$$

$$\mathrm{sp}(\tau_2\rho)=\begin{cases} 0 & \text{if } p_1 \text{ is even},\\ 1 & \text{if } p_1 \text{ is odd and } g \text{ is even},\\ 2 & \text{if } p_1 \text{ and } g \text{ are odd}.\end{cases}$$

□

We observe that

- in case c) if $2r + p_1 + p_2 + p_3 = 0$ then $X = \{y^2 = x^N - 1\}$, which is not of class $(\mathrm{IVc})_N$. Indeed, it is the Accola-Maclachlan curve of genus $g = N/2 - 1$, which has $8g + 8 = 4N$ analytic automorphisms; it is a curve of class $(\mathrm{IVb})_{2N}$.
- in case e) if $g = 2$ then $N = 8$ and $X = \{y^2 = x(x^4 - 1)\}$, which is not of class IV. Indeed, it is a curve of class VIIc, as we shall see, with 48 analytic automorphisms.
- in case f) if $2r + p_1 + p_2 + p_3 = 0$ then $X = \{y^2 = x(x^N - 1)\}$, which is not of class $(\mathrm{IVf})_N$. Indeed, it admits $8g = 4N$ analytic automorphisms and, in fact, it is of class $(\mathrm{IVe})_{2N}$.

3.5. Symmetry types of hyperelliptic algebraic curves of class V

These are the curves whose induced group $\widehat{G}$ of Möbius transformations is $\widehat{G} = \langle \widehat{\tau}_1, \widehat{a} \,|\, \widehat{\tau}_1^2, \widehat{a}^3, (\widehat{\tau}_1\widehat{a}\widehat{\tau}_1\widehat{a}^{-1})^2 \rangle$, where $\widehat{\tau}_1$ represents an orientation reversing involution and $\widehat{a}$ an orientation preserving one. The signature of Λ is one of the following:

Va: $(0; +; [2^r, 3]; \{(2^p, 2)\})$, with $2r + p = (g+1)/6$,
Vb: $(0; +; [2^r, 6]; \{(2^p, 2)\})$, with $2r + p = (g-3)/6$ and $g > 3$,
Vc: $(0; +; [2^r, 3]; \{(2^p, 4)\})$, with $2r + p = (g-2)/6$ and $g > 2$,
Vd: $(0; +; [2^r, 6]; \{(2^p, 4)\})$, with $2r + p = (g-6)/6$ and $g > 6$.

The group $\widehat{G}$ has also the following presentation

$$\widehat{G} = \langle \widehat{a}, \widehat{b} \,|\, \widehat{a}^{\,3}, \widehat{b}^{\,2}, (\widehat{a}\widehat{b})^3 \rangle \oplus \langle \widehat{\tau}_2 \rangle = \mathrm{A}_4 \oplus \mathrm{Z}_2,$$

where $\widehat{b} := \widehat{\tau}_1\widehat{a}\widehat{\tau}_1\widehat{a}^{-1}$ and $\widehat{\tau}_2 := (\widehat{\tau}_1\widehat{a})^3$. The factor A_4 represents the group of orientation preserving isometries of a regular tetrahedron, whilst $\widehat{\tau}_2$ is the antipodal map since it is an orientation reversing involution which commutes with all the elements of A_4. A fundamental set for the action of $\widehat{G}$ is obtained from a spherical triangle $\triangle V_0V_1V_2$ with angles $\pi/4$, $\pi/4$ and $2\pi/3$. The vertices V_0, V_1 and V_2 are fixed points of $\widehat{b}$, $\widehat{a}^{-1}\widehat{b}\widehat{a}$ and $\widehat{a}$, respectively. The action of $\widehat{a}$ identifies the sides V_2V_1 and V_2V_0 and so the quotient $S^2/\widehat{G}$ is a topological closed disc whose boundary consists of the side V_0V_1. According to the signature of Λ, the boundary contains $p+1$ branch points of the canonical projection $\mathcal{H} \to \mathcal{H}/\Lambda$, whilst the interior contains $r+1$. Figure 5 represents the stereographic projection of the tiling of S^2 by the images of triangle $\triangle V_0V_1V_2$ under the elements of $\widehat{G}$.

The branching order of each V_i with respect to the projection $\mathcal{H} \to \mathcal{H}/\Lambda$ may be read off from the signature of Λ. That of V_2 is 3 in cases Va and Vc, and 6 otherwise; that of V_1 is 2 in cases Va and Vb and 4 otherwise. ($V_0 = V_1$ in the quotient $\mathcal{H}/\Lambda$.) This is the geometric distinction among different classes which will be used for computing equations.

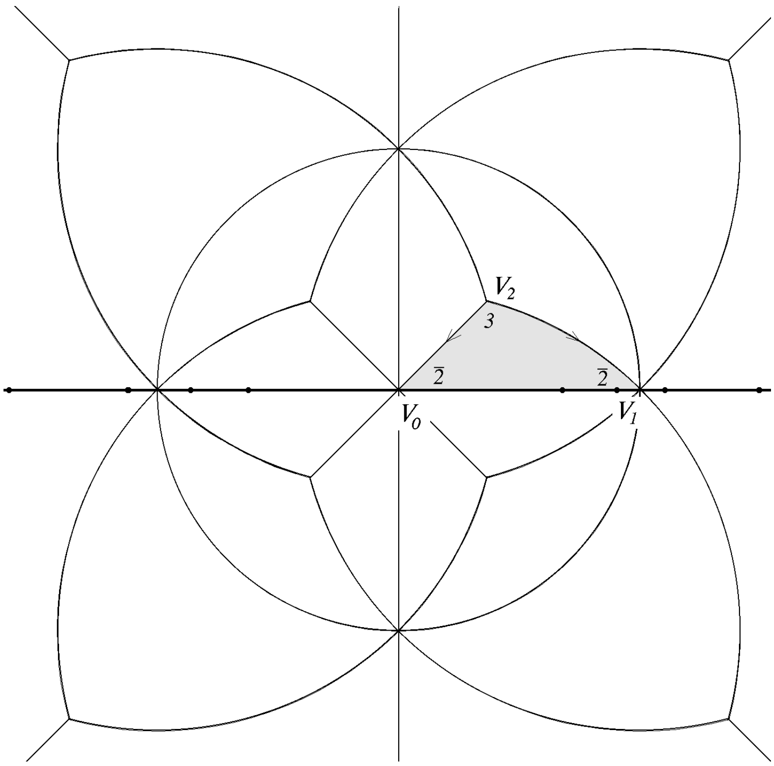


FIGURE 5

We now proceed to compute the species of the liftings of the conjugacy classes of symmetries of $\widehat{G}$. There are two such classes, represented by the reflection $\widehat{\tau}_1$ and the antipodal map $\widehat{\tau}_2$. The species of the liftings τ_2 and $\tau_2\rho$ of $\widehat{\tau}_2$ (if symmetries) are

Class V
$\mathrm{sp}(\tau_2) = \mathrm{sp}(\tau_2\rho) = 0$

In order to compute the species of τ_1 and $\tau_1\rho$ we have to calculate the number of points e_j lying on $\mathrm{Fix}(\widehat{\tau}_1)$, where $e_1, \dots, e_{2g+2}$ are the branch points of the projection $\mathcal{H} \to S^2$. It is easy to see that $\widehat{\tau}_1$ fixes V_0 and V_1 and, in fact, $\widehat{\tau}_1$ is the reflection with respect to the axis containing the arc V_0V_1. Figure 5 shows that $\mathrm{Fix}(\widehat{\tau}_1)$ is divided into four parts, each one being the image of the arc V_0V_1 under elements of $\widehat{G}$. So, the same arguments used in the preceding sections show that the number of branch points e_j lying on $\mathrm{Fix}(\widehat{\tau}_1)$ is four times the number of those lying on the arc V_0V_1. Moreover, V_0 (and so V_1) is one of the e_j in cases Vc and Vd and is not in cases Va and Vb. As a consequence, in cases Va and Vb there are $4p$ branch points e_j lying on $\mathrm{Fix}(\widehat{\tau}_1)$ and $4p+4$ in cases Vc and Vd. This gives us the species of τ_1 and $\tau_1\rho$. Note that there always exists some e_j lying outside $\mathrm{Fix}(\widehat{\tau}_1)$ and so the species cannot attain the maximal value $g+1$. Only in cases Va and Vb, in which the genus is odd, there may be no branch point on $\mathrm{Fix}(\widehat{\tau}_1)$. Consequently,

Va, Vb	$p=0$	$p>0$
$\{\,\mathrm{sp}(\tau_1),\,\mathrm{sp}(\tau_1\rho)\,\}$	$\{2,0\}$	$\{-2p,-2p\}$

Vc, Vd
$\mathrm{sp}(\tau_1)=-(2p+2)$

In cases Vc and Vd we have not included $\tau_1\rho$ because it is conjugate to τ_1. We now collect all symmetry types of curves of class V and list them in the following theorems.

Recall that there exist genus g curves of class Va if and only if $g\equiv 5\,(\mathrm{mod}\ 6)$. They have four real forms, with representatives τ_1, $\tau_1\rho$, τ_2 and $\tau_2\rho$.

Theorem 3.5.1. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* Va *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integer p appearing in the list is positive $\leqslant (g+1)/6$.*

(1) *If* $g\equiv 11\,(\mathrm{mod}\ 12)$:
- $\{2,0,0,0\}$*;*
- $\{-2p,-2p,0,0\}$ *with p even.*

(2) *If* $g\equiv 5\,(\mathrm{mod}\ 12)$:
- $\{-2p,-2p,0,0\}$ *with p odd.*

The necessary and sufficient condition for the existence of genus g curves of class Vb is $g\equiv 3\,(\mathrm{mod}\ 6)$ with $g>3$. They have four real forms, with representatives τ_1, $\tau_1\rho$, τ_2 and $\tau_2\rho$.

Theorem 3.5.2. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* Vb *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integer p is positive $\leqslant (g-3)/6$.*

(1) *If* $g\equiv 3\,(\mathrm{mod}\ 12)$:
- $\{2,0,0,0\}$*;*
- $\{-2p,-2p,0,0\}$ *with p even.*

(2) *If* $g\equiv 9\,(\mathrm{mod}\ 12)$:
- $\{-2p,-2p,0,0\}$ *with p odd.*

Genus g curves of class Vc exist if and only if $g \equiv 2 \pmod 6$ with $g > 2$. They only have one real form, with representative τ_1.

THEOREM 3.5.3. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* Vc *is*

- $\{-(2p+2)\}$ *with* $p \leqslant (g-2)/6$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 2 \pmod{12}, \\ \textit{odd} & \textit{if } g \equiv 8 \pmod{12}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class Vc.

Genus g curves of class Vd exist if and only if $g \equiv 0 \pmod 6$ with $g > 6$. They only have one real form, with representative τ_1.

