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MASAKI KASHIWARA
PIERRE SCHAPIRA

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Moderate and formal cohomology associated with constructible sheaves

Masaki KASHIWARA[1]   Pierre SCHAPIRA[2]

Abstract — On a complex manifold X, we construct the functors $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $\mathcal{T}hom(\cdot\,, \mathcal{O}_X)$ of formal and moderate cohomology from the category of $\mathbb{R}$-constructible sheaves to that of $\mathcal{D}_X$-modules. It allows us to treat functorially and in a unified manner C^∞ functions, distributions, formal completion and local algebraic cohomology.

The behavior of these functors under the usual operations on $\mathcal{D}$-modules is systematically studied, and adjunction formulas for correspondences of complex manifolds are obtained.

This theory provides a natural tool to treat integral transformations with growth conditions such as Radon, Poisson and Laplace transforms.

Résumé — Sur une variété complexe X, nous construisons les foncteurs $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ et $\mathcal{T}hom(\cdot\,, \mathcal{O}_X)$ de cohomologie formelle et modérée de la catégorie des faisceaux $\mathbb{R}$-constructibles à valeurs dans celle des $\mathcal{D}_X$-modules. Cela permet de traiter fonctoriellement et de manière unifiée les fonctions C^∞, les distributions, la complétion formelle et la cohomologie locale algébrique.

On étudie systématiquement le comportement de ces foncteurs pour les opérations usuelles sur les $\mathcal{D}$-modules, et on obtient des formules d'adjonction pour les correspondances de variétés complexes.

Cette théorie fournit les outils naturels pour traiter les transformations intégrales avec conditions de croissance comme les transformations de Radon, Poisson et Laplace.

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[1]Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan
[2]Institut de Mathématiques, Université Paris VI, Case 247, 4 Place Jussieu 75252 Paris, France, `schapira@mathp6.jussieu.fr`

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Introduction

"Algebraic analysis", following Mikio Sato's terminology, is an attempt to treat classical analysis with the methods and tools of Algebra, in particular, sheaf theory and homological algebra. This approach has proved its efficiency, especially when applied to the theory of linear partial differential equations (see [S-K-K]), which has become, in some sense, a simple application of the microlocal theory of sheaves (see [K-S]). However, while this sheaf theoretical approach perfectly works when dealing with holomorphic functions and the various sheaves associated to it (hyperfunctions, ramified holomorphic functions, etc.), some important difficulties appear when treating growth conditions, which is quite natural since such conditions are obviously not of local nature. However, as is commonly known, classical analysis is better concerned with distributions and C^∞-functions than with hyperfunctions and real analytic functions.

These difficulties have been overcome by the introduction of the functor $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ of temperate cohomology in [Ka3] and its microlocalization, the functor $\mathcal{T}\mu hom(\cdot, \mathcal{O}_X)$ of Andronikof [An]. The idea of $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ is quite natural: the usual functor $R\,\mathcal{H}om(F, \mathcal{O}_X)$ may be calculated by applying $\mathcal{H}om(F, \cdot)$ to $\mathcal{B}^\bullet_X$, the Dolbeault complex with hyperfunction coefficients, which is an injective resolution of $\mathcal{O}_X$. If $\mathcal{B}^\bullet_X$ is replaced by $\mathcal{D}b^\bullet_X$, the Dolbeault complex with distribution coefficients, one gets a new functor which is well-defined and behaves perfectly with respect to F when F is $\mathbb{R}$-constructible. If X is a complexification of a real analytic manifold M and if one chooses for F the orientation sheaf on M (shifted by the dimension), then the sheaf of distributions on M is recovered (this was already noticed by Martineau [Mr]). If Y is a closed complex analytic subset of X and if one chooses $F = \mathbb{C}_Y$, one recovers $R\Gamma_{[Y]}(\mathcal{O}_X)$, the algebraic cohomology of $\mathcal{O}_X$ with support in Y. The functor $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ is an inverse to the functor $\mathcal{S}ol(\cdot) := R\,\mathcal{H}om_{\mathcal{D}_X}(\cdot, \mathcal{O}_X)$ in the Riemann-Hilbert correspondence, and this was the motivation for its introduction in [Ka3]. However, as we shall see below, it has many other applications.

The functor $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ being well understood, and corresponding —roughly speaking— to Schwartz's distributions, it was natural to look for its dual. This is one of the aims of this paper in which we shall introduce the new functor $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ of formal cohomology. In fact, we shall treat in a unified way both functors, $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ and $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$, starting with an abstract result. We show that a functor ψ defined on the category $\mathcal{S}_X$ of open relatively compact subanalytic subsets of a real analytic manifold X with values in an abelian category and satisfying a kind of Mayer-Vietoris property, extends naturally to an exact functor on the category $\mathbb{R}$-Cons(X) of $\mathbb{R}$-constructible sheaves (see Theorem 1.1 for a precise statement). The functor $U \mapsto \mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_X) := \mathcal{D}b_X/\Gamma_{(X\setminus U)}\mathcal{D}b_X$ as well as the functor $U \mapsto \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty :=$ the subsheaf of $\mathcal{C}_X^\infty$ consisting of sections vanishing up to infinite order on $X \setminus U$ satisfy the required properties, and thus extend as exact functors on $\mathbb{R}$-Cons(X). When X is a complex manifold, the functors $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ and $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ are the Dolbeault complexes of the preceding ones. When X is a complexification of a real analytic manifold M, $\mathbb{C}_M \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ is nothing but $\mathcal{C}_M^\infty$ and if Y is a closed complex analytic subset of X, $\mathbb{C}_Y \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ is the formal completion of $\mathcal{O}_X$ along Y. Moreover, if F is an $\mathbb{R}$-constructible sheaf, then $R\Gamma(X; F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$ and $R\Gamma_c(X; \mathcal{T}hom(F, \Omega_X[d_X]))$ are well-defined objects of the derived categories of FN-spaces and DFN-spaces respectively, and are dual to each other (see Proposition 5.2, and its generalization to solution sheaves of $\mathcal{D}$-modules, Theorem 6.1).

In this paper, we present a detailed study of the usual operations (external product, inverse and direct images) on these functors. Of course, the results concerning $\mathcal{T}hom$ were already obtained in [Ka$_3$], but our treatment is slightly different and more systematic. Our main results are the adjunction formulas in Theorems 7.2, 7.3 and 10.8. In order to prove Theorem 7.3 we have made use of the theory of $\mathcal{O}_X$-modules of type FN or DFN of Ramis-Ruget [R-R] (see also [Ho]) and we thank J-P. Schneiders for communicating his proof of Theorem 8.1.

Applications of our functors will not be given here. Let us simply mention that the adjunction formulas appear as extremely useful tools in integral geometry (see [D'A-S$_1$], [D'A-S$_2$]) and representation theory (in the spirit of [K-Sm]) and the specialization of the functor of formal cohomology leads to a functorial treatment of "asymptotic developments" (see [Co]). Finally, in a forthcoming paper, we shall apply this theory to the study of integral transforms with exponential kernels, and particularly to the Laplace transform.

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[1]M. Kashiwara and P. Schapira, *Integral Transforms with Exponential Kernels and Laplace Transform*, RIMS-1102 (1996).

1. Functors on $\mathbb{R}$-Constructible Sheaves

We shall mainly follow the notations of [K-S] for derived categories and sheaf theory. In particular, if $\mathbf{A}$ is an additive category, we denote by $C^b(\mathbf{A})$ the additive category of bounded complexes of $\mathbf{A}$, and by $K^b(\mathbf{A})$ the category obtained by identifying with 0 the morphisms in $C^b(\mathbf{A})$ homotopic to 0. If $\mathbf{A}$ is abelian we denote by $\mathbf{D}^b(\mathbf{A})$ its derived category with bounded cohomologies, the localization of $K^b(\mathbf{A})$ by exact complexes. We denote by Q the canonical functor from $K^b(\mathbf{A})$ to $\mathbf{D}^b(\mathbf{A})$. We define similarly $C^*(\mathbf{A})$ or $K^*(\mathbf{A})$ ($* = +$ or $-$) by considering complexes bounded from above or below. If R is a ring or a sheaf of rings, we write for short $C^b(R)$, etc. instead of $C^b(\mathrm{Mod}(R))$, etc.. For example, if X is a topological space, $\mathbf{D}^b(\mathbb{C}_X)$ is the derived category with bounded cohomologies of sheaves of $\mathbb{C}$-vector spaces on X.

Let X be a real analytic manifold and denote by $\mathbb{R}\text{-Cons}(X)$ the abelian category of $\mathbb{R}$-constructible sheaves of $\mathbb{C}$-vector spaces (see [K-S] for an exposition). Denote by $\mathbb{R}\text{-Cons}_c(X)$ the thick subcategory consisting of sheaves with compact support.

Let $\mathscr{S}_X$ be the family of open relatively compact subanalytic subsets of X and let us denote by the same letter $\mathscr{S}_X$ the category whose objects are the elements of $\mathscr{S}_X$ and the morphisms $U \to V$ are the inclusions $U \subset V$, U and V in $\mathscr{S}_X$. Then $U \mapsto \mathbb{C}_U$ gives a faithful functor

$$\mathscr{S}_X \longrightarrow \mathbb{R}\text{-Cons}_c(X)\,.$$

Let $\mathbf{A}$ be an abelian category over $\mathbb{C}$. This means that $\mathrm{Hom}_{\mathbf{A}}(M, N)$ has a structure of $\mathbb{C}$-vector space for M, $N \in \mathbf{A}$, and the composition of morphisms is $\mathbb{C}$-bilinear. Let $\psi : \mathscr{S}_X \longrightarrow \mathbf{A}$ be a functor, and consider the conditions:

(1.1) $\psi(\emptyset) = 0\,;$

(1.2) for any U, V in $\mathscr{S}_X$, the sequence

$$\psi(U \cap V) \to \psi(U) \oplus \psi(V) \to \psi(U \cup V) \to 0$$

is exact;

(1.3) for any open inclusion $U \subset V$ in $\mathcal{S}_X$, $\psi(U) \to \psi(V)$ is a monomorphism.

Theorem 1.1. —

(a) *Assume* (1.1) *and* (1.2). *Then there is a right exact functor, unique up to an isomorphism,*
$$\Psi : \mathbb{R}\text{-}\mathrm{Cons}_{\mathrm{c}}(X) \longrightarrow \mathbf{A}$$
such that $\Psi(\mathbb{C}_U) \simeq \psi(U)$ *functorially in* $U \in \mathcal{S}_X$.

(b) *Assume* (1.1), (1.2) *and* (1.3). *Then* Ψ *is exact.*

(c) *Let* ψ_1 *and* ψ_2 *be two functors from* $\mathcal{S}_X$ *to* $\mathbf{A}$ *both satisfying* (1.1) *and* (1.2), *and let* Ψ_1 *and* Ψ_2 *be the corresponding functors given in* (a). *Let* $\theta : \psi_1 \to \psi_2$ *be a morphism of functors. Then* θ *extends uniquely to a morphism of functors*
$$\Theta : \Psi_1 \longrightarrow \Psi_2.$$

(d) *In the situation of* (a), *assume that* $\mathbf{A}$ *is a subcategory of the category* $\mathrm{Mod}(\mathbb{C}_X)$ *of sheaves of* $\mathbb{C}$*-vector spaces on* X, *and that* $\mathbf{A}$ *is local, that is: an object* F *of* $\mathrm{Mod}(\mathbb{C}_X)$ *belongs to* $\mathbf{A}$ *if for any relatively compact open subset* U *there exists* F' *in* $\mathbf{A}$ *such that* $F|_U \simeq F'|_U$.
Assume further that ψ *is local, that is:* $\mathrm{supp}(\psi(U)) \subset \bar{U}$ *for any* $U \in \mathcal{S}_X$.
Then ψ *extends uniquely to* $\mathbb{R}\text{-}\mathrm{Cons}(X)$ *as a right exact functor* Ψ *which is local, that is,* $\Psi(F)|_U \simeq \Psi(F_U)|_U$ *for any* $F \in \mathbb{R}\text{-}\mathrm{Cons}(X)$ *and* $U \in \mathcal{S}_X$. *Moreover the assertion* (b) *remains valid, as well as* (c), *provided that both* ψ_1 *and* ψ_2 *are local.*

Proof. Let Vect denote the category of $\mathbb{C}$-vector spaces and let ${\mathcal{S}_X}^\vee$ be the category of contravariant functors from $\mathcal{S}_X$ to Vect. Let $\xi : \mathbb{R}\text{-}\mathrm{Cons}(X) \to {\mathcal{S}_X}^\vee$ denote the canonical functor. Let P be an object of ${\mathcal{S}_X}^\vee$ satisfying the following two conditions similar to (1.1–2).

(1.4) $P(\emptyset) = 0$;

(1.5) for any $U_1, U_2 \in \mathcal{S}_X$,
$$0 \to P(U_1 \cup U_2) \to P(U_1) \oplus P(U_2) \to P(U_1 \cap U_2)$$
is an exact sequence.

Lemma 1.2. — *Assume that* $P \in {\mathcal{S}_X}^\vee$ *satisfies* (1.4) *and* (1.5). *Then for any* $V \in \mathcal{S}_X$, *the composition*
(1.6)
$$\mathrm{Hom}_{{\mathcal{S}_X}^\vee}(\xi(\mathbb{C}_V), P) \to \mathrm{Hom}_{\mathrm{Vect}}(\xi(\mathbb{C}_V)(V), P(V)) \to \mathrm{Hom}_{\mathrm{Vect}}(\mathbb{C}, P(V)) \simeq P(V)$$
is an isomorphism.

Proof. Let us first remark that $P(\sqcup U_j) \simeq \oplus P(U_j)$ for a finite disjoint family $\{U_j\}$ of objects in $\mathcal{S}_X$. Also recall that any relatively compact subanalytic subset has a finite number of connected components.

Let us prove the injectivity of (1.6). For $U \subset V$ let us denote by 1_U the canonical element of $\xi(\mathbb{C}_V)(U)$. Then the map (1.6) is given by $\mathrm{Hom}_{\mathcal{S}_X{}^\vee}(\xi(\mathbb{C}_V), P) \ni \alpha \mapsto \alpha(V)(1_V) \in P(V)$. Let α be an element of $\mathrm{Hom}_{\mathcal{S}_X{}^\vee}(\xi(\mathbb{C}_V), P)$. Assuming that $\alpha(V)(1_V) \in P(V)$ vanishes, we shall prove that $\alpha(U) : \Gamma(U; \mathbb{C}_V) \to P(V)$ vanishes for any $U \in \mathcal{S}_X$. By the above remark, we may assume that U is connected. If U is not contained in V then $\xi(\mathbb{C}_V)(U) = 0$ and hence $\alpha(U) = 0$. If U is contained in V, then $\xi(\mathbb{C}_V)(U)$ is a one-dimensional vector space generated by 1_U. Then $\alpha(U) = 0$ follows by the commutative diagram

$$\begin{array}{ccc} \xi(\mathbb{C}_V)(V) & \longrightarrow & P(V) \\ \downarrow & & \downarrow \\ \xi(\mathbb{C}_V)(U) & \longrightarrow & P(U) \end{array}$$

in which the left vertical arrow sends 1_V to 1_U.

Let us prove the surjectivity by tracing backwards the arguments above. Let a be an element of $P(V)$. For a connected $U \in \mathcal{S}_X$, define $\alpha(U)$ as follows. When U is not contained in V, set $\alpha(U) = 0$. When U is contained in V, define $\alpha(1_U)$ to be the image of a by the restriction map $P(V) \to P(U)$. For a general $U \in \mathcal{S}_X$, letting $U = \sqcup U_j$ be the decomposition of U into connected components, we set $\alpha(U) = \oplus \alpha(U_j)$. Then we can see easily that α belongs to $\mathrm{Hom}_{\mathcal{S}_X{}^\vee}(\xi(\mathbb{C}_V), P)$ and the map (1.6) sends α to a. □

Now we are ready to prove Theorem 1.1. First we assume that ψ satisfies the condition (1.1) and (1.2), and we shall prove (a) in Theorem 1.1.

For an object $M \in \mathbf{A}$ and $U \in \mathcal{S}_X$, we set

$$P(M)(U) = \mathrm{Hom}_{\mathbf{A}}(\psi(U), M) .$$

Then $P(M)$ is an object of $\mathcal{S}_X{}^\vee$ and it satisfies the conditions (1.4) and (1.5). Now we shall show

(1.7) For any $F \in \mathbb{R}\text{-Cons}_c(X)$, the functor $\Psi(F) : M \mapsto \mathrm{Hom}_{\mathcal{S}_X{}^\vee}(\xi(F), P(M))$ is representable by an object of $\mathbf{A}$.

If $F = \mathbb{C}_V$ for $V \in \mathcal{S}_X$, then $\Psi(F)$ is represented by $\psi(V)$ by Lemma 1.2. Hence if F is a finite direct sum of sheaves of the form $\mathbb{C}_V$, then $\Psi(F)$ is representable. Every $F \in \mathbb{R}\text{-Cons}_c(X)$ is the cokernel of a morphism $F_1 \to F_2$ in $\mathbb{R}\text{-Cons}_c(X)$, where F_1 and F_2 are finite direct sums of sheaves of the form $\mathbb{C}_V$. Since $\Psi(F_1)$ and

$\Psi(F_2)$ are representable, $\Psi(F)$ is represented by the cokernel of $\Psi(F_1) \to \Psi(F_2)$. This completes the proof of (1.7).

Thus we obtained the functor $\Psi : \mathbb{R}\text{-}\mathrm{Cons}_c(X) \to \mathbf{A}$ and it is obvious that Ψ satisfies the desired condition.

We shall show (b). Namely assuming (1.1), (1.2) and (1.3), we shall show that $\Psi(F) \to \Psi(F')$ is a monomorphism if $F \to F'$ is a monomorphism in $\mathbb{R}\text{-}\mathrm{Cons}_c(X)$. There is a finite family of $\{U_j\}_{j=1,\dots,n}$ of relatively open subanalytic sets and morphisms $f_j : \mathbb{C}_{U_j} \to F'$ such that $F' = \sum_j \operatorname{Im} f_j$. Set $F_k = F + \sum_{j=1}^{k} \operatorname{Im} f_j$. It is enough to show that $\Psi(F_k) \to \Psi(F_{k+1})$ is a monomorphism. Hence replacing F and F' with F_k and F_{k+1}, we may assume from the beginning that $F' = F + \operatorname{Im} f$ for some $f : \mathbb{C}_U \to F'$. Let us consider the commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F \oplus \mathbb{C}_U & \longrightarrow & F' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \mathbb{C}_U & \longrightarrow & F'/F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since K is a subobject of $\mathbb{C}_U$, it is equal to $\mathbb{C}_V$ for some subanalytic open subset $V \subset U$. Applying Ψ to the diagram above, we obtain a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \psi(F) & \longrightarrow & \psi(F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \psi(V) & \longrightarrow & \psi(F) \oplus \psi(U) & \longrightarrow & \psi(F') & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \psi(V) & \longrightarrow & \psi(U) & \longrightarrow & \psi(F'/F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The rows are exact by (1.3) and the right exactitude of Ψ, and the columns are exact except the right one. Hence the right column is also exact.

The property (c) is obvious by the construction above. The assertion (d) follows easily from $\mathrm{supp}(\Psi(F)) \subset \mathrm{supp}(F)$. This completes the proof of Theorem 1.1. $\square$

Now we consider a stronger condition than (1.2)

(1.8) For any U, V in $\mathscr{S}_X$, the sequence

$$0 \to \psi(U \cap V) \to \psi(U) \oplus \psi(V) \to \psi(U \cup V) \to 0$$

is exact.

Proposition 1.3. — *Assume* (1.1) *and* (1.8). *Then for any* $U \in \mathscr{S}_X$ *and any exact sequence in* $\mathbb{R}\text{-Cons}_c(X)$

$$0 \to G \to F \to \mathbb{C}_U \to 0\,,$$

the sequence $0 \to \Psi(G) \to \Psi(F) \to \Psi(\mathbb{C}_U) \to 0$ *is exact.*

Proof. We shall prove this in two steps.

(Step 1) Assume that $F = \oplus_{j=1}^r \mathbb{C}_{U_j}$ for connected subsets U_j in $\mathscr{S}_X$.

We shall prove the proposition by induction on r. We may assume that $\mathbb{C}_{U_j} \to \mathbb{C}_U$ is given by 1. For $r = 2$, this is nothing but (1.8). Set $U' = \bigcup_{j=2}^r U_j$. Then we have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G_1 \arrow[r, "u"] \arrow[d] & G_2 \arrow[d] & & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & \mathbb{C}_U \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C}_{U' \cap U_1} \arrow[r] \arrow[d] & \mathbb{C}_{U'} \oplus \mathbb{C}_{U_1} \arrow[r] \arrow[d] & \mathbb{C}_U \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

We can see easily that u is an isomorphism. By applying the right exact functor Ψ

we obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Psi(G_1) \arrow[r, "u"] \arrow[d] & \Psi(G_2) \arrow[d] & & \\
0 \arrow[r] & \Psi(G) \arrow[r] \arrow[d] & \Psi(F) \arrow[r] \arrow[d] & \Psi(\mathbb{C}_U) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Psi(\mathbb{C}_{U'\cap U_1}) \arrow[r] \arrow[d] & \Psi(\mathbb{C}_{U'}) \oplus \Psi(\mathbb{C}_{U_1}) \arrow[r] \arrow[d] & \Psi(\mathbb{C}_U) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

In this diagram, the bottom row is exact by (1.8) and the columns are exact by the induction hypothesis. Hence the middle row is exact.

(Step 2) In the general case, we can find an epimorphism $F' \to F$, where $F' = \oplus \mathbb{C}_{U_j}$. Then we have a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, "\sim"] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & G' \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d] & \mathbb{C}_U \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & \mathbb{C}_U \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By applying Ψ, we obtain

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Psi(K) & \xrightarrow{\sim} & \Psi(K) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Psi(G') & \longrightarrow & \Psi(F') & \longrightarrow & \Psi(\mathbb{C}_U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Psi(G) & \longrightarrow & \Psi(F) & \longrightarrow & \Psi(\mathbb{C}_U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since the columns are exact as well as the middle row by (Step 1), the bottom row is also exact. □

Proposition 1.4. — (i) *Assume* (1.1) *and* (1.8). *Then the functor* Ψ : $\mathbb{R}\text{-Cons}_c(X) \to \mathbf{A}$, *which is right exact, is left derivable. Let* $L\Psi$ *denote the left derived functor and set* $L_j\Psi = H^{-j} \circ L\Psi$. *Then* $L_j\Psi = 0$ *for* $j > 1$ *and* $L_1\Psi(\mathbb{C}_U) = 0$ *for any* $U \in \mathcal{S}_X$.

(ii) *Under the locality condition as in Theorem 1.1* (d), Ψ, *as a functor on* $\mathbb{R}\text{-Cons}(X)$ *is left derivable.*

Proof. Let us denote by $\mathcal{P}$ the subcategory of $\mathbb{R}\text{-Cons}_c(X)$ consisting of objects P such that for any exact sequence $0 \to G \to F \to P \to 0$ in $\mathbb{R}\text{-Cons}_c(X)$, the sequence $0 \to \Psi(G) \to \Psi(F) \to \Psi(P) \to 0$ remains exact. One checks easily that if $0 \to P' \to P \to P'' \to 0$ is exact and if P' and P'' belong to $\mathcal{P}$, then so does P.

Now, let K be a subobject of $\oplus_{j=1}^r \mathbb{C}_{U_j}$. Arguing by induction on r, one gets that $K \in \mathcal{P}$. Then the proof follows. □

Proposition 1.5. — *Let* Ψ_1 *and* Ψ_2 *be two functors of triangulated categories from* $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ *to a triangulated category, and let* $\Theta : \Psi_1 \to \Psi_2$ *be a morphism of functors of triangulated categories. We assume the following conditions:*

(i) *for any* $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$, $\Theta(F)$ *is an isomorphism if* $\Theta(F_Z)$ *is an isomorphism for any compact subanalytic subset* Z *of* X,

(ii) *for any closed (resp. open) subanalytic subset* Z *(resp.* U*) of* X, $\Theta(\mathbb{C}_Z)$ *(resp.* $\Theta(\mathbb{C}_U)$*) is an isomorphism.*

Then Θ *is an isomorphism.*

Proof. It is enough to show that $\Theta(F)$ is an isomorphism for any $F \in \mathbb{R}\text{-Cons}(X)$ with compact support. For such an F, there exists a finite filtration $X = X_0 \supset X_1 \supset \cdots X_N = \emptyset$ such that $F|_{X_j \setminus X_{j+1}}$ is a constant sheaf. Since there exist exact sequences $0 \to F_{X_j \setminus X_{j+1}} \to F_{X_j} \to F_{X_{j+1}} \to 0$, it is enough to show that $\Theta(\mathbb{C}_Z)$ is an isomorphism for any locally closed subanalytic subset Z of X. Since Z may be written as the difference of two closed (resp. open) subanalytic subsets, the assertion follows. □

2. The Functors $\cdot \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty$ and $\mathcal{T}hom(\cdot\,, \mathcal{D}b_X)$

In this section and the two subsequent ones, X denotes a real analytic manifold. We denote by $\mathscr{A}_X, \mathscr{C}_X^\infty, \mathcal{D}b_X, \mathfrak{B}_X$ the sheaves on X of complex-valued real analytic functions, C^∞-functions, Schwartz's distributions and Sato's hyperfunctions. We denote by or_X the orientation sheaf on X, by Ω_X the sheaf of real analytic differential forms of maximal degree and we define the sheaf of real analytic densities:

$$\mathscr{A}_X^\vee = \Omega_X \otimes \mathrm{or}_X\,.$$

If $\mathscr{F}$ is an $\mathscr{A}_X$-module, we set

$$\mathscr{F}^\vee = \mathscr{A}_X^\vee \otimes_{\mathscr{A}_X} \mathscr{F}\,.$$

We denote by $\mathscr{D}_X$ the sheaf of rings on X of finite-order differential operators with coefficients in $\mathscr{A}_X$. Recall that $\mathrm{Mod}(\mathscr{D}_X)$ (resp. $\mathrm{Mod}(\mathscr{D}_X^{\mathrm{opp}})$) denotes the category of left (resp. right) $\mathscr{D}_X$-modules, and $\mathbf{D}^b(\mathscr{D}_X)$ (resp. $\mathbf{D}^b(\mathscr{D}_X^{\mathrm{opp}})$) its derived category with bounded cohomologies.

We denote by $\omega_X(\simeq \mathrm{or}_X[\dim X])$ the topological dualizing complex on X, and for $F \in \mathbf{D}^b(\mathbb{C}_X)$, we set:

$$\begin{aligned} \mathrm{D}'_X(F) &= R\,\mathcal{H}om(F, \mathbb{C}_X)\,,\\ \mathrm{D}_X(F) &= R\,\mathcal{H}om(F, \omega_X)\,. \end{aligned}$$

Let U be an open subanalytic subset of X and $Z = X \setminus U$. We shall denote by $\mathscr{I}_{X,Z}^\infty$ the subsheaf of $\mathscr{C}_X^\infty$ consisting of functions which vanish on Z up to infinite order. We set:

$$\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty = \mathscr{I}_{X,Z}^\infty \tag{2.1}$$

and we define $\mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_X)$ by the exact sequence:

$$0 \to \Gamma_Z \mathcal{D}b_X \to \mathcal{D}b_X \to \mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_X) \to 0\,. \tag{2.2}$$

Let us recall the following result, due to Lojaciewicz (see [Lo], [Ma]), which will be a basic tool for all our constructions.

Theorem 2.1. (Lojaciewicz) — *Let U_1 and U_2 be two subanalytic open subsets of X. Then the two sequences below are exact:*

$$0 \to \mathbb{C}_{U_1 \cap U_2} \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty \to (\mathbb{C}_{U_1} \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty) \oplus (\mathbb{C}_{U_2} \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty) \to \mathbb{C}_{U_1 \cup U_2} \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty \to 0,$$
$$0 \to \mathcal{T}\!hom(\mathbb{C}_{U_1 \cup U_2}, \mathcal{D}b_X) \to \mathcal{T}\!hom(\mathbb{C}_{U_1}, \mathcal{D}b_X) \oplus \mathcal{T}\!hom(\mathbb{C}_{U_2}, \mathcal{D}b_X)$$
$$\to \mathcal{T}\!hom(\mathbb{C}_{U_1 \cap U_2}, \mathcal{D}b_X) \to 0.$$

By this result, the condition (1.2) is satisfied and (1.1) is obvious as well as (1.3). Applying Theorem 1.1, we obtain two exact local functors :

$$\cdot \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty : \mathbb{R}\text{-Cons}(X) \to \mathrm{Mod}(\mathscr{D}_X)\,, \tag{2.3}$$

$$\mathcal{T}\!hom(\cdot\,, \mathcal{D}b_X) : (\mathbb{R}\text{-Cons}(X))^{\mathrm{opp}} \to \mathrm{Mod}(\mathscr{D}_X). \tag{2.4}$$

We call the first functor the Whitney functor and the second one the Schwartz functor. Of course this last functor is nothing but the functor $TH_X(\cdot\ )$ of [Ka3]. Notice that for $F \in \mathbb{R}\text{-Cons}(X)$, the sheaves $F \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty$ and $\mathcal{T}\!hom(F, \mathcal{D}b_X)$ are $\mathscr{C}_X^\infty$-modules, hence are soft sheaves.