THEOREM 3.5.4. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* Vd *is*

- $\{-(2p+2)\}$ *with* $p \leqslant (g-6)/6$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 6 \pmod{12}, \\ \textit{odd} & \textit{if } g \equiv 0 \pmod{12}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class Vd.

3.5.1. Equations of curves of class V and their real forms. — Explicit polynomial equations of these curves are more involved than those in the preceding classes. The same happens in classes VI, VII and VIII.

As generators of the group $\widehat{G} = \langle \widehat{a}, \widehat{b} \rangle \oplus \langle \widehat{\tau}_2 \rangle = \mathrm{A}_4 \oplus \mathrm{Z}_2$ we choose the following

$$\widehat{a} : x \longmapsto \frac{-ix+i}{x+1}, \quad \widehat{b} : x \longmapsto -x \quad \text{and} \quad \widehat{\tau}_2 : x \longmapsto \frac{-1}{\overline{x}}.$$

The fixed points of $\widehat{a}$ are $V_2 = (\sqrt{3}-1)(1+i)/2$ and $-1/\overline{V}_2 = (-\sqrt{3}-1)(1+i)/2$; those of $\widehat{b}$ are $V_0 = 0$ and ∞ and those of $\widehat{a}^{-1}\widehat{b}\widehat{a} : x \mapsto 1/x$ are $V_1 = 1$ and -1. It turns out that the following "triangle" F with vertices V_0, V_1 and V_2 is a fundamental set for the action of $\widehat{G}$ on $\overline{\mathbb{C}}$ (see Figure 5):

$$F = \{z \in \mathbb{C} : |z+i|^2 < 2,\ \arg(z) \in [0, \pi/4]\} \cup \{V_2\}.$$

The $\widehat{G}$-orbit of the vertex V_0 consists of ∞ and the roots of the polynomial $x(x^4-1)$; that of V_2 consists of the roots of $x^8 + 14x^4 + 1$. For other points $\alpha \in F$ we denote by P_α the polynomial whose roots constitute the $\widehat{G}$-orbit of α :

$$P_\alpha(x) = \prod_{t \in \mathrm{orbit}(\alpha)} (x-t).$$

Note that P_α has degree 24 if α is fixed by no element of $\widehat{G}$ and degree 12 if α lies on $\mathrm{Fix}(\widehat{\tau}_1) - \{V_0, V_1\}$, that is, if $0 < \alpha < 1$.

THEOREM 3.5.5. — *With the above notations, for almost every choice of pairwise different points*

- $\{\omega_1, \dots, \omega_r\} \subset \{z \in F : \arg(z) \neq 0\}$ *with* $\omega_j \neq V_2$,
- $0 < \lambda_1 < \cdots < \lambda_p < 1$,

with $r+p>0$, the equation

$$y^2 = (x^5 - x)^{\varepsilon_1} \cdot (x^8 + 14x^4 + 1)^{\varepsilon_2} \cdot \prod_{j=1}^{r} P_{\omega_j}(x) \cdot \prod_{j=1}^{p} P_{\lambda_j}(x),$$

where $\varepsilon_i \in \{0,1\}$, defines a hyperelliptic complex curve X of class

a) Va *and genus $g = 12r + 6p - 1$ if $\varepsilon_1 = \varepsilon_2 = 0$;*
b) Vb *and genus $g = 12r + 6p + 3$ if $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$;*
c) Vc *and genus $g = 12r + 6p + 2$ if $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$;*
d) Vd *and genus $g = 12r + 6p + 6$ if $\varepsilon_1 = \varepsilon_2 = 1$.*

Conversely, each genus g hyperelliptic curve of class V *is (isomorphic to another) of the above form for some r, p, ε_1 and ε_2.*

Representatives of all the real forms of X are the following

Classes Va *and* Vb
τ_1, $\tau_1\rho$, τ_2, $\tau_2\rho$

Classes Vc *and* Vd
τ_1

where

$$\tau_1 : (x,y) \longmapsto (\overline{x}, \overline{y}) \quad \text{and} \quad \tau_2 : (x,y) \longmapsto \left(\frac{-1}{\overline{x}}, \frac{\overline{y}}{\overline{x}^{g+1}}\right).$$

In all cases the species of each real form is given.

Proof. — It is easy to check that the transformation $x \mapsto 1/x$ belongs to $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ and so $P_{\omega_j}(0) = P_{\lambda_j}(0) = 1$ for all j. It follows that the formula of τ_2 is the given above. In particular,

$$\mathrm{sp}(\tau_2) = \mathrm{sp}(\tau_2\rho) = 0.$$

By construction of the branch point set of X, there are $4p + 4\varepsilon_1$ branch points on $\mathrm{Fix}(\widehat{\tau}_1) = \mathbb{R} \cup \{\infty\}$. Since $\widehat{\tau}_1$ is complex conjugation and $\varepsilon_1 = 0$ implies g odd, Theorem 1.3.4 directly gives

$$\mathrm{sp}(\tau_1) = \begin{cases} -2(p+\varepsilon_1) & \text{if } p + \varepsilon_1 > 0, \\ 2 & \text{if } p = \varepsilon_1 = 0. \end{cases} \qquad \mathrm{sp}(\tau_1\rho) = \begin{cases} -2(p+\varepsilon_1) & \text{if } p + \varepsilon_1 > 0, \\ 0 & \text{if } p = \varepsilon_1 = 0. \end{cases}$$

□

We observe that the condition $r+p>0$ in the statement of the theorem is necessary. Indeed, for $r = p = 0$ we get the curves

- $y^2 = x^8 + 14x^4 + 1$ in case b,
- $y^2 = x(x^4 - 1)$ in case c and
- $y^2 = x(x^4 - 1)(x^8 + 14x^4 + 1)$ in case d.

However, none of them is of class V since in all cases the induced group $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ of Möbius transformations contains $x \mapsto ix$, which has order 4. Therefore, it cannot coincide with $\mathrm{A}_4 \oplus \mathrm{Z}_2$. We will see that they are curves of class VIIb, VIIc and VIId respectively.

In order to obtain examples of curves of low genus we have to consider the orbit of some $\lambda \in (0,1)$. For example, an easy computation shows that the orbit of $\lambda = \sqrt{3}/3$ consists of the roots of the polynomial

$$P_{\sqrt{3}/3}(x) = (x^4 - \frac{10}{3}x^2 + 1)(x^4 + 14x^2 + 1)(x^4 + x^2 + 1).$$

Hence,

- $y^2 = P_{\sqrt{3}/3}(x)$ is a genus 5 curve of class Va;
- $y^2 = (x^8 + 14x^4 + 1)P_{\sqrt{3}/3}(x)$ is a genus 9 curve of class Vb;
- $y^2 = x(x^4 - 1)P_{\sqrt{3}/3}(x)$ is a genus 8 curve of class Vc;
- $y^2 = x(x^4 - 1)(x^8 + 14x^4 + 1)P_{\sqrt{3}/3}(x)$ is a genus 12 curve of class Vd.

3.6. Symmetry types of hyperelliptic algebraic curves of class VI

These are the curves whose induced group $\widehat{G}$ of Möbius transformations is $\widehat{G} = \langle \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3, \mid \widehat{\tau}_1^2, \widehat{\tau}_2^2, \widehat{\tau}_3^2, (\widehat{\tau}_1\widehat{\tau}_2)^2, (\widehat{\tau}_2\widehat{\tau}_3)^3, (\widehat{\tau}_1\widehat{\tau}_3)^3 \rangle$, where each $\widehat{\tau}_i$ represents an orientation reversing involution. The signature of Λ is one of the following:

VIa: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},3)\})$, with $2r+p_1+p_2+p_3 = (g+1)/6$,
VIb: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},6)\})$, with $2r+p_1+p_2+p_3 = (g-1)/6$,
VIc: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},6)\})$, with $2r+p_1+p_2+p_3 = (g-3)/6 > 0$,
VId: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},3)\})$, with $2r+p_1+p_2+p_3 = (g-2)/6 > 0$,
VIe: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},6)\})$, with $2r+p_1+p_2+p_3 = (g-4)/6$,
VIf: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},6)\})$, with $2r+p_1+p_2+p_3 = (g-6)/6 > 0$.

Since each generator of $\widehat{G}$ is a reflection it follows from its presentation that a fundamental set for the $\widehat{G}$-action is a spherical triangle $\triangle V_0V_1V_2$ with angles $\pi/2$, $\pi/3$ and $\pi/3$. Its sides V_0V_1, V_0V_2 and V_1V_2 are contained in $\mathrm{Fix}(\widehat{\tau}_1)$, $\mathrm{Fix}(\widehat{\tau}_2)$ and $\mathrm{Fix}(\widehat{\tau}_3)$ respectively. So, the vertices V_0, V_1 and V_2 are fixed points of the rotations $\widehat{\tau}_1\widehat{\tau}_2$, $\widehat{\tau}_1\widehat{\tau}_3$ and $\widehat{\tau}_2\widehat{\tau}_3$ respectively. ($\widehat{G}$ is the full group of isometries of a regular tetrahedron.) Figure 6 represents the stereographic projection of the tiling of S^2 by the images of triangle $\triangle V_0V_1V_2$ under the elements of $\widehat{G}$.

The vertices V_0, V_1 and V_2 are boundary branch points of the projection $\mathcal{H} \to \mathcal{H}/\Lambda$. The branching order of V_0 is 2 in classes VIa, VIb and VIc and 4 otherwise; that of V_1 is 3 in classes VIa, VIb, VId and VIe and 6 otherwise; that of V_2 is 3 in classes VIa and VId and 6 otherwise. This gives a geometric distinction between curves of class VI in terms of the distribution of their branch points $e_1, \ldots, e_{2g+2}$. In concrete,

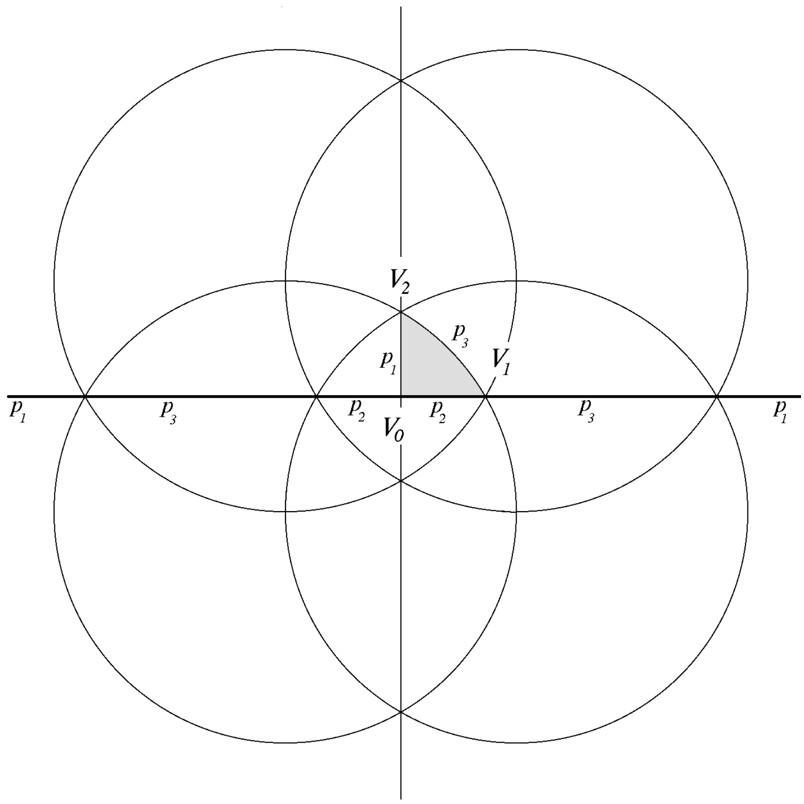


Figure 6

	Class VIa	Class VIb	Class VIc	Class VId	Class VIe	Class VIf
V_0	no	no	no	yes	yes	yes
V_1	no	no	yes	no	no	yes
V_2	no	yes	yes	no	yes	yes

where a "yes" means that the vertex V_i is one of the branch points e_j of a curve of the corresponding class. According also to the signature of Λ, there are p_2 boundary branch points in V_0V_1, p_3 in V_1V_2 and p_1 in V_0V_2, all with branching order 2.

There is only one conjugacy class of symmetries in $\widehat{G}$, with representative $\widehat{\tau}_1$. Since it is the reflection with respect to the line containing the side V_0V_1 we have to calculate the number of branch points e_j lying on this line. It is easy to check that the number of such points on $\mathrm{Fix}(\widehat{\tau}_1)$ but not on the orbits of V_0, V_1 or V_2 is $2(p_1+p_2+p_3)$ (see Figure 6). Since the orbit of each V_i has 2 points on $\mathrm{Fix}(\widehat{\tau}_1)$, we conclude that the number of e_j fixed by $\widehat{\tau}_1$ is

- $2(p_1 + p_2 + p_3)$ in class VIa,
- $2(p_1 + p_2 + p_3 + 1)$ in classes VIb and VId,
- $2(p_1 + p_2 + p_3 + 2)$ in classes VIc and VIe,
- $2(p_1 + p_2 + p_3 + 3)$ in class VIf.

Note that there always exist branch points lying outside $\mathrm{Fix}(\widehat{\tau}_1)$. Thus the species of τ_1 and $\tau_1\rho$ cannot attain the maximal value $g+1$. Only in class VIa, in which the genus is odd, there may be no branch point fixed by $\widehat{\tau}_1$. Consequently, the species of τ_1 and $\tau_1\rho$ are the following.

VIa	$p_1 + p_2 + p_3 = 0$	$p_1 + p_2 + p_3 > 0$
$\{\,\mathrm{sp}(\tau_1),\, \mathrm{sp}(\tau_1\rho)\,\}$	$\{2, 0\}$	$\{-(p_1 + p_2 + p_3), -(p_1 + p_2 + p_3)\}$

VIb, VId
$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho) = -(p_1 + p_2 + p_3 + 1)$

VIc, VIe
$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho) = -(p_1 + p_2 + p_3 + 2)$

VIf
$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho) = -(p_1 + p_2 + p_3 + 3)$

Recall that curves of class VIa, VIb or VIc have two real forms, with representatives τ_1 and $\tau_1\rho$. Those of class VId, VIe or VIf have just one, with representative τ_1. The complete list of symmetry types of curves of class VI is given along the following theorems. For brevity, we write p instead of $p_1 + p_2 + p_3$.

There exist genus g curves of class VIa if and only if $g \equiv 5 \pmod 6$.

THEOREM 3.6.1. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIa *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The integer p appearing in the list is positive* $\leqslant (g+1)/6$.

(1) *If* $g \equiv 11 \pmod{12}$:
  - $\{2, 0\}$*;*
  - $\{-p, -p\}$ *with p even.*

(2) *If* $g \equiv 5 \pmod{12}$:
  - $\{-p, -p\}$ *with p odd.*

There exist genus g curves of class VIb if and only if $g \equiv 1 \pmod 6$.

THEOREM 3.6.2. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIb *is*

- $\{-(p+1),-(p+1)\}$ *with* $p \leqslant (g-1)/6$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 1 \pmod{12}, \\ \textit{odd} & \textit{if } g \equiv 7 \pmod{12}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIb.

There exist genus g curves of class VIc if and only if $g \equiv 3 \pmod 6$ with $g > 3$.

THEOREM 3.6.3. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIc *is*

- $\{-(p+2),-(p+2)\}$ *with* $p \leqslant (g-3)/6$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 3 \pmod{12}, \\ \textit{odd} & \textit{if } g \equiv 9 \pmod{12}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIc.

There exist genus g curves of class VId if and only if $g \equiv 2 \pmod 6$ with $g > 2$.

THEOREM 3.6.4. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VId *is*

- $\{-(p+1)\}$ *with* $p \leqslant (g-2)/6$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 2 \pmod{12}, \\ \textit{odd} & \textit{if } g \equiv 8 \pmod{12}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VId.

There exist genus g curves of class VIe if and only if $g \equiv 4 \pmod 6$.

THEOREM 3.6.5. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIe *is*

- $\{-(p+2)\}$ *with* $p \leqslant (g-4)/6$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 4 \pmod{12}, \\ \textit{odd} & \textit{if } g \equiv 10 \pmod{12}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIe.

There exist genus g curves of class VIf if and only if $g \equiv 0 \pmod 6$ with $g > 6$.

THEOREM 3.6.6. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIf *is*

- $\{-(p+3)\}$ *with* $p \leqslant (g-6)/6$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 6 \pmod{12}, \\ \textit{odd} & \textit{if } g \equiv 0 \pmod{12}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIf.

3.6.1. Equations of curves of class VI and their real forms. — As generators of the group $\widehat{G} = \langle \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3, \mid \widehat{\tau}_1^2, \widehat{\tau}_2^2, \widehat{\tau}_3^2, (\widehat{\tau}_1\widehat{\tau}_2)^2, (\widehat{\tau}_2\widehat{\tau}_3)^3, (\widehat{\tau}_1\widehat{\tau}_3)^3 \rangle$, we choose the following

$$\widehat{\tau}_1 : x \longmapsto \overline{x}, \qquad \widehat{\tau}_2 : x \longmapsto -\overline{x} \quad \text{and} \quad \widehat{\tau}_3 : x \longmapsto \frac{(1+i)\overline{x} - 2}{-\overline{x} - 1 + i}.$$

The fixed point set of $\widehat{\tau}_1$, $\widehat{\tau}_2$ and $\widehat{\tau}_3$ are the real axis $\mathbb{R} \cup \{\infty\}$, the imaginary axis $i\mathbb{R} \cup \{\infty\}$ and the circumference $\{|z + 1 + i| = 2\}$, respectively. Thus the "triangle" with vertices $V_0 = 0 \in \operatorname{Fix}(\widehat{\tau}_1) \cap \operatorname{Fix}(\widehat{\tau}_2)$, $V_1 = \sqrt{3} - 1 \in \operatorname{Fix}(\widehat{\tau}_1) \cap \operatorname{Fix}(\widehat{\tau}_3)$ and $V_2 = (\sqrt{3} - 1)i \in \operatorname{Fix}(\widehat{\tau}_2) \cap \operatorname{Fix}(\widehat{\tau}_3)$ is a fundamental set for the action of $\widehat{G}$ on $\widehat{\mathbb{C}}$ (see Figure 6). Explicitly,

$$F = \{z \in \mathbb{C} : |z + 1 + i| \leqslant 2, \ \arg(z) \in [\,0, \pi/2\,]\,\}.$$

The $\widehat{G}$-orbit of the vertex V_0 consists of ∞ and the roots of the polynomial $x(x^4 + 4)$; that of V_1 consists of the roots of $x^4 + 4\sqrt{3}x^2 - 4$ and that of V_2 consists of the roots of $x^4 - 4\sqrt{3}x^2 - 4$. For other points $\alpha \in F$ we denote by P_α the polynomial whose roots constitute the orbit of α:

$$P_\alpha(x) = \prod_{t \,\in\, \operatorname{orbit}(\alpha)} (x - t).$$

Note that $\deg P_\alpha = 24$ if α is fixed by no element of $\widehat{G}$ and $\deg P_\alpha = 12$ if $\alpha \neq V_i$ is fixed by some $\widehat{\tau}_i$.

These computations, together with the distinction between curves of class VI in terms of V_0, V_1 and V_2 given at the beginning of this section, give the following.

THEOREM 3.6.7. — *With the above notations, for almost every choice of pairwise different points*

- $\{\omega_1, \ldots, \omega_r\}$ *in the interior of* F,
- $\{\lambda_1, \ldots, \lambda_p\}$ *in the boundary of* F *with* $\lambda_j \neq V_0, V_1, V_2$,

the equation

$$y^2 = (x^5 + 4x)^{\varepsilon_0} \cdot (x^4 + 4\sqrt{3}x^2 - 4)^{\varepsilon_1} \cdot (x^4 - 4\sqrt{3}x^2 - 4)^{\varepsilon_2} \prod_{j=1}^{r} P_{\omega_j}(x) \cdot \prod_{j=1}^{p} P_{\lambda_j}(x),$$

where $\varepsilon_i \in \{0, 1\}$, *defines a hyperelliptic complex curve* X *of class*

a) VIa *and genus* $g = 12r + 6p - 1$ *if* $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$;
b) VIb *and genus* $g = 12r + 6p + 1$ *if* $\varepsilon_0 = \varepsilon_1 = 0$, $\varepsilon_2 = 1$;
c) VIc *and genus* $g = 12r + 6p + 3$ *if* $\varepsilon_0 = 0$, $\varepsilon_1 = \varepsilon_2 = 1$ *and* $2r + p > 0$;
d) VId *and genus* $g = 12r + 6p + 2$ *if* $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = 0$ *and* $2r + p > 0$;
e) VIe *and genus* $g = 12r + 6p + 4$ *if* $\varepsilon_0 = \varepsilon_2 = 1$, $\varepsilon_1 = 0$;
f) VIf *and genus* $g = 12r + 6p + 6$ *if* $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$ *and* $2r + p > 0$.