If $\mathscr{L}$ be a locally free $\mathscr{A}_X$-module of finite rank, we set:

$$F \overset{\mathrm{w}}{\otimes} (\mathscr{C}_X^\infty \otimes_{\mathscr{A}_X} \mathscr{L}) = (F \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty) \otimes_{\mathscr{A}_X} \mathscr{L},$$
$$\mathcal{T}\!hom(F, \mathcal{D}b_X \otimes_{\mathscr{A}_X} \mathscr{L}) = \mathcal{T}\!hom(F, \mathcal{D}b_X) \otimes_{\mathscr{A}_X} \mathscr{L}.$$

For the notions on topological vector spaces that we shall use now, we refer to Grothendieck [Gr1]. In particular we say that a vector space is of type FN (resp. DFN) if it is Fréchet nuclear (resp. the dual of a Fréchet nuclear space).

Proposition 2.2. — *Let $F \in \mathbb{R}\text{-Cons}(X)$. There exist natural topologies of type FN on $\Gamma(X; F \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty)$ and of type DFN on $\Gamma_c(X; \mathcal{T}\!hom(F, \mathcal{D}b_X^\vee))$ and they are dual to each other.*

Proof. (a) We first prove the result when $F = \mathbb{C}_U, U$ an open subanalytic subset of X. Set $Z = X \setminus U$ and consider the two sequences:

$$0 \longrightarrow \Gamma(X; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty) \longrightarrow \Gamma(X; \mathscr{C}_X^\infty) \longrightarrow \Gamma(X; \mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty) \longrightarrow 0, \tag{2.5}$$

$$0 \leftarrow \Gamma_c(X; \mathcal{T}\!hom(\mathbb{C}_U, \mathcal{D}b_X^\vee)) \leftarrow \Gamma_c(X; \mathcal{D}b_X^\vee) \leftarrow \Gamma_c(X; \mathcal{T}\!hom(\mathbb{C}_Z, \mathcal{D}b_X^\vee)) \leftarrow 0\,. \tag{2.6}$$

These two sequences are exact since they are obtained by applying the functors $\Gamma(X; \cdot\ )$ or $\Gamma_c(X; \cdot\ )$ to exact sequences of soft sheaves. Moreover $\Gamma(X; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty) = \Gamma(X; \mathscr{I}_{X,Z}^\infty)$ is a closed subspace of the FN-space $\Gamma(X; \mathscr{C}_X^\infty)$, hence inherits a structure of an FN-space as well as the third term of (2.5). The space

$\Gamma_c(X;\mathcal{D}b_X^\vee)$ is the topological dual space of $\Gamma(X;\mathcal{C}_X^\infty)$. Hence in order to see that $\Gamma_c(X;\mathcal{T}hom(\mathbb{C}_U,\mathcal{D}b_X^\vee))$ is the dual space of $\Gamma(X;\mathbb{C}_U\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^\infty)$, it is enough to show that

$$\Gamma_c\left(X;\Gamma_Z(\mathcal{D}b_X^\vee)\right)=\left\{f\in\Gamma_c(X;\mathcal{D}b_X^\vee)\,;\int uf=0\text{ for any }u\in\Gamma(X;\mathbb{C}_U\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^\infty)\right\}.$$

This is easily obtained by the following result.

Lemma 2.3. — *For any open subanalytic subset U of X, $\Gamma_c(U;\mathcal{C}_X^\infty)$ is dense in $\Gamma(X;\mathbb{C}_U\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^\infty)$.*

The proof is given in Chapter I, Lemma 4.3 of [Ma].

(b) We shall say that two complexes $V^\bullet$ and $W^\bullet$ of topological vector spaces of type FN and DFN respectively are dual to each other if:

$$V^\bullet:\ \cdots\to V^i\underset{v^i}{\longrightarrow}V^{i+1}\to\cdots\tag{2.7}$$

$$W^\bullet:\ \cdots\to W^{-i-1}\underset{w^i}{\longrightarrow}W^{-i}\to\cdots\tag{2.8}$$

W^{-i} is the topological dual of V^i and w^i is the transpose of v^i.

(c) Let us prove the proposition when $F\in\mathbb{R}\text{-Cons}_c(X)$. In such a case F is quasi-isomorphic to a bounded complex:

$$F^\bullet:\cdots\to F^{-1}\to F^0\to 0$$

where F^0 is in degree 0 and each F^j is a finite direct sum of sheaves of type $\mathbb{C}_U$, U being open relatively compact and subanalytic (see [K-S, Chap.VIII]). Applying the functors $\Gamma(X;\cdot\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^\infty)$ and $\Gamma_c(X;\mathcal{T}hom(\cdot\,,\mathcal{D}b_X^\vee))$, we obtain two complexes $V^\bullet$ and $W^\bullet$ of type FN and DFN, dual to each other. Moreover $V^i=0$ for $i>0$, $W^i=0$ for $i<0$ and these complexes are exact except in degree 0. Hence all w^i have closed range and consequently their adjoints v^i have also closed range. Therefore, $H^0(V^\bullet)$ and $H^0(W^\bullet)$ are of type FN and DFN respectively, and dual to each other. It follows from the closed graph theorem that the topologies we have defined by this procedure do not depend on the choice of the resolution of F.

(d) Finally consider the general case where $F\in\mathbb{R}\text{-Cons}(X)$. Let us take an increasing sequence $\{Z_n\}_n$ of compact subanalytic subsets such that X is the union of the interiors of Z_n. Then $\Gamma(X;F\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^\infty)$ is the projective limit of $\Gamma(X;F_{Z_n}\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^\infty)$ with surjective projections and $\Gamma_c(X;\mathcal{T}hom(F,\mathcal{D}b_X^\vee))$ is the inductive limit of $\Gamma_c(X;\mathcal{T}hom(F_{Z_n},\mathcal{D}b_X^\vee))$. Then the result follows from (c). ☐

Corollary 2.4. — *Let $u:F\to G$ be a morphism in $\mathbb{R}$-Cons(X). Then the associated morphisms $\Gamma(X;F\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^\infty)\to\Gamma(X;G\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^\infty)$ and $\Gamma_c(X;\mathcal{T}hom(G,\mathcal{D}b_X^\vee))\to\Gamma_c(X;\mathcal{T}hom(F,\mathcal{D}b_X^\vee))$ have closed ranges.*

From now on, we shall work in $\mathbf{D}^b(\mathbb{R}\text{-Cons}(X))$, the derived category of $\mathbb{R}\text{-Cons}(X)$. Recall that $\mathbf{D}^b(\mathbb{R}\text{-Cons}(X))$ is equivalent to the full triangulated subcategory $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ of $\mathbf{D}^b(\mathbb{C}_X)$ consisting of objects whose cohomology groups belong to $\mathbb{R}\text{-Cons}(X)$ (see [Ka3]). The functors $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X$ and $\mathcal{T}hom(\cdot\ , \mathcal{D}b_X)$ being exact, they extend to functors from $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ to $\mathbf{D}^b(\mathcal{D}_X)$. We keep the same notations for these functors on the derived categories.

Proposition 2.5. — *Let F and G be in $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. There are natural morphisms in $\mathbf{D}^b(\mathcal{D}_X)$, functorial with respect to F and G:*

$$F \otimes \mathcal{C}^\infty_X \to F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X, \tag{2.9}$$

$$(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{A}_X} (G \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \to (F \otimes G) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X, \tag{2.10}$$

$$(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{A}_X} \mathcal{T}hom(G, \mathcal{D}b_X) \to \mathcal{T}hom(R\,\mathcal{H}om(F, G), \mathcal{D}b_X). \tag{2.11}$$

Proof.

(i) First let us construct (2.9). Applying Theorem 1.1, we may assume $F = \mathbb{C}_U$, for an open subanalytic subset U of X. In this case, the construction is clear.

(ii) Let us construct (2.10). For F, G in $\mathbb{R}\text{-Cons}(X)$, the morphism:

$$(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes (G \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \to (F \otimes G) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X,$$

is easily constructed, by using Theorem 1.1, and reducing to the case where $F = \mathbb{C}_U$ and $G = \mathbb{C}_V$, for U and V two open subanalytic subsets of X. Since this morphism is $\mathcal{A}_X$-bilinear, it defines a morphism of $\mathcal{D}_X$-modules:

$$(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes_{\mathcal{A}_X} (G \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \to (F \otimes G) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X.$$

Using the natural morphism $\mathfrak{M}^\bullet \overset{\mathrm{L}}{\otimes}_{\mathcal{A}_X} \mathfrak{N}^\bullet \to \mathfrak{M}^\bullet \otimes_{\mathcal{A}_X} \mathfrak{N}^\bullet$ for complexes of $\mathcal{D}_X$-modules $\mathfrak{M}^\bullet$, $\mathfrak{N}^\bullet$, we obtain the desired morphism.

(ii) In order to construct (2.11), we need several lemmas.

Lemma 2.6. — *Let U be an open subanalytic subset of X. Then the composition of morphisms:*

$$(\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes \Gamma_{(X \setminus U)} \mathcal{D}b_X \to \mathcal{C}^\infty_X \otimes \mathcal{D}b_X \to \mathcal{D}b_X$$

is zero.

This follows immediately from Lemma 2.3.

Lemma 2.7. — *Let $G \in \mathbb{R}\text{-Cons}(X)$ and let U be an open subanalytic subset of X. There exists a natural morphism:*

$$(\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes \mathcal{T}hom(G_U, \mathcal{D}b_X) \to \mathcal{T}hom(G, \mathcal{D}b_X)\,.$$

Proof. Using Theorem 1.1, we may reduce the proof to the case where $G = \mathbb{C}_V$ for a subanalytic open subset V of X. Consider the diagram in which we set $S = X \setminus (U \cap V)$:

$$\begin{tikzcd}
 & (\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes \Gamma_S \mathcal{D}b_X \arrow[r] & (\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes \mathcal{D}b_X \arrow[r] \arrow[d, "\alpha"] & (\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes \mathcal{T}hom(\mathbb{C}_{U\cap V}, \mathcal{D}b_X) \arrow[r] & 0 \\
0 \arrow[r] & \Gamma_{(X\setminus V)} \mathcal{D}b_X \arrow[r] & \mathcal{D}b_X \arrow[r] & \mathcal{T}hom(\mathbb{C}_V, \mathcal{D}b_X) \arrow[r] & 0.
\end{tikzcd}$$

Here α is given by the multiplication. Then it is enough to check that α sends $(\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes \Gamma_S \mathcal{D}b_X$ to $\Gamma_{(X\setminus V)} \mathcal{D}b_X$. This follows from Lemma 2.6. □

End of the proof of Proposition 2.5. Let $j : U \hookrightarrow X$ denote the embedding. In Lemma 2.7, we replace G by $j_* j^{-1} G$ and use the isomorphism $(j_* j^{-1} G)_U \simeq G_U$. Applying the morphism $G_U \to G$, we get:

$$(\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes \mathcal{T}hom(G, \mathcal{D}b_X) \to (\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes \mathcal{T}hom(G_U, \mathcal{D}b_X) \to \mathcal{T}hom(j_* j^{-1} G, \mathcal{D}b_X).$$

We can write $j_* j^{-1} G$ as $\mathcal{H}om(\mathbb{C}_U, G)$. Then, applying Theorem 1.1, we have constructed a morphism, for F and G in $\mathbb{R}$-Cons(X):

$$(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes \mathcal{T}hom(G, \mathcal{D}b_X) \to \mathcal{T}hom(\mathcal{H}om(F, G), \mathcal{D}b_X).$$

(Notice that both terms are right exact in F.) This morphism being $\mathcal{A}_X$-bilinear, it defines:

$$(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes_{\mathcal{A}_X} \mathcal{T}hom(G, \mathcal{D}b_X) \to \mathcal{T}hom(\mathcal{H}om(F, G), \mathcal{D}b_X).$$

This construction extends naturally to a morphism in $K^b(\mathcal{D}_X)$ for F,$G \in K^b(\mathbb{R}\text{-Cons}(X))$.

For F and G given in $\mathbb{R}$-Cons(X), there exists a simplicial set $\mathfrak{S}$ and a homeomorphism $i : \mathfrak{S} \to X$, such that F and G are the images of simplicial sheaves (see [Ka3] or [K-S]). On the category $\mathbb{R}$-Cons$(\mathfrak{S})$, the functor $\mathcal{H}om(F, G)$ admits a right derived functor with respect to F, and it coincides with the usual $R\,\mathcal{H}om(F, G)$. Now recall that Q denotes the functor from K^b to $\mathbf{D}^b$ and that "$\varinjlim$" and "$\varprojlim$" denote ind-objects and pro-objects (see [K-S] Chapter 1, §11). Then we obtain "$\varinjlim\limits_{F' \to F}$" $Q(\mathcal{H}om(F', G)) \simeq R\,\mathcal{H}om(F, G)$. where $F' \to F$ ranges over the family of quasi-isomorphisms in $K^b(\mathbb{R}\text{-Cons}(X))$. Thus we obtain

$$\begin{aligned}
Q(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{A}_X} Q(\mathcal{T}hom(G, \mathcal{D}b_X)) &\to \text{“}\varprojlim_{F' \to F}\text{”}\, Q\left((F' \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \otimes_{\mathcal{A}_X} \mathcal{T}hom(G, \mathcal{D}b_X)\right) \\
&\to \text{“}\varprojlim_{F' \to F}\text{”}\, Q(\mathcal{T}hom(\mathcal{H}om(F', G), \mathcal{D}b_X)) \\
&\simeq \mathcal{T}hom(R\,\mathcal{H}om(F, G), \mathcal{D}b_X).
\end{aligned}$$

This completes the proof of Proposition 2.5. □

Proposition 2.8. — *Let F and G be in $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. There are natural morphisms in $\mathbf{D}^b(\mathscr{D}_X)$, functorial with respect to F and G:*

$$\mathrm{D}'_X F \otimes \mathscr{C}^\infty_X \to \mathrm{D}'_X F \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X \to \mathcal{T}hom(F, \mathcal{D}b_X) \to R\,\mathcal{H}om(F, \mathcal{D}b_X), \tag{2.12}$$

$$G \otimes (F \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X) \to (G \otimes F) \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X, \tag{2.13}$$

$$\mathcal{T}hom(G \otimes F, \mathcal{D}b_X) \to R\,\mathcal{H}om\,(G, \mathcal{T}hom(F, \mathcal{D}b_X)), \tag{2.14}$$

$$\mathrm{D}'_X(F \otimes G) \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X \to R\,\mathcal{H}om(G, \mathrm{D}'_X F \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X), \tag{2.15}$$

$$\mathrm{D}'_X G \otimes \mathcal{T}hom(F, \mathcal{D}b_X) \to \mathcal{T}hom(G \otimes F, \mathcal{D}b_X). \tag{2.16}$$

Proof. The first morphism in (2.12) is (2.9). The second one is obtained by choosing $G = \mathbb{C}_X$ in (2.11). The third morphism is equivalent to $F \otimes \mathcal{T}hom(F, \mathcal{D}b_X) \to \mathcal{D}b_X$. This last morphism is obtained by:

$$(F \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathcal{T}hom(F, \mathcal{D}b_X) \to \mathcal{T}hom(R\,\mathcal{H}om(F, F), \mathcal{D}b_X).$$

The morphism (2.13) follows from (2.9) and (2.10). The morphism (2.14) follows from (2.9), (2.11) and $F \to R\,\mathcal{H}om(G, G \otimes F)$. The morphism (2.15) follows from (2.13) and $G \otimes \mathrm{D}'_X(F \otimes G) \to \mathrm{D}'_X F$. Finally, the morphism (2.16) follows from (2.14) and $\mathrm{D}'_X G \otimes (G \otimes F) \to F$. $\square$

Remark 2.9. Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. Then there is a commutative diagram in $\mathbf{D}^b(\mathscr{D}_X)$:

$$\begin{tikzcd}
\mathrm{D}'_X(F) \otimes \mathscr{A}_X \arrow[rr] \arrow[d] & & R\,\mathcal{H}om(F, \mathscr{A}_X) \arrow[d] \\
\mathrm{D}'_X(F) \otimes \mathscr{C}^\infty_X \arrow[r] \arrow[d] & \mathrm{D}'_X(F) \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X \arrow[r] \arrow[d] & R\,\mathcal{H}om(F, \mathscr{C}^\infty_X) \arrow[d] \\
\mathrm{D}'_X(F) \otimes \mathcal{D}b_X \arrow[r] \arrow[d] & \mathcal{T}hom(F, \mathcal{D}b_X) \arrow[r] & R\,\mathcal{H}om(F, \mathcal{D}b_X) \arrow[d] \\
\mathrm{D}'_X(F) \otimes \mathfrak{B}_X \arrow[rr] & & R\,\mathcal{H}om(F, \mathfrak{B}_X).
\end{tikzcd} \tag{2.17}$$

3. Operations on $\cdot \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X$

We follow the notations of [K-S]. In particular we denote by $\underline{f}^{-1}, \underline{f}_!, \underline{f}_*$, $\boxtimes$ the operations of inverse image, proper direct image, direct image and external product in $\mathscr{D}$-modules theory. Let $f\colon Y \to X$ be a morphism of real analytic manifolds. We denote by $\mathrm{or}_{Y/X}$ the relative orientation sheaf $\mathrm{or}_Y \otimes f^{-1}\mathrm{or}_X$. Let $\mathscr{D}_{Y\to X}$ and $\mathscr{D}_{X\leftarrow Y}$ be the "transfer bimodules". Recall that they are defined by

$$
\begin{aligned}
\mathscr{D}_{Y\to X} &= \mathscr{A}_Y \otimes_{f^{-1}\mathscr{A}_X} f^{-1}\mathscr{D}_X,\\
\mathscr{D}_{X\leftarrow Y} &= \mathscr{A}_Y^\vee \otimes_{\mathscr{A}_Y} \mathscr{D}_{Y\to X} \otimes_{f^{-1}\mathscr{A}_X} (f^{-1}\mathscr{A}_X^\vee)^{\otimes(-1)}
\end{aligned}
$$

and they are a $(\mathscr{D}_Y, f^{-1}\mathscr{D}_X)$-bimodule and an $(f^{-1}\mathscr{D}_X, \mathscr{D}_Y)$-bimodule, respectively. For a left $\mathscr{D}_X$-module $\mathfrak{M}$ (or more generally, an object of $\mathbf{D}^b(\mathscr{D}_X)$), we define

$$
\underline{f}^{-1}\mathfrak{M} = \mathscr{D}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_X} f^{-1}\mathfrak{M}
$$

and for a left $\mathscr{D}_Y$-module $\mathfrak{N}$ (or more generally, an object of $\mathbf{D}^b(\mathscr{D}_Y)$), we define

$$
\begin{aligned}
\underline{f}_!\mathfrak{N} &= R f_!(\mathscr{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathfrak{N}),\\
\underline{f}_*\mathfrak{N} &= R f_*(\mathscr{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathfrak{N}).
\end{aligned}
$$

We can define the same functors for right $\mathscr{D}$-modules. For example for $\mathfrak{N} \in \mathbf{D}^b(\mathscr{D}_Y^{\mathrm{opp}})$

$$
\begin{aligned}
\underline{f}_!\mathfrak{N} &= R f_!(\mathfrak{N} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X}),\\
\underline{f}_*\mathfrak{N} &= R f_*(\mathfrak{N} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X}).
\end{aligned}
$$

Proposition 3.1. — *Let X and Y be two real analytic manifolds. Then there exists a natural morphism in $\mathbf{D}^b(\mathscr{D}_{X\times Y})$, functorial with respect to $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ and $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$:*

$$
(F \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X) \underline{\boxtimes} (G \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_Y) \to (F \boxtimes G) \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{X\times Y}. \tag{3.1}
$$

Proof. First assume $G = \mathbb{C}_V$ for an open subanalytic subset V of Y. Denote by ψ_1 and ψ_2 the two functors on $\mathcal{S}_X$ defined by:

$$\psi_1(U) = (\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \boxtimes (\mathbb{C}_V \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_Y),$$
$$\psi_2(U) = \mathbb{C}_{U\times V} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_{X\times Y}.$$

There is a natural morphism $\psi_1 \to \psi_2$. Applying Theorem 1.1, we get the result in case $G = \mathbb{C}_V$. Now let $F \in \mathbb{R}\text{-Cons}(X)$. We apply the same argument to the functors:

$$\psi_1(V) = (F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \boxtimes (\mathbb{C}_V \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_Y)$$
$$\psi_2(V) = (F \boxtimes \mathbb{C}_V) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_{X\times Y}$$

and the result follows. □

Remark that morphism (3.1) is not an isomorphism in general. To have an isomorphism, one has to consider the topological tensor product $\cdot \mathbin{\widehat{\boxtimes}} \cdot$ of [Gr1].

Proposition 3.2. — *Let $F \in \mathbb{R}\text{-Cons}(X)$ and $G \in \mathbb{R}\text{-Cons}(Y)$. Then:*

$$\Gamma(X \times Y\,;\,(F \boxtimes G) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_{X\times Y}) \simeq \Gamma(X; F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \mathbin{\widehat{\boxtimes}} \Gamma(Y; G \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_Y)\,. \tag{3.2}$$

Proof. The functor $\cdot \mathbin{\widehat{\boxtimes}} \cdot$ being exact on the category of vector spaces of type FN, one may reduce the proof (using Theorem 1.1) to the case $F = \mathbb{C}_{Z_1}$, $G = \mathbb{C}_{Z_2}$, where Z_1 and Z_2 are closed subanalytic subsets of X and Y respectively. Then it is enough to prove:

$$\Gamma(X \times Y; \mathcal{I}^\infty_{X\times Y, Z_1\times Z_2}) \simeq \Gamma(X; \mathcal{I}^\infty_{X,Z_1}) \mathbin{\widehat{\boxtimes}} \Gamma(Y; \mathcal{I}^\infty_{Y,Z_2}).$$

It is well-known that

$$\Gamma(X \times Y; \mathcal{C}^\infty_{X\times Y}) \simeq \Gamma(X; \mathcal{C}^\infty_X) \mathbin{\widehat{\boxtimes}} \Gamma(Y; \mathcal{C}^\infty_Y).$$

For $x \in X$ (resp. $y \in Y$) let us denote by E_x (resp. F_y) the set of C^∞-functions on X (resp. Y) that vanish at x (resp. y) to infinite order. Then we can see easily that $E_x \mathbin{\widehat{\otimes}} F_y$ is the set of C^∞-functions on $X \times Y$ that vanish at (x, y) to infinite order. Now we remark that for an FN-space E and a complete space F and a family of closed subspaces F_j of F, we have

$$\bigcap_j (E \mathbin{\widehat{\otimes}} F_j) = E \mathbin{\widehat{\otimes}} \big(\bigcap_j F_j\big)\,,$$

since $E \widehat{\otimes} F$ coincides with the space of continuous maps from E^* to F. Applying this remark, we obtain

$$\begin{aligned}\Gamma(X \times Y; \mathscr{I}^{\infty}_{X\times Y, Z_1\times Z_2}) &= \bigcap_{\substack{x\in Z_1\\ y\in Z_2}} E_x \widehat{\otimes} F_y = (\bigcap_{x\in Z_1} E_x) \widehat{\otimes} (\bigcap_{y\in Z_2} F_y)\\ &= \Gamma(X; \mathscr{I}^{\infty}_{X,Z_1}) \widehat{\boxtimes} \Gamma(Y; \mathscr{I}^{\infty}_{Y,Z_2}).\end{aligned}$$

□

Now, let $f : Y \to X$ be a morphism of real analytic manifolds.

Theorem 3.3. — *Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$.*

(i) *There exists a natural morphism in $\mathbf{D}^b(\mathscr{D}_Y)$, functorial in F:*

$$f^{-1}(F \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_X) \to f^{-1}F \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_Y. \tag{3.3}$$

(ii) *This morphism is equivalent to the morphism in $\mathbf{D}^b(f^{-1}\mathscr{D}_X)$:*

$$f^{-1}(F \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_X) \to R\,\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{D}_{Y\to X}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_Y)\,. \tag{3.4}$$

(iii) *If f is a closed embedding,* (3.3) *is an isomorphism.*

(iv) *If f is smooth,* (3.4) *is an isomorphism.*

Proof. (i) For $U \in \mathscr{S}_X$, set:

$$\begin{aligned}\psi_1(U) &= \mathscr{D}_{Y\to X} \otimes_{f^{-1}\mathscr{D}_X} f^{-1}(\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_X),\\ \psi_2(U) &= \mathbb{C}_{f^{-1}(U)} \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_Y.\end{aligned}$$

These two functors satisfy conditions (1.1) and (1.2). Let $Z = X \setminus U$. The natural morphism

$$\mathscr{A}_Y \otimes_{f^{-1}\mathscr{A}_X} f^{-1}\mathscr{I}^{\infty}_{X,Z} \to \mathscr{I}^{\infty}_{Y,f^{-1}(Z)}$$

defines the morphism:

$$\theta(U) \colon \psi_1(U) \to \psi_2(U)\,.$$

Theorem 1.1 gives a morphism

$$\mathscr{D}_{Y\to X} \otimes_{f^{-1}\mathscr{D}_X} f^{-1}(F \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_X) \to (f^{-1}F) \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_Y.$$

Then to obtain (i), it remains to use

$$\underline{f}^{-1}(F \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_X) \to \mathscr{D}_{Y\to X} \otimes_{f^{-1}\mathscr{D}_X} f^{-1}(F \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_X)\,.$$

(ii) follows from the adjunction formula:

$$\mathrm{Hom}_{D^b(\mathcal{D}_Y)}(\mathcal{D}_{Y\to X}\overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_X}\mathfrak{M},\mathfrak{N})\simeq \mathrm{Hom}_{D^b(f^{-1}\mathcal{D}_X)}(\mathfrak{M},R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X},\mathfrak{N}))$$

applied with $\mathfrak{M}=f^{-1}(F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X)$ and $\mathfrak{N}=f^{-1}F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_Y$.

(iii) We may assume that Y is a closed submanifold of X. Arguing by induction on codim Y, we may assume that Y is a hypersurface defined by the equation $g=0$, with $dg\neq 0$. Using Proposition 1.3, we may also assume $F=\mathbb{C}_U$ for an open subanalytic subset U of X. Let $Z=X\setminus U$. We have to show that the natural morphism:

$$\theta\colon \mathcal{I}^\infty_{X,Z}/g\mathcal{I}^\infty_{X,Z}\to\mathcal{I}^\infty_{Y,Z\cap Y}$$

is an isomorphism.

Since $\mathcal{I}^\infty_{X,Z}\cap g\mathcal{C}^\infty_X=g\mathcal{I}^\infty_{X,Z}$, θ is injective. On the other hand, any $h\in\mathcal{I}^\infty_{Y,Z\cap Y}$ may be extended to $\tilde h\in\mathcal{I}^\infty_{X,Z\cap Y}$. By Theorem 2.1, we may decompose $\tilde h$ as $\tilde h=\tilde h_1+\tilde h_2$, with $\tilde h_1\in\mathcal{I}^\infty_{X,Z}$, $\tilde h_2\in\mathcal{I}^\infty_{X,Y}$. Hence θ sends $\tilde h_1$ to h.

(iv) We may argue locally on Y and make an induction on $\dim Y-\dim X$. Hence we may assume that $Y=X\times\mathbb{R}$ and f is the projection. Moreover, by Proposition 1.3, we may assume $F=\mathbb{C}_U$ for an open subanalytic subset U of X. Let $Z=X\setminus U$. Denoting by t the coordinate of $\mathbb{R}$, it is enough to show that

$$0\to f^{-1}\mathcal{I}^\infty_{X,Z}\to\mathcal{I}^\infty_{Y,f^{-1}(Z)}\xrightarrow{\partial/\partial t}\mathcal{I}^\infty_{Y,f^{-1}(Z)}\to 0$$

is exact. This is an easy exercise. □

Remark 3.4. If f is smooth, the isomorphism (3.4) defines a morphism:

$$\underline{f}_!(f^{-1}F\overset{\mathrm{w}}{\otimes}\mathcal{C}^{\infty\vee}_Y)\to F\overset{\mathrm{w}}{\otimes}\mathcal{C}^{\infty\vee}_X. \tag{3.5}$$

In fact we may write (3.4) as

$$\mathcal{D}_{X\leftarrow Y}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y}(f^{-1}F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_Y\otimes\mathrm{or}_Y)[-d]\simeq f^{-1}(F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X\otimes\mathrm{or}_X),$$

where $d=\dim Y-\dim X$, or equivalently:

$$(f^{-1}F\overset{\mathrm{w}}{\otimes}\mathcal{C}^{\infty\vee}_Y)\overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y}\mathcal{D}_{Y\to X}\simeq f^!(F\overset{\mathrm{w}}{\otimes}\mathcal{C}^{\infty\vee}_X).$$

Then (3.5) follows by adjunction.

The morphism (3.5) is also constructed as in Proposition 4.3 below by using the integration along the fiber $f_!(\mathcal{C}^{\infty\vee}_Y)\to\mathcal{C}^{\infty\vee}_X$.

Theorem 3.5. — *Let $G\in\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$ and assume that f is proper on* supp(G). *Then there is a natural isomorphism in $\mathbf{D}^b(\mathcal{D}_X)$, functorial with respect to G:*

$$Rf_!G\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X\xrightarrow{\sim}Rf_!(R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X},G\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_Y)). \tag{3.6}$$

Proof.
(i) Using morphism (3.4) with $F = Rf_!G$, we get the morphism:

$$Rf_!G \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty \to Rf_*R\,\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{D}_{Y\to X}, f^{-1}Rf_*G \overset{\mathrm{w}}{\otimes} \mathscr{C}_Y^\infty)\,.$$

By composing with $f^{-1}Rf_*G \to G$, we get morphism (3.6). Let us prove that this is an isomorphism. By decomposing f as a product of a smooth map and a closed embedding, we may argue separately in these cases.

(ii) First assume that f is smooth. We may suppose supp(G) is contained in an arbitrarily small open subset of Y (if $Z = \mathrm{supp}(G)$ and $Z = Z_1 \cup Z_2$, use the distinguished triangle $G \to G_{Z_1} \oplus G_{Z_2} \to G_{Z_1 \cap Z_2} \xrightarrow{+1}$). Hence we may assume that $Y = X \times \mathbb{R}^p$ and f is the projection. Arguing by induction, we may assume $p = 1$. Moreover, by Proposition 1.3, we may assume $G = \mathbb{C}_Z$, where Z is a closed subanalytic subset of Y.

Lemma 3.6. — *There exists a disjoint locally finite family $\{Z_j\}$ of locally closed subanalytic subsets of Y satisfying the following properties:*

(i) $Z = \sqcup_j Z_j$,

(ii) *$f(Z_j)$ is locally closed and Z_j is closed in $f^{-1}f(Z_j)$ for any j,*

(iii) *for any j and $x \in f(Z_j)$, $f^{-1}(x) \cap Z_j$ is connected,*

(iv) *for any j, $\overline{Z_j} \setminus Z_j$ is a union of Z_k's.*

Proof. Since $f_*(\mathbb{C}_Z)$ is a constructible sheaf, there exists a subanalytic stratification $X = \sqcup_\alpha X_\alpha$ such that $f_*(\mathbb{C}_Z)\big|_{X_\alpha}$ is locally constant of rank N_α. Then for any $x \in X_\alpha$, $f^{-1}(x) \bigcap Z$ has exactly N_α connected components, say $\{Z_j(x)\}_{j=1,\cdots,N_\alpha}$. We order them so that if we take $z_j \in Z_j(x)$ then $z_j < z_{j'}$ for $j < j'$. Set $Z_{\alpha,j} = \bigcup_{x\in X_\alpha} Z_j(x)$. Hence Z is a disjoint union of $Z_{\alpha,j}$.

Let us show that $Z_{\alpha,j}$ is closed in $Z \bigcap f^{-1}(X_\alpha)$. Take $x_0 \in X_\alpha$. There exists a disjoint family $\{U_j\}_{j=1,\cdots,N_\alpha}$ of open subsets of Y such that $Z_j(x_0) \subset U_j$. Then there exists a neighborhood W of x_0 such that $Z \bigcap f^{-1}(W) \subset \bigcup_j U_j$. Since $f_*(\mathbb{C}_Z) \simeq \bigoplus_j f_*(\mathbb{C}_{Z\cap U_j})$ on W, $f_*(\mathbb{C}_{Z\cap U_j})\big|_{W\cap X_\alpha}$ is a locally constant sheaf of rank 1, by taking W such that $W \cap X_\alpha$ is connected. Then the fiber of $Z \bigcap U_j \to X$ is connected over $W \bigcap X_\alpha$ and hence $Z_{\alpha,j} \bigcap f^{-1}W = Z \bigcap U_j \bigcap f^{-1}(X_\alpha \bigcap W)$. This shows that $Z_{\alpha,j}$ is closed in $Z \bigcap f^{-1}(X_\alpha)$. Therefore $Z_{\alpha,j}$ is subanalytic. The family $\{Z_{\alpha,j}\}_{\alpha,j}$ satisfies the desired property. □

By this lemma, we may assume $G = \mathbb{C}_Z$ where Z is a locally closed subanalytic subset of Y satisfying the following properties:

$$\tag{3.7} \begin{cases} T = f(Z) \text{ is a locally closed subanalytic subset of } X, \\ \text{for any } x \in T,\ Z \bigcap f^{-1}(x) \text{ is connected,} \\ Z \text{ is closed in } f^{-1}(T), \\ \bar{Z} \to X \text{ is proper.} \end{cases}$$

Moreover we may assume that Z is contained in $X \times \{t \in \mathbb{R}; -1 < t < 1\}$. Set $S = (\bar{T} \setminus T) \times \{t \in \mathbb{R}; -1 \leq t \leq 1\}$. Then $Z_1 = S \bigcup Z$ is a closed analytic subset with connected fibers over X. Then it is enough to prove the theorem for $G = \mathbb{C}_S$ and $G = \mathbb{C}_{Z_1}$. Hence we reduced the theorem to the case $G = \mathbb{C}_Z$ where Z is a closed subanalytic subset satisfying the following two properties:

$$\tag{3.8} \begin{cases} \text{for any } x \in f(Z),\ Z \bigcap f^{-1}(x) \text{ is connected,} \\ Z \subset X \times \{t \in \mathbb{R}; 0 < t < 1\}. \end{cases}$$

Let $p_\pm : Y \times \mathbb{R}_{\geq 0} \to Y$ be the map $((x,t),s) \mapsto (x, t \pm s)$. Set $Z_\pm = p_\pm(Z \times \mathbb{R}_{\geq 0}) \bigcap X \times [0,1]$. Then $Z_\pm$ is a closed subanalytic set and $Z = Z_+ \bigcap Z_-$ and $T \times [0,1] = Z_+ \bigcup Z_-$. Therefore we have an exact sequence

$$0 \to \mathbb{C}_{T \times [0,1]} \to \mathbb{C}_{Z_+} \oplus \mathbb{C}_{Z_-} \to \mathbb{C}_Z \to 0\,.$$

Hence it is enough to check the theorem for $G = \mathbb{C}_{Z_\pm}, \mathbb{C}_{T \times [0,1]}$.

Thus we have finally reduced the theorem to the case $G = \mathbb{C}_Z$, Z being a closed subanalytic subset of Y satisfying:

$$\tag{3.9} \begin{cases} Z \text{ is proper over } X, \\ \text{for any } x \in f(Z),\ Z \bigcap f^{-1}(x) \text{ is a closed interval containing } 0. \end{cases}$$

Set $T = f(Z)$. Then $Rf_*(G) = \mathbb{C}_T$. We have a commutative diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{I}^\infty_{X,T} \arrow[r] \arrow[d] & f_*(\mathscr{I}^\infty_{Y,Z}) \arrow[r, "\partial/\partial t"] \arrow[d] & f_*(\mathscr{I}^\infty_{Y,Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{C}^\infty_X \arrow[r] \arrow[d] & f_*(\mathscr{C}^\infty_Y) \arrow[r, "\partial/\partial t"] \arrow[d] & f_*(\mathscr{C}^\infty_Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}_T \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X \arrow[r] \arrow[d] & f_*(\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_Y) \arrow[r, "\partial/\partial t"] \arrow[d] & f_*(\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_Y) \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $\partial/\partial t$ has a right inverse given by $u(x,t) \mapsto \int_0^t u(x,t)\,dt$, the top and the middle rows are exact and hence the bottom row is exact.

(iii) Finally assume that f is a closed embedding. Arguing by induction, we may assume $Y = \{x_n = 0\}$, where $(x_1, \dots, x_n)$ is a local coordinate system. Moreover, by Proposition 1.3, we may assume $G = \mathbb{C}_Z$, Z being a closed subanalytic subset of Y. Then we have $\mathscr{D}_{Y\to X} \simeq \bigoplus_{k\geq 0} \mathscr{D}_Y(\partial/\partial x_n)^k/k!$. Hence for a $\mathscr{D}_Y$-module $\mathfrak{N}$, we have the isomorphism

$$\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{D}_{Y\to X}, \mathfrak{N}) \simeq \mathfrak{N}[[x_n]] = \prod_{k=0}^{\infty} \mathfrak{N} \otimes \mathbb{C}x_n^k$$

given by

$$\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{D}_{Y\to X}, \mathfrak{N}) \ni f \mapsto \sum_{k=0}^{\infty} x_n^k f\left((\partial/\partial x_n)^k/k!\right) .$$

Hence taking $\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_Y$ as $\mathfrak{N}$, (3.6) reduces to the bijectivity of:

$$\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X \to (\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_Y)[[x_n]] \tag{3.10}$$

Let us consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}^\infty_{X,Z} \arrow[r] \arrow[d, "\alpha"] & \mathscr{C}^\infty_X \arrow[r] \arrow[d, "\beta"] & \mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X \arrow[r] \arrow[d, "(3.10)"] & 0 \\
0 \arrow[r] & \mathscr{I}^\infty_{Y,Z}[[x_n]] \arrow[r] & \mathscr{C}^\infty_Y[[x_n]] \arrow[r] & (\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_Y)[[x_n]] \arrow[r] & 0
\end{tikzcd}$$

Then $\operatorname{Ker}\alpha \simeq \operatorname{Ker}\beta \simeq \mathcal{I}^{\infty}_{X,Y}$ and α and β are surjective. Hence (3.10) is an isomorphism. $\square$

Remark 3.7 Note that Theorem 3.5 does not remain true if we replace $\overset{\mathrm{w}}{\otimes}$ with $\otimes$.

4. Operations on $\mathcal{T}hom(\cdot, \mathcal{D}b_X)$

The results of this section already appeared in [Ka3], but our construction of the direct image morphism is slightly different.

Proposition 4.1. — *Let X and Y be two real analytic manifolds. Then there exists a natural morphism in $\mathbf{D}^b(\mathcal{D}_{X\times Y})$, functorial in $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ and $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$:*

$$\mathcal{T}hom(F, \mathcal{D}b_X) \underline{\boxtimes}\, \mathcal{T}hom(G, \mathcal{D}b_Y) \to \mathcal{T}hom(F \boxtimes G, \mathcal{D}b_Y). \tag{4.1}$$

The proof is similar to the one of Proposition 3.1 and we do not repeat it.

Remark that the morphism (4.1) is not an isomorphism in general. Similarly to Proposition 3.2, we have:

Proposition 4.2. — *For $F \in \mathbb{R}\text{-Cons}(X)$ and $G \in \mathbb{R}\text{-Cons}(Y)$, we have*

(4.2)
$$\Gamma_c\,(X \times Y; \mathcal{T}hom(F \boxtimes G, \mathcal{D}b_{X\times Y})) \simeq \Gamma_c\,(X; \mathcal{T}hom(F, \mathcal{D}b_X)) \,\widehat{\boxtimes}\, \Gamma_c\,(Y; \mathcal{T}hom(G, \mathcal{D}b_Y))\,.$$

Proof. This follows by duality (Proposition 2.2) from Proposition 3.2. □

Now let $f : Y \to X$ be a morphism of real analytic manifolds.

Proposition 4.3. — *There is a natural morphism in $\mathbf{D}^b(\mathcal{D}_X^{\mathrm{opp}})$, functorial in $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$:*

$$\underline{f}_!\, \mathcal{T}hom(f^{-1}F, \mathcal{D}b_Y^\vee) \to \mathcal{T}hom(F, \mathcal{D}b_X^\vee)\,. \tag{4.3}$$

Proof. Let Z be a closed subanalytic subset of X. For a $\mathcal{D}_Y$-module $\mathfrak{M}$, we have the Spencer sequence $Sp_\bullet(\mathfrak{M})$ and a quasi-isomorphism $Sp_\bullet(\mathfrak{M}) \to \mathfrak{M}$. Denoting by Θ_Y the sheaf of real analytic vector fields on Y, we have $Sp_k(\mathfrak{M}) = \mathcal{D}_Y \otimes_{\mathcal{A}_Y} \bigwedge^k \Theta_Y \otimes_{\mathcal{A}_Y} \mathfrak{M}$. Then $Sp_\bullet(\mathcal{D}_{Y\to X})$ gives a resolution of $\mathcal{D}_{Y\to X}$ as a $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$-bimodule locally free over $\mathcal{D}_Y$. Hence $\Gamma_{f^{-1}(Z)}\mathcal{D}b_Y^\vee \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ is represented by the complex $\mathcal{K}_\bullet = \Gamma_{f^{-1}(Z)}\mathcal{D}b_Y^\vee \otimes_{\mathcal{D}_Y} Sp_\bullet(\mathcal{D}_{Y\to X})$. We have $\mathcal{K}_k = \Gamma_{f^{-1}(Z)}\mathcal{D}b_Y^\vee \otimes_{\mathcal{A}_Y} \bigwedge^k \Theta_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{D}_X$. Hence we have $f_!(\mathcal{K}_0) = \Gamma_Z f_!(\mathcal{D}b_Y^\vee) \otimes_{\mathcal{A}_X} \mathcal{D}_X$. The integration of distributions gives a morphism $\int_f : f_!(\mathcal{D}b_Y^\vee) \to \mathcal{D}b_X^\vee$. Since $\mathcal{D}b_X^\vee$ is a right $\mathcal{D}_X$-module, we obtain the morphism $u : f_!(\mathcal{K}_0) \to \Gamma_Z\mathcal{D}b_X^\vee$. We shall show that the composition

$$f_!(\mathcal{K}_1) \xrightarrow{d_1} f_!(\mathcal{K}_0) \xrightarrow{u} \Gamma_Z\mathcal{D}b_X^\vee$$

vanishes. The homomorphism

$$d_1 : \mathcal{K}_1 = \Gamma_{f^{-1}(Z)}\mathcal{D}b_Y^\vee \otimes_{\mathcal{A}_Y} \Theta_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{D}_X \to \mathcal{K}_0 = \Gamma_{f^{-1}(Z)}\mathcal{D}b_Y^\vee \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{D}_X$$

is given explicitly as follows. For $\varphi \in \mathcal{D}b_Y^\vee$, $v \in \Theta_Y$ and $P \in \mathcal{D}_X$, writing the image of v by the morphism $\Theta_Y \to \mathcal{A}_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\Theta_X$ as $\sum a_j \otimes w_j$ ($a_j \in \mathcal{A}_Y$, $w_j \in \Theta_X$), we have

$$d_1(\varphi \otimes v \otimes P) = \varphi v \otimes P - \sum_j \varphi a_j \otimes w_j P\,.$$

Let s be a section of $f_!(\mathcal{K}_1)$. We may assume $s = \varphi \otimes v \otimes P$, where the support of φ is small enough. In order to see that $ud_1(s) = 0$, it is enough to show that $(\int_f \varphi v)P - \sum_j(\int_f \varphi a_j)w_j P = 0$. For any C^∞-function g on X we have

$$\begin{aligned}\int_X \Big(\Big(\int_f \varphi v\Big)P - \sum_j \Big(\int_f \varphi a_j\Big) w_j P\Big) g \\ = \int_X \Big(\int_f \varphi v\Big)(Pg) - \sum_j \int_X \Big(\int_f \varphi a_j\Big) w_j P g \\ = \int_Y \varphi\Big(v f^*(Pg) - \sum_j a_j f^*(w_j P g)\Big) = 0\,.\end{aligned}$$

Hence we obtain $ud_1 = 0$. Thus we have constructed a morphism of complexes

$$f_!\left(\Gamma_{f^{-1}Z}(\mathcal{D}b_Y^\vee) \otimes_{\mathcal{D}_Y} Sp_\bullet(\mathcal{D}_{Y\to X})\right) \to \Gamma_Z(\mathcal{D}b_X^\vee)\,.$$

Since $F \mapsto f_!\left(\mathcal{T}hom(f^{-1}F, \mathcal{D}b_Y^\vee) \otimes_{\mathcal{D}_Y} Sp_\bullet(\mathcal{D}_{Y\to X})\right)$ is an exact functor from $\mathbb{R}$-Cons(X) to the category of complexes of $\mathcal{D}_X$-modules, we may apply Theorem 1.1 and define a natural morphism:

$$f_!\left(\mathcal{T}hom(f^{-1}F, \mathcal{D}b_Y^\vee) \otimes_{\mathcal{D}_Y} Sp_\bullet(\mathcal{D}_{Y\to X})\right) \to \mathcal{T}hom(F, \mathcal{D}b_X^\vee)$$

for $F \in \mathbb{R}$-Cons(X) and hence for $F \in K^b(\mathbb{R}\text{-Cons}(X))$. Thus we get (4.3) since $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ is the derived category of $\mathbb{R}$-Cons(X). □

Theorem 4.4. — *Let $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$ and assume that f is proper on* $\operatorname{supp} G$. *Then there is a natural isomorphism in $\mathbf{D}^b(\mathcal{D}_X)$, functorial with respect to G:*

$$\underline{f}_! \, \mathcal{T}\!hom(G, \mathcal{D}b_Y) \xrightarrow{\sim} \mathcal{T}\!hom(R f_* G, \mathcal{D}b_X). \tag{4.4}$$

Proof. The morphism is constructed by applying Proposition 4.3 with $F = R f_* G$, and then using $f^{-1} R f_* G \to G$. By using the graph embedding, it is enough to prove the theorem in the case of a closed embedding and the case of a smooth morphism.

When f is a closed embedding, applying Proposition 1.3, we can reduce to the case $G = \mathbb{C}_Z$ for a closed subanalytic subset of Y, and then one easily sees that (4.4) is an isomorphism, using the local structure theorem of distributions supported by a submanifold:

$$\Gamma_Y(\mathcal{D}b_X) \simeq \mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{D}b_Y \, .$$

If f is smooth, the proof that (4.4) is an isomorphism goes as in Theorem 3.5, and one can reduce the theorem to the case where $Y = X \times \mathbb{R}$ and f the projection to X, $G = \mathbb{C}_Z$ where Z satisfies the condition (3.9). Thus we have to check the exactitude of

$$0 \to f_! \Gamma_Z \mathcal{D}b_Y \xrightarrow{\partial/\partial t} f_! \Gamma_Z \mathcal{D}b_Y \xrightarrow{\int \cdot \, dt} \Gamma_{f(Z)} \mathcal{D}b_X \to 0.$$

This is an easy verification (cf. [Ka3, Lemma 4.5]). □

For $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$, the morphism (4.3) defines the morphisms

$$\mathcal{D}_{X \leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{T}\!hom(f^{-1}F, \mathcal{D}b_Y) \to f^! \, \mathcal{T}\!hom(F, \mathcal{D}b_X) \tag{4.5}$$

$$\mathcal{T}\!hom(f^{-1}F, \mathcal{D}b_Y) \to R \, \mathcal{H}om_{f^{-1}\mathcal{D}_X}(\mathcal{D}_{X \leftarrow Y}, f^! \, \mathcal{T}\!hom(F, \mathcal{D}b_X)). \tag{4.6}$$

Theorem 4.5. — *Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$.*

(i) *Assume that f is smooth. Then* (4.5) *defines the isomorphism:*

$$R \, \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y \to X}, \mathcal{T}\!hom(f^{-1}F, \mathcal{D}b_Y)) \xrightarrow{\sim} f^{-1} \, \mathcal{T}\!hom(F, \mathcal{D}b_X) \, . \tag{4.7}$$

(ii) *Assume that f is a closed embedding. Then* (4.6) *defines the isomorphism:*

$$\mathcal{T}\!hom(f^{-1}F, \mathcal{D}b_Y) \xrightarrow{\sim} R \, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X \leftarrow Y}, \mathcal{T}\!hom(F, \mathcal{D}b_X)). \tag{4.8}$$

Proof. (i) Set $d = \dim Y - \dim X$. Since f is smooth, $f^! S \simeq f^{-1} S \otimes \mathrm{or}_{Y/X}[d]$ for any sheaf S on X, and $\mathcal{D}_{X \leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{N} \simeq R \, \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y \to X}, \mathcal{N}) \otimes \mathrm{or}_{Y/X}[d]$ for any $\mathcal{D}_Y$-module $\mathcal{N}$. This defines the morphism (4.7). To prove that it is an isomorphism, we may reduce the proof to the case $Y = X \times \mathbb{R}$, f is the projection and $F = \mathbb{C}_Z$, Z being a closed subanalytic subset of X. Then one checks that the sequence:

$$0 \to f^{-1}\Gamma_Z \mathcal{D}b_X \to \Gamma_{f^{-1}(Z)}\mathcal{D}b_Y \xrightarrow{\partial/\partial t} \Gamma_{f^{-1}(Z)}\mathcal{D}b_Y \to 0$$

is exact. Here t denotes the coordinate of $\mathbb{R}$.
(ii) Let us prove first

$$R\,\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{D}_{X\leftarrow Y}, \mathcal{T}hom(F_{X\setminus Y}, \mathcal{D}b_X)\right) = 0. \tag{4.9}$$

The question being local, we can write $Y = \{x = (x_1, \dots, x_n); x_1 = \cdots = x_l = 0\}$. Set $Y_i = \{x; x_i = 0\}$. Then we have an exact sequence

$$0 \leftarrow F_{X\setminus Y} \leftarrow \oplus_i F_{X\setminus Y_i} \leftarrow \oplus_{i\neq j} F_{X\setminus(Y_i\cup Y_j)} \leftarrow \cdots .$$

Hence by replacing F with $F_{X\setminus Y_i}$, $F_{X\setminus(Y_i\cup Y_j)}$, etc., we may assume that $F_{X\setminus Y} = F_{X\setminus Y_i}$ for some i. Since we have

$$\begin{aligned}
&R\,\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{D}_{X\leftarrow Y}, \mathcal{T}hom(F_{X\setminus Y}, \mathcal{D}b_X)\right)\\
&\qquad\simeq R\,\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{D}_{X\leftarrow Y_i} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{Y_i}} \mathcal{D}_{Y_i\leftarrow Y}, \mathcal{T}hom(F_{X\setminus Y}, \mathcal{D}b_X)\right)\\
&\qquad\simeq R\,\mathcal{H}om_{\mathcal{D}_{Y_i}}\left(\mathcal{D}_{Y_i\leftarrow Y}, R\,\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{D}_{X\leftarrow Y_i}, \mathcal{T}hom(F_{X\setminus Y}, \mathcal{D}b_X)\right)\right)
\end{aligned}$$

we can reduce to the case when Y is a hypersurface defined by the equation $\{g = 0\}$ with $dg \neq 0$. We may also assume $F = \mathbb{C}_U$, U being an open subanalytic subset of X. The multiplication by g on $\mathcal{T}hom(\mathbb{C}_{U\setminus Y}, \mathcal{D}b_X)$ is surjective (resp. injective) since it is a quotient of $\mathcal{D}b_X$ (resp. a subsheaf of $j_*\mathcal{D}b_{U\setminus Y}$ where $j : U \setminus Y \to X$ is the open embedding). This shows (4.9).

Using (4.9) and the distinguished triangle $F_{X\setminus Y} \to F \to F_Y \xrightarrow{+1}$, it remains to prove (4.8) when $F = f_*G$ with $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$. Then by Theorem 4.4,

$$\begin{aligned}
R\,\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{D}_{X\leftarrow Y}, \mathcal{T}hom(F, \mathcal{D}b_X)\right) &\simeq R\,\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{D}_{X\leftarrow Y}, \mathcal{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{T}hom(G, \mathcal{D}b_Y)\right)\\
&\simeq R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y}, \mathcal{D}_{X\leftarrow Y}) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{T}hom(f^{-1}F, \mathcal{D}b_Y)
\end{aligned}$$

and the result follows from

$$R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y}, \mathcal{D}_{X\leftarrow Y}) \simeq \mathcal{D}_Y .$$

□

5. The functors $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $\mathcal{T}hom(\cdot\,, \mathcal{O}_X)$

From now on, all manifolds and morphisms of manifolds will be complex analytic. If X is a complex manifold, one denotes by $\mathcal{O}_X$ its structural sheaf and by $\mathcal{O}_X^{(p)}$ the sheaf of holomorphic p-forms. One denotes by d_X the complex dimension of X, and we also write Ω_X instead of $\mathcal{O}_X^{(d_X)}$. We denote by $X_\mathbb{R}$ the underlying real analytic manifold of X and by $\overline{X}$ the complex conjugate of X, i.e. the complex manifold with real underlying manifold $X_\mathbb{R}$ and structural sheaf $\mathcal{O}_{\overline{X}}$, the sheaf of anti-holomorphic functions on X. Then, $X \times \overline{X}$ is a complexification of $X_\mathbb{R}$ by the diagonal embedding $X_\mathbb{R} \hookrightarrow X \times \overline{X}$. If $f : Y \to X$ is a morphism of complex manifolds, we denote by $f_\mathbb{R}$ the real analytic underlying morphism. However, if there is no risk of confusion, we often write X or f instead of $X_\mathbb{R}$ or $f_\mathbb{R}$. For example, we shall always write $\mathcal{C}_X^\infty$ instead of $\mathcal{C}_{X_\mathbb{R}}^\infty$, or $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ instead of $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_{X_\mathbb{R}})$. We denote by $\mathcal{D}_X$ the sheaf of rings of finite order holomorphic differential operators on X, and by $\underline{f}^{-1}$, $\underline{f}_!$, $\underline{f}_*$, $\boxtimes$ the operations on holomorphic $\mathcal{D}$-modules. We denote by $\mathcal{D}_{Y\to X}$ and $\mathcal{D}_{X\leftarrow Y}$ the "transfer bimodules". Notice that $\mathcal{D}_X$ and $\mathcal{D}_{\overline{X}}$ are two subrings of $\mathcal{D}_{X_\mathbb{R}}$ and if $P \in \mathcal{D}_X$, $Q \in \mathcal{D}_{\overline{X}}$, then $[P, Q] = 0$.

Definition 5.1 — *Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. We set:*

$$F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X = R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty)\,,$$
$$\mathcal{T}hom(F, \mathcal{O}_X) = R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{T}hom(F, \mathcal{D}b_X))\,.$$

We call $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $\mathcal{T}hom(\cdot\,, \mathcal{O}_X)$ the functors of formal and moderate cohomology, respectively. The objects $F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $\mathcal{T}hom(F, \mathcal{O}_X)$ belong to $\mathbf{D}^b(\mathcal{D}_X)$. If $\mathcal{G}$ is a locally free $\mathcal{O}_X$-module of finite rank, we set:

$$F \overset{\mathrm{w}}{\otimes} \mathcal{G} = (F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G}\,,$$
$$\mathcal{T}hom(F, \mathcal{G}) = \mathcal{T}hom(F, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G}\,.$$

Notice that:

$$F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \simeq \Omega_{\overline{X}} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{\overline{X}}} (F \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty)[-d_X]$$

and similarly for $\mathcal{T}hom(F, \mathcal{O}_X)$.

Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. Applying Proposition 2.8, we get a sequence of morphisms:

$$\mathrm{D}'_X F \otimes \mathcal{O}_X \to \mathrm{D}'_X F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \to \mathcal{T}hom(F, \mathcal{O}_X) \to R\,\mathcal{H}om(F, \mathcal{O}_X)\,.$$

Moreover, if $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$, there are natural morphisms:

$$G \otimes (F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \to (G \otimes F) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X\,,$$
$$\mathcal{T}hom(G \otimes F, \mathcal{O}_X) \to R\,\mathcal{H}om\,(G, \mathcal{T}hom(F, \mathcal{O}_X))\,.$$

We shall have to work in the derived categories of FN or DFN-spaces. Let us recall their constructions. Denote by $C^b(FN)$ the additive category of bounded complexes of topological vector spaces of type FN and linear continuous morphisms and by $K^b(FN)$ the category obtained by identifying to 0 a morphism homotopic to zero. Then $\mathbf{D}^b(FN)$ is the localization of $K^b(FN)$ by the complexes which are algebraically exact. The construction of $\mathbf{D}^b(DFN)$ is similar. The duality functors between FN and DFN spaces being exact, they extend to duality functors between the derived categories.