Conversely, each genus g hyperelliptic curve of class VI *is (isomorphic to another) of the above form for some r, p, ε_0, ε_1 and ε_2.*

Representatives of all the real forms of X are the following

Classes VIa, VIb *and* VIc
τ_1, $\tau_1\rho$

Classes VId, VIe *and* Vf
τ_1

where $\tau_1 : (x,y) \mapsto (\overline{x}, \overline{y})$. *Their species are*

$$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho) = -(p+\varepsilon_0+\varepsilon_1+\varepsilon_2) \quad \textit{if } p+\varepsilon_0+\varepsilon_1+\varepsilon_2 > 0,$$

$$\mathrm{sp}(\tau_1) = 2, \qquad \mathrm{sp}(\tau_1\rho) = 0 \qquad \textit{if } p=\varepsilon_0=\varepsilon_1=\varepsilon_2=0.$$

Observe that for $r = p = 0$ we obtain

- $y^2 = x^8 - 56x^4 + 16$ in case c,
- $y^2 = x(x^4 + 4)$ in case d and
- $y^2 = x(x^4 + 4)(x^8 - 56x^4 + 16)$ in case f.

However, none of them is of class VI since in all cases the induced group $\mathrm{Aut}^{\pm} X_{\widehat{\mathbb{C}}}$ of Möbius transformations contains $x \mapsto ix$, which has order 4. Therefore, it cannot coincide with the group $\widehat{G}$ we are working with since $\widehat{G}$ contains no orientation preserving transformation of order 4. A lifting of $x \mapsto x/(1+i)$ transforms the above curves into $y^2 = x^8 + 14x^4 + 1$, $y^2 = x(x^4 - 1)$ and $y^2 = x(x^4 - 1)(x^8 + 14x^4 + 1)$ respectively. We will see that these are curves of class VIIb, VIIc and VIId respectively.

The lowest genus of a curve of class VI is $g = 4$, which is attained by the curve of class VIe given by $y^2 = x(x^4 + 4)(x^4 - 4\sqrt{3}x^2 - 4)$.

3.7. Symmetry types of hyperelliptic algebraic curves of class VII

These are the curves whose induced group $\widehat{G}$ of Möbius transformations is $\widehat{G} = \langle \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3, \mid \widehat{\tau}_1^2, \widehat{\tau}_2^2, \widehat{\tau}_3^2, (\widehat{\tau}_1\widehat{\tau}_2)^2, (\widehat{\tau}_2\widehat{\tau}_3)^3, (\widehat{\tau}_1\widehat{\tau}_3)^4 \rangle$, where each $\widehat{\tau}_i$ represents an orientation reversing involution. The signature of Λ is one of the following:

VIIa: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},4)\})$, with $2r+p_1+p_2+p_3 = (g+1)/12$,
VIIb: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},4)\})$, with $2r+p_1+p_2+p_3 = (g-3)/12$,
VIIc: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},8)\})$, with $2r+p_1+p_2+p_3 = (g-2)/12$,
VIId: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},8)\})$, with $2r+p_1+p_2+p_3 = (g-6)/12$,
VIIe: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},4)\})$, with $2r+p_1+p_2+p_3 = (g-5)/12$,
VIIf: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},4)\})$, with $2r+p_1+p_2+p_3 = (g-9)/12$.
VIIg: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},8)\})$, with $2r+p_1+p_2+p_3 = (g-8)/12$,
VIIh: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},8)\})$, with $2r+p_1+p_2+p_3 = (g-12)/12$.

Since each generator of $\widehat{G}$ is a reflection it follows from its presentation that a fundamental set for the $\widehat{G}$-action is a spherical triangle $\triangle V_0V_1V_2$ with angles $\pi/4$, $\pi/2$

and $\pi/3$. Its sides V_0V_1, V_1V_2 and V_0V_2 are contained in $\mathrm{Fix}(\widehat{\tau}_1)$, $\mathrm{Fix}(\widehat{\tau}_2)$ and $\mathrm{Fix}(\widehat{\tau}_3)$ respectively. So, the vertices V_0, V_1 and V_2 are fixed points of the rotations $\widehat{\tau}_1\widehat{\tau}_3$, $\widehat{\tau}_1\widehat{\tau}_2$ and $\widehat{\tau}_2\widehat{\tau}_3$ respectively. ($\widehat{G}$ is the full group of isometries of a regular cube.) Figure 7 represents the stereographic projection of the tiling of S^2 by the images of triangle $\triangle V_0V_1V_2$ under the elements of $\widehat{G}$.

According to the signature of Λ, there are p_1 boundary branch points of the projection $\mathcal{H} \to \mathcal{H}/\Lambda$ in V_0V_1, p_2 in V_1V_2 and p_3 in V_0V_2. They all have branching order 2. There are three more boundary branch points, namely V_0, V_1 and V_2. Their branching orders vary according to the signature of Λ. This gives a geometric distinction among curves of class VII in terms of the distribution of their branch points $e_1, \ldots, e_{2g+2}$. In concrete,

	VIIa	VIIb	VIIc	VIId	VIIe	VIIf	VIIg	VIIh
V_0	no	no	yes	yes	no	no	yes	yes
V_1	no	no	no	no	yes	yes	yes	yes
V_2	no	yes	no	yes	no	yes	no	yes

where a "yes" means that the vertex V_i is one of the e_j of a curve of the corresponding class.

The group $\widehat{G}$ has exactly three conjugacy classes of symmetries, with representatives $\widehat{\tau}_1$, $\widehat{\tau}_3$ and $\widehat{\tau}_4 := (\widehat{\tau}_1\widehat{\tau}_2\widehat{\tau}_3)^3$. The last one is the antipodal map since it is central in $\widehat{G}$, as is easy to check. So the species of its liftings τ_4 and $\tau_4\rho$ (if symmetries) are

Class VII
$\mathrm{sp}(\tau_4) = \mathrm{sp}(\tau_4\rho) = 0$

The species of the liftings of $\widehat{\tau}_1$ depend on the number of points e_j lying on $\mathrm{Fix}(\widehat{\tau}_1)$. Since $\widehat{\tau}_1$ is the reflection with respect to the line containing the side V_0V_1 it follows from Figure 7 that, apart from the orbits of V_0, V_1 and V_2, there are $8p_1$ branch points e_j lying on $\mathrm{Fix}(\widehat{\tau}_1)$. Now, each of the orbits of V_0 and V_1 has 4 points on $\mathrm{Fix}(\widehat{\tau}_1)$, whilst that of V_2 has no point on it. It is also clear that there always exist branch points not fixed by $\widehat{\tau}_1$ and so the species of τ_1 and $\tau_1\rho$ cannot attain the value $g+1$. Only in cases VIIa and VIIb, in which the genus is odd, there may be no branch point fixed by $\widehat{\tau}_1$. Consequently, the species of τ_1 and $\tau_1\rho$ are the following.

VIIa, VIIb	$p_1 = 0$	$p_1 > 0$
$\{\mathrm{sp}(\tau_1), \mathrm{sp}(\tau_1\rho)\}$	$\{2, 0\}$	$\{-4p_1, -4p_1\}$

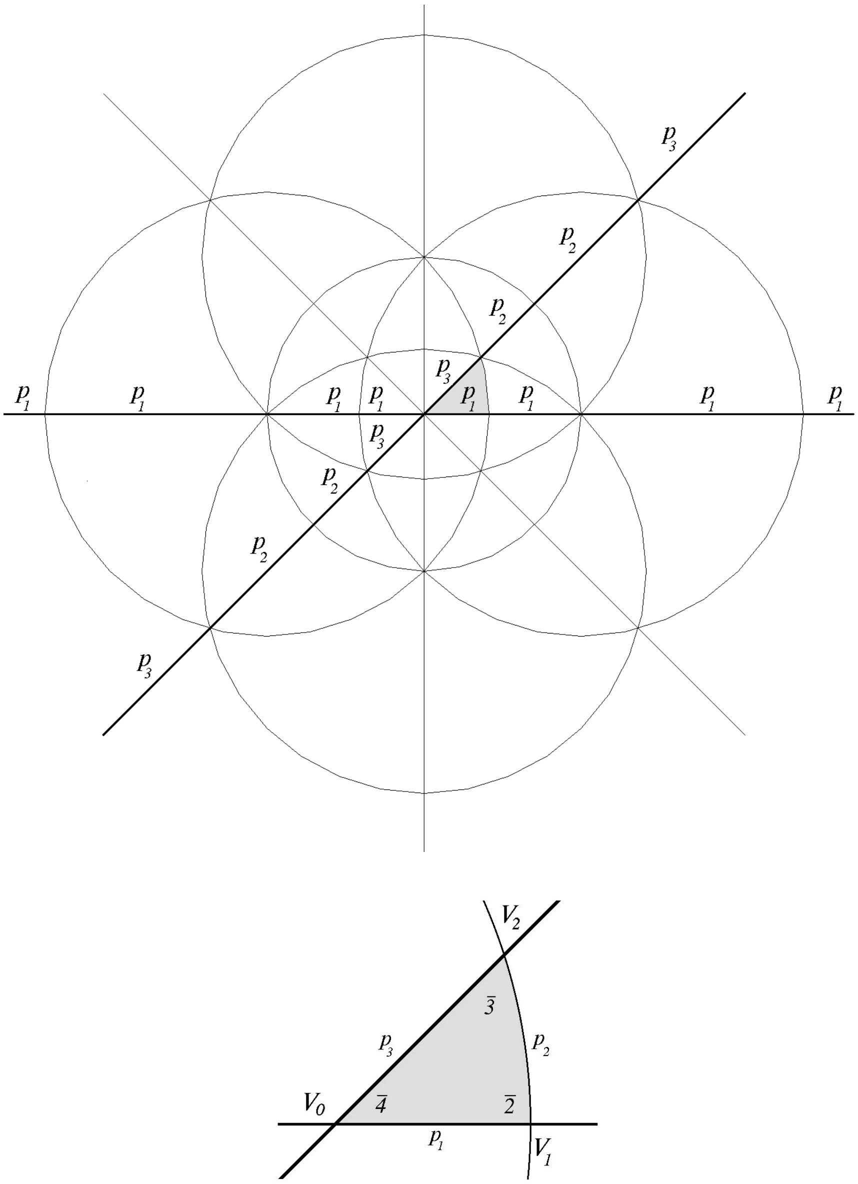

p_3
p_2
p_2
p_3
p_1 p_1 p_1 p_1 p_1 p_1 p_1 p_1
p_3
p_2
p_2
p_3
V_2
$\bar{3}$
p_3
p_2
V_0 $\bar{4}$ $\bar{2}$
p_1
V_1

FIGURE 7

VIIc, VIId, VIIe, VIIf
$\mathrm{sp}(\tau_1) = -(4p_1 + 2)$

VIIg, VIIh
$\mathrm{sp}(\tau_1) = -(4p_1 + 4)$

Note that in cases VIIc to VIIh we have omitted the species of $\tau_1\rho$ because this symmetry is conjugate to τ_1.