The bifunctor $\cdot \mathbin{\widehat{\boxtimes}} \cdot$ on the category of FN-spaces (resp. DFN-spaces) being exact, it extends to the derived category:

$$\widehat{\boxtimes}\colon \mathbf{D}^b(FN) \times \mathbf{D}^b(FN) \to \mathbf{D}^b(FN),$$
$$\widehat{\boxtimes}\colon \mathbf{D}^b(DFN) \times \mathbf{D}^b(DFN) \to \mathbf{D}^b(DFN)\,.$$

Proposition 5.2. — *Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. Then we can define*

$$R\Gamma(X; F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \text{ and } R\Gamma_c(X; \mathcal{T}hom(F, \Omega_X)[d_X])$$

as objects of $\mathbf{D}^b(FN)$ and $\mathbf{D}^b(DFN)$ respectively, and they are dual to each other.

This proposition will be generalized to the case of solutions of $\mathcal{D}$-modules in § 6.

Proof. First assume $F \in \mathbb{R}\text{-Cons}(X)$. Set:

$$V^i = \Gamma(X; F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty(0,i)}_X)$$
$$W^i = \Gamma_c(X; \mathcal{T}hom(F, \mathcal{D}b^{(d_X, d_X+i)}_X)).$$

By Proposition 2.2, the space V^i (resp. W^{-i}) is naturally endowed with a topology of type FN (resp. DFN) and these two spaces are dual to each other. The complexes $R\Gamma(X; F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$ and $R\Gamma_c(X; \mathcal{T}hom(F, \Omega_X[d_X]))$ are represented by the complexes:

$$0 \longrightarrow V^0 \underset{\bar\partial}{\rightarrow} V^1 \longrightarrow \cdots \longrightarrow V^{d_X} \longrightarrow 0$$
$$0 \longrightarrow W^{-d_X} \underset{\bar\partial}{\rightarrow} W^{-d_X+1} \longrightarrow \cdots \longrightarrow W^0 \longrightarrow 0,$$

respectively. Now let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. By [Ka$_3$, Theorem 2.8], F is represented by a bounded complex of $\mathbb{R}$-constructible sheaves, and the proof is similar. $\square$

We shall now study the functorial operations on the functors of formal and moderate cohomology.

Proposition 5.3. — *Let X and Y be two complex manifolds. Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$, $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$. Then there exist natural morphisms in $\mathbf{D}^b(\mathcal{D}_{X\times Y})$, functorial with respect to F and G:*

$$(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)\underline{\boxtimes}(G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y) \longrightarrow (F \boxtimes G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{X\times Y}, \tag{5.1}$$

$$\mathcal{T}hom(F, \mathcal{O}_X)\underline{\boxtimes}\,\mathcal{T}hom(G, \mathcal{O}_Y) \longrightarrow \mathcal{T}hom(F \boxtimes G, \mathcal{O}_{X\times Y}). \tag{5.2}$$

Proof. Apply $R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}\times\overline{Y}}}(\mathcal{O}_{\overline{X}\times\overline{Y}}, \cdot\,)$ to the morphisms (3.1) and (4.1). $\square$

Proposition 5.4. — *Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ and $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$. Then there are natural isomorphisms:*

(5.3)
$$R\Gamma(X \times Y; (F \boxtimes G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{X\times Y}) \simeq R\Gamma(X; F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \mathbin{\widehat{\boxtimes}} R\Gamma(Y; G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y),$$

(5.4)
$$R\Gamma_c(X \times Y; \mathcal{T}hom(F \boxtimes G, \mathcal{O}_{X\times Y})) \simeq R\Gamma_c(X; \mathcal{T}hom(F, \mathcal{O}_X)) \mathbin{\widehat{\boxtimes}} R\Gamma_c(Y; \mathcal{T}hom(G, \mathcal{O}_Y)).$$

Proof. The results follow from the corresponding ones with $\mathcal{O}$ replaced by $\mathcal{C}^\infty$ or $\mathcal{D}b$ in Propositions 3.2 and 4.2. $\square$

Now let $f : Y \to X$ be a morphism of complex manifolds. We shall often make use of the following morphisms.

Lemma 5.5. —

(i) *For $\mathcal{N} \in \mathbf{D}^b(\mathcal{D}_{Y_\mathbb{R}})$, we have the canonical isomorphisms:*

$$\begin{aligned} R\,\mathcal{H}om_{f^{-1}\mathcal{D}_{\overline{X}}}\Big(f^{-1}\mathcal{O}_{\overline{X}},\, R\,\mathcal{H}om_{\mathcal{D}_{Y_\mathbb{R}}}(\mathcal{D}_{Y_\mathbb{R}\to X_\mathbb{R}}, \mathcal{N})\Big) \\ \simeq R\,\mathcal{H}om_{\mathcal{D}_Y}\left(\mathcal{D}_{Y\to X}, R\,\mathcal{H}om_{\mathcal{D}_{\overline{Y}}}(\mathcal{O}_{\overline{Y}}, \mathcal{N})\right)\,, \end{aligned} \tag{5.5}$$

$$\underline{f}_* R\,\mathcal{H}om_{\mathcal{D}_{\overline{Y}}}(\mathcal{O}_{\overline{Y}}, \mathcal{N})[d_Y] \simeq R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \underline{f_\mathbb{R}}_* \mathcal{N})[d_X]\,, \tag{5.6}$$

$$\underline{f}_! R\,\mathcal{H}om_{\mathcal{D}_{\overline{Y}}}(\mathcal{O}_{\overline{Y}}, \mathcal{N})[d_Y] \simeq R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \underline{f_\mathbb{R}}_! \mathcal{N})[d_X]\,. \tag{5.7}$$

(ii) *For $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_{X_\mathbb{R}})$, we have a canonical morphism:*

$$\underline{f}^{-1} R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{M}) \to R\,\mathcal{H}om_{\mathcal{D}_{\overline{Y}}}(\mathcal{O}_{\overline{Y}}, \underline{f_\mathbb{R}}^{-1}\mathcal{M})\,. \tag{5.8}$$

Proof. Let us prove first (5.6). For a $\mathcal{D}_{Y_\mathbb{R}}$-module $\mathcal{N}$, we have

$$\mathcal{D}_{X_\mathbb{R}\leftarrow Y_\mathbb{R}} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{Y_\mathbb{R}}} \mathcal{N} \simeq \mathcal{D}_{\overline{X}\leftarrow\overline{Y}} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{\overline{Y}}} (\mathcal{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{N}) .$$

Hence we have

$$\begin{aligned}
\Omega_{\overline{X}} &\overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{\overline{X}}} \underline{f_{\mathbb{R}}}_* \mathcal{N} \\
&\simeq R f_* \left(f^{-1}\Omega_{\overline{X}} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_{\overline{X}}} \mathcal{D}_{\overline{X}\leftarrow\overline{Y}} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{\overline{Y}}} (\mathcal{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{N})\right) \\
&\simeq R f_* \left(\mathcal{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} (f^{-1}\Omega_{\overline{X}} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_{\overline{X}}} \mathcal{D}_{\overline{X}\leftarrow\overline{Y}} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{\overline{Y}}} \mathcal{N})\right) .
\end{aligned}$$

Hence (5.6) follows from $f^{-1}\Omega_{\overline{X}} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_{\overline{X}}} \mathcal{D}_{\overline{X}\leftarrow\overline{Y}} \simeq \Omega_{\overline{Y}}$.

The proof of (5.5) is similar. We have

$$\begin{aligned}
R\,\mathcal{H}om_{f^{-1}\mathcal{D}_{\overline{X}}} &\left(f^{-1}\mathcal{O}_{\overline{X}},\, R\,\mathcal{H}om_{\mathcal{D}_{Y_\mathbb{R}}}(\mathcal{D}_{Y_\mathbb{R}\to X_\mathbb{R}}, \mathcal{N})\right) \\
&\simeq R\,\mathcal{H}om_{\mathcal{D}_Y}\left(\mathcal{D}_{Y\to X}, R\,\mathcal{H}om_{f^{-1}\mathcal{D}_{\overline{X}}}(f^{-1}\mathcal{O}_{\overline{X}},\, R\,\mathcal{H}om_{\mathcal{D}_{\overline{Y}}}(\mathcal{D}_{\overline{Y}\to\overline{X}}, \mathcal{N}))\right) \\
&\simeq R\,\mathcal{H}om_{\mathcal{D}_Y}\left(\mathcal{D}_{Y\to X},\, R\,\mathcal{H}om_{\mathcal{D}_{\overline{Y}}}(\mathcal{D}_{\overline{Y}\to\overline{X}} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_{\overline{X}}} f^{-1}\mathcal{O}_{\overline{X}}, \mathcal{N})\right) .
\end{aligned}$$

Then (5.5) follows from $\mathcal{D}_{\overline{Y}\to\overline{X}} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_{\overline{X}}} f^{-1}\mathcal{O}_{\overline{X}} \simeq \mathcal{O}_{\overline{Y}}$.

The isomorphism (5.7) is obtained by the same method as for (5.6).

Let us prove (5.8). There is a morphism

$$\begin{aligned}
f^{-1}R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{M}) &\to R\,\mathcal{H}om_{\mathcal{D}_{\overline{Y}}}(\mathcal{D}_{\overline{Y}\to\overline{X}} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_{\overline{X}}} f^{-1}\mathcal{O}_{\overline{X}}, \mathcal{D}_{\overline{Y}\to\overline{X}} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_{\overline{X}}} f^{-1}\mathcal{M}) \\
&\simeq R\,\mathcal{H}om_{\mathcal{D}_{\overline{Y}}}(\mathcal{O}_{\overline{Y}}, \mathcal{D}_{\overline{Y}\to\overline{X}} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_{\overline{X}}} f^{-1}\mathcal{M}) .
\end{aligned}$$

Applying the functor $\mathcal{D}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_X} \cdot$, we obtain the desired morphism. □

Theorem 5.6. — *Functorially in $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$, there are a natural morphism in $\mathbf{D}^b(\mathcal{D}_Y)$:*

$$f^{-1}(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \longrightarrow f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y , \tag{5.9}$$

and a natural morphism in $\mathbf{D}^b(\mathcal{D}_X)$:

$$\underline{f}_!\, \mathcal{T}hom(f^{-1}F, \mathcal{O}_Y[d_Y]) \longrightarrow \mathcal{T}hom(F, \mathcal{O}_X[d_X]) . \tag{5.10}$$

Proof. In order to get (5.9), we apply (5.8) with $\mathcal{M} = F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X$ and apply Theorem 3.3. In order to get (5.10), we apply (5.7) with $\mathcal{N} = \mathcal{T}hom(f^{-1}F, \mathcal{D}b_X)$ and use Proposition 4.3. □

Theorem 5.7. — *Let $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$ and assume that f is proper on* supp G. *Then there are natural isomorphisms in $\mathbf{D}^b(\mathcal{D}_X)$, functorial with respect to G:*

$$R f_* R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y) \xleftarrow{\sim} R f_! G \overset{\mathrm{w}}{\otimes} \mathcal{O}_X , \tag{5.11}$$

$$\underline{f}_!\, \mathcal{T}hom(G, \mathcal{O}_Y[d_Y]) \xrightarrow{\sim} \mathcal{T}hom(R f_! G, \mathcal{O}_X[d_X]) . \tag{5.12}$$

Proof. In order to get (5.11), apply (5.5) with $\mathcal{M} = G \overset{\mathrm{w}}{\otimes} \mathcal{C}_Y^\infty$ and use the isomorphism (3.6). Similarly, to obtain (5.12), apply (5.7) with $\mathcal{M} = \mathcal{T}hom(G, \mathcal{D}b_Y)$ and use the isomorphism (4.4). □

Theorem 5.8. —

(i) *If f is smooth, there are natural isomorphisms in $\mathbf{D}^b(f^{-1}\mathcal{D}_X)$:*

$$f^{-1}(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \xrightarrow{\sim} R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y)\,, \tag{5.13}$$

$$R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, \mathcal{T}hom(f^{-1}F, \mathcal{O}_Y)) \xrightarrow{\sim} f^{-1}\,\mathcal{T}hom(F, \mathcal{O}_X)\,. \tag{5.14}$$

(ii) *If f is a closed embedding, there are natural isomorphisms in $\mathbf{D}^b(\mathcal{D}_Y)$:*

$$\underline{f}^{-1}(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \xrightarrow{\sim} f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y\,, \tag{5.15}$$

$$\mathcal{T}hom(f^{-1}F, \mathcal{O}_Y) \xrightarrow{\sim} \underline{f}^{-1}\,\mathcal{T}hom(F, \mathcal{O}_X)\,. \tag{5.16}$$

Proof. (i) Assume that f is smooth. To obtain the isomorphism (5.13), we apply (5.5) with $\mathcal{M} = f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{C}_Y^\infty$ and then Theorem 3.3 (iv). Similarly to obtain the isomorphism (5.14), we apply (5.5) with $\mathcal{M} = \mathcal{T}hom(f^{-1}F, \mathcal{D}b_Y)$ and then Theorem 4.5 (i).

(ii) Assume that f is a closed embedding. First, let us prove

$$\mathcal{D}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} (F_{X\setminus Y} \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty) = 0\,, \tag{5.17}$$

$$\mathcal{D}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{T}hom(F_{X\setminus Y}, \mathcal{D}b_X) = 0\,. \tag{5.18}$$

As in the proof of Theorem 4.5, we can reduce to the case where Y is a hypersurface defined by a holomorphic equation $\{g = 0\}$ with $dg \neq 0$. Using Proposition 1.3, we may assume that $F = \mathbb{C}_U$, U being open subanalytic in X. Let $Z = X \setminus U$. Then we have to check that g acting on $\mathcal{I}_{X,Z\cup Y}^\infty$ as well as g acting on $\mathcal{T}hom(\mathbb{C}_{U\setminus Y}, \mathcal{D}b_X)$ are isomorphisms, which is clear. Applying $R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \cdot\,)$ to (5.17) and (5.18), we get

$$\underline{f}^{-1}(F_{X\setminus Y} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) = 0\,,$$
$$\underline{f}^{-1}\,\mathcal{T}hom(F_{X\setminus Y}, \mathcal{O}_X) = 0\,.$$

Using the distinguished triangle $F_{X\setminus Y} \to F \to F_Y \xrightarrow{+1}$, we may assume $F = f_*G$ for some $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$. Then the isomorphisms (5.15) and (5.16) follow from Theorem 5.7 by applying $\mathcal{D}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \cdot$ to (5.11) and (5.12), noticing that:

$$\mathcal{D}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\leftarrow Y} \simeq \mathcal{D}_Y[d_X - d_Y],$$
$$\mathcal{D}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, \mathcal{M}) \simeq \mathcal{M}.$$

□

Proposition 5.9. — *Functorially in $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$, there are a natural morphism in $\mathbf{D}^b(\mathcal{D}_X)$:*

$$\underline{f}_!(f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y[d_Y]) \to F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X[d_X] \tag{5.19}$$

and a natural morphism in $\mathbf{D}^b(\mathcal{D}_Y)$:

$$\underline{f}^{-1}\mathcal{T}hom(F, \mathcal{O}_X) \to \mathcal{T}hom(f^{-1}F, \mathcal{O}_Y). \tag{5.20}$$

Proof. By decomposing f as the product of the graph embedding $Y \to X \times Y$ and the projection $X \times Y \to X$, it is enough to define those morphisms for a closed embedding and a smooth morphism.

(i) Closed embedding case. We have by (5.5)

$$\begin{aligned}
\underline{f}_!(f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y[d_Y]) &\xleftarrow{\sim} \underline{f}_!(\underline{f}^{-1}(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)[d_Y]) \\
&\simeq \mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} (\mathcal{D}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} (F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)[d_Y]) \\
&\simeq \mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y}, (F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)[d_X]) \\
&\to (F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)[d_X].
\end{aligned}$$

We get (5.19). The morphism (5.20) is nothing but (5.16).

(ii) Smooth case. We have by (5.14)

$$\begin{aligned}
\underline{f}^{-1}\mathcal{T}hom(F, \mathcal{O}_X) &= \mathcal{D}_{Y\to X} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{T}hom(F, \mathcal{O}_X) \\
&\xleftarrow{\sim} \mathcal{D}_{Y\to X} \otimes_{f^{-1}\mathcal{D}_X} R\,\mathcal{H}om_{\mathcal{D}_Y}\big(\mathcal{D}_{Y\to X}, \mathcal{T}hom(f^{-1}F, \mathcal{O}_Y)\big) \\
&\to \mathcal{T}hom(f^{-1}F, \mathcal{O}_Y)\,.
\end{aligned}$$

Similarly by (5.13)

$$\begin{aligned}
\underline{f}_!(f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y[d_Y]) &= Rf_!\,\big(\mathcal{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} (f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y[d_Y])\big) \\
&\simeq Rf_! R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y[d_Y])[d_Y - d_X] \\
&\xleftarrow{\sim} Rf_! f^{-1}(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)[2d_Y - d_X] \\
&\simeq Rf_! f^!(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)[d_X] \\
&\to F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X[d_X]\,.
\end{aligned}$$

□

As a consequence of the stability by external product (Proposition 5.3) and by inverse image (Theorem 5.8), we get natural morphisms for F and G in $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$

$$(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} (G \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \to (F \otimes G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X, \tag{5.21}$$

$$\mathcal{T}hom(F, \mathcal{O}_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathcal{T}hom(G, \mathcal{O}_X) \to \mathcal{T}hom(F \otimes G, \mathcal{O}_X). \tag{5.22}$$

Let us give a few applications of the preceding results.

Let M be a real analytic manifold, X a complexification of M, $i : M \hookrightarrow X$ the embedding.

Theorem 5.10. — *Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_M)$. Then we have*

$$i_*F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \simeq i_*(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M)\,, \tag{5.23}$$

$$\mathcal{T}hom(i_*F, \Omega_X[d_X]) \simeq i_*\mathcal{T}hom(F, \mathcal{D}b^\vee_M)\,. \tag{5.24}$$

In particular:

$$\mathbb{C}_M \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \simeq \mathcal{C}^\infty_M,$$
$$\mathcal{T}hom(\mathrm{D}'_X\mathbb{C}_M, \mathcal{O}_X) \simeq \mathcal{D}b_M.$$

Notice that (5.24) is a result of Andronikof [An], and the last formula is due to Martineau [Mr].

Proof. Let us identify X and $X_\mathbb{R}$ for simplicity. Then

$$i_*F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X = R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, i_*F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X)$$

by the definition, and

$$i_*F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X \simeq R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to X\times\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M)$$

by Theorem 3.5. Hence we have

$$\begin{aligned}
i_*F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X &\simeq R\,\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to X\times\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M))\\
&\simeq R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to X\times\overline{X}} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{\overline{X}}} \mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M)\\
&\simeq R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M)\\
&\simeq i_*(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M)\,.
\end{aligned}$$

The proof of (5.24) is similar, using Theorem 4.4. □

Next, we consider a closed complex analytic subset Z of X. Let $\mathcal{I}_Z$ denote the defining ideal of Z in X. Recall ([Gr$_2$]) that one sets for an $\mathcal{O}_X$-module $\mathcal{F}$:

$$\mathcal{F}\widehat{|}_Z = \varprojlim_k \mathcal{F}/\mathcal{I}^k_Z\mathcal{F}\,,$$
$$\Gamma_{[Z]}(\mathcal{F}) = \varinjlim_k \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}/\mathcal{I}^k_Z, \mathcal{F}).$$

One denotes by $R\Gamma_{[Z]}(\cdot\,)$ the derived functor of $\Gamma_{[Z]}(\cdot\,)$. One calls $\mathcal{F}\widehat{|}_Z$ the formal completion of $\mathcal{F}$ along Z, and $R\Gamma_{[Z]}(\mathcal{F})$ the algebraic cohomology of $\mathcal{F}$ supported by Z.

It is a well-known fact that $\mathcal{O}_X\widehat{|}_Z$ is a flat $\mathcal{O}_X$-module and $\mathcal{F}\widehat{|}_Z \simeq \mathcal{F}\otimes_{\mathcal{O}_X}(\mathcal{O}_X\widehat{|}_Z)$ for a coherent $\mathcal{O}_X$-module $\mathcal{F}$.

Lemma 5.11. — *For a closed submanifold Z of X, we have the isomorphism*

$$\mathcal{O}_X\widehat{|}_Z \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_Z}(\mathcal{D}_{Z\to X}, \mathcal{O}_Z)\,. \tag{5.25}$$

Proof. We have the homomorphism $\mathcal{D}_{Z\to X}\otimes_{\mathcal{D}_X}(\mathcal{O}_X\widehat{|}_Z) \simeq \mathcal{O}_Z\otimes_{\mathcal{O}_X}((\mathcal{O}_X\widehat{|}_Z)\to\mathcal{O}_Z$. Since it is $\mathcal{D}_Z$-linear, we obtain the $\mathcal{D}_X$-linear homomorphism

$$\mathcal{O}_X\widehat{|}_Z \to \mathcal{H}om_{\mathcal{D}_Z}(\mathcal{D}_{Z\to X}, \mathcal{O}_Z)\,. \tag{5.26}$$

We shall show that it is an isomorphism. The question being local, we may assume $X=\{(x,y); x\in\mathbb{C}^n, y\in\mathbb{C}^m\}$ and Z is given by $x=0$. For $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}^n_{\geq 0}$, let us denote by D_x^α the differential operator $(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. Then we have

$$\mathcal{D}_{Z\to X}\simeq\oplus_\alpha\mathcal{D}_Z D_x^\alpha\,.$$

This implies

$$\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{D}_{Z\to X}, \mathcal{O}_Z)\simeq\prod_\alpha\mathcal{O}_Z\otimes(\mathbb{C}D^\alpha)^*\,,$$

and the homomorphism (5.26) is given by $\mathcal{O}_X\widehat{|}_Z \ni u\mapsto (D_x^\alpha u|_Z)_\alpha\in\prod_\alpha\mathcal{O}_Z\otimes(\mathbb{C}D^\alpha)^*$. It is obvious that this is an isomorphism. $\square$

Theorem 5.12. — *Let Z be a closed complex analytic subset of X. There are natural isomorphisms:*

$$\mathbb{C}_Z\overset{\mathrm{w}}{\otimes}\mathcal{O}_X\simeq\mathcal{O}_X\widehat{|}_Z\,, \tag{5.27}$$

$$\mathcal{T}hom(\mathbb{C}_Z,\mathcal{O}_X)\simeq R\Gamma_{[Z]}(\mathcal{O}_X)\,. \tag{5.28}$$

In particular, $\mathbb{C}_Z\overset{\mathrm{w}}{\otimes}\mathcal{O}_X$ is concentrated in degree 0.

Notice that Dufresnoy [Du] already proved that $\mathbb{C}_Z\overset{\mathrm{w}}{\otimes}\mathcal{O}_X$ is concentrated in degree zero.

Proof. (i) Let us prove (5.27). The morphism $\mathcal{O}_X\simeq\mathbb{C}_X\overset{\mathrm{w}}{\otimes}\mathcal{O}_X\to\mathbb{C}_Z\overset{\mathrm{w}}{\otimes}\mathcal{O}_X$ induces a morphism

$$\mathcal{O}_X\longrightarrow H^0(\mathbb{C}_Z\overset{\mathrm{w}}{\otimes}\mathcal{O}_X)\,. \tag{5.29}$$

Set $\mathcal{F} = H^0(\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$. Then applying the functor $\cdot\widehat{|}_Z$ to (5.29), we obtain

(5.30) $$\mathcal{O}_X\widehat{|}_Z \to \mathcal{F}\widehat{|}_Z\,.$$

Hence in order to see $\mathcal{O}_X\widehat{|}_Z \simeq \mathcal{F}$, it is enough to show that this morphism and the morphism

(5.31) $$\mathcal{F} \to \mathcal{F}\widehat{|}_Z$$

are isomorphisms.

Now the question being local, we can find a closed embedding $f : X \to X'$ from X into a smooth manifold X' and a closed smooth submanifold Z' of X' such that $Z = f^{-1}(Z')$. Theorem 5.7 and Lemma 5.11 imply

$$\mathcal{O}_{X'}\widehat{|}_{Z'} \simeq \mathbb{C}_{Z'} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{X'}\,.$$

Theorem 5.8 implies

$$\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \simeq \underline{f}^{-1}(\mathbb{C}_{Z'} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{X'})\,.$$

On the other hand, $\underline{f}^{-1}(\mathcal{O}_{X'}\widehat{|}_{Z'}) = \mathcal{O}_X \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_{X'}} (\mathcal{O}_{X'}\widehat{|}_{Z'}) \simeq \mathcal{O}_X\widehat{|}_Z$. Hence we have

$$\mathcal{O}_X\widehat{|}_Z \simeq \mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{O}_X\,.$$

Then to see that (5.30) and (5.31) are isomorphisms, it is enough to remark that

$$(\mathcal{O}_X\widehat{|}_Z)\widehat{|}_Z \simeq \mathcal{O}_X\widehat{|}_Z\,.$$

(ii) Let us prove (5.28). It is enough to show a similar result with $\mathcal{O}_X$ replaced by $\mathcal{D}b_X$. Since the germ of $\mathcal{D}b_X$ is injective over the germ of $\mathcal{O}_X$ [Ma, Chapter VII, Theorem 2.4], $R\Gamma_{[Z]}(\mathcal{D}b_X) \simeq \Gamma_{[Z]}(\mathcal{D}b_X)$. Hence it is enough to prove

$$\mathcal{T}\!hom(\mathbb{C}_Z, \mathcal{D}b_X) \simeq \Gamma_{[Z]}(\mathcal{D}b_X)\,,$$

that is,

$$\Gamma_Z(\mathcal{D}b_X) \simeq \Gamma_{[Z]}(\mathcal{D}b_X)\,.$$

This is equivalent to saying that a distribution with support in Z is locally annihilated by $\mathcal{I}_Z^k$ for $k \gg 0$. We can reduce this to the case where Z is a hypersurface and it is well-known in this case. $\square$

6. Duality Theorem

Let X be a complex manifold of complex dimension d_X. As usual, one denotes by $\mathbf{D}^b_{\mathrm{q-coh}}(\mathscr{D}_X)$ (resp. $\mathbf{D}^b_{\mathrm{coh}}(\mathscr{D}_X)$) the full triangulated subcategory of $\mathbf{D}^b(\mathscr{D}_X)$ consisting of objects having quasi-coherent (resp. coherent) cohomologies.

The following theorem generalizes Proposition 5.2.

Theorem 6.1. — *Let $\mathfrak{M} \in \mathbf{D}^b_{coh}(\mathscr{D}_X)$ and let $F, G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. Then we can define $R\Gamma\,(X; R\,\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M} \otimes G, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X))$ and $R\Gamma_c\,(X; \mathscr{T}hom(F, \Omega_X)[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathfrak{M} \otimes G))$ as objects of $\mathbf{D}^b(FN)$ and $\mathbf{D}^b(DFN)$, functorially with respect to $\mathfrak{M}$, F and G. Moreover, these two objects are dual to each other.*

Proof. We shall use the results of the appendix. Following the notations there, $\mathbf{D}^-_{coh}(\mathbf{P}(\mathscr{D}_X))$ is equivalent to $\mathbf{D}^-_{coh}(\mathscr{D}_X)$. Here we take as $\mathscr{S}$ in A.2 the set of relatively compact open subsets. Also $\mathbf{D}^-(\mathbf{P}(X))$ is equivalent to $\mathbf{D}^-_{\mathbb{R}-c}(\mathbb{C}_X)$. Here we take as $\mathscr{S}$ in A.3 the set of relatively compact open subanalytic subsets. As in the appendix, for a locally finite family $\mathfrak{U} = \{U_i\}_{i\in I}$ of relatively compact open subsets, set $L_D(\mathfrak{U}) = \oplus_{i\in I}(\mathscr{D}_X)_{U_i}$. For a locally finite family $\mathfrak{V} = \{V_j\}_{j\in J}$ of relatively compact open subanalytic subsets, set $L_C(\mathfrak{V}) = \oplus_{j\in J}\mathbb{C}_{V_j}$. Then for $F \in \mathbb{R}\text{-Cons}(X)$, we have

$$\Gamma\Big(X; \mathscr{H}om_{\mathscr{D}_X}\left(L_D(\mathfrak{U}) \otimes L_C(\mathfrak{V}), F \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^{(0,k)}\right)\Big) \simeq \prod_{i,j} \Gamma\left(U_i \cap V_j; F \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^{(0,k)}\right)$$

and

$$(6.1) \quad \Gamma_c\Big(X; \mathscr{T}hom(F, \mathscr{D}b_X^{(d_X, d_X-k)}) \otimes_{\mathscr{D}_X} (L_D(\mathfrak{U}) \otimes L_C(\mathfrak{V}))\Big) \simeq \bigoplus_{i,j} \Gamma_c\left(U_i \cap V_j; \mathscr{T}hom(F, \mathscr{D}b_X^{(d_X, d_X-k)})\right).$$

They are an FN-space and a DFN-space respectively and are dual to each other.

For a complex $\mathfrak{U}^\bullet \in C^-(\mathbf{P}(\mathscr{D}_X))$, a complex $\mathfrak{V}^\bullet \in C^-(\mathbf{P}(X))$ and a bounded complex $F^\bullet$ of $\mathbb{R}$-constructible sheaves,

$$A(\mathfrak{U}^\bullet, \mathfrak{V}^\bullet, F^\bullet) = \Gamma\Big(X\,;\,\mathcal{H}om_{\mathscr{D}_X}\Big(L_D(\mathfrak{U}^\bullet)\otimes L_C(\mathfrak{V}^\bullet)\,,\,F^\bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^{(0,\cdot)}\Big)\Big)$$

and

$$B(\mathfrak{U}^\bullet, \mathfrak{V}^\bullet, F^\bullet) = \Gamma_c\Big(X\,;\,\mathcal{T}hom(F^\bullet, \mathcal{D}b_X^{(d_X,d_X+\cdot)}) \otimes_{\mathscr{D}_X} (L_D(\mathfrak{U}^\bullet)\otimes L_C(\mathfrak{V}^\bullet))\Big)$$

are a complex of FN-spaces and a complex of DFN-spaces respectively, and they are dual to each other. Hence they give an object of $\mathbf{D}^+(FN)$ and an object of $\mathbf{D}^-(DFN)$ dual to each other. Forgetting the topology, they become

$$R\Gamma\Big(X; R\,\mathcal{H}om_{\mathscr{D}_X}(L_D(\mathfrak{U}^\bullet)\otimes L_C(\mathfrak{V}^\bullet), F^\bullet \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\Big)$$

and

$$R\Gamma_c\Big(X; \mathcal{T}hom(F^\bullet, \Omega_X)[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (L_D(\mathfrak{U}^\bullet)\otimes L_C(\mathfrak{V}^\bullet))\Big).$$

Hence the functors A and B send quasi-isomorphisms to quasi-isomorphisms, and they induce the functors

$$\mathbf{D}^-_{coh}(\mathbf{P}(\mathscr{D}_X))^{\mathrm{opp}} \times \mathbf{D}^-(\mathbf{P}(X))^{\mathrm{opp}} \times \mathbf{D}^b(\mathbb{R}\text{-Cons}(X)) \to \mathbf{D}^+(FN)$$

and

$$\mathbf{D}^-_{coh}(\mathbf{P}(\mathscr{D}_X)) \times \mathbf{D}^-(\mathbf{P}(X)) \times \mathbf{D}^b(\mathbb{R}\text{-Cons}(X))^{\mathrm{opp}} \to \mathbf{D}^-(DFN).$$

To obtain the theorem, it is enough to recall that

$$\mathbf{D}^-_{coh}(\mathbf{P}(\mathscr{D}_X)) \simeq \mathbf{D}^-_{coh}(\mathscr{D}_X) \text{ and } \mathbf{D}^-(\mathbf{P}(X)) \simeq \mathbf{D}^-_{\mathbb{R}-c}(\mathbb{C}_X). \qquad \square$$

Let us derive an easy corollary. Let $\mathfrak{M}$ be a regular holonomic $\mathscr{D}_X$-module, and let F be an object of $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. It is proved in [Ka3] that the natural morphism:

$$\mathcal{T}hom(F, \Omega_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathfrak{M} \to R\,\mathcal{H}om(F, \Omega_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathfrak{M} \tag{6.1}$$

is an isomorphism.

Corollary 6.2. — *Let $\mathfrak{M}$ be a regular holonomic $\mathscr{D}_X$-module, and let F be an object of $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. Then, the natural morphism:*

$$R\,\mathcal{H}om_{\mathscr{D}_X}(\mathfrak{M}, F\otimes \mathscr{O}_X) \to R\,\mathcal{H}om_{\mathscr{D}_X}(\mathfrak{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \tag{6.2}$$

is an isomorphism.

Proof. We shall deduce (6.2) from (6.1) by duality. Let U be an open relatively compact subanalytic subset of X. Set

$$\begin{aligned}
A_1 &= R\Gamma\,(U; R\,\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, F\otimes\mathcal{O}_X))\\
A_2 &= R\Gamma\,(U; R\,\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, F\overset{\mathrm{w}}{\otimes}\mathcal{O}_X))\\
B_2 &= R\Gamma_c\,(U; \mathcal{T}hom(F, \Omega_X[d_X])\overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X}\mathfrak{M})\\
B_1 &= R\Gamma_c\,(U; R\,\mathcal{H}om(F, \Omega_X[d_X])\overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X}\mathfrak{M})
\end{aligned}$$

Then we have morphisms $A_1 \to A_2$ and $B_2 \to B_1$ in $\mathbf{D}^b(\mathrm{Vect})$. By (6.1), $B_2 \to B_1$ is an isomorphism. In order to prove the assertion, it is enough to show that $A_1 \to A_2$ is an isomorphism. There are pairings $A_1 \otimes B_1 \to \mathbb{C}$ and $A_2 \otimes B_2 \to \mathbb{C}$, which are compatible, namely, the following diagram commutes:

$$\begin{array}{ccc}
A_1\otimes B_2 & \longrightarrow & A_2\otimes B_2\\
\downarrow & & \downarrow\\
A_1\otimes B_1 & \longrightarrow & \mathbb{C}.
\end{array}$$

By [Ka$_1$], the cohomology groups of A_1 and B_1 are finite-dimensional and they are dual to each other in $\mathbf{D}^b(\mathrm{Vect})$. Since $B_2 \to B_1$ is an isomorphism in $\mathbf{D}^b(\mathrm{Vect})$, the cohomology groups of B_2 are finite-dimensional. By Theorem 6.1, A_2 is the dual of B_2 in $\mathbf{D}^b(FN)$ and hence the cohomology groups of A_2 are finite-dimensional and A_2 is isomorphic to the dual of B_1 in $\mathbf{D}^b(\mathrm{Vect})$. Therefore $A_1 \to A_2$ is an isomorphism. □

7. Adjunction Formulas

The purpose of this section is to give adjunction formulas for the functors $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $\mathcal{T}\!hom(\cdot, \mathcal{O}_X)$, using $\mathcal{D}$-modules theory. Some of the proofs will be given in §9.

We say that a quasi-coherent $\mathcal{D}_X$-modules $\mathfrak{M}$ is good (resp. quasi-good) if, on every relatively compact open subset of X, it admits a filtration $\{\mathfrak{M}_k\}$ by coherent $\mathcal{D}_X$-submodules such that each quotient $\mathfrak{M}_k/\mathfrak{M}_{k-1}$ admits a good filtration and $\mathfrak{M}_k = 0$ for $|k| \gg 0$ (resp. $k \ll 0$). One defines the full triangulated subcategory $\mathbf{D}^b_{\mathrm{good}}(\mathcal{D}_X)$ (resp. $\mathbf{D}^b_{\mathrm{q-good}}(\mathcal{D}_X)$) of $\mathbf{D}^b(\mathcal{D}_X)$ consisting of objects with good (resp. quasi-good) cohomologies. One defines similarly $\mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_X^{\mathrm{opp}})$, $\mathbf{D}^b_{\mathrm{q-coh}}(\mathcal{D}_X^{\mathrm{opp}})$, $\mathbf{D}^b_{\mathrm{good}}(\mathcal{D}_X^{\mathrm{opp}})$ and $\mathbf{D}^b_{\mathrm{q-good}}(\mathcal{D}_X^{\mathrm{opp}})$ for right $\mathcal{D}$-modules.

Let $\mathfrak{M}$ be an object of $\mathbf{D}^b_{coh}(\mathcal{D}_X)$. We define its dual by the formula:

$$\mathbb{D}_X\mathfrak{M} = R\,\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{D}_X[d_X]). \tag{7.1}$$

This is an object of $\mathbf{D}^b_{coh}(\mathcal{D}_X^{\mathrm{opp}})$.

Let $f : Y \to X$ be a morphism of complex manifolds. We set:

$$d_{Y/X} = d_Y - d_X = \dim Y - \dim X.$$

Let us recall the following well-known results.

Theorem 7.1. —

(i) *Let $\mathfrak{M} \in \mathbf{D}^b(\mathcal{D}_X)$ and $\mathfrak{N} \in \mathbf{D}^b(\mathcal{D}_X^{\mathrm{opp}})$. Then there is a natural isomorphism in $\mathbf{D}^b(\mathbb{C}_X)$:*

$$Rf_!(\mathfrak{N} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \underline{f}^{-1}\mathfrak{M}) \simeq \underline{f}_!\mathfrak{N} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathfrak{M}. \tag{7.2}$$

(ii) *Assume $\mathfrak{M} \in \mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. $\mathbf{D}^b_{\mathrm{good}}(\mathcal{D}_X)$) and f is non characteristic for $\mathfrak{M}$.*

   (a) *We have $\underline{f}^{-1}\mathfrak{M} \in \mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_Y)$ (resp. $\mathbf{D}^b_{\mathrm{good}}(\mathcal{D}_Y)$) and*

$$\underline{f}^{-1}\mathbb{D}_X\mathfrak{M} \simeq \mathbb{D}_Y\underline{f}^{-1}\mathfrak{M}.$$

(b) *Moreover, for* $\mathcal{L} \in \mathbf{D}^b(\mathcal{D}_X)$, *we have the isomorphism:*

$$R f_! R\ \mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathfrak{M}, \mathcal{L}\,[d_{Y/X}]) \simeq R\ \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \underline{f}_!\mathcal{L}).$$

(iii) *Let* $\mathfrak{N} \in \mathbf{D}^b_{\mathrm{good}}(\mathcal{D}_Y)$ *and assume that* f *is proper of* $\mathrm{supp}(\mathfrak{N})$. *Then* $\underline{f}_*\mathfrak{N} \in \mathbf{D}^b_{\mathrm{good}}(\mathcal{D}_X)$ *and:*

$$\underline{f}_*\mathbb{D}_Y\mathfrak{N} \simeq \mathbb{D}_X\underline{f}_*\mathfrak{N}.$$

(iv) *For* $\mathfrak{N} \in \mathbf{D}^b_{\mathrm{q-good}}(\mathcal{D}_Y)$ *and* $\mathcal{L} \in \mathbf{D}^b(\mathcal{D}_Y)$, *there is a natural isomorphism:*

$$R f_* R\ \mathcal{H}om_{\mathcal{D}_Y}(\mathfrak{N}, \underline{f}^{-1}\mathcal{L}\,[d_{Y/X}]) \xrightarrow{\sim} R\ \mathcal{H}om_{\mathcal{D}_X}(\underline{f}_*\mathfrak{N}, \mathcal{L}).$$

Proof. (i) is obvious, (ii.a) is proved in [S-K-K] and (ii.b) follows immediately, (iii) is proved in [Ka$_2$], [Sc] (see also [S-Sc]). The morphism $\underline{f}_!\underline{f}^{-1}\mathcal{O}_X[d_{Y/X}] \to \mathcal{O}_X$ defines the morphism $\underline{f}_!\underline{f}^{-1}\mathcal{L}[d_{Y/X}] \to \mathcal{L}$ which defines the morphism in (iv). To prove that it is an isomorphism, we first reduce this to the case where $\mathfrak{N}$ is quasi-good, then to the case where it is good. Then it remains to apply (iii). □

We can now state our adjunction formulas.

Theorem 7.2. — *Let* $\mathfrak{M} \in \mathbf{D}^b(\mathcal{D}_X)$ *and let* $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$. *Assume that* f *is proper on* $\mathrm{supp}(G)$. *Then there are natural isomorphisms:*

$$R f_! R\ \mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathfrak{M}, G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y) \xleftarrow{\sim} R\ \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, R f_! G \overset{\mathrm{w}}{\otimes} \mathcal{O}_X), \tag{7.3}$$

$$R f_!\left(\mathcal{T}hom(G, \Omega_Y[d_Y]) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \underline{f}^{-1}\mathfrak{M}\right) \xrightarrow{\sim} \mathcal{T}hom(R f_! G, \Omega_X[d_X]) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathfrak{M}. \tag{7.4}$$

Notice that if $\mathfrak{M} \in \mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$ and f is non characteristic for $\mathfrak{M}$, (7.4) is equivalent to the isomorphism:

(7.5)
$$R f_! R\ \mathcal{H}om_{\mathcal{D}_Y}\left(\underline{f}^{-1}\mathfrak{M}, \mathcal{T}hom(G, \mathcal{O}_Y)\right)[2d_{Y/X}] \xrightarrow{\sim} R\ \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{T}hom(R f_! G, \mathcal{O}_X)).$$

Proof. By Theorem 5.7, we have the isomorphism:

$$R\ \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, R f_! G \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \xrightarrow{\sim} R\ \mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, R f_* R\ \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y)).$$

Then (7.3) follows by adjunction.
The isomorphism (7.4) follows from Theorem 5.7 and the formula (7.2). □

Theorem 7.3. — *Let* $\mathfrak{N} \in \mathbf{D}^b_{\mathrm{q-good}}(\mathcal{D}_Y)$ *and assume that* f *is proper on* $\mathrm{supp}(\mathfrak{N})$. *Let* $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. *Then there are natural isomorphisms:*

$$R f_* R\ \mathcal{H}om_{\mathcal{D}_Y}(\mathfrak{N}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y)[d_Y] \xleftarrow{\sim} R\ \mathcal{H}om_{\mathcal{D}_X}(\underline{f}_*\mathfrak{N}, F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)[d_X], \tag{7.6}$$

$$R f_!\left(\mathcal{T}hom(f^{-1}F, \Omega_Y) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathfrak{N}\right) \xrightarrow{\sim} \mathcal{T}hom(F, \Omega_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \underline{f}_!\mathfrak{N}. \tag{7.7}$$

The proof will be given in chapter 9.

8. $\mathcal{O}_X$-Modules of Type FN or DFN

We shall recall here some constructions and results of Ramis-Ruget [R-R] and Houzel [Ho].

A sheaf $\mathcal{F}$ on a real manifold X is said of type FN (resp. DFN) if for each open (resp. compact) subset U (resp. K) of X, the space $\Gamma(U;\mathcal{F})$ (resp. $\Gamma(K;\mathcal{F})$) is endowed with a topology of type FN (resp. DFN), and the restriction mappings are continuous. For example, if X is real analytic and $F \in \mathbb{R}\text{-Cons}(X)$, then $F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X$ is a sheaf of type FN. However, one shall take care that $\mathcal{T}hom(F, \mathcal{D}b_X)$ is not of type DFN in general.

Let X be a complex manifold. Following [Ho], we consider $\mathcal{O}_X$ as a sheaf of complete bornological algebras and deal with $\text{Born}(\mathcal{O}_X)$, the category of complete bornological $\mathcal{O}_X$-modules. Houzel (loc. cit.) has defined a tensor product bifunctor $\cdot \mathbin{\widehat{\otimes}}_{\mathcal{O}_X} \cdot$ on this category. This category contains the category of $\mathcal{O}_X$-modules of type FN and that of type DFN as its full subcategories.

On the other-hand, [R-R] defined the notion of an FN-free (resp. DFN-free) $\mathcal{O}_X$-module as an $\mathcal{O}_X$-module of type FN (resp. DFN) isomorphic to $E \widehat{\otimes} \mathcal{O}_X$ for some FN (resp. DFN) vector space E. This is an object of $\text{Born}(\mathcal{O}_X)$.

Let $E \mathbin{\widehat{\otimes}} \mathcal{O}_X$ be an FN-free (resp. DFN-free) $\mathcal{O}_X$-module and let $\mathcal{G}$ be an $\mathcal{O}_X$-module of type FN (resp. DFN). Then one has the isomorphism:

$$(E \mathbin{\widehat{\otimes}} \mathcal{O}_X) \mathbin{\widehat{\otimes}}_{\mathcal{O}_X} \mathcal{G} \simeq E \mathbin{\widehat{\otimes}} \mathcal{G}. \tag{8.1}$$

Notice that $E \mathbin{\widehat{\otimes}} \mathcal{G}$, as defined by [Ho] is the same as that defined by [R-R]. For example, in the FN-case, $E \mathbin{\widehat{\otimes}} \mathcal{G}$ is the sheaf $U \mapsto E \mathbin{\widehat{\otimes}} \Gamma(U;\mathcal{G})$.

In particular, for a continuous $\mathcal{O}_X$-linear homomorphism $E_1 \mathbin{\widehat{\otimes}} \mathcal{O}_X \to E_2 \mathbin{\widehat{\otimes}} \mathcal{O}_X$ of FN-free (resp. DFN-free) $\mathcal{O}_X$-modules and an $\mathcal{O}_X$-module $\mathcal{G}$ of type FN (resp. DFN), we can define a continuous $\mathcal{O}_X$-linear homomorphism $E_1 \mathbin{\widehat{\otimes}} \mathcal{G} \to E_2 \mathbin{\widehat{\otimes}} \mathcal{G}$.

Let $\mathcal{E} = E\hat{\otimes}\mathcal{O}_X$ be an FN-free or DFN-free $\mathcal{O}_X$-module, and let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Then we have the natural isomorphism: $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{G} \simeq E\hat{\otimes}\mathcal{G}$. This implies that the functor $\mathcal{E} \otimes_{\mathcal{O}_X} \cdot$ is exact on the category of coherent $\mathcal{O}_X$-modules. Hence $\mathcal{E}$ is $\mathcal{O}_X$-flat. In other words, FN-free and DFN-free $\mathcal{O}_X$-modules are flat over $\mathcal{O}_X$.

Let U (resp. K) be an open (resp. compact) subset of X, and let $E\hat{\otimes}\mathcal{O}_X$ be an FN-free (resp. a DFN-free) $\mathcal{O}_X$-module. Then $R\Gamma(U;E\hat{\otimes}\mathcal{O}_X) \simeq E\hat{\otimes}R\Gamma(U;\mathcal{O}_X)$ (resp. $R\Gamma(K;E\hat{\otimes}\mathcal{O}_X) \simeq E\hat{\otimes}R\Gamma(K;\mathcal{O}_X)$).

Examples of FN or DFN-free $\mathcal{O}_X$-modules may be obtained as follows. Let Z be a Stein complex manifold, K a Stein compact subset of Z, f_Z (resp. f_K) the projection $Z \times X \to X$ (resp. $K \times X \to X$). Then $Rf_{Z*}(\mathcal{O}_{Z\times X}) \simeq \Gamma(Z;\mathcal{O}_Z)\hat{\otimes}\mathcal{O}_X$ is FN-free, and $Rf_{K*}(\mathcal{O}_{Z\times X}|_{K\times X}) \simeq \Gamma(K;\mathcal{O}_Z)\hat{\otimes}\mathcal{O}_X$ is DFN-free.

The following theorem is an essential tool in the proof of Theorem 7.3. Although it has already been used in [S-Sc], its proof, due to J-P. Schneiders, was not written down in this paper and for the reader convenience we include it here. This proof is an adaption of the techniques developed by Ramis-Ruget [R-R].

Theorem 8.1. — *Let $\mathcal{R}^\bullet$ be a complex of FN-free (resp. DFN-free) $\mathcal{O}_X$-modules and let $\mathcal{G}$ be an $\mathcal{O}_X$-module of type FN (resp. DFN). Assume that $\mathcal{R}^\bullet$ has bounded $\mathcal{O}_X$-coherent cohomology groups. Then the natural homomorphism*

$$\mathcal{R}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{R}^\bullet \mathbin{\widehat{\otimes}}_{\mathcal{O}_X} \mathcal{G}$$

is a quasi-isomorphism.

We shall only treat the case of sheaves of type FN, the other case being similar. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of type FN. Define the $\mathcal{O}_X$-module:

$$\mathbb{S}_n(\mathcal{F}) = \mathcal{O}_X \mathbin{\widehat{\otimes}} \mathcal{O}_X(X) \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} \mathcal{O}_X(X) \mathbin{\widehat{\otimes}} \mathcal{F}(X)$$

where $\mathcal{F}(X) = \Gamma(X;\mathcal{F})$, $\mathcal{O}_X(X) = \Gamma(X;\mathcal{O}_X)$ and $\mathcal{O}_X(X)$ appears n-times. The $\mathcal{O}_X$-module structure of $\mathbb{S}_n(\mathcal{F})$ is defined by the first factor. Define for $n \geq 1$:

$$\delta_n : \mathbb{S}_n(\mathcal{F}) \to \mathbb{S}_{n-1}(\mathcal{F})$$

by:

$$f_0 \otimes \cdots \otimes f_{n+1} \mapsto \sum_{j=0}^{n} (-1)^j f_0 \otimes \cdots \otimes f_j f_{j+1} \otimes \cdots \otimes f_{n+1}$$

and define:

$$\varepsilon : \mathbb{S}_0(\mathcal{F}) \to \mathcal{F}$$

by:

$$h \otimes f \mapsto hf.$$

One checks that $\delta_{n-1} \circ \delta_n = 0$. Hence we get a complex $\mathbb{S}_\bullet(\mathcal{F}) \in C^-(\mathcal{O}_X)$.

Lemma 8.2. — *Assume $\mathcal{F}$ is FN-free. Then ε induces an isomorphism:*

$$\varepsilon : \mathbb{S}_\bullet(\mathcal{F}) \xrightarrow{\sim} \mathcal{F} \text{ in } K^-(\mathcal{O}_X).$$

Proof. First assume $\mathcal{F} = \mathcal{O}_X$. We construct the homotopy operators:

$$h_n : \mathbb{S}_n(\mathcal{O}_X) \to \mathbb{S}_{n+1}(\mathcal{O}_X)$$

by:

$$f_0 \otimes \cdots \otimes f_{n+1} \mapsto (-1)^{n+1} f_0 \otimes \cdots \otimes f_{n+1} \otimes 1$$

and

$$\eta : \mathcal{O}_X \to \mathbb{S}_0(\mathcal{O}_X)$$

by:

$$f \mapsto f \otimes 1.$$

One checks that:

(i) for $n > 0$, $\delta_{n+1} \circ h_n + h_{n-1} \circ \delta_n = id$,

(ii) for $n = 0$, $\delta_1 \circ h_0 + \eta \circ \varepsilon = id$,

(iii) $\varepsilon \circ \eta = id$.

This proves the lemma in case $\mathcal{F} = \mathcal{O}_X$. The case $\mathcal{F} = E \mathbin{\widehat{\otimes}} \mathcal{O}_X$ follows by applying the exact functor $E \mathbin{\widehat{\otimes}} \cdot$ to the preceding complexes.

□

Lemma 8.3. — *Let $\mathcal{F}^\bullet$ be a complex of FN-free $\mathcal{O}_X$-modules, and let $\mathcal{G}$ be an $\mathcal{O}_X$-module of type FN. Assume $\mathcal{F}^\bullet$ is exact. Then $\mathcal{F}^\bullet \mathbin{\widehat{\otimes}}_{\mathcal{O}_X} \mathcal{G}$ is exact.*

Proof. Since the problem is local, we may assume X is Stein. For a double complex $H^{\bullet\bullet}$, we denote by $s(H^{\bullet\bullet})$ the associated simple complex : $s(H^{\bullet\bullet})^n = \oplus_{n=p+q} H^{p,q}$. Remark the following well-known property:

(8.2) if $H^{p,\bullet}$ is exact for every p, then $s(H^{\bullet\bullet})$ is exact.

By Lemma 8.2 we have $\mathbb{S}_\bullet(\mathscr{F}^k) \simeq \mathscr{F}^k$ in $K(\mathscr{O}_X)$ for any k. Hence tensoring by $\mathscr{G}$, we have $\mathbb{S}_\bullet(\mathscr{F}^k) \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G} \simeq \mathscr{F}^k \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G}$ in $K(\mathscr{O}_X)$. Hence, by applying (8.2) to the double complex $\mathbb{S}_\bullet(\mathscr{F}^\bullet) \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G} \to \mathscr{F}^\bullet \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G}$,

(8.3) $\quad s(\mathbb{S}_\bullet(\mathscr{F}^\bullet) \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G}) \to \mathscr{F}^\bullet \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G}$ is a quasi-isomorphism.

We set $\mathscr{F}^\bullet(X) = \Gamma(X;\mathscr{F}^\bullet)$. Since the $\mathscr{F}^j$'s are FN-free and X is Stein, one has $H^k(X;\mathscr{F}^j) = 0$ for $k \neq 0$. This shows that $R\Gamma(X;\mathscr{F}^\bullet) \simeq \mathscr{F}^\bullet(X)$, that is, $\mathscr{F}^\bullet(X)$ is exact. This implies:

$$\mathscr{O}_X \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{O}_X(X) \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} \mathscr{O}_X(X) \mathbin{\widehat{\otimes}} \mathscr{F}^\bullet(X) \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G} \quad \text{is exact.}$$

Hence by applying again (8.2)

(8.4) $\quad s(\mathbb{S}(\mathscr{F}^\bullet) \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G}) \quad$ is exact.

Then the lemma follows from (8.3) and (8.4). □

Lemma 8.4. — *Let $u : \mathscr{F}_1^\bullet \to \mathscr{F}_2^\bullet$ be a morphism of complexes of FN-free $\mathscr{O}_X$-modules, and assume that u is a quasi-isomorphism. Let $\mathscr{G}$ be an $\mathscr{O}_X$-module of type FN. Then $u \mathbin{\widehat{\otimes}} \mathscr{G} : \mathscr{F}_1^\bullet \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G} \to \mathscr{F}_2^\bullet \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G}$ is a quasi-isomorphism.*

Proof. Let $M(u)$ denote the mapping cone of u. This is a bounded from above complex of FN-free $\mathscr{O}_X$-modules quasi-isomorphic to 0. Then $M(u) \mathbin{\widehat{\otimes}} \mathscr{G}$ is quasi-isomorphic to 0 by Lemma 8.3, and it remains to notice that $M(u) \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G}$ is the mapping cone of $u \mathbin{\widehat{\otimes}} \mathscr{G}$. □

Proof of Theorem 8.1. Since $\mathscr{R}^\bullet$ has bounded and coherent cohomology, locally on X, there exist a bounded complex $\mathscr{L}^\bullet$ of free $\mathscr{O}_X$-modules of finite type and a quasi-isomorphism

$$u : \mathscr{L}^\bullet \underset{qis}{\simeq} \mathscr{R}^\bullet.$$

Since any FN-free $\mathscr{O}_X$-module is flat, we have:

$$\mathscr{L}^\bullet \otimes_{\mathscr{O}_X} \mathscr{G} \underset{qis}{\simeq} \mathscr{R}^\bullet \otimes_{\mathscr{O}_X} \mathscr{G}.$$

On the other hand we have by Lemma 8.4:

$$\mathscr{L}^\bullet \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G} \underset{qis}{\simeq} \mathscr{R}^\bullet \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G}.$$

Since $\mathscr{L}^\bullet \otimes_{\mathscr{O}_X} \mathscr{G} \simeq \mathscr{L}^\bullet \mathbin{\widehat{\otimes}}_{\mathscr{O}_X} \mathscr{G}$, the proof is complete. □

9. Proof of Theorem 7.3

We begin by the proof of (7.6) and we shall deduce later (7.7) by duality. Notice that since $\mathcal{T}hom(F, \mathcal{D}b_X)$ is not a sheaf of type DFN in general, it would not have been possible to copy the argument of the proof of (7.6) (in particular, when using Theorem 8.1 as we shall do), to obtain (7.7).