The fixed point set of the reflection $\widehat{\tau}_3$ is the line containing the side V_0V_2. It follows that, apart from the orbits of V_0, V_1 and V_2, there are $4p_2 + 4p_3$ branch points e_j lying on $\mathrm{Fix}(\widehat{\tau}_3)$. Each of the orbits of V_0 and V_1 has 2 points on $\mathrm{Fix}(\widehat{\tau}_3)$ and that of V_2 has 4. Again, not all the e_j are fixed by $\widehat{\tau}_3$ and so the species of τ_3 and $\tau_3\rho$ cannot attain the value $g + 1$. Only in case VIIa, in which the genus is odd, there may be no e_j on $\mathrm{Fix}(\widehat{\tau}_3)$. Consequently, the species of τ_3 and $\tau_3\rho$ are the following.

VIIa	$p_2 + p_3 = 0$	$p_2 + p_3 > 0$
$\{\mathrm{sp}(\tau_3), \mathrm{sp}(\tau_3\rho)\}$	$\{2, 0\}$	$\{-2(p_2 + p_3), -2(p_2 + p_3)\}$

VIIb, VIIg
$\mathrm{sp}(\tau_3) = \mathrm{sp}(\tau_3\rho) = -(2p_2 + 2p_3 + 2)$

VIIc, VIIe
$\mathrm{sp}(\tau_3) = -(2p_2 + 2p_3 + 1)$

VIId, VIIf
$\mathrm{sp}(\tau_3) = -(2p_2 + 2p_3 + 3)$

VIIh
$\mathrm{sp}(\tau_3) = -(2p_2 + 2p_3 + 4)$

Note that we have joined cases VIIb and VIIg although in the latter case τ_3 is conjugate to $\tau_3\rho$. We now collect all symmetry types of curves of class VII and list them in the following theorems.

There exist genus g curves of class VIIa if and only if $g \equiv 11 \pmod{12}$. They have six real forms, with representatives τ_1, $\tau_1\rho$, τ_3, $\tau_3\rho$, τ_4 and $\tau_4\rho$.

Theorem 3.7.1. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIa *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The sum of all the distinct integers p_i appearing in a symmetry type is* $\leqslant (g+1)/12$ *and* $\equiv (g+1)/12 \pmod 2$*. Moreover,* $p_1 > 0$ *and* $p_2 + p_3 > 0$.

- $\{2, 0, 2, 0, 0, 0\}$, *only if* $g \equiv 23 \pmod{24}$*;*
- $\{2, 0, -2(p_2 + p_3), -2(p_2 + p_3), 0, 0\}$*;*
- $\{-4p_1, -4p_1, 2, 0, 0, 0\}$;
- $\{-4p_1, -4p_1, -2(p_2 + p_3), -2(p_2 + p_3), 0, 0\}$.

There exist genus g curves of class VIIb if and only if $g \equiv 3 \pmod{12}$. They have six real forms, with representatives τ_1, $\tau_1\rho$, τ_3, $\tau_3\rho$, τ_4 and $\tau_4\rho$.

Theorem 3.7.2. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIb *is one of the listed below. Conversely, each of them is the symmetry type of such a curve. The sum of all the distinct integers p_i appearing in a symmetry type is* $\leqslant (g-3)/12$ *and* $\equiv (g-3)/12 \pmod 2$*. Moreover,* $p_1 > 0$.

- $\{2, 0, -(2p_2 + 2p_3 + 2), -(2p_2 + 2p_3 + 2), 0, 0\}$*;*
- $\{-4p_1, -4p_1, -(2p_2 + 2p_3 + 2), -(2p_2 + 2p_3 + 2), 0, 0\}$.

There exist genus g curves of class VIIc if and only if $g \equiv 2 \pmod{12}$. They have two real forms, with representatives τ_1 and τ_3.

Theorem 3.7.3. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIc *is*

$$\{-(4p_1 + 2), -(2p_2 + 2p_3 + 1)\}$$

with $p_1 + p_2 + p_3 \leqslant (g-2)/12$ *and* $\equiv (g-2)/12 \pmod 2$*. Conversely, this is the symmetry type of some genus g curve of class* VIIc.

There exist genus g curves of class VIId if and only if $g \equiv 6 \pmod{12}$. They have two real forms, with representatives τ_1 and τ_3.

Theorem 3.7.4. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIId *is*

$$\{-(4p_1 + 2), -(2p_2 + 2p_3 + 3)\}$$

with $p_1 + p_2 + p_3 \leqslant (g-6)/12$ *and* $\equiv (g-6)/12 \pmod 2$*. Conversely, this is the symmetry type of some genus g curve of class* VIId.

There exist genus g curves of class VIIe if and only if $g \equiv 5 \pmod{12}$. They have three real forms, with representatives τ_1, τ_3 and τ_4.

THEOREM 3.7.5. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIe *is*

$$\{-(4p_1+2), -(2p_2+2p_3+1), 0\}$$

with $p_1+p_2+p_3 \leqslant (g-5)/12$ *and* $\equiv (g-5)/12 \pmod 2$. *Conversely, this is the symmetry type of some genus g curve of class* VIIe.

There exist genus g curves of class VIIf if and only if $g \equiv 9 \pmod{12}$. They have three real forms, with representatives τ_1, τ_3 and τ_4.

THEOREM 3.7.6. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIf *is*

$$\{-(4p_1+2), -(2p_2+2p_3+3), 0\}$$

with $p_1+p_2+p_3 \leqslant (g-9)/12$ *and* $\equiv (g-9)/12 \pmod 2$. *Conversely, this is the symmetry type of some genus g curve of class* VIIf.

There exist genus g curves of class VIIg if and only if $g \equiv 8 \pmod{12}$. They have two real forms, with representatives τ_1 and τ_3.

THEOREM 3.7.7. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIg *is*

$$\{-(4p_1+4), -(2p_2+2p_3+2)\}$$

with $p_1+p_2+p_3 \leqslant (g-8)/12$ *and* $\equiv (g-8)/12 \pmod 2$. *Conversely, this is the symmetry type of some genus g curve of class* VIIg.

There exist genus g curves of class VIIh if and only if $g \equiv 0 \pmod{12}$. They have two real forms, with representatives τ_1 and τ_3.

THEOREM 3.7.8. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIh *is*

$$\{-(4p_1+4), -(2p_2+2p_3+4)\}$$

with $p_1+p_2+p_3 \leqslant (g-12)/12$ *and* $\equiv (g-12)/12 \pmod 2$. *Conversely, this is the symmetry type of some genus g curve of class* VIIh.

3.7.1. Equations of curves of class VII and their real forms. — As generators of the group $\widehat{G} = \langle \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3, \mid \widehat{\tau}_1^2, \widehat{\tau}_2^2, \widehat{\tau}_3^2, (\widehat{\tau}_1\widehat{\tau}_2)^2, (\widehat{\tau}_2\widehat{\tau}_3)^3, (\widehat{\tau}_1\widehat{\tau}_3)^4 \rangle$, we choose the following

$$\widehat{\tau}_1 : x \longmapsto \overline{x}, \qquad \widehat{\tau}_2 : x \longmapsto \frac{-\overline{x}+1}{\overline{x}+1} \quad \text{and} \quad \widehat{\tau}_3 : x \longmapsto i\overline{x}.$$

The fixed point sets of $\widehat{\tau}_1$, $\widehat{\tau}_2$ and $\widehat{\tau}_3$ are the real axis, the circumference $\{|z+1|^2 = 2\}$ and the axis $\{\arg(z) = \pi/4 \text{ or } 5\pi/4\}$, respectively. Thus the "triangle" with vertices

$V_0 = 0 \in \mathrm{Fix}(\widehat{\tau}_1)\cap\mathrm{Fix}(\widehat{\tau}_3)$, $V_1 = \sqrt{2}-1 \in \mathrm{Fix}(\widehat{\tau}_1)\cap\mathrm{Fix}(\widehat{\tau}_2)$ and $V_2 = (\sqrt{3}-1)(1+i)/2 \in \mathrm{Fix}(\widehat{\tau}_2)\cap\mathrm{Fix}(\widehat{\tau}_3)$ is a fundamental set for the action of $\widehat{G}$ on $\widehat{\mathbb{C}}$ (see Figure 7). Explicitly,

$$F = \{z \in \mathbb{C} : |z+1|^2 \leqslant 2,\ \arg(z) \in [\,0, \pi/4\,]\,\}.$$

The $\widehat{G}$-orbit of the vertex V_0 consists of ∞ and the roots of the polynomial $x(x^4-1)$; that of V_1 consists of the roots of $x^{12} - 33x^8 - 33x^4 + 1$ and that of V_2 consists of the roots of $x^8 + 14x^4 + 1$. For other points $\alpha \in F$ we denote by P_α the polynomial whose roots constitute the orbit of α:

$$P_\alpha(x) = \prod_{t\,\in\,\mathrm{orbit}(\alpha)} (x - t).$$

Note that $\deg P_\alpha = 48$ if α is fixed by no element of $\widehat{G}$ and $\deg P_\alpha = 24$ if $\alpha \neq V_i$ is fixed by some $\widehat{\tau}_i$.

Theorem 3.7.9. — *With the above notations, for every choice of pairwise distinct points*

- $\{\omega_1, \ldots, \omega_r\}$ *in the interior of* F,
- $\{\lambda_1, \ldots, \lambda_{p_1}\}$ *in* V_0V_1 *with* $\lambda_j \neq V_0$, V_1,
- $\{\mu_1, \ldots, \mu_{p_2+p_3}\}$ *in* $V_1V_2 \cup V_0V_2$ *with* $\mu_j \neq V_0$, V_1, V_2,

the equation $y^2 = P(x)$ *where* $P(x)$ *is the polynomial*

$$(x^5-x)^{\varepsilon_0}\cdot(x^{12}-33x^8-33x^4+1)^{\varepsilon_1}\cdot(x^8+14x^4+1)^{\varepsilon_2}\cdot\prod_{j=1}^{r} P_{\omega_j}(x)\cdot\prod_{j=1}^{p_1} P_{\lambda_j}(x)\cdot\prod_{j=1}^{p_2+p_3} P_{\mu_j}(x)$$

with $\varepsilon_i \in \{0,1\}$ *defines a hyperelliptic complex curve* X *of class*

a) VIIa *and genus* $g = 12(2r+p_1+p_2+p_3) - 1$ *if* $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$;
b) VIIb *and genus* $g = 12(2r+p_1+p_2+p_3) + 3$ *if* $\varepsilon_0 = \varepsilon_1 = 0$, $\varepsilon_2 = 1$;
c) VIIc *and genus* $g = 12(2r+p_1+p_2+p_3) + 2$ *if* $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = 0$;
d) VIId *and genus* $g = 12(2r+p_1+p_2+p_3) + 6$ *if* $\varepsilon_0 = \varepsilon_2 = 1$, $\varepsilon_1 = 0$;
e) VIIe *and genus* $g = 12(2r+p_1+p_2+p_3) + 5$ *if* $\varepsilon_0 = \varepsilon_2 = 0$, $\varepsilon_1 = 1$;
f) VIIf *and genus* $g = 12(2r+p_1+p_2+p_3) + 9$ *if* $\varepsilon_0 = 0$, $\varepsilon_1 = \varepsilon_2 = 1$;
g) VIIg *and genus* $g = 12(2r+p_1+p_2+p_3) + 8$ *if* $\varepsilon_0 = \varepsilon_1 = 1$, $\varepsilon_2 = 0$;
h) VIIh *and genus* $g = 12(2r+p_1+p_2+p_3) + 12$ *if* $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$.