In view of Theorem 7.1 (iv), we have to prove that the morphism defined by (5.9)

$$(9.1) \qquad R f_! R \mathcal{H}om_{\mathcal{D}_Y} \big(\mathcal{M}, \underline{f}^{-1}(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)\big) \to R f_! R \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y)$$

is an isomorphism. By Theorem 5.8, this morphism is an isomorphism if f is a closed embedding. Hence, using the graph decomposition of f, we may assume from the beginning that $Y = Z \times X$ and f is the second projection. Moreover we may assume $F \in \mathbb{R}\text{-Cons}(X)$ and $\mathcal{M}$ admits a good filtration. Then we can reduce to the case where $\mathcal{M} = \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}$ for a coherent $\mathcal{O}_Y$-module $\mathcal{F}$ with proper support over X. Now the left hand side of (9.1) is isomorphic to

$$R f_! R \mathcal{H}om_{\mathcal{O}_Y} \big(\mathcal{F}, \mathcal{O}_Y \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{O}_X} f^{-1}(F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)\big) \simeq R f_! R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} (F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X).$$

Hence it is enough to show that

$$(9.2) \qquad R f_! R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} (F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \to R f_! R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y)$$

is an isomorphism. Let us introduce the sheaf:

$$f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}\mathcal{C}^{\infty}_{Y/X} = R \mathcal{H}om_{\mathcal{D}_{\overline{Z}}}(\mathcal{O}_{\overline{Z}}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_Y).$$

Instead of proving (9.2), it is enough to prove that

$$(9.3) \quad R f_* R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} (F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_X) \to R f_* R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathcal{O}\mathcal{C}^{\infty}_{Y/X})$$

is an isomorphism. The morphism (9.2) is obtained by applying $R \mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \cdot)$ to (9.3).

For $x_0 \in X$, we shall prove that (9.3) is an isomorphism on a neighborhood of x_0. Let us take an open neighborhood W of x_0 and a subanalytic Stein compact subset K

such that $W \subset K$. Let $p : Z\times K \to Z$ be the projection. Then $\mathscr{A} = p_*(\mathscr{O}_Y|_{Z\times K})$ is a coherent ring on Z. The category of coherent $\mathscr{O}_Y|_{Z\times K}$-modules is equivalent to the category of coherent $\mathscr{A}$-modules by the functor $\mathscr{G} \to p_*(\mathscr{G})$. Hence $\tilde{\mathscr{F}} = p_*(\mathscr{F}|_{Z\times K})$ is a coherent $\mathscr{A}$-module. Now let us apply the results in §A.2 in the appendix. Let us take as $\mathscr{S}$ in §A.2 the set of relatively compact Stein open subsets in Z. Then $\mathscr{S}$ satisfies the conditions (A.7) and (A.8). Hence there exists $\mathfrak{U}^\bullet \in C^-(\mathbf{P}(\mathscr{A}))$ and a quasi-isomorphism $L_{\mathscr{A}}(\mathfrak{U}^\bullet) \to \tilde{\mathscr{F}}$. Writing $\mathfrak{U}^k = \{U_{k,i}\}_{i\in I(k)}$, we set $\tilde{\mathfrak{U}}^k = \{U_{k,i} \times W\}_{i\in I(k)}$. Then there is a quasi-isomorphism

$$L_{\mathscr{O}_Y}(\tilde{\mathfrak{U}}^\bullet)\big|_{Z\times W} \to \mathscr{F}\big|_{Z\times W}.$$

For any relatively compact Stein open subset V of Z we have

$$Rf_*R\,\mathscr{H}om_{\mathscr{O}_Y}((\mathscr{O}_Y)_{V\times X}, \mathscr{O}_Y) \simeq \Gamma(V;\mathscr{O}_Z)\ \widehat{\otimes}\ \mathscr{O}_X \tag{9.4}$$

and

(9.5)
$$\begin{aligned} Rf_*R\,\mathscr{H}om_{\mathscr{O}_Y}((\mathscr{O}_Y)_{V\times X}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathscr{O}\mathscr{C}^\infty_{Y/X}) &\simeq \Gamma(V;\mathscr{O}_Z)\ \widehat{\otimes}\ (F\overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X) \\ &\simeq (\Gamma(V;\mathscr{O}_Z)\ \widehat{\otimes}\ \mathscr{O}_X)\ \widehat{\otimes}_{\mathscr{O}_X}\ (F\overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X). \end{aligned}$$

We set $\mathscr{R}^\bullet = f_*\,\mathscr{H}om_{\mathscr{O}_Y}(L_{\mathscr{O}_Y}(\tilde{\mathfrak{U}}^\bullet), \mathscr{O}_Y)\big|_W$. By (9.4), each $\mathscr{R}^k$ is an FN-free $\mathscr{O}_W$-module. In the derived category, $\mathscr{R}^\bullet$ is isomorphic to $Rf_*R\,\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{O}_Y)\big|_W$. Hence $\mathscr{R}^\bullet$ has bounded coherent cohomology groups. The object

$$Rf_*R\,\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{O}_Y) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} (F \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X)$$

is represented by $\mathscr{R}^\bullet \otimes_{\mathscr{O}_X} (F\overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X)$, and by (9.5),

$$Rf_*R\,\mathscr{H}om_{\mathscr{O}_Y}\left(\mathscr{F}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathscr{O}\mathscr{C}^\infty_{Y/X}\right)$$

is represented by $\mathscr{R}^\bullet\ \widehat{\otimes}_{\mathscr{O}_X}\ (F\overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X)$ on W. Hence to prove that (9.3) is an isomorphism, it is sufficient to apply Theorem 8.1.

Finally, let us prove (7.7). Set:

$$\begin{aligned} \mathscr{H}_1 &= Rf_*R\,\mathscr{H}om_{\mathscr{D}_Y}(\mathfrak{N}, f^{-1}F \overset{\mathrm{w}}{\otimes} \mathscr{O}_Y)[d_Y] \\ \mathscr{H}_2 &= R\,\mathscr{H}om_{\mathscr{D}_X}(\underline{f}_*\mathfrak{N}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)[d_X] \\ \mathscr{K}_1 &= Rf_!\left(\mathscr{T}hom(f^{-1}F, \Omega_Y) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathfrak{N}\right), \\ \mathscr{K}_2 &= \mathscr{T}hom(F, \Omega_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \underline{f}_!\mathfrak{N}. \end{aligned}$$

The morphism

$$\mathcal{H}_1 \to \mathcal{H}_2 \tag{9.6}$$

is equivalent to the morphism:

$$R\, f_! R\ \mathcal{H}om_{\mathcal{D}_Y}\left(\underline{\mathbb{D}}_Y \mathfrak{N}, \mathcal{T}hom(f^{-1}F, \Omega_Y[d_Y])\right) \\ \to R\ \mathcal{H}om_{\mathcal{D}_X}\left(\underline{f}_! \underline{\mathbb{D}}_Y \mathfrak{N}, \mathcal{T}hom(F, \Omega_X[d_X])\right),$$

which follows from Proposition 5.6. Hence, to prove that (9.6) is an isomorphism, it is enough to prove that for each open subset U of X, the morphism:

$$R\Gamma_c(U; \mathcal{H}_1) \to R\Gamma_c(U; \mathcal{H}_2) \tag{9.7}$$

is an isomorphism. Consider the morphism deduced from (7.6):

$$R\Gamma(U; \mathcal{H}_2) \to R\Gamma(U; \mathcal{H}_1)\,. \tag{9.8}$$

By its construction, this last morphism is well-defined in the category $\mathbf{D}^b(FN)$, and is dual to (9.7) by Theorem 6.1. By (7.6) and the closed graph theorem, (9.8) is an isomorphism in $\mathbf{D}^b(FN)$. Hence (9.7) is an isomorphism and the proof is complete. □

10. Integral Transformations

10.1 Tempered C^∞ Functions

In this section, in order to study a multiplicative structure of $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $\mathcal{T}hom(\cdot, \mathcal{O}_X)$, we shall construct an auxiliary functor $\mathcal{T}hom(F, \mathcal{C}_X^\infty)$. It is not exact in F but left exact. We show that, for a complex manifold X, $\mathcal{T}hom(F, \mathcal{O}_X)$ can be also calculated by the Dolbeault complex of $\mathcal{T}hom(F, \mathcal{C}_X^\infty)$.

Let X be a real analytic manifold. Let U be an open subanalytic set. A function $f \in \mathcal{C}^\infty(U)$ is called *with polynomial growth* at $p \in X$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} (\operatorname{dist}(x, K \setminus U))^N |f(x)| < \infty . \tag{10.1}$$

Here, $\operatorname{dist}(x, K \setminus U)$ is the distance from x to $K \setminus U$. It is obvious that f has polynomial growth at any point of U. We say that f is *tempered* at p if all its derivatives are with polynomial growth at p. We say that f is tempered on an open set Ω if it is tempered at any point of Ω.

Remark that in this case f can be extended to a distribution defined on Ω.

Proposition 10.1. — *Let $X = \mathbb{R}^n$ and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$. Let u be a distribution on X. Assume that Δu is C^∞ on an open subanalytic subset U and that $\Delta u|_U$ is tempered at $p \in X$. Then $u|_U$ is also tempered at p.*

Proof. By the ellipticity of Δ, u is C^∞ on U. Let us take a distribution $K(x)$ and a C^∞ function $R(x)$ such that

$$\delta(x) = \Delta K(x) + R(x)$$

and the support of $K(x)$ and the support of $R(x)$ are contained in $\{x \in X; |x| \leq 1\}$. Then $K(x)$ is integrable. For $c > 0$, set

$$K_c(x) = c^{2-n} K(c^{-1}x) \text{ and } R_c(x) = c^{-n} R(c^{-1}x).$$

Then we have again

$$\delta(x) = \Delta K_c(x) + R_c(x).$$

Hence we have

$$u(x) = \int K_c(x-y)(\Delta u)(y)dy + \int R_c(x-y)u(y)dy.$$

Now we take $x \in U$ and set $c = \mathrm{dist}(x, X \setminus U)/2$. Then we have

$$\left|\int K_c(x-y)(\Delta u)(y)dy\right| \leq \left(\sup_{|x-y|\leq c} |(\Delta u)(y)|\right) \int |K_c(x-y)|dy \leq \mathrm{const.}\, c^{-N_1}$$

for some N_1. On the other hand, we have

$$\left|\int R_c(x-y)u(y)dy\right| \leq \mathrm{const.} \sum_{|\alpha|\leq N} \sup_{y\in X} |D_y^\alpha R_c(x-y)| \leq \mathrm{const.}\, c^{-N}$$

for some N. Thus $u|_U$ has polynomial growth at p.

Since $\Delta D_x^\alpha u(x) = D_x^\alpha \Delta u(x)$, any derivative of $u|_U$ has polynomial growth at p and hence $u|_U$ is tempered at p. □

10.2 The Functor $\mathcal{T}hom(\cdot, \mathcal{C}_X^\infty)$

Let X be a real analytic manifold. For a subanalytic open subset U, we shall define the $\mathcal{D}_X$-module $\mathcal{T}hom(\mathbb{C}_U, \mathcal{C}_X^\infty)$ as follows. For an open subset Ω, $\Gamma(\Omega; \mathcal{T}hom(\mathbb{C}_U, \mathcal{C}_X^\infty))$ is the set of C^∞ functions on $\Omega \cap U$ which are tempered on Ω. Then $U \mapsto \mathcal{T}hom(\mathbb{C}_U, \mathcal{C}_X^\infty)$ is a contravariant functor from $\mathcal{S}_X$ to the category of $\mathcal{D}_X$-modules.

Proposition 10.2. — *For any subanalytic open subsets U and V,*

$$0 \to \mathcal{T}hom(\mathbb{C}_{U\cup V}, \mathcal{C}_X^\infty) \to \mathcal{T}hom(\mathbb{C}_U, \mathcal{C}_X^\infty) \oplus \mathcal{T}hom(\mathbb{C}_V, \mathcal{C}_X^\infty) \to \mathcal{T}hom(\mathbb{C}_{U\cap V}, \mathcal{C}_X^\infty) \to 0$$

is exact.

Proof. It is enough to show the exactness of the following sequence, assuming that $X = \mathbb{R}^n$ and that U and V are relatively compact:

$$\begin{aligned} 0 \to \Gamma(X; \mathcal{T}hom(\mathbb{C}_{U\cup V}, \mathcal{C}_X^\infty)) \longrightarrow & \Gamma(X; \mathcal{T}hom(\mathbb{C}_U, \mathcal{C}_X^\infty)) \oplus \Gamma(X; \mathcal{T}hom(\mathbb{C}_V, \mathcal{C}_X^\infty)) \\ & \xrightarrow{\alpha} \Gamma(X; \mathcal{T}hom(\mathbb{C}_{U\cap V}, \mathcal{C}_X^\infty)) \to 0. \end{aligned}$$

The property $\mathrm{Ker}(\alpha) = \Gamma(X; \mathcal{T}hom(\mathbb{C}_{U\cup V}, \mathcal{C}_X^\infty))$ easily follows from the existence of a positive integer N and $C > 0$ such that

$$\mathrm{dist}(x, X \setminus (U \cup V))^N \leq C(\mathrm{dist}(x, X \setminus U) + \mathrm{dist}(x, X \setminus V)) \text{ for any } x \in U \cup V.$$

Let us prove the surjectivity of α. Set $F_0 = \{x \in \bar{U}; \mathrm{dist}(x, X \setminus V) \leq \mathrm{dist}(x, X \setminus U)/2\} \subset \overline{U \setminus V}$ and $F_1 = \{x \in \bar{V}; \mathrm{dist}(x, X \setminus U) \leq \mathrm{dist}(x, X \setminus V)/2\} \subset \overline{V \setminus U}$. Then $U \cap V \subset X \setminus (F_0 \cap F_1)$.

Now recall the following lemma on cut-off functions.

Lemma 10.3. — ([Hö, Cor. 1.4.11]) *Let F_0 and F_1 be closed subanalytic subsets. Then there exists $\psi \in \mathscr{C}^\infty(X \setminus (F_0 \bigcap F_1))$ such that*

(10.2) $\psi = 0$ *on a neighborhood of* $F_0 \setminus F_1$*;*

(10.3) $\psi = 1$ *on a neighborhood of* $F_1 \setminus F_0$*;*

(10.4) ψ *is tempered at any points of* $X \setminus (F_0 \bigcap F_1)$.

Take $\psi \in \Gamma(X; \mathcal{T}\!hom(\mathbb{C}_{X \setminus (F_0 \cap F_1)}, \mathscr{C}^\infty_X))$ as in the lemma above.

For $f \in \Gamma(X; \mathcal{T}\!hom(\mathbb{C}_{U \cap V}, \mathscr{C}^\infty_X))$, define $f_0 \in \mathscr{C}^\infty(U)$ by

$$f_0(x) = \begin{cases} \psi(x) f(x) & \text{if } x \in U \cap V, \\ 0 & \text{if } x \in U \setminus V. \end{cases}$$

For $x \in U \cap V \cap \mathrm{supp}(\psi) \subset (U \cap V) \setminus F_0$, we have

$$\mathrm{dist}(x, X \setminus U) \leq 2 \min(\mathrm{dist}(x, X \setminus U), \mathrm{dist}(x, X \setminus V)) \leq \mathrm{dist}(x, X \setminus (U \cap V)).$$

Therefore f_0 belongs to $\Gamma(X; \mathcal{T}\!hom(\mathbb{C}_U, \mathscr{C}^\infty_X))$. Similarly define $f_1 \in \mathscr{C}^\infty(V)$ by

$$f_1(x) = \begin{cases} (1 - \psi(x)) f(x), & \text{if } x \in U \cap V, \\ 0, & \text{if } \in V \setminus U. \end{cases}$$

Then f_1 belongs to $\Gamma(X; \mathcal{T}\!hom(\mathbb{C}_V, \mathscr{C}^\infty_X))$ and $f = \alpha(f_0 \oplus f_1)$. □

By the proposition above and Proposition 1.4, we can extend the functor $\mathcal{T}\!hom(\mathbb{C}_U, \mathscr{C}^\infty_X)$ to

$$\text{(10.5)} \qquad \mathcal{T}\!hom(\cdot, \mathscr{C}^\infty_X) : \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X) \simeq \mathbf{D}^b(\mathbb{R}\text{-Cons}(X)) \longrightarrow \mathbf{D}^b(\mathscr{D}_X).$$

Namely, the functor $\psi(U) = \mathcal{T}\!hom(\mathbb{C}_U, \mathscr{C}^\infty_X)$ can be extended to a contravariant functor $\Psi \colon \mathbb{R}\text{-Cons}(X) \to \mathrm{Mod}(\mathscr{D}_X)$ and $\mathcal{T}\!hom(\cdot, \mathscr{C}^\infty_X)$ is its right derived functor. By Proposition 1.4, we have:

$$\text{(10.6)} \qquad H^j(\mathcal{T}\!hom(F, \mathscr{C}^\infty_X)) = 0 \text{ for any } F \in \mathbb{R}\text{-Cons}(X) \text{ and } j \neq 0,1,$$

$$\text{(10.7)} \qquad H^j(\mathcal{T}\!hom(\mathbb{C}_U, \mathscr{C}^\infty_X)) = 0 \text{ for any open subanalytic set } U \text{ and } j \neq 0.$$

We can see easily that there is a sequence of morphisms

$$\mathbb{C}_{\bar{U}} \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_X \to \mathcal{T}\!hom(\mathbb{C}_U, \mathscr{C}^\infty_X) \to \mathcal{T}\!hom(\mathbb{C}_U, \mathcal{D}b_X).$$

This induces functorial morphisms in $\mathbf{D}^b(\mathcal{D}_X)$

$$\mathrm{D}'_X(F)\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X \to \mathcal{T}hom(F,\mathcal{C}^\infty_X) \to \mathcal{T}hom(F,\mathcal{D}b_X)\,. \tag{10.8}$$

Proposition 10.4. — *We have a functorial morphism in $F,G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$*

$$\mathcal{T}hom(F,\mathcal{C}^\infty_X)\overset{\mathrm{L}}{\otimes}_{\mathcal{A}_X}((F\otimes G)\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X) \to G\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X\,. \tag{10.9}$$

Proof. We can easily reduce the proof to the case where $F=\mathbb{C}_U$ and $G=\mathbb{C}_V$ for open subanalytic subsets U and V. Then we have

$$\begin{aligned}\mathcal{T}hom(F,\mathcal{C}^\infty_X)\otimes_{\mathcal{A}_X}((F\otimes G)\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X) &= \mathcal{T}hom(\mathbb{C}_U,\mathcal{C}^\infty_X)\otimes_{\mathcal{A}_X}(\mathbb{C}_{U\cap V}\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X)\\ &\to \mathcal{T}hom(\mathbb{C}_{U\cap V},\mathcal{C}^\infty_X)\otimes_{\mathcal{A}_X}(\mathbb{C}_{U\cap V}\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X)\,.\end{aligned} \tag{10.10}$$

For $f \in \mathcal{T}hom(\mathbb{C}_{U\cap V},\mathcal{C}^\infty_X)$ and $g \in \mathbb{C}_{U\cap V}\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X$, the product fg belongs to $\mathbb{C}_{U\cap V}\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X$. Hence it defines

(10.11)
$$\mathcal{T}hom(\mathbb{C}_{U\cap V},\mathcal{C}^\infty_X)\otimes_{\mathcal{A}_X}(\mathbb{C}_{U\cap V}\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X) \to \mathbb{C}_{U\cap V}\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X \to \mathbb{C}_V\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X = G\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X\,.$$

Composing (10.10), (10.11) and

$$\mathcal{T}hom(F,\mathcal{C}^\infty_X)\overset{\mathrm{L}}{\otimes}_{\mathcal{A}_X}((F\otimes G)\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X) \to \mathcal{T}hom(F,\mathcal{C}^\infty_X)\otimes_{\mathcal{A}_X}((F\otimes G)\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X)\,,$$

we obtain the desired morphism. □

10.3 Complex Case

Now we assume that X is a complex manifold.

Theorem 10.5. — *For any $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$, the morphism*

$$R\,\mathcal{H}om_{\mathcal{D}_{\bar X}}(\mathcal{O}_{\bar X},\mathcal{T}hom(F,\mathcal{C}^\infty_X)) \to R\,\mathcal{H}om_{\mathcal{D}_{\bar X}}(\mathcal{O}_{\bar X},\mathcal{T}hom(F,\mathcal{D}b_X))$$

is an isomorphism in $\mathbf{D}^b(\mathcal{D}_X)$.

Proof. The morphism is constructed in (10.8). As the question is then local, we may assume that $X=\mathbb{C}^n$ and $F=\mathbb{C}_U$ for a subanalytic open subset U. Let Δ be the differential operator $\sum_{i=1}^n \partial^2/\partial x_i\partial\bar x_i$. There exists an exact sequence of $\mathcal{D}_{X_\mathbb{R}}$-modules:

$$0 \longleftarrow \mathcal{D}_{X_\mathbb{R}}\otimes_{\mathcal{D}_{\bar X}}\mathcal{O}_{\bar X} \longleftarrow (\mathcal{D}_{X_\mathbb{R}}/\mathcal{D}_{X_\mathbb{R}}\Delta)^{\oplus N_0} \longleftarrow (\mathcal{D}_{X_\mathbb{R}}/\mathcal{D}_{X_\mathbb{R}}\Delta)^{\oplus N_1} \longleftarrow \cdots\,.$$

This sequence is constructed from a free resolution of $\mathbb{C}[\partial_1, \dots, \partial_n, \bar\partial_1, \dots, \bar\partial_n]/(\bar\partial_1, \dots, \bar\partial_n)$ as a module over $\mathbb{C}[\partial_1, \dots, \partial_n, \bar\partial_1, \dots, \bar\partial_n]/(\Delta)$. Hence it is enough to show that the vertical arrows in the following diagram give a quasi-isomorphism from the complex of the top row to the one of the bottom row.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathcal{T}hom(\mathbb{C}_U, \mathcal{C}^\infty_X) \arrow[r, "\Delta"] \arrow[d] & \mathcal{T}hom(\mathbb{C}_U, \mathcal{C}^\infty_X) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_X) \arrow[r, "\Delta"] & \mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_X) \arrow[r] & 0. \end{tikzcd}$$

It is well-known that $\mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_X) \xrightarrow{\Delta} \mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_X)$ is an epimorphism. Let us prove the surjectivity of $\Delta : \mathcal{T}hom(\mathbb{C}_U, \mathcal{C}^\infty_X) \to \mathcal{T}hom(\mathbb{C}_U, \mathcal{C}^\infty_X)$. For $g \in \mathcal{T}hom(\mathbb{C}_U, \mathcal{C}^\infty_X)$ let us take $f \in \mathcal{D}b_X$ such that $g = \Delta f$. Then by Proposition 10.1, f belongs to $\mathcal{T}hom(\mathbb{C}_U, \mathcal{C}^\infty_X)$.

Hence it is enough to show that if $f \in \mathcal{T}hom(\mathbb{C}_U, \mathcal{D}b_X)$ satisfies $\Delta f = 0$ then f belongs to $\mathcal{T}hom(\mathbb{C}_U, \mathcal{C}^\infty_X)$. This also follows from the same proposition. □

This proposition says that to define $\mathcal{T}hom(F, \mathcal{O}_X)$, we can use the Dolbeault complex of $\mathcal{T}hom(F, \mathcal{C}^\infty_X)$ instead of $\mathcal{T}hom(F, \mathcal{D}b_X)$.

Proposition 10.6. — *There exist functorial morphisms in $F, G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$:*

$$\mathcal{T}hom(F, \mathcal{O}_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} ((F \otimes G) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \to G \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X ,$$
$$\mathcal{T}hom(F, \mathcal{O}_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} ((F \otimes G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \to G \overset{\mathrm{w}}{\otimes} \mathcal{O}_X .$$

Proof. It is enough to apply the functor $R\,\mathcal{H}om_{\mathcal{D}_{\bar X}}(\mathcal{O}_{\bar X}, \cdot\,)$ to the morphism in Proposition 10.4. □

In the following theorem, (10.14) and (10.15) are due to J. E. Björk [Bj, Th. 7.9.11]. We denote by $\mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_X)$ the full subcategory of $\mathbf{D}^b(\mathcal{D}_X)$ consisting of objects with regular holonomic $\mathcal{D}_X$-modules as cohomologies. We set $\mathcal{S}ol(\mathfrak{M}) = R\,\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)$. Then $\mathcal{S}ol$ is a contravariant functor from $\mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_X)$ to $\mathbf{D}^b_{\mathbb{C}-c}(\mathbb{C}_X)$.

Theorem 10.7. — *Let $\mathfrak{M} \in \mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_X)$ and $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. We have canonical isomorphisms in $\mathbf{D}^b(\mathcal{D}_X)$*

$$R\,\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{M}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \simeq (\mathcal{S}ol(\mathfrak{M}) \otimes F) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X, \tag{10.12}$$
$$R\,\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{M}, F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \simeq (\mathcal{S}ol(\mathfrak{M}) \otimes F) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X, \tag{10.13}$$

and

$$\mathfrak{M} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathcal{T}hom(F, \mathcal{D}b_X) \simeq \mathcal{T}hom(\mathcal{S}ol(\mathfrak{M}) \otimes F, \mathcal{D}b_X), \tag{10.14}$$
$$\mathfrak{M} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathcal{T}hom(F, \mathcal{O}_X) \simeq \mathcal{T}hom(\mathcal{S}ol(\mathfrak{M}) \otimes F, \mathcal{O}_X) . \tag{10.15}$$

Proof. The isomorphisms (10.14) and (10.15) are proved in [Bj]. Let us prove the others by duality. Set $G = \mathcal{S}ol(\mathfrak{M})$. Then $\mathfrak{M} = \mathcal{T}hom(G, \mathcal{O}_X)$ by [Ka3]. By Proposition 10.6, there exists a morphism $\mathfrak{M} \overset{L}{\otimes}_{\mathcal{O}_X} ((G \otimes F) \overset{w}{\otimes} \mathcal{C}_X^\infty) \to F \overset{w}{\otimes} \mathcal{C}_X^\infty$. This gives

$$(G \otimes F) \overset{w}{\otimes} \mathcal{C}_X^\infty \to R\,\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{M}, F \overset{w}{\otimes} \mathcal{C}_X^\infty)\,. \tag{10.16}$$

Let us prove that this is an isomorphism.

For any open subset U, $R\Gamma(U; (G \otimes F) \overset{w}{\otimes} \mathcal{C}_X^\infty)$ is the dual of $R\Gamma_c(U; \mathcal{T}hom(G \otimes F, \mathcal{D}b_X^\vee))$. If U is sufficiently small, there exists a bounded exact complex of $\mathcal{O}_X$-modules on U

$$0 \longleftarrow \mathfrak{M} \longleftarrow \mathcal{O}_X^{\oplus I_0} \longleftarrow \mathcal{O}_X^{\oplus I_1} \longleftarrow \cdots,$$

where $I_0, I_1, \ldots$ are countable sets. Hence $R\Gamma(U; R\,\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{M}, F \overset{w}{\otimes} \mathcal{C}_X^\infty))$ is the dual of $R\Gamma_c(U; \mathfrak{M} \overset{L}{\otimes}_{\mathcal{O}_X} Thom(F, \mathcal{D}b_X^\vee))$. Since (10.14) implies that

$$R\Gamma_c(U; \mathcal{T}hom(G \otimes F, \mathcal{D}b_X^\vee)) \leftarrow R\Gamma_c(U; \mathfrak{M} \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{T}hom(F, \mathcal{D}b_X^\vee))$$

is an isomorphism, we conclude by duality that

$$R\Gamma(U; (G \otimes F) \overset{w}{\otimes} \mathcal{C}_X^\infty) \to R\Gamma(U; R\,\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{M}, F \overset{w}{\otimes} \mathcal{C}_X^\infty))$$

is an isomorphism. This shows that (10.16) is an isomorphism. Thus we obtained (10.12). To obtain (10.13), it is enough to apply the functor $R\,\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \cdot\,)$ to (10.12). $\square$

10.4 Integral Transformations

Let us consider the following situation. Let X, Y and S be complex manifolds, and let d_X, d_Y and d_S be their dimension. Let us consider a diagram of morphisms of complex manifolds.

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd}$$

Let $\mathfrak{M} \in \mathbf{D}^b_{\text{q-good}}(\mathcal{D}_X)$, $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$ and $\mathfrak{L} \in \mathbf{D}^b_{\text{rh}}(\mathcal{D}_S)$. Set $L = \mathcal{S}ol(\mathfrak{L})$. We assume that

$$\begin{cases} f^{-1}\operatorname{supp}(\mathfrak{M}) \cap \operatorname{supp}(\mathfrak{L}) \text{ is proper over } Y, \\ g^{-1}\operatorname{supp}(G) \cap \operatorname{supp}(\mathfrak{L}) \text{ is proper over } X. \end{cases} \tag{10.17}$$

We define

$$\mathfrak{M} \circ \mathfrak{L} = \underline{g}_!\,(\underline{f}^{-1}\mathfrak{M} \overset{L}{\otimes}_{\mathcal{O}_S} \mathfrak{L}) \tag{10.18}$$

and

$$L \circ G = R\, f_!(L \otimes g^{-1}G)\,. \tag{10.19}$$

Theorem 10.8. — *We have isomorphisms:*

$$\begin{aligned} &R\Gamma\Big(X; R\,\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, (L\circ G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)\Big)[d_S] \\ &\qquad \simeq R\Gamma\Big(Y; R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathfrak{M}\circ\mathfrak{L}, G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y)\Big)[d_Y], \end{aligned} \tag{10.18}$$

$$\begin{aligned} &R\Gamma_c\Big(X; \mathcal{T}hom\,(L\circ G, \Omega_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathfrak{M}\Big)[d_X] \\ &\qquad \simeq R\Gamma_c\Big(Y; \mathcal{T}hom(G, \Omega_Y) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} (\mathfrak{M}\circ\mathfrak{L})\Big)[d_S]\,. \end{aligned} \tag{10.19}$$

and there are similar formulas by exchanging Γ *and* Γ_c.

Proof. Theorem 7.3 implies

$$\begin{aligned} &R\Gamma\Big(Y; R\,\mathcal{H}om_{\mathcal{D}_Y}\Big(\underline{g}_!(\underline{f}^{-1}\mathfrak{M}\otimes_{\mathcal{O}_S}\mathfrak{L}), G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y\Big)\Big)[d_Y] \\ &\qquad \simeq R\Gamma\Big(S; R\,\mathcal{H}om_{\mathcal{D}_S}\big(\underline{f}^{-1}\mathfrak{M}\otimes_{\mathcal{O}_S}\mathfrak{L}, g^{-1}G \overset{\mathrm{w}}{\otimes} \mathcal{O}_S\big)\Big)[d_S]\,. \end{aligned}$$

We have

$$R\,\mathcal{H}om_{\mathcal{D}_S}(\underline{f}^{-1}\mathfrak{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{O}_S}\mathfrak{L}, g^{-1}G \overset{\mathrm{w}}{\otimes} \mathcal{O}_S) \simeq R\,\mathcal{H}om_{\mathcal{D}_S}\big(\underline{f}^{-1}\mathfrak{M}, R\,\mathcal{H}om_{\mathcal{O}_S}(\mathfrak{L}, g^{-1}G \overset{\mathrm{w}}{\otimes} \mathcal{O}_S)\big)\,.$$

Theorem 10.7 implies

$$R\,\mathcal{H}om_{\mathcal{O}_S}(\mathfrak{L}, g^{-1}G \overset{\mathrm{w}}{\otimes} \mathcal{O}_S) \simeq (L\otimes g^{-1}G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_S.$$

Hence we obtain

$$\begin{aligned} &R\Gamma\Big(Y; R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathfrak{M}\circ\mathfrak{L}, G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y)\Big)[d_Y] \\ &\qquad \simeq R\Gamma\Big(S; R\,\mathcal{H}om_{\mathcal{D}_S}\big(\underline{f}^{-1}\mathfrak{M}, (L\otimes g^{-1}G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_S\big)\Big)[d_S]\,. \end{aligned}$$

We have by Theorem 7.2

$$\begin{aligned} &R\Gamma\Big(S; R\,\mathcal{H}om_{\mathcal{D}_S}\big(\underline{f}^{-1}\mathfrak{M}, (L\otimes g^{-1}G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_S\big)\Big) \\ &\qquad \simeq R\Gamma\Big(X; R\,\mathcal{H}om_{\mathcal{D}_X}\big(\mathfrak{M}, R\,f_*(L\otimes g^{-1}G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X\big)\Big)\,. \end{aligned}$$

Thus we obtain (10.18). The other isomorphism is similarly proved. □

Remark 10.9. By replacing $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ with $\cdot \otimes \mathcal{O}_X$ and $R\,\mathcal{H}om(\cdot, \mathcal{O}_X)$, the similar formulas to those in Theorem 10.8 hold under conditions different from (10.17). Instead of (10.17), assume that $\mathfrak{M} \in \mathbf{D}^b_{\mathrm{good}}(\mathcal{D}_X)$ and

$$\text{(10.20)} \qquad \begin{cases} f^{-1}\,\mathrm{supp}(\mathfrak{M}) \cap \mathrm{supp}(\mathfrak{L}) \text{ is proper over } Y, \\ \mathfrak{M} \text{ is non characteristic with respect to } f, \\ \mathrm{Char}(\underline{f}^{-1}\mathfrak{M}) \cap \mathrm{Char}(\mathfrak{L}) \subset T^*_S S. \end{cases}$$

Then we have

$$\text{(10.21)} \qquad \begin{aligned} &R\Gamma_c\Big(X; R\,\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, (L \circ G) \otimes \mathcal{O}_X)\Big)[d_S] \\ &\qquad \simeq R\Gamma_c\Big(Y; R\,\mathcal{H}om_{\mathcal{D}_Y}(\mathfrak{M} \circ \mathfrak{L}\,,\, G \otimes \mathcal{O}_Y)\Big)[d_Y]\,, \end{aligned}$$

$$\text{(10.22)} \qquad \begin{aligned} &R\Gamma\Big(X; R\,\mathcal{H}om\,(L \circ G, \Omega_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathfrak{M}\Big)[d_X] \\ &\qquad \simeq R\Gamma\Big(Y; R\,\mathcal{H}om(G, \Omega_Y) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} (\mathfrak{M} \circ \mathfrak{L})\Big)[d_S]\,. \end{aligned}$$

In the case where $\mathfrak{L} = \mathcal{O}_S$ (10.21–22) was obtained in [D'A-S$_1$]. Such formulas have nice applications (see e.g. [D'A-S$_1$], [D'A-S$_2$]).

A. Almost Free Resolutions

A.1 General Theory

In this appendix, we shall show that a complex with coherent cohomology groups has a resolution by "almost free" modules. In order to see this, we first discuss the problem in a general setting.

Let us denote by $\mathbf{Ab}$ the category of abelian groups. Let $\mathbf{P}$ be an additive category and $\mathbf{A}$ an abelian category. We are given an additive functor $L : \mathbf{P} \to \mathbf{A}$, an additive bifunctor $H : \mathbf{P}^{\mathrm{opp}} \times \mathbf{A} \to \mathbf{Ab}$, and a morphism of bifunctors $\alpha_{X,M} : H(X, M) \to \mathrm{Hom}_{\mathbf{A}}(L(X), M)$ in $X \in \mathbf{P}$ and $M \in \mathbf{A}$.

For $X \in \mathbf{P}$ and $M \in \mathbf{A}$, we call an element $\psi \in H(X, M)$ a morphism from X to M and write $\psi : X \to M$. Then we can consider the composition $\psi \circ f : Y \to M$ for a morphism $f : Y \to X$ in $\mathbf{P}$ and the composition $u \circ \psi : X \to N$ for a morphism $u : M \to N$. In fact $\psi \circ f = H(L(f), M)(\psi)$ and $u \circ \psi = H(X, u)(\psi)$. We have $(u \circ \psi) \circ f = u \circ (\psi \circ f)$. In another word, $\mathbf{P} \sqcup \mathbf{A}$ is a category. We have $\alpha(u \circ \psi) = u \circ \alpha(\psi)$ and $\alpha(\psi \circ f) = \alpha(\psi) \circ L(f)$.

For morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{P}$, we say that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if $g \circ f = 0$ and $L(X) \xrightarrow{L(f)} L(Y) \xrightarrow{L(g)} L(Z)$ is exact. Similarly for a morphism $f : X \to Y$ in $\mathbf{P}$ and $\varphi : Y \to M$ with $M \in \mathbf{A}$, we say that $X \xrightarrow{f} Y \xrightarrow{\varphi} M$ is exact if $\varphi \circ f = 0$ and $L(X) \xrightarrow{L(f)} L(Y) \xrightarrow{\alpha(\varphi)} M$ is exact. For a morphism $f : X \to Y$ in $\mathbf{P}$, we say that X is a cover of Y if $L(X) \xrightarrow{L(f)} L(Y)$ is an epimorphism. Similarly for $X \in \mathbf{P}$, $M \in \mathbf{A}$ and $\varphi : X \to M$, we say that X is a cover of M if $L(X) \xrightarrow{\alpha(\varphi)} M$ is an epimorphism.

We assume that these data satisfy the following four axioms.

(A.1) For any $X \in \mathbf{P}$, the functor $H(X, M)$ is left exact in $M \in \mathbf{A}$.

(A.2) For any morphism $g : Y \to Z$ in $\mathbf{P}$, there exists a morphism $f : X \to Y$ in $\mathbf{P}$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact.

(A.3) For any epimorphism $u : M \to N$ in $\mathbf{A}$, $Y \in \mathbf{P}$ and $\psi \in H(Y, N)$, there exist a cover $g : X \to Y$ of Y and $\varphi \in H(X, M)$ such that $\psi \circ g = u \circ \varphi$.

(A.4) For any $X,Y \in \mathbf{P}$ and $\psi \in H(X, L(Y))$ there exist a cover $f : X' \to X$ of X and a morphism $g : X' \to Y$ such that $L(g) = \alpha(\psi \circ f)$ in $\mathrm{Hom}_{\mathbf{A}}(L(X'), L(Y))$.

We say that an object M of $\mathbf{A}$ is $\mathbf{P}$-coherent if M satisfies the following two conditions.

(A.5) There exists a cover $f : X \to M$ of M.

(A.6) For any $Y \to M$ in $H(Y, M)$, there exists a morphism $X \to Y$ in $\mathbf{P}$ such that $X \to Y \to M$ is exact.

We shall denote by $\mathscr{C}$ the full subcategory of $\mathbf{A}$ consisting of $\mathbf{P}$-coherent objects.

Proposition A.1. — *$\mathscr{C}$ is stable by kernels, cokernels and extensions.*

Proof. Let $0 \to K \xrightarrow{u} M \xrightarrow{v} N$ be an exact sequence in $\mathbf{A}$ and assume that M and N are $\mathbf{P}$-coherent. Let us show that K is $\mathbf{P}$-coherent. Let us take a cover $\psi : X \to M$ of M. Then there exists $Y \in \mathbf{P}$ and an exact sequence $Y \xrightarrow{g} X \to N$. By (A.1) there exists $\varphi : Y \to K$ such that $u \circ \varphi = \psi \circ g$. It is easy to see that $\alpha(\varphi) : L(Y) \to K$ is an epimorphism. Therefore K satisfies (A.5).

Now $X \in \mathbf{P}$ and $\varphi : X \to K$ are given. Then there exists $f : Y \to X$ such that $Y \to X \to M$ is exact. Then by (A.1), $\varphi \circ f = 0$ and $L(Y) \to L(X) \to K$ is exact. Hence K is in $\mathscr{C}$.

To see that $\mathscr{C}$ is stable by taking the cokernel, it is enough to show that for an exact sequence $0 \to K \xrightarrow{u} M \xrightarrow{v} N \to 0$, if K and M are $\mathbf{P}$-coherent, then N is $\mathbf{P}$-coherent. It is obvious that N satisfies the condition (A.5).

To see (A.6), let $X \in \mathbf{P}$ and $\psi : X \to N$. Then by (A.3), there exists a cover $f : Y \to X$ of X and $\varphi : Y \to M$ such that $\psi \circ f = v \circ \varphi$. Let us take $\xi : Z \to K$ such that $L(Z) \to K$ is an epimorphism. Let us consider $Z \oplus Y \to M$ given by ξ and φ. Then there exists $h : W \to Z \oplus Y$ such that $W \to Z \oplus Y \to M$ is exact. Then $W \to X \to N$ is exact. Hence N is $\mathbf{P}$-coherent.

Finally let us show that $\mathscr{C}$ is stable by extensions. Let $0 \to K \xrightarrow{u} M \xrightarrow{v} N \to 0$ be an exact sequence and assume that K and N are $\mathbf{P}$-coherent. Let us show that M satisfies (A.5). There exists a cover $X \to N$ of N. By (A.3), replacing X with its cover, we may assume that $X \to N$ decomposes into $X \to M \to N$. Let us take a cover $Z \to K$ of K. Then $L(Z \oplus X) \to M$ is an epimorphism. Hence M satisfies (A.5).

In order to see that M satisfies (A.6), let $\varphi : X \to M$ be an element of $H(X, M)$. Let us take $Y \to X$ such that $Y \to X \xrightarrow{v \circ \varphi} N$ is exact. Then by (A.1), $Y \to M$ decomposes into $Y \to K \to M$. Let us take an exact sequence $Z \to Y \to K$. Then $Z \to X \to M$ is exact. □

The functor $L : \mathbf{P} \to \mathbf{A}$ induces a functor $K^-(\mathbf{P}) \to K^-(\mathbf{A})$. Let us denote by $\mathcal{N}(\mathbf{P})$ the full subcategory of $K^-(\mathbf{P})$ consisting of complexes X such that $L(X)$ is exact. Then we can easily see that $\mathcal{N}(\mathbf{P})$ is a null system (see [K-S, Def. 1.6.6]). We define $\mathbf{D}^-(\mathbf{P})$ the quotient of $K^-(\mathbf{P})$ by $\mathcal{N}(\mathbf{P})$. The category $\mathbf{D}^-(\mathbf{P})$ is described as follows. We say that a morphism $f : X \to Y$ in $K^-(\mathbf{P})$ is a quasi-isomorphism if $H^n(L(X)) \to H^n(L(Y))$ is an isomorphism for every n. The set of objects of $\mathbf{D}^-(\mathbf{P})$ is the same as the one of $K^-(\mathbf{P})$ and

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^-(\mathbf{P})}(X, Y) &= \varinjlim_{X' \to X} \mathrm{Hom}_{K^-(\mathbf{P})}(X', Y) \\ &= \varinjlim_{X' \to X, Y \to Y'} \mathrm{Hom}_{K^-(\mathbf{P})}(X', Y') \\ &= \varinjlim_{Y \to Y'} \mathrm{Hom}_{K^-(\mathbf{P})}(X, Y') .\end{aligned}$$

Here $X' \to X$ and $Y \to Y'$ range over the sets of quasi-isomorphisms. Then L induces a functor

$$L : \mathbf{D}^-(\mathbf{P}) \to \mathbf{D}^-(\mathbf{A}) .$$

Let us denote by $\mathbf{D}^-_{coh}(\mathbf{A})$ the full subcategory of $\mathbf{D}^-(\mathbf{A})$ consisting of the objects whose cohomology groups are $\mathbf{P}$-coherent. By the preceding proposition, $\mathbf{D}^-_{coh}(\mathbf{A})$ is a triangulated category. Similarly, let us denote by $\mathbf{D}^-_{coh}(\mathbf{P})$ the full subcategory of $\mathbf{D}^-(\mathbf{P})$ consisting of objects X such that $H^n(L(X))$ is $\mathbf{P}$-coherent for every n. Then it is also a triangulated category and we have a functor

$$L : \mathbf{D}^-_{coh}(\mathbf{P}) \to \mathbf{D}^-_{coh}(\mathbf{A}) .$$

We shall show that it is an equivalence of categories. The following proposition says that it is essentially surjective.

Proposition A.2. — *Let $M^\bullet$ be a complex in $\mathbf{A}$. Assume that $H^n(M^\bullet)$ is $\mathbf{P}$-coherent for every n and $H^n(M^\bullet) = 0$ for $n \gg 0$. Then there exists $X^\bullet \in C^-(\mathbf{P})$ and $\psi : X^\bullet \to M^\bullet$ such that $\alpha(\psi) : L(X^\bullet) \to M^\bullet$ is a quasi-isomorphism.*

Proof. Let us denote by Z^n the kernel of $d^n_M : M^n \to M^{n+1}$ and by B^n the image of $d^{n-1}_M : M^{n-1} \to M^n$. Assume that we have constructed a commutative diagram

$$\begin{tikzcd} & & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[r] \arrow[d] & X^{n+2} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & M^{n+2} \arrow[r] & \cdots \end{tikzcd}$$

such that $H^k(X^\bullet) \to H^k(M^\bullet)$ is an isomorphism for $k > n$ and an epimorphism for $k = n$. Let us take an exact sequence $W \to X^n \to X^{n+1}$. Then $W \to X^n \to M^n$ decomposes into $W \to Z^n \to M^n$. By the assumption $W \to Z^n \to H^n(X^\bullet)$ is an epimorphism. Since $H^n(M^\bullet)$ is $\mathbf{P}$-coherent, there is an exact sequence $Y \to W \to H^n(M^\bullet)$. Then $Y \to W \to Z^n$ decomposes into $Y \to B^n \to Z^n$. By (A.3), replacing Y with its cover, we may assume that $Y \to B^n$ factors through $Y \to M^{n-1} \to B^n$.

Take a cover $U \to H^{n-1}(M^\bullet)$ of $H^{n-1}(M^\bullet)$. By (A.3), replacing U with its cover, we may assume that $U \to H^{n-1}(M^\bullet)$ decomposes into $U \to Z^{n-1} \to H^{n-1}(M^\bullet)$. We set $X^{n-1} = U \oplus Y$. We define $d_X^{n-1} : X^{n-1} \to X^n$ by the zero morphism $U \to X^n$ and $Y \to W \to X^n$ on Y. Define $\psi^{n-1} : X^{n-1} \to M^{n-1}$ by $U \to Z^{n-1} \to M^{n-1}$ and $Y \to M^{n-1}$. Then $\psi^n \circ d_X^{n-1} = d_M^{n-1} \circ \psi^{n-1}$. Furthermore, $H^k(X^\bullet) \to H^k(M^\bullet)$ is an isomorphism for $k = n$ and an epimorphism for $k = n - 1$. Thus the induction proceeds and we can construct a desired complex $X^\bullet$ and $X^\bullet \to M^\bullet$. □

Proposition A.3. — *Let $Y^\bullet, Z^\bullet \in C^-(\mathbf{P})$. Let $u : L(Y^\bullet) \to L(Z^\bullet)$ be a morphism in $C^-(\mathbf{A})$. Assume that the cohomology groups of $L(Y^\bullet)$ are $\mathbf{P}$-coherent. Then there are $X^\bullet \in C^-(\mathbf{P})$ and a quasi-isomorphism $f : X^\bullet \to Y^\bullet$ and $g : X^\bullet \to Z^\bullet$ such that $L(g) = u \circ L(f) \in \mathrm{Hom}_{\mathbf{A}}(L(X^\bullet), L(Z^\bullet))$.*

Proposition A.4. — *Let $g : Y^\bullet \to Z^\bullet$ be a morphism in $C^-(\mathbf{P})$. Assume that the cohomology groups of $L(Y^\bullet)$ are $\mathbf{P}$-coherent. If $L(g) : L(Y^\bullet) \to L(Z^\bullet)$ is homotopic to 0, then there exists a quasi-isomorphism $f : X^\bullet \to Y^\bullet$ such that $g \circ f : X^\bullet \to Z^\bullet$ is homotopic to 0.*

We shall give the proofs of these two propositions in §A.4.

Now we are ready to prove the following main result in this subsection.

Theorem A.5. — $\mathbf{D}^-_{coh}(\mathbf{P}) \to \mathbf{D}^-_{coh}(\mathbf{A})$ *is an equivalence of triangulated categories.*

Proof. We saw already that this functor is essentially surjective. Hence it is enough to show that for any $X^\bullet, Y^\bullet \in C^-(\mathbf{P})$,

$$\mathrm{Hom}_{\mathbf{D}^-_{\mathrm{coh}}(\mathbf{P})}(X^\bullet, Y^\bullet) \to \mathrm{Hom}_{\mathbf{D}^-_{\mathrm{coh}}(\mathbf{A})}(L(X^\bullet), L(Y^\bullet))$$

is bijective.

Injectivity. Let $f : X^\bullet \to Y^\bullet$ be a morphism in $C^-(\mathbf{P})$ such that $L(f)$ vanishes as an element of $\mathrm{Hom}_{\mathbf{D}^-_{\mathrm{coh}}(\mathbf{A})}(L(X^\bullet), L(Y^\bullet))$. Then there exists a quasi-isomorphism $u : M^\bullet \to L(X^\bullet)$ in $C^-(\mathbf{A})$ such that the composition $M^\bullet \xrightarrow{u} L(X^\bullet) \xrightarrow{L(f)} L(Y^\bullet)$ is homotopic to 0. By Proposition A.2, we may assume that $M^\bullet = L(Z^\bullet)$ for some

$Z^\bullet \in C^-(\mathbf{P})$. By Proposition A.3, there exist a quasi-isomorphism $g : W^\bullet \to Z^\bullet$ and a morphism $h : W^\bullet \to X^\bullet$ such that

$$L(h) = u \circ L(g) : L(W^\bullet) \to L(X^\bullet).$$

Then $L(f \circ h) = L(f) \circ u \circ L(g)$ is homotopic to 0. Hence by Proposition A.4, there exists a quasi-isomorphism $U^\bullet \to W^\bullet$ such that $U^\bullet \to W^\bullet \to Y^\bullet$ is homotopic to 0. Since the composition $U^\bullet \to W^\bullet \to Y^\bullet$ is equal to $U^\bullet \to W^\bullet \to X^\bullet \to Y^\bullet$ and $U^\bullet \to W^\bullet \to X^\bullet$ is a quasi-isomorphism, f is 0 as an element of $\mathrm{Hom}_{\mathbf{D}^-_{\mathrm{coh}}(\mathbf{P})}(X^\bullet, Y^\bullet)$.

Surjectivity. Let us consider a morphism $L(X^\bullet) \to L(Y^\bullet)$ in $\mathbf{D}^-_{\mathrm{coh}}(\mathbf{A})$. Then there is a quasi-isomorphism $u : M^\bullet \to L(X^\bullet)$ and a morphism $v : M^\bullet \to L(Y^\bullet)$ in $C^-(\mathbf{A})$ such that $v \circ u^{-1}$ is the given morphism $L(X^\bullet) \to L(Y^\bullet)$ in $\mathbf{D}^-_{\mathrm{coh}}(\mathbf{A})$. There exist $Z^\bullet \in C^-(\mathbf{P})$ and a quasi-isomorphism $w : L(Z^\bullet) \to M^\bullet$. Then by using Proposition A.3, there is a quasi-isomorphism $f : W^\bullet \to Z^\bullet$ together with morphisms $g : W^\bullet \to X^\bullet$ and $h : W^\bullet \to Y^\bullet$ such that $L(g) = u \circ w \circ L(f) : L(W^\bullet) \to L(X^\bullet)$ and $L(h) = v \circ w \circ L(f) : L(W^\bullet) \to L(X^\bullet)$. Then g is a quasi-isomorphism and the morphism $h \circ g^{-1} : X^\bullet \to Y^\bullet$ in $\mathbf{D}^-_{\mathrm{coh}}(\mathbf{P})$ is sent to $v \circ u^{-1}$ in $\mathbf{D}^-_{\mathrm{coh}}(\mathbf{A})$. □

A.2 Almost Free Resolutions of Coherent Modules

Let us apply the theory above to the situation of coherent modules. Let X be a paracompact and locally compact space and $\mathscr{A}$ a sheaf of rings on X (with 1 but not necessarily commutative) which is coherent as a left $\mathscr{A}$-module. Let us take a set $\mathscr{S}$ of relatively compact open subsets of X. We assume the following two conditions on $\mathscr{S}$.

(A.7) For any $x \in X$, $\{U \in \mathscr{S}; x \in U\}$ is a neighborhood system of x.

(A.8) For $U,V \in \mathscr{S}$, $U \cap V$ is a finite union of open subsets belonging to $\mathscr{S}$.

Let us take $\mathrm{Mod}(\mathscr{A})$ as $\mathbf{A}$ in the situation of the last subsection. We define $\mathbf{P}(\mathscr{A})$ as follows. The set of objects of $\mathbf{P}(\mathscr{A})$ is the set of locally finite families of open subsets in $\mathscr{S}$. For two objects $\mathfrak{U} = \{U_i\}_{i\in I}$ and $\mathfrak{V} = \{V_j\}_{j\in J}$ of $\mathbf{P}(\mathscr{A})$, we define

$$\begin{aligned}\mathrm{Hom}_{\mathbf{P}(\mathscr{A})}(\mathfrak{U}, \mathfrak{V}) &= \prod_{i\in I}\Big(\bigoplus_{U_i \subset V_j} \Gamma(\overline{U_i}; \mathscr{A})\Big) \\ &= \big\{(a_{i,j})_{i\in I, j\in J}; a_{i,j} \in \Gamma(\overline{U_i}; \mathscr{A}) \text{ and } a_{i,j} = 0 \text{ unless } U_i \subset V_j\big\}.\end{aligned}$$

Note that for any $i \in I$, $\{j \in J; U_i \subset V_j\}$ is a finite set. For $\mathfrak{W} = \{W_k\}_{k\in K}$, we define the composition $c = (c_{i,k}) \in \mathrm{Hom}_{\mathbf{P}(\mathscr{A})}(\mathfrak{U}, \mathfrak{W})$ of $a = (a_{i,j}) \in \mathrm{Hom}_{\mathbf{P}(\mathscr{A})}(\mathfrak{U}, \mathfrak{V})$ and $b = (b_{j,k}) \in \mathrm{Hom}_{\mathbf{P}(\mathscr{A})}(\mathfrak{V}, \mathfrak{W})$ by

$$c_{i,k} = \sum_j a_{i,j}(b_{j,k}|_{\overline{U_i}}) \in \Gamma(\overline{U_i}; \mathscr{A}).$$

The sum ranges over the $j \in J$ with $U_i \subset V_j \subset W_k$. It is easy to see that $\mathbf{P}(\mathscr{A})$ is an additive category.

We define the functor $L_{\mathscr{A}} : \mathbf{P}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A})$ by

$$L_{\mathscr{A}}(\mathfrak{U}) = \bigoplus_{i \in I} \mathscr{A}_{U_i}$$

for $\mathfrak{U} = \{U_i\}_{i \in I}$. We can easily see that it is well defined.

We define the bifunctor $H : \mathbf{P}(\mathscr{A})^{\mathrm{opp}} \times \mathrm{Mod}(\mathscr{A}) \to \mathbf{Ab}$ by

$$H(\mathfrak{U}, M) = \prod_{i \in I} \Gamma(\overline{U_i}; M).$$

We can easily see that it is a well-defined functor. We define

$$\alpha_{\mathfrak{U},M} : H(\mathfrak{U}, M) \to \mathrm{Hom}_{\mathscr{A}}(L_{\mathscr{A}}(\mathfrak{U}), M)$$

by the restriction map $\prod_{i \in I} \Gamma(\overline{U_i}; M) \to \prod_{i \in I} \Gamma(U_i; M) \cong \mathrm{Hom}_{\mathscr{A}}(L_{\mathscr{A}}(\mathfrak{U}), M)$.

Proposition A.6. — *The axioms* (A.1)–(A.4) *hold.*

Proof. The axiom (A.1) is obvious.

In order to prove the other axioms, we shall prepare the following lemma.

Lemma A.7. — *Let K be a compact subset of X and W a neighborhood of K. Then for any $U \in \mathscr{S}$, there exists a finite family $\{V_j\}$ of open subsets belonging to $\mathscr{S}$ such that*

$$U \cap K \subset \cup_j V_j \subset U \cap W.$$

Proof. By (A.7) , there exists a finite family $\{V_j\}$ of open sets in $\mathscr{S}$ such that

$$K \subset \cup_j V_j \subset W.$$

Since $U \cap V_j$ is a union of finite subsets belonging to $\mathscr{S}$ by (A.8), we obtain the desired result. □

Proof of (A.2). Let us take $\mathfrak{U} = \{U_i\}_{i \in I}$ and $\mathfrak{V} = \{V_j\}_{j \in J}$ and a morphism $f = (a_{i,j}) : \mathfrak{U} \to \mathfrak{V}$. For any $x \in X$, set $I(x) = \{i \in I; x \in \overline{U_i}\}$. Then there exists a neighborhood $W(x)$ of x such that $W(x) \cap \overline{U_i} = \emptyset$ for any $i \in I \setminus I(x)$. By shrinking $W(x)$, we may assume that $a_{i,j}$ extends to $\tilde{a}_{i,j} \in \Gamma(\overline{U_i} \cup \overline{W(x)}; \mathscr{A})$. Then for any subset G of $I(x)$, $\tilde{a}_{i,j}$ defines a morphism $\mathscr{A}^{\oplus G}|_{W(x)} \to \mathscr{A}^{\oplus J}|_{W(x)}$. Since $\mathscr{A}$ is coherent, its kernel is finitely generated on a neighborhood of x. Hence shrinking $W(x)$ if necessary, we may assume that there are a finite index set $N(G, x)$ and an exact sequence

$$\mathscr{A}^{\oplus N(G,x)}\big|_{W(x)} \xrightarrow{h_G} \mathscr{A}^{\oplus G}\big|_{W(x)} \to \mathscr{A}^{\oplus J}\big|_{W(x)}.$$

There exists a locally finite covering $\{W_k\}_{k\in K}$ of X such that $W_k \in \mathcal{S}$ and there exists x_k with $\overline{W_k} \subset W(x_k)$. Write $W_k \bigcap(\bigcap_{i\in G} U_i) = \bigcup_{m\in C(k,G)} W(k,G,m)$ for a finite index set $C(k,G)$ and $W(k,G,m) \in \mathcal{S}$. We set

$$K' = \{(k,G,m,n); k \in K, G \subset I(x_k),\ m \in C(G,k),\ n \in N(G,x_k)\}$$

and $W(k,G,m,n) = W(k,G,m)$. Then $\mathfrak{W} = \{W(k,G,m,n)\}_{(k,G,m,n)\in K'}$ is an object of $\mathbf{P}(\mathcal{A})$. The morphism

$$h_G : \mathcal{A}^{\oplus\{n\}}|_{W(x_k)} \to \mathcal{A}^{\oplus N(G,x_k)}|_{W(x_k)} \to \mathcal{A}^{\oplus G}|_{W(x_k)} \to \mathcal{A}^{\oplus\{i\}}|_{W(x_k)}$$

gives $c_{(k,G,n),i} \in \Gamma(\overline{W(k,G,m,n)}\,;\mathcal{A})$. This defines a morphism from $\mathfrak{W} \to \mathfrak{U}$. By the construction, it satisfies the desired conditions: $\mathfrak{W} \to \mathfrak{U} \to \mathfrak{V}$ vanishes and $L_{\mathcal{A}}(\mathfrak{W}) \to L_{\mathcal{A}}(\mathfrak{U}) \to L_{\mathcal{A}}(\mathfrak{V})$ is exact.