Conversely, each genus g hyperelliptic curve of class VII *is (isomorphic to another) of the above form for some intergers* r, p_i *and* ε_j.

Representatives of all the real forms of X *are the following*

Class	*Representatives*
VIIa, VIIb	τ_1, $\tau_1\rho$, τ_3, $\tau_3\rho$, τ_4, $\tau_4\rho$
VIIc, VIId, VIIg, VIIh	τ_1, τ_3
VIIe, VIIf	τ_1, τ_3, τ_4

where

$$\tau_1 : (x,y) \longmapsto (\overline{x},\overline{y}),\ \ \tau_3 : (x,y) \longmapsto (i\overline{x}, \overline{y}(e^{\pi i/4})^{\varepsilon_0})\ \ \textit{and}\ \ \tau_4 : (x,y) \longmapsto \left(\frac{-1}{\overline{x}}, \frac{\overline{y}}{\overline{x}^{g+1}}\right).$$

In all cases the species of each real form is given.

Note that the theorem holds for *every* choice of the parameters ω_j, λ_j and μ_j since there is no finite group of Möbius transformations containing $\mathrm{Aut}X_{\widehat{\mathbb{C}}}$ properly.

Proof. — It is easy to check that the formulae of the lifings τ_1 and τ_3 are those given above. The formula of τ_4 can be computed by using that $P_{\omega_j}(0) = P_{\lambda_j}(0) = P_{\mu_j}(0) = 1$ for all ω_j, λ_j and μ_j. Clearly, $\mathrm{sp}(\tau_4) = \mathrm{sp}(\tau_4\rho) = 0$. By construction, the number of branch points of X fixed by $\widehat{\tau}_1$ is $8p_1 + 4\varepsilon_0 + 4\varepsilon_1$ ($\neq 2g+2$). So

$$\begin{aligned} \mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho) = -2(2p_1+\varepsilon_0+\varepsilon_1) \quad &\text{if } p_1+\varepsilon_0+\varepsilon_1 > 0,\\ \mathrm{sp}(\tau_1) = 2, \quad \mathrm{sp}(\tau_1\rho) = 0 \quad &\text{if } p_1 = \varepsilon_0 = \varepsilon_1 = 0. \end{aligned}$$

The number of branch points fixed by $\widehat{\tau}_3$ is $4p_2 + 4p_3 + 2\varepsilon_0 + 2\varepsilon_1 + 4\varepsilon_2$ ($\neq 2g+2$). So

$$\mathrm{sp}(\tau_3) = \mathrm{sp}(\tau_3\rho) = -(2p_2+2p_3+\varepsilon_0+\varepsilon_1+2\varepsilon_2) \quad \text{if } p_2+p_3+\varepsilon_0+\varepsilon_1+\varepsilon_2 > 0.$$

If $p_2 = p_3 = \varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$ then X is of class VIIa and so either $\mathrm{sp}(\tau_3) = 2$ or 0. Consider the curve X' whose branch point set $B_{X'}$ is $m^{-1}(B_X)$ where $m : x \mapsto e^{\pi i/4}x$. Since m conjugates $\widehat{\tau}_1$ to $\widehat{\tau}_3$ it follows that $\tau_1' : (x,y) \mapsto (\overline{x},\overline{y})$ is a symmetry of X'. Moreover, τ_1' coincides with $f^{-1}\circ\tau_3\circ f$ where $f : X' \to X$ is a lifting of m. Hence $\mathrm{sp}(\tau_3) = \mathrm{sp}(\tau_1') \neq 0$ and we conclude that

$$\mathrm{sp}(\tau_3) = 2, \quad \mathrm{sp}(\tau_3\rho) = 0 \quad \text{if } p_2 = p_3 = \varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0.$$

□

As announced in preceding classes, the curves $y^2 = x(x^4-1)$, $y^2 = x^8 + 14x^4 + 1$ and $y^2 = x(x^4-1)(x^8+14x^4+1)$ are curves of class VIIc, VIIb and VIId respectively.

3.8. Symmetry types of hyperelliptic algebraic curves of class VIII

These are the curves whose induced group $\widehat{G}$ of Möbius transformations is $\widehat{G} = \langle \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3, \mid \widehat{\tau}_1^2, \widehat{\tau}_2^2, \widehat{\tau}_3^2, (\widehat{\tau}_1\widehat{\tau}_2)^2, (\widehat{\tau}_2\widehat{\tau}_3)^3, (\widehat{\tau}_1\widehat{\tau}_3)^5 \rangle$, where each $\widehat{\tau}_i$ represents an orientation reversing involution. The signature of Λ is one of the following:

VIIIa: $(0; +; [2^r]; \{(2^{p_1}, 2, 2^{p_2}, 3, 2^{p_3}, 5)\})$, with $2r + p_1 + p_2 + p_3 = (g+1)/30$,

VIIIb: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},3,2^{p_3},10)\})$, with $2r+p_1+p_2+p_3=(g-5)/30$,
VIIIc: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},10)\})$, with $2r+p_1+p_2+p_3=(g-15)/30$,
VIIId: $(0;+;[2^r];\{(2^{p_1},2,2^{p_2},6,2^{p_3},5)\})$, with $2r+p_1+p_2+p_3=(g-9)/30$,
VIIIe: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},5)\})$, with $2r+p_1+p_2+p_3=(g-14)/30$,
VIIIf: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},3,2^{p_3},10)\})$, with $2r+p_1+p_2+p_3=(g-20)/30$,
VIIIg: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},5)\})$, with $2r+p_1+p_2+p_3=(g-24)/30$,
VIIIh: $(0;+;[2^r];\{(2^{p_1},4,2^{p_2},6,2^{p_3},510\})$, with $2r+p_1+p_2+p_3=(g-30)/30$.

A fundamental set for the $\widehat{G}$-action is a spherical triangle $\triangle V_0V_1V_2$ with angles $\pi/5$, $\pi/2$ and $\pi/3$. Its sides V_0V_1, V_1V_2 and V_0V_2 are contained in $\mathrm{Fix}(\widehat{\tau}_1)$, $\mathrm{Fix}(\widehat{\tau}_2)$ and $\mathrm{Fix}(\widehat{\tau}_3)$ respectively. So, the vertices V_0, V_1 and V_2 are fixed points of the rotations $\widehat{\tau}_1\widehat{\tau}_3$, $\widehat{\tau}_1\widehat{\tau}_2$ and $\widehat{\tau}_2\widehat{\tau}_3$ respectively. ($\widehat{G}$ is the full group of isometries of a regular icosahedron.) Figure 8 represents the stereographic projection of the tiling of S^2 by the images of triangle $\triangle V_0V_1V_2$ under the elements of $\widehat{G}$.

According to the signature of Λ, there are p_1 (boundary) branch points of the projection $\mathcal{H}\to\mathcal{H}/\Lambda$ in V_0V_1, p_2 in V_1V_2 and p_3 in V_0V_2. The vertices V_0, V_1 and V_2 are also boundary branch points, and their presence or not among the branch points of a curve of class VIII gives the following distinction within this class:

	VIIIa	VIIIb	VIIIc	VIIId	VIIIe	VIIIf	VIIIg	VIIIh
V_0	no	yes	yes	no	no	yes	no	yes
V_1	no	no	no	no	yes	yes	yes	yes
V_2	no	no	yes	yes	no	no	yes	yes

where a "yes" means that the vertex V_i is one of the branch points e_j of a curve of the corresponding class.

The group $\widehat{G}$ has exactly two conjugacy classes of symmetries, with representatives $\widehat{\tau}_1$ and $\widehat{\tau}_4 := (\widehat{\tau}_1\widehat{\tau}_2\widehat{\tau}_3)^5$. The last one is the antipodal map since it is central in $\widehat{G}$, as is easy to check. So the species of its liftings τ_4 and $\tau_4\rho$ (if symmetries) are

Class VIII
$\mathrm{sp}(\tau_4)=\mathrm{sp}(\tau_4\rho)=0$

The species of the liftings of $\widehat{\tau}_1$ depend on the number of e_j lying on $\mathrm{Fix}(\widehat{\tau}_1)$. Since $\widehat{\tau}_1$ is the reflection with respect to the line containing the side V_0V_1 it follows from Figure 8 that, apart from the orbits of V_0, V_1 and V_2, there are $4p_1+4p_2+4p_3$ branch points e_j lying on $\mathrm{Fix}(\widehat{\tau}_1)$. It also contains 4 points of each of the orbits of V_0, V_1 and V_2. There always exist branch points not fixed by $\widehat{\tau}_1$ and only in case VIIIa, in which the genus is odd, there may be no branch point on $\mathrm{Fix}(\widehat{\tau}_1)$. Consequently, the species of τ_1 and $\tau_1\rho$ are the following.

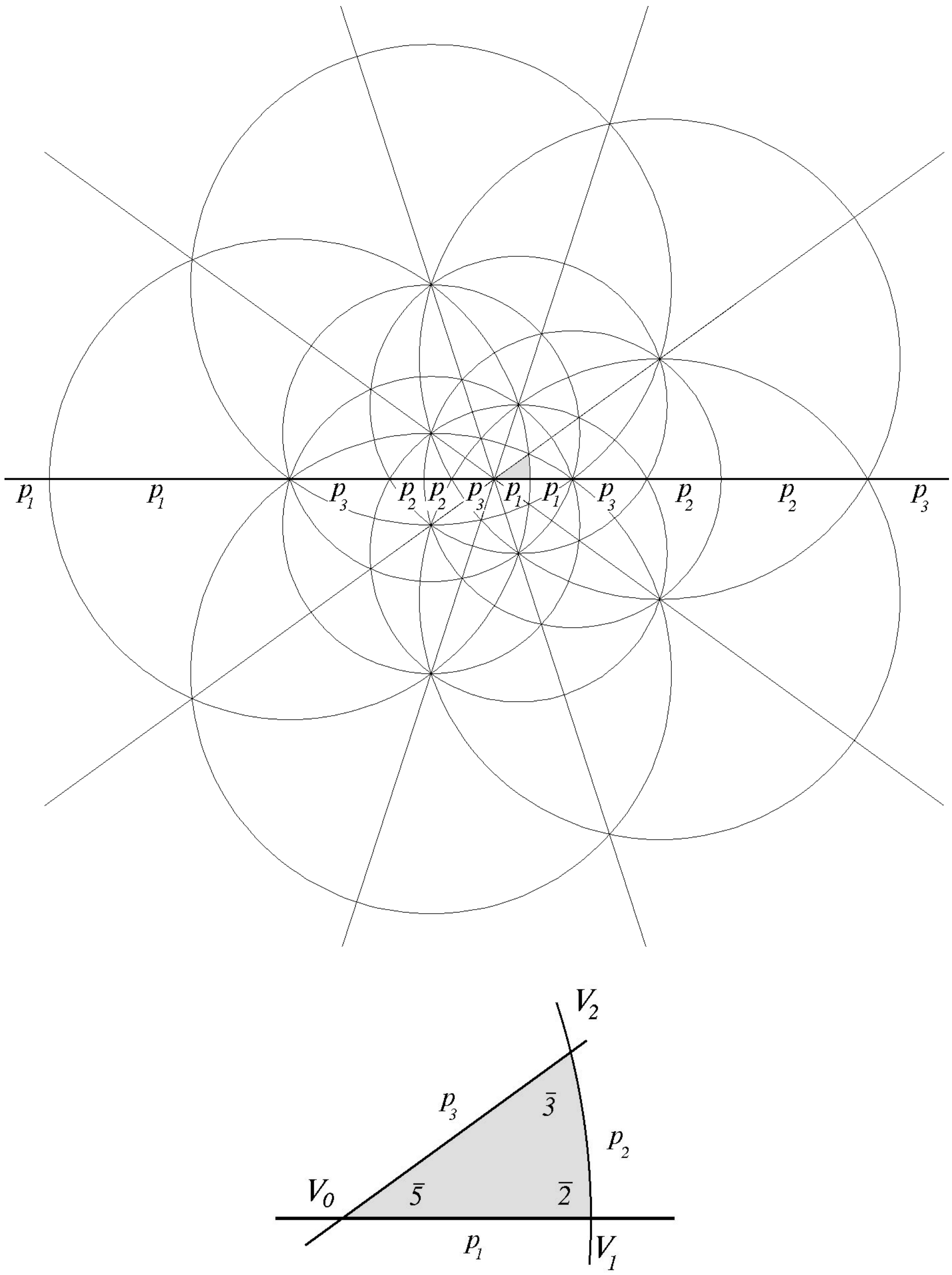


FIGURE 8

VIIIa	$p_1+p_2+p_3=0$	$p_1+p_2+p_3>0$
$\{\,\mathrm{sp}(\tau_1),\,\mathrm{sp}(\tau_1\rho)\,\}$	$\{2,0\}$	$\{-2(p_1+p_2+p_3),-2(p_1+p_2+p_3)\}$

VIIIb, VIIId, VIIIe
$\mathrm{sp}(\tau_1)=\mathrm{sp}(\tau_1\rho)=-2(p_1+p_2+p_3+1)$

VIIIc, VIIIf, VIIIg
$\mathrm{sp}(\tau_1)=\mathrm{sp}(\tau_1\rho)=-2(p_1+p_2+p_3+2)$

VIIIh
$\mathrm{sp}(\tau_1)=-2(p_1+p_2+p_3+3)$

In cases VIIIe, VIIIf, VIIIg and VIIIh the symmetry $\tau_1\rho$ is conjugate to τ_1, which is a representative of the unique conjugacy class of symmetries. In the remaining cases there are four conjugacy classes of symmetries, with representatives τ_1, $\tau_1\rho$, τ_4 and $\tau_4\rho$. Collecting their species we obtain the following theorems, where for brevity we write p instead of $p_1+p_2+p_3$.

There exist genus g curves of class VIIIa if and only if $g\equiv 29 \pmod{30}$.

THEOREM 3.8.1. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIIa *is one of the listed below. Conversely, each of them is the symmetry type of such a curve.*

- $\{2,0,0,0\}$, *only if* $g\equiv 59 \pmod{60}$*;*
- $\{-2p,-2p,0,0\}$ *with* $0<p\leqslant(g+1)/30$ *and* $\begin{cases} \textit{even} & \textit{if } g\equiv 59 \pmod{60},\\ \textit{odd} & \textit{if } g\equiv 29 \pmod{60}.\end{cases}$

There exist genus g curves of class VIIIb if and only if $g\equiv 5 \pmod{30}$.

THEOREM 3.8.2. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIIb *is*

- $\{-2(p+1),-2(p+1),0,0\}$ *with* $p\leqslant(g-5)/30$ *and* $\begin{cases} \textit{even} & \textit{if } g\equiv 5 \pmod{60},\\ \textit{odd} & \textit{if } g\equiv 35 \pmod{60}.\end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIIIb.

There exist genus g curves of class VIIIc if and only if $g\equiv 15 \pmod{30}$.

THEOREM 3.8.3. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIIc *is*

- $\{-2(p+2), -2(p+2), 0, 0\}$ *with* $p \leqslant (g-15)/30$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 15 \pmod{60}, \\ \textit{odd} & \textit{if } g \equiv 45 \pmod{60}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIIIc.

There exist genus g curves of class VIIId if and only if $g \equiv 9 \pmod{30}$.

Theorem 3.8.4. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIId *is*

- $\{-2(p+1), -2(p+1), 0, 0\}$ *with* $p \leqslant (g-9)/30$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 9 \pmod{60}, \\ \textit{odd} & \textit{if } g \equiv 39 \pmod{60}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIIId.

There exist genus g curves of class VIIIe if and only if $g \equiv 14 \pmod{30}$.

Theorem 3.8.5. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIIe *is*

- $\{-2(p+1)\}$ *with* $p \leqslant (g-14)/30$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 14 \pmod{60}, \\ \textit{odd} & \textit{if } g \equiv 44 \pmod{60}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIIIe.

There exist genus g curves of class VIIIf if and only if $g \equiv 20 \pmod{30}$.

Theorem 3.8.6. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIIf *is*

- $\{-2(p+2)\}$ *with* $p \leqslant (g-20)/30$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 20 \pmod{60}, \\ \textit{odd} & \textit{if } g \equiv 50 \pmod{60}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIIIf.

There exist genus g curves of class VIIIg if and only if $g \equiv 24 \pmod{30}$.

Theorem 3.8.7. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIIg *is*

- $\{-2(p+2)\}$ *with* $p \leqslant (g-24)/30$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 24 \pmod{60}, \\ \textit{odd} & \textit{if } g \equiv 54 \pmod{60}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIIIg.

There exist genus g curves of class VIIIh if and only if $g \equiv 0 \pmod{30}$.

Theorem 3.8.8. — *The symmetry type of a genus g hyperelliptic algebraic curve of class* VIIIh *is*

- $\{-2(p+3)\}$ *with* $p \leqslant (g-30)/30$ *and* $\begin{cases} \textit{even} & \textit{if } g \equiv 0 \pmod{60}, \\ \textit{odd} & \textit{if } g \equiv 30 \pmod{60}. \end{cases}$

Conversely, this is the symmetry type of some genus g curve of class VIIIh.

3.8.1. Equations of curves of class VIII and their real forms. — As generators of the group $\widehat{G} = \langle \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3, \mid \widehat{\tau}_1^2, \widehat{\tau}_2^2, \widehat{\tau}_3^2, (\widehat{\tau}_1\widehat{\tau}_2)^2, (\widehat{\tau}_2\widehat{\tau}_3)^3, (\widehat{\tau}_1\widehat{\tau}_3)^5 \rangle$ we choose the following

$$\widehat{\tau}_1 : x \longmapsto \overline{x}, \qquad \widehat{\tau}_2 : x \longmapsto \frac{-2\overline{x} + \sqrt{5} - 1}{(\sqrt{5} - 1)\overline{x} + 2} \quad \text{and} \quad \widehat{\tau}_3 : x \longmapsto \overline{x} e^{2\pi i/5}.$$

The fixed point sets of $\widehat{\tau}_1$, $\widehat{\tau}_2$ and $\widehat{\tau}_3$ are the real axis, the circumference $\{|z + 2\cos\pi/5| = 2\sin 2\pi/5\}$ and the axis $\{\arg(z) = \pi/5 \text{ or } 6\pi/5\}$, respectively. Thus

$$F = \{z \in \mathbb{C} : |z + 2\cos\pi/5| \leqslant 2\sin 2\pi/5,\ \arg(z) \in [\,0, \pi/5\,]\,\}$$

is a fundamental set for the action of $\widehat{G}$ on $\widehat{\mathbb{C}}$. Its vertices are $V_0 = 0 \in \mathrm{Fix}(\widehat{\tau}_1) \cap \mathrm{Fix}(\widehat{\tau}_3)$, $V_1 = 2(\sin 2\pi/5 - \cos\pi/5) \in \mathrm{Fix}(\widehat{\tau}_1) \cap \mathrm{Fix}(\widehat{\tau}_2)$ and $V_2 =$ the point of $\mathrm{Fix}(\widehat{\tau}_2) \cap \mathrm{Fix}(\widehat{\tau}_3)$ with positive imaginary part. The orbit of V_0 consists of ∞ and the roots of the polynomial $x(x^{10} + 11x^5 - 1)$; that of V_1 consists of the roots of $x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1$ and that of V_2 consists of the roots of $x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1$. For other points $\alpha \in F$ let

$$P_\alpha(x) = \prod_{t \,\in\, \mathrm{orbit}(\alpha)} (x - t),$$

which has degree 120 if α is fixed by no element of $\widehat{G}$ or 60 if $\alpha \neq V_i$ is fixed by some $\widehat{\tau}_i$.

It is now easy to prove the following.

Theorem 3.8.9. — *With the above notations, for every choice of pairwise distinct points*

- $\{\omega_1, \ldots, \omega_r\}$ *in the interior of* F,
- $\{\lambda_1, \ldots, \lambda_p\}$ *in the boundary of* F *with* $\lambda_j \neq V_0, V_1, V_2$,

the equation

$$y^2 = (x^{11} + 11x^6 - x)^{\varepsilon_0} \cdot (x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1)^{\varepsilon_1} \cdot (x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)^{\varepsilon_2} \cdot \prod_{j=1}^{r} P_{\omega_j}(x) \cdot \prod_{j=1}^{p} P_{\lambda_j}(x),$$

where $\varepsilon_i \in \{0, 1\}$, *defines a hyperelliptic complex curve* X *of class*

a) VIIIa *and genus* $g = 60r + 30p - 1$ *if* $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$;
b) VIIIb *and genus* $g = 60r + 30p + 5$ *if* $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = 0$;
c) VIIIc *and genus* $g = 60r + 30p + 15$ *if* $\varepsilon_0 = \varepsilon_2 = 1$, $\varepsilon_1 = 0$;
d) VIIId *and genus* $g = 60r + 30p + 9$ *if* $\varepsilon_0 = \varepsilon_1 = 0$, $\varepsilon_2 = 1$;
e) VIIIe *and genus* $g = 60r + 30p + 14$ *if* $\varepsilon_0 = \varepsilon_2 = 0$, $\varepsilon_1 = 1$;
f) VIIIf *and genus* $g = 60r + 30p + 20$ *if* $\varepsilon_0 = \varepsilon_1 = 1$, $\varepsilon_2 = 0$;
g) VIIIg *and genus* $g = 60r + 30p + 24$ *if* $\varepsilon_0 = 0$, $\varepsilon_1 = \varepsilon_2 = 1$;
h) VIIIh *and genus* $g = 60r + 30p + 30$ *if* $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$;

Conversely, each genus g hyperelliptic curve of class VIII *is (isomorphic to another) of the above form for some integers* r, p *and* ε_j.