Proof of (A.3). Let $u : M \to N$ be an epimorphism in $\mathrm{Mod}(\mathcal{A})$, $\mathfrak{U} = \{U_i\}_{i\in I}$ an object of $\mathbf{P}(\mathcal{A})$ and $\varphi : \mathfrak{U} \to N$ an element of $H(\mathfrak{U}, N)$. Set $\varphi = (s_i)_{i\in I}$ with $s_i \in \Gamma(\overline{U_i}; N)$. For any x, we define $I(x) \subset I$ as above and take an open neighborhood $W(x)$ of x such that $W(x) \cap \overline{U_i} = \emptyset$ for $i \notin I(x)$. Shrinking $W(x)$ again, there exists $t_{(i,x)} \in \Gamma(W(x); M)$ such that $u(t_{(i,x)})|_{W(x)\cap\overline{U_i}} = s_i|_{W(x)\cap\overline{U_i}}$. Then take a locally finite covering $\{W_k\}_{k\in K}$ of X such that $\overline{W_k} \subset W(x_k)$ for some x_k and $W_k \in \mathcal{S}$. Write $W_k \cap U_i = \bigcup_{n\in C(k,i)} W(k,i,n)$ with a finite index set $C(k,i)$ and $W(k,i,n) \in \mathcal{S}$. Then set $K' = \{(k,i,n); k \in K,\ i \in I(x_k),\ n \in C(k,i)\}$ and $\mathfrak{W} = \{W(k,i,n)\}_{(k,i,n)\in K'}$. Then $t_{(i,x_k)}$ gives a morphism $\mathfrak{W} \to M$ and $a_{(k,i,n),i'} = \delta_{ii'} \in \Gamma(\overline{W(k,i,n)}; \mathcal{A})$ defines a morphism $\mathfrak{W} \to \mathfrak{U}$. We can easily see that

$$\begin{array}{ccc} \mathfrak{W} & \longrightarrow & \mathfrak{U} \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

is commutative and $L_{\mathcal{A}}(\mathfrak{W}) \to L_{\mathcal{A}}(\mathfrak{U})$ is an epimorphism.

Proof of (A.4). Let us take objects $\mathfrak{U} = \{U_i\}_{i\in I}$ and $\mathfrak{V} = \{V_j\}_{j\in J}$ of $\mathbf{P}(\mathcal{A})$ and $\varphi : \mathfrak{U} \to L_{\mathcal{A}}(\mathfrak{V})$. We have $H(\mathfrak{U}, L_{\mathcal{A}}(\mathfrak{V})) = \prod_i \Gamma(\overline{U_i}; \oplus_j \mathcal{A}_{V_j}) \simeq \prod_{i,j} \Gamma(\overline{U_i}; \mathcal{A}_{V_j})$. Let $a_{i,j} \in \Gamma(\overline{U_i}; \mathcal{A}_{V_j})$ be the element corresponding to φ. Then $\mathrm{supp}(a_{i,j})$ is a compact subset of V_j. Hence by Lemma A.7, there exists a finite family $\{W_{i,j,n}\}_{n\in K(i,j)}$ such that $W_{i,j,n} \in \mathcal{S}$ and

$$U_i \bigcap \mathrm{supp}(a_{i,j}) \subset \bigcup_{n\in K(i,j)} W_{i,j,n} \subset U_i \bigcap V_j\,.$$

By the same lemma, there is also a finite covering $\{W'_{i,j,m}\}_{m\in K'(i,j)}$ such that $W'_{i,j,m} \in \mathscr{S}$ and

$$U_i \setminus \left(\bigcup_{n\in K(i,j)} W_{i,j,n} \right) \subset \bigcup_{m\in K'(i,j)} W'_{i,j,m} \subset U_i \setminus \operatorname{supp}(a_{i,j})\,.$$

Set $K = \{(i,j,n); \overline{U_i} \cap V_j \neq \emptyset,\ n \in K(i,j)\}$ and $K' = \{(i,j,m); \overline{U_i} \cap V_j \neq \emptyset,\ m \in K'(i,j)\}$. Set $\mathfrak{W} = \{W_{i,j,n}\}_{(i,j,n)\in K}$ and $\mathfrak{W}' = \{W'_{i,j,m}\}_{(i,j,m)\in K'}$. They are objects of $\mathbf{P}(\mathscr{A})$. Define $\mathfrak{W} \to \mathfrak{U}$ by $b_{(i,j,n),i'} = \delta_{ii'} \in \Gamma(\overline{W_{i,j,n}};\mathscr{A})$ and $\mathfrak{W} \to \mathfrak{V}$ by $c_{(i,j,n),j'} = \delta_{jj'}a_{i,j} \in \Gamma(\overline{W_{i,j,n}};\mathscr{A})$. Define $\mathfrak{W}' \to \mathfrak{U}$ by $b'_{(i,j,n),i'} = \delta_{ii'} \in \Gamma(\overline{W'_{i,j,n}};\mathscr{A})$ and $\mathfrak{W}' \to \mathfrak{V}$ by 0. Then $\mathfrak{W} \oplus \mathfrak{W}' \to \mathfrak{U}$ and $\mathfrak{W} \oplus \mathfrak{W}' \to \mathfrak{V}$ satisfy the desired conditions. □

Proposition A.8. — *An $\mathscr{A}$-module M is coherent if and only if M is $\mathbf{P}(\mathscr{A})$-coherent.*

Proof. First let us show that a coherent $\mathscr{A}$-module M is $\mathbf{P}(\mathscr{A})$-coherent. The property (A.5) for coherent sheaves is obvious. Let us show (A.6). The proof is similar to the proof of (A.2). Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an object of $\mathbf{P}(\mathscr{A})$ and let $\varphi : \mathfrak{U} \to M$ be given by $s_i \in \Gamma(\overline{U_i}; M)$. For $x \in X$, let us define $I(x)$ as in the proof of (A.2) and a neighborhood $W(x)$ of x such that $W(x) \cap U_i = \emptyset$ for $i \notin I(x)$. We may assume that s_i is extended to $\overline{W(x)} \cup \overline{U_i}$. For $G \subset I(x)$, let us take an exact sequence, by shrinking $W(x)$ if necessary, $\mathscr{A}^{N(G,x)}\big|_{W(x)} \to \mathscr{A}^G\big|_{W(x)} \to M\big|_{W(x)}$. As the rest of the arguments is similar to the proof of (A.2), we shall omit it.

Let us show that a $\mathbf{P}(\mathscr{A})$-coherent $\mathscr{A}$-module M is coherent. Let us take $\mathfrak{U} = (U_i)_{i\in I}$ and a cover $\varphi = (s_i)_{i\in I} : \mathfrak{U} \to M$. For any x in X, $s_i \in \Gamma(\overline{U_i}; M)$ extends to a neighborhood W of x. Then $L_{\mathscr{A}}(\mathfrak{U})\big|_W \to M\big|_W$ decomposes as $L_{\mathscr{A}}(\mathfrak{U})\big|_W \to \mathscr{A}^{\oplus N}\big|_W \to M\big|_W$ for some integer N. Hence M is locally finitely generated. We may assume further that W is in $\mathscr{S}$. Set $\mathfrak{W} = \{W\}$. Then we have $\mathfrak{W}^{\oplus N} \to M$, which is surjective on W. There is an exact sequence $\mathfrak{V} \to \mathfrak{W}^{\oplus N} \to M$. By a similar argument as above, the kernel of $L_{\mathscr{A}}(\mathfrak{W}^{\oplus N}) \to M$ is finitely generated on a neighborhood of x. Hence M is coherent. □

Let us denote by $\mathbf{D}^-_{\mathrm{coh}}(\mathscr{A})$ the full subcategory of $\mathbf{D}^-(\mathscr{A})$ consisting of objects with coherent cohomology groups. Similarly, we denote by $\mathbf{D}^-_{\mathrm{coh}}(\mathbf{P}(\mathscr{A}))$ the full subcategory of $\mathbf{D}^-(\mathbf{P}(\mathscr{A}))$ consisting of objects Y such that $L_{\mathscr{A}}(Y)$ has coherent cohomology groups. Then Theorem A.5 implies the following theorem.

Theorem A.9. — *$\mathbf{D}^-_{coh}(\mathbf{P}(\mathscr{A})) \to \mathbf{D}^-_{coh}(\mathscr{A})$ is an equivalence of triangulated categories.*

Let us define the additive category $\tilde{\mathbf{P}}(\mathscr{A})$ by $\mathrm{Ob}(\tilde{\mathbf{P}}(\mathscr{A})) = \mathrm{Ob}(\mathbf{P}(\mathscr{A}))$ and

$$\mathrm{Hom}_{\tilde{\mathbf{P}}(\mathscr{A})}(\mathfrak{U}, \mathfrak{V}) = \mathrm{Hom}_{\mathscr{A}}(L(\mathfrak{U}), L(\mathfrak{V})).$$

Then $\tilde{\mathbf{P}}(\mathscr{A})$ is a full subcategory of $\mathrm{Mod}(\mathscr{A})$. We can define similarly $\mathbf{D}^-_{\mathrm{coh}}(\tilde{\mathbf{P}}(\mathscr{A}))$. The following theorem is also easy to prove.

Theorem A.10. — $\mathbf{D}^-_{coh}(\tilde{\mathbf{P}}(\mathscr{A})) \to \mathbf{D}^-_{coh}(\mathscr{A})$ *is an equivalence of triangulated categories.*

We call a complex $M^\bullet$ of $\mathscr{A}$-modules almost free if each component M^n is isomorphic to $\oplus_i \mathscr{A}_{U_i}$ for a locally finite family $\{U_i\}$ of relatively compact open subsets of X in $\mathscr{S}$. Then the above theorem says that any complex of $\mathscr{A}$-modules with coherent cohomology groups is quasi-isomorphic to an almost free complex.

A.3 $\mathbb{R}$-Constructible Case

Let X be a real analytic manifold of dimension d_X. Let $\mathscr{S}$ be a set of open subanalytic subsets of X. We assume that any relatively compact open subanalytic subset is a finite union of open subsets in $\mathscr{S}$. For example we can take as $\mathscr{S}$ the set of open subanalytic subsets U of X such that $(\overline{U}, \partial U)$ is homeomorphic to (B^{d_X}, S^{d_X}) (by the subanalytic triangulation theorem). Here B^{d_X} is the d_X-dimensional ball and S^{d_X} is its boundary. Let us take $\mathbb{R}$-Cons(X) as $\mathbf{A}$. We define the category $\mathbf{P}(X)$ as follows. The set of objects of $\mathbf{P}(X)$ is the set of locally finite families of open subsets belonging to $\mathscr{S}$. For $\mathfrak{U} = \{U_i\}_{i \in I} \in \mathbf{P}(X)$, we set

$$L_C(\mathfrak{U}) = \oplus_{i \in I} \mathbb{C}_{U_i}$$

and set

$$\mathrm{Hom}_{\mathbf{P}(X)}(\mathfrak{U}, \mathfrak{V}) = \mathrm{Hom}\,(L(\mathfrak{U}), L(\mathfrak{V}))$$

and

$$H(\mathfrak{U}, F) = \mathrm{Hom}(L(\mathfrak{U}), F)$$

for $\mathfrak{U}, \mathfrak{V} \in \mathbf{P}(X)$ and $F \in \mathbb{R}$-Cons(X). Hence $\mathbf{P}(X)$ is a full subcategory of $\mathbb{R}$-Cons(X). Remark that any $F \in \mathbb{R}$-Cons(X) has an epimorphism $L_C(\mathfrak{U}) \to F$ for some $\mathfrak{U} \in \mathbf{P}(X)$. By this, we can easily check that (A.1)–(A.4) are satisfied. We see also that any $\mathbb{R}$-constructible sheaf is $\mathbf{P}(X)$-coherent. Thus we obtain the following proposition.

Theorem A.11. — $\mathbf{D}^-(\mathbf{P}(X)) \to \mathbf{D}^-(\mathbb{R}\text{-Cons}(X)) \to \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ *are equivalences of categories.*

Remark that we have

$$\mathrm{D}'_X(L(\mathfrak{U})) \simeq \oplus_{i\in I}\mathbb{C}_{\overline{U_i}}$$

for $\mathfrak{U} = \{U_i\}_{i\in I} \in \mathbf{P}(X)$ such that every $(\overline{U_i}, \partial U_i)$ is homeomorphic to (B^{d_X}, S^{d_X}).

A.4 Proofs of Propositions A.3 and A.4

We shall remark first the following lemma.

Lemma A.12. — *Let $f, g : X \to Y$ be morphisms in $\mathbf{P}$. If $L(f) = L(g)$, there exists a cover $h : X' \to X$ such that $f \circ h = g \circ f$.*

Proof. By (A.2), there exists an exact sequence

$$X' \xrightarrow{h} X \xrightarrow{f-g} Y\,.$$

Then $L(h) : L(X') \to L(X)$ is an epimorphism and $f \circ h = g \circ h$. □

Proof of Proposition A.3. We shall construct $X^\bullet \in C^-(\mathbf{P})$, a quasi-isomorphism $f : X^\bullet \to Y^\bullet$, $\varphi : X^\bullet \to L(Y^\bullet)$ and $g : X^\bullet \to Z^\bullet$ such that

(A.9) $$L(g) = u \circ L(f) : L(X^\bullet) \to L(Z^\bullet)$$

and

(A.10) $$L(f) = \alpha(\varphi) : L(X^\bullet) \to L(Y^\bullet)\,.$$

Assume that we are given

$$\begin{array}{ccccccccc} & & & & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\ & & & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & Y^{n-1} & \longrightarrow & Y^n & \longrightarrow & Y^{n+1} & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{ccccccccc} & & & & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\ & & & & \downarrow{\scriptstyle \varphi^n} & & \downarrow{\scriptstyle \varphi^{n+1}} & & \\ \cdots & \longrightarrow & L(Y^{n-1}) & \longrightarrow & L(Y^n) & \longrightarrow & L(Y^{n+1}) & \longrightarrow & \cdots \end{array}$$

and

$$\begin{array}{ccccccccc} & & & & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\ & & & & \downarrow{\scriptstyle g^n} & & \downarrow{\scriptstyle g^{n+1}} & & \\ \cdots & \longrightarrow & Z^{n-1} & \longrightarrow & Z^n & \longrightarrow & Z^{n+1} & \longrightarrow & \cdots \end{array}$$

such that they satisfy (A.9) and (A.10). We assume further that $H^k(L(X^\bullet)) \to H^k(L(Y^\bullet))$ is an isomorphism for $k > n$ and an epimorphism for $k = n$. Let us take an exact sequence $U \to X^n \to X^{n+1}$. Set $Z^n(L(Y^\bullet)) = \operatorname{Ker}\left(d^n_{L(Y^\bullet)} : L(Y^n) \to L(Y^{n+1})\right)$ and
$B^n(L(Y^\bullet)) = \operatorname{Im}\left(d^{n-1}_{L(Y^\bullet)} : L(Y^{n-1}) \to L(Y^n)\right)$. Then $U \to X^n \xrightarrow{\varphi^n} L(Y^n)$ decomposes into $U \to Z^n(L(Y^\bullet)) \to L(Y^n)$. By the assumption, the composition $U \to Z^n(L(Y^\bullet)) \to H^n(L(Y^\bullet))$ is an epimorphism. Since $H^n(L(Y^\bullet))$ is $\mathbf{P}$-coherent, there is an exact sequence $V \to U \to H^n(L(Y^\bullet))$. Then $V \to U \to Z^n(L(Y^\bullet))$ decomposes into $V \to B^n(L(Y^\bullet)) \to Z^n(L(Y^\bullet))$. Hence by replacing V with its cover, we may assume that $V \to B^n(L(Y^\bullet))$ decomposes into $V \xrightarrow{\xi} L(Y^{n-1}) \to B^n(L(Y^\bullet))$. By (A.4), by replacing V with its cover, we may assume that there exists $h : V \to Y^{n-1}$ such that $L(h) = \alpha(\xi)$. We have $L(d_Y^{n-1} \circ h) = L(V \to U \to X^n \to Y^n) \in \operatorname{Hom}_{\mathbf{A}}(L(V), L(Y^n))$. Hence by Lemma A.12, replacing V with its cover, we may assume that

$$\begin{array}{ccc} V & \longrightarrow & X^n \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle f^n} \\ Y^{n-1} & \longrightarrow & Y^n \end{array}$$

commutes. By the similar arguments, by replacing V with its cover, we may assume that there exists $b : V \to Z^{n-1}$ such that $L(b) = u^{n-1} \circ L(h) : L(V) \to L(Z^{n-1})$ and

$$\begin{array}{ccc} V & \longrightarrow & X^n \\ \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle g^n} \\ Z^{n-1} & \longrightarrow & Z^n \end{array}$$

commutes.

Since $H^{n-1}(L(Y^\bullet))$ is $\mathbf{P}$-coherent, there is a cover $G \to H^{n-1}(L(Y^\bullet))$. By replacing G with its cover we may assume that $G \to H^{n-1}(L(Y^\bullet))$ decomposes into $G \xrightarrow{\eta} Z^{n-1}(L(Y^\bullet)) \to H^{n-1}(L(Y^\bullet))$. Then by the similar arguments as above we may assume that, after replacing G with its cover, there exists $G \xrightarrow{g} Y^{n-1}$ such that the composition $G \to Y^{n-1} \to Y^n$ vanishes and $L(G) \xrightarrow{\alpha(\eta)} Z^{n-1} \to L(Y^{n-1})$ coincides with L(g). Replacing again G with its cover we may assume that there exists $c : G \to Z^{n-1}$ such that $G \xrightarrow{c} Z^{n-1} \to Z^n$ vanishes and $L(c) = u^{n-1} \circ L(u) : L(G) \to L(Z^{n-1})$.

We set $X^{n-1} = G \oplus V$. Define $f^{n-1} : X^{n-1} \to Y^{n-1}$ by $g : G \to Y^{n-1}$ and $h : V \to Y^{n-1}$. Define $\varphi^{n-1} : X^{n-1} \to L(Y^{n-1})$ by $\xi : V \to L(Y^{n-1})$ and $G \xrightarrow{\eta} Z^{n-1}(L(Y^\bullet)) \to L(Y^{n-1})$. We define $g^{n-1} : X^{n-1} \to Z^{n-1}$ by $b : V \to Z^{n-1}$ and $c : G \to Z^{n-1}$. Then $H^n(L(X^\bullet)) \to H^n(L(Y^\bullet))$ is an isomorphism and $H^{n-1}(L(X^\bullet)) \to H^{n-1}(L(Y^\bullet))$ is an epimorphism. Thus the induction proceeds.

Proof of Proposition A.4. The proof is similar to the above proof. Let $s^n : L(Y^n) \to L(Z^{n-1})$ be a homotopy. We shall construct $X \in C^-(\mathbf{P})$ and a quasi-isomorphism $f : X^\bullet \to Y^\bullet$, $\varphi : X^\bullet \to L(Y^\bullet)$ and $t^n : X^n \to Z^{n-1}$ such that

$$g^n \circ f^n = d_Z^{n-1} \circ t^n + t^{n+1} \circ d_X^n \,, \tag{A.11}$$

$$L(f) = \alpha(\varphi) : L(X^\bullet) \to L(Y^\bullet)\,, \tag{A.12}$$

and

$$L(t^n) = s^n \circ L(f^n)\,. \tag{A.13}$$

Assume that we are given

$$\begin{array}{ccccccccc} & & & & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\ & & & & \big\downarrow{\scriptstyle f^n} & & \big\downarrow{\scriptstyle f^{n+1}} & & , \\ \cdots & \longrightarrow & Y^{n-1} & \longrightarrow & Y^n & \longrightarrow & Y^{n+1} & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{ccccccccc} & & & & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\ & & & & \big\downarrow{\scriptstyle \varphi^n} & & \big\downarrow{\scriptstyle \varphi^{n+1}} & & \\ \cdots & \longrightarrow & L(Y^{n-1}) & \longrightarrow & L(Y^n) & \longrightarrow & L(Y^{n+1}) & \longrightarrow & \cdots \end{array}$$

and $t^k : X^k \to Z^{k-1}$ $(k \geq n)$ satisfying the conditions (A.11)–(A.13). We assume further that $H^k(L(X^\bullet)) \to H^k(L(Y^\bullet))$ is an isomorphism for $k > n$ and an epimorphism for $k = n$. By the similar arguments with the above proof, we can construct $a : V \to X^n$, $h : V \to Y^{n-1}$, $\xi : V \to L(Y^{n-1})$ such that $L(h) = \alpha(\xi) : L(V) \to L(Y^{n-1})$, the composition $V \to X^n \to X^{n+1}$ vanishes,

$$\begin{array}{ccc} V & \xrightarrow{a} & X^n \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle f^n} \\ Y^{n-1} & \longrightarrow & Y^n \end{array}$$

commutes and the cohomology of $L(V) \to L(X^n) \to L(X^{n-1})$ is isomorphic to $H^n(L(Y^\bullet))$. By replacing V with its cover, we may assume that there exists $t' : V \to Z^{n-2}$ such that $L(t') = s^{n-1} \circ L(h)$. We have $L(g^{n-1} \circ h - d_Z^{n-2} \circ t' - t^n \circ a) = L(g^{n-1} \circ h) - L(d_Z^{n-2}) \circ s^{n-1} \circ L(h) - s^n \circ L(f^n) \circ L(a) = L(g^{n-1}) \circ L(h) - L(d_Z^{n-2}) \circ s^{n-1} \circ L(h) - s^n \circ L(d_Y^{n-1}) \circ L(h) = 0$. Hence by Lemma A.12, by replacing V with its cover, we may assume that $g^{n-1} \circ h - d_Z^{n-2} \circ t' - t^n \circ a = 0$.

As in the above proof, we can construct $g : G \to Y^{n-1}$ and $\eta : G \to Z^{n-1}(L(Y^\bullet))$ such that the composition $G \xrightarrow{\eta} Z^{n-1}(L(Y^\bullet)) \to H^{n-1}(L(Y^\bullet))$ is a cover of $H^{n-1}(L(Y^\bullet))$ and $L(G) \xrightarrow{L(g)} L(Y^{n-1})$ coincides with $L(G) \xrightarrow{\alpha(\eta)} Z^{n-1}(L(Y^\bullet)) \to$

$L(Y^{n-1})$. By replacing G with its cover, we may assume that there is $t'' : G \to Z^{n-2}$ such that $L(t'') : L(G) \to L(Z^{n-2})$ coincides with $L(G) \xrightarrow{L(g)} L(Y^{n-1}) \xrightarrow{s^{n-1}} L(Z^{n-2})$. Set $X^{n-1} = V \oplus G$. Define $d_{X^\cdot}^{n-1} : X^{n-1} \to X^n$ by $a : V \to X^n$ and zero on G. Define $f^{n-1} : X^{n-1} \to Y^{n-1}$ by $h : V \to Y^{n-1}$ and $g : G \to Y^{n-1}$. Define $t^{n-1} : X^{n-1} \to Z^{n-2}$ by $t' : V \to Z^{n-2}$ and $t'' : G \to Z^{n-2}$. Then, $H^n(L(X^\bullet)) \to H^n(L(Y^\bullet))$ is an isomorphism and $H^{n-1}(L(X^\bullet)) \to H^{n-1}(L(Y^\bullet))$ is an epimorphism. We have also $g^{n-1} \circ f^{n-1} = d_Z^{n-2} \circ t^{n-1} + t^n \circ d_X^{n-1}$, $L(f^{n-1}) = \alpha(\varphi^{n-1})$ and $L(t^{n-1}) = s^{n-1} \circ L(f^{n-1})$. Hence the induction proceeds. This completes the proof of Proposition A.4.

Bibliography

[An] E. Andronikof, *Microlocalisation tempérée*, Mém. Soc. Math. France t. 57 (1994)

[Bj] J-E. Björk, *Analytic $\mathcal{D}$-modules and Applications*, Kluwer Academic Publisher, Dordrecht-Boston-London (1993)

[Co] V. Colin, *Spécialisation du foncteur de Whitney.* C. R. Acad. Sci. t. 322 (1996)

[D'A-S$_1$] A. D'Agnolo and P. Schapira, *The Radon-Penrose transform for $\mathcal{D}$-modules*, Journal of Functional Analysis, 139, 349–382 (1996)

[D'A-S$_2$] A. D'Agnolo and P. Schapira, *Leray's quantization of projective duality*, Duke Math. J. 84, 453–496 (1996)

[Du] A. Dufresnoy, *Sur l'opérateur d'' et les fonctions différentiables au sens de Whitney*, Ann. Inst. Fourier Grenoble t. 29, 229–238 (1979)

[Gr$_1$] A. Grothendieck, *Produits Tensoriels Topologiques et Espaces Nucléaires*, Mem. Amer. Math. Soc. 16 (1955)

[Gr$_2$] A. Grothendieck, *Local Cohomology*, Lecture Notes in Math. 41, Springer-Verlag (1967)

[Hö] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Grundlehren Math. Wiss. 256, Springer-Verlag (1983)

[Ho] C. Houzel, *Espaces analytiques relatifs et théorème de finitude*, Math. Annalen, vol. 205, 13–54 (1973)

[Ka$_1$] M. Kashiwara, *On the maximally overdetermined system of linear differential equations I*, Publ. RIMS Kyoto Univ. vol. 10, 563–579 (1975)

[Ka$_2$] M. Kashiwara, *B-functions and holonomic systems*, Inventiones Math. vol. 38, 33–53 (1976)

[Ka$_3$] M. Kashiwara, *The Riemann-Hilbert problem for holonomic systems*, Publ. RIMS, Kyoto Univ. vol. 20, 319 - 365 (1984)

[K-S] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grundlehren Math. Wiss. 292, Springer-Verlag (1990)

[K-Sm] M. Kashiwara and W. Schmid, *Quasi-equivariant $\mathscr{D}$-modules, equivariant derived category, and representations of reductive groups*, in *Lie theory and Geometry in honor of Bertram Kostant*, Progress in Mathematics, 123, 457–488 (1994)

[Lo] S. Lojaciewicz, *Sur le problème de la division*, Studia Math. vol. 8, 87-136 (1959)

[Ma] B. Malgrange, *Ideals of Differentiable Functions*, Tata Institute of Fundamental Research, Oxford Univ. Press (1966)

[Mr] A. Martineau, *Distributions et valeurs au bord des fonctions holomorphes*, Proceeding Int. Summer School Lisbon (1964)

[R-R] J-P. Ramis and G. Ruget, *Residus et dualité*, Inventiones Math. 26, 89-131 (1974)

[S-K-K] M. Sato, T. Kawai and M. Kashiwara, *Hyperfunctions and pseudo-differential equations*, in *Proceedings Katata 1971*, Lecture Notes in Math. Springer-Verlag 287, 265–529 (1973)

[S-Sc] P. Schapira and J.-P. Schneiders, *Index theorem for elliptic pairs I: Relative finiteness and duality*, Astérisque, Soc. Math. France (1994)

[Sc] J.-P. Schneiders, *Un théorème de dualité pour les modules différentiels*, C. R. Acad. Sci., vol. 303, 235–238 (1986)