Representatives of all the real forms of X *are the following*

Classes VIIIa, VIIIb, VIIIc, VIIId
τ_1, $\tau_1\rho$, τ_4, $\tau_4\rho$

Classes VIIIe, VIIIf, VIIIg, VIIIh
τ_1

where

$$\tau_1 : (x, y) \longmapsto (\overline{x}, \overline{y}) \quad \textit{and} \quad \tau_4 : (x, y) \longmapsto \left(\frac{-1}{\overline{x}}, \frac{\overline{y}}{\overline{x}^{g+1}} \cdot \sqrt{k} \right)$$

with $k = \prod P_{\omega_j}(0) \prod P_{\lambda_j}(0)$. *Their species are*

$$\begin{array}{ll} \mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho) = -2(p + \varepsilon_0 + \varepsilon_1 + \varepsilon_2) & \textit{if } p + \varepsilon_0 + \varepsilon_1 + \varepsilon_2 > 0, \\ \mathrm{sp}(\tau_1) = 2, \quad \mathrm{sp}(\tau_1\rho) = 0 & \textit{if } p = \varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0; \\ \mathrm{sp}(\tau_4) = \mathrm{sp}(\tau_4\rho) = 0. & \end{array}$$

3.9. Symmetry types of hyperelliptic algebraic curves of class IX

These are the curves whose induced group $\widehat{G}$ of Möbius transformations is $\widehat{G} = \langle \widehat{\tau} \mid \widehat{\tau}^{2N} \rangle = \mathrm{Z}_{2N}$, where $\widehat{\tau}$ is an orientation reversing transformation. Recall that these curves have odd genus. The signature of Λ is one of the following:

IXa: $(1; -; [2^r, N]; \{-\})$, with $r = (g+1)/N$ and N odd,
IXb: $(1; -; [2^r, 2N]; \{-\})$, with $r = g/N > 1$.

Another presentation of $\widehat{G}$ is the following

$$\widehat{G} = \langle \widehat{a} \rangle \oplus \langle \widehat{\tau}_1 \rangle = \mathrm{Z}_N \oplus \mathrm{Z}_2,$$

where $\widehat{a} = \widehat{\tau}^2$ is a rotation and $\widehat{\tau}_1 = \widehat{\tau}^N$ is an orientation reversing involution. It turns out that $\widehat{\tau}_1$ is the antipodal map. Otherwise it would be a reflection with respect to a plane orthogonal to the axis of $\widehat{a}$, and hence the quotient of S^2 under the action of $\widehat{G}$ would have non-empty boundary, contrary to the signature of Λ.

Therefore the unique real forms of curves of class IX are the liftings of the antipodal map and so their species are

Class IX
$\mathrm{sp}(\tau_1) = \mathrm{sp}(\tau_1\rho) = 0$

THEOREM 3.9.1. — *The symmetry type of a genus g (odd) hyperelliptic algebraic curve of class* IX *is* $\{0, 0\}$. *Conversely, this is the symmetry type of any such a curve.*

3.9.1. Equations of curves of class IX and their real forms. — As generators of $\widehat{G} = \langle \widehat{a} \rangle \oplus \langle \widehat{\tau}_1 \rangle = \mathrm{Z}_N \oplus \mathrm{Z}_2$, we choose the following

$$\widehat{a} : x \longmapsto x e^{2\pi i/N} \quad \text{and} \quad \widehat{\tau}_1 : x \longmapsto \frac{-1}{\overline{x}}.$$

A fundamental set F for the action of $\widehat{G}$ is the sector of angle $2\pi/N$ of the unit disc illustrated in Figure 9.

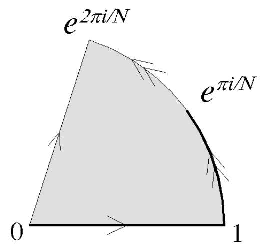


FIGURE 9

Note that the lines joining 0 with $e^{2\pi i/N}$ and $e^{2\pi i/N}$ with $e^{\pi i/N}$ are excluded because they are identified with the lines joining 0 with 1 and 1 with $e^{\pi i/N}$ respectively. Explicitly,

$$F = \{z \in \mathbb{C} : |z| < 1,\ \arg(z) \in [\,0, 2\pi/N\,)\} \cup \{z \in \mathbb{C} : |z| = 1,\ \arg(z) \in [\,0, \pi/N\,)\}.$$

The $\widehat{G}$-orbit of a point $\omega \in F$ is $\{0, \infty\}$ if $\omega = 0$ and the roots of $(x^N - \omega^N)(x^N + 1/\overline{\omega}^N)$ otherwise. Given $\omega_1, \ldots, \omega_r \in F - \{0\}$ the product $\prod \omega_j/\overline{\omega}_j$ is an unimodular number and so a suitable rotation of the form $x \mapsto xe^{i\alpha}$ transforms that number into 1. Note that such rotation commutes with $\widehat{a}$ and $\widehat{\tau}_1$. It follows that the branch point set of a curve of class IX may be chosen so that it satisfies the condition $\prod \omega_j/\overline{\omega}_j = 1$.

According to the signature of Λ, the fixed points 0 and ∞ of the rotation $\widehat{a}$ are among the branch points e_j of a curve of class IX only if the curve is of class IXb.

THEOREM 3.9.2. — *With F as above, for almost every choice of r pairwise different points $\omega_1, \ldots, \omega_r \in F - \{0\}$ with $\prod_{j=1}^{r} \omega_j/\overline{\omega}_j = 1$ and for $\varepsilon \in \{0, 1\}$, the equation*

$$y^2 = x^{\varepsilon} \cdot \prod_{j=1}^{r} (x^N - \omega_j^N)(x^N + 1/\overline{\omega}_j^N)$$

with N odd, defines a hyperelliptic complex curve of class

a) IXa *and genus $g = rN - 1$ if $\varepsilon = 0$ and r even,*
b) IXb *and genus $g = rN$ if $\varepsilon = 1$ and r odd > 1.*

Conversely, each hyperelliptic curve of class IX *is (isomorphic to another) of the above form for some* r*,* N *and* ε*. It has exactly two real forms, with representatives*

$$\tau_1 : (x,y) \longmapsto \left(\frac{-1}{\overline{x}}, \frac{\overline{y}}{\overline{x}^{g+1}}\right) \quad \text{and} \quad \tau_1\rho.$$

Their species are 0*.*

We observe that in class IXb condition $r > 1$ is necessary. Indeed, if $r = 1$ then $X = \{y^2 = x(x^N - \omega^N)(x^N + 1/\overline{\omega}^N)\}$ and from condition $\omega/\overline{\omega} = 1$ it follows that X admits the symmetry $(x,y) \mapsto (\overline{x}, \overline{y})$.

3.10. Symmetry types of hyperelliptic algebraic curves of class X

For the sake of completeness we conclude this chapter giving the defining equations and the formulae of the symmetries of those curves whose induced group $\widehat{G}$ is generated by a single symmetry. So $\widehat{G} = \langle \widehat{\tau} \rangle = \mathrm{Z}_2$ where $\widehat{\tau}$ is either complex conjugation or the antipodal map. In the first case we say that the curve is of class Xa, and of class Xb in the second. Recall that the symmetry types of these curves were calculated by Klein in [**35**].

3.10.1. Class Xa.

— A fundamental set for the action of complex conjugation is the closed upper halfsphere. We may assume that ∞ is not a branch point of curves of class Xa. Thus they have an even number of real branch points.

THEOREM 3.10.1. — *For almost every choice of*

- $\{\omega_1, \ldots, \omega_r\} \subset \mathbb{C}^+$ *with* $\omega_i \neq \omega_j$ *if* $i \neq j$*,*
- $\lambda_1 < \cdots < \lambda_{2p} \subset \mathbb{R}$*,*

with $r + p \geqslant 3$*, the equation*

$$y^2 = \prod_{j=1}^{r}(x - \omega_j)(x - \overline{\omega}_j) \cdot \prod_{j=1}^{2p}(x - \lambda_j)$$

defines a hyperelliptic complex curve of class Xa *and genus* $g = r + p - 1$*. Conversely, each such a curve is (isomorphic to another) of the above form. It has exactly two real forms, with representatives* $\tau : (x,y) \mapsto (\overline{x}, \overline{y})$ *and* $\tau\rho$*. Their species are the following.*

	$p = 0$	$0 < p < g+1$	$p = g+1$
$\mathrm{sp}(\tau)$	1 *if* g *is even* 2 *if* g *is odd*	$-p$	$g+1$
$\mathrm{sp}(\tau\rho)$	0	$-p$	$g+1$

3.10.2. Class Xb. — A fundamental set F for the action of the antipodal map is the union of the open halfplane $\mathbb{C}^+$ and the non-negative real axis $\{r \in \mathbb{R} : r \geqslant 0\}$. We may assume that ∞ and 0 are not branch points of curves of class Xb. As in class IX, we may also assume that the product $\prod(\omega_j/\overline{\omega}_j)$ of all branch points equals 1.

Since a lifting of the antipodal map is a symmetry, curves of class Xb have odd genus.

Theorem 3.10.2. — *For almost every choice of $g+1$ different points $\omega_1, \dots, \omega_{g+1} \in \mathbb{C}^+ \cup \{r \in \mathbb{R} : r > 0\}$ with $\prod_{j=1}^{g+1} \omega_j/\overline{\omega}_j = 1$ and g odd the equation*

$$y^2 = \prod_{j=1}^{g+1} (x - \omega_j)(x + 1/\overline{\omega}_j)$$

defines a hyperelliptic complex curve of class Xb *and genus g. Conversely, each such a curve is (isomorphic to another) of the above form. It has exactly two real forms, with representatives $\tau : (x, y) \mapsto (-1/\overline{x}, \overline{y}/\overline{x}^{g+1})$ and $\tau\rho$. Their species are* 0.

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