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SET–THEORETIC GENERATION OF IDEALS

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(Dedicated to Professor P. Samuel)

SUMMARY

We study the problem of whether a given surface in affine space is a set–theoretic complete intersection. We show, in particular, that surfaces which are birational to a product of curves are set–theoretic complete intersections.

RESUME

On étudie le problème de savoir si une surface donnée dans un espace affine est une intersection complète ensembliste. On démontre en particulier qu'une surface birationellement équivalente à un produit de courbes est une telle intersection.

§0. Introduction.

In this paper, we study set–theoretic generators of ideals in affine algebras. We will be working over an algebraically closed field k. We will prove a sufficient condition for a smooth surface X to be a set–theoretic complete intersection in $\mathbb{A}^n$ $n \geq 5$). This condition is trivially satisfied by a birationally ruled surface. We will show that this condition is satisfied by surfaces birational to product of curves. Spencer Bloch has recently shown to me that this condition is also satisfied by surfaces birational to abelian surfaces.

Another problem we attempt in this article is whether a codimension one subvariety of a smooth affine variety X of dimension n is set–theoretically defined by $n-1$ equations. The main interest in this problem, at least for the author, is that if this were not so, then one can find stably trivial non–trivial bundles of rank $n-1$ on such varieties. To see why this case is interesting, the reader may see [3]. Of course, the problem is easy when $n=1$ or 2. The real difficulty is from $n=3$. We will show that when $n \geq 3$, a subvariety as above is set–theoretically the zeroes of a section of a stably free, rank $n-1$ module. For a precise statement, see Theorem 2.

I thank Professors M. Raynaud and L. Szpiro for including me in the Samuel Colloquium. I thank Professor M.P. Murthy for many discussions on the subject matter of this article and Professor Spencer Bloch for showing me how my results apply to the case of surfaces birational to abelian surfaces as well.

§1. Surfaces.

Let $X \subset \mathbb{A}^n$ be a smooth affine surface. Let A denote the coordinate ring of X. Let $P =$ the conormal module of X in $\mathbb{A}^n$.

THEOREM (Boratynski [1]) *$X \subset \mathbb{A}^n$ is a set-theoretic complete intersection if and only if the ideal $S_+(P)$ = positively graded elements in $R = S(P)$, the symmetric algebra of P over A, is a set-theoretic complete intersection in R.*

We say that A satisfies $(\star)$ if for any $z \in A_0(A) =$ zero–cycles modulo rational equivalence, there exists $L_1,\dots,L_n \in \operatorname{Pic} A$ such that $z = \Sigma_{i=1}^n (L_i.L_i)$, where $(L.L)$ denotes the intersection product in the Chow–ring.

THEOREM 1. *Let A be the co-ordinate ring of a smooth surface. Let P be any A-projective module with rang $P \geq 3$. Let $R = S(P)$ = symmetric algebra of P over A and $I = S_+(P)$, the ideal of positively graded elements. If A satisfies $(\star)$, then I is a set-theoretic complete intersection in R.*

To prove this theorem, we introduce the notion of modifications. Let the notation be as in the theorem. A projective module Q over A is said to be a modification of P, written $Q[P]$, if

i) rank $Q =$ rank P,

ii) there exists an A–algebra homomorphism $f : S(Q) \longrightarrow S(P)$, such that $\mathrm{rad}(f(S_+Q)) = S_+(P)$.

REMARKS :

i) If $Q_2[Q_1]$ and $Q_3[Q_2]$ then $Q_3[Q_1]$.

ii) If $P \approx Q \oplus L$ where $L \in \mathrm{Pic}\, A$ then $(Q \oplus L^m)[P]$ for any $m \geq 1$.

The first remark is obvious and the second remark follows, once we use the natural map $S(L^m) \longrightarrow S(L)$ for any $m \geq 1$.

PROOF OF THE THEOREM : We need only to show that P can be modified to a free module. Let $L = \det P$. Since $\dim A = 2$ and rank $P \geq 3$, by Serre's theorem [9], there exists a projective module Q such that $P \approx Q \oplus L^{-1}$. Then $\det Q = L^{\otimes 2}$. By remark ii), $Q \oplus L^{-\otimes 2}$ is a modification of P. Also $\det(Q \oplus L^{-\otimes 2}) = A$. Thus we may assume that $\det P = A$. Let $c_2(P) \in A_0(A)$ be the second chern class of P. $A_0(A)$ is divisible [see e.g. [6], Lemma 2.3]. So we may write $c_2(P) = 3z$. Since A satisfies $(\star)$, we may write $z = \Sigma_{i=1}^{n}(L_i.L_i)$ with $L_i \in \mathrm{Pic}\, A$. Now, the proof is by induction on n. If $n = 0$, then $z = 0$ and by [5], P is free.

We will show that P can be modified to a projective module P' with $\det P' = A$ and $c_2(P) = 3z'$, where $z' = \Sigma_{i=1}^{n-1}(L_i.L_i)$. This will complete the proof.

For notational simplicity let $M = L_n$. As before we may write $P = P_1 \oplus M$. Let c denote the total chern class. Then we have

a) $c(p) = c(P_1).(1+c_1(M))$.

By Remark ii), $P_1 \oplus M^{\otimes 2}$ is a modification of P. Again we may write $P_1 \oplus M^{\otimes 2} = P_2 \oplus M^{-\otimes 1}$. Then we have

b) $c(P_1).(1+2c_1(M)) = c(P_2).(1-c_1(M))$.

Again by Remark ii), $P_2 \oplus M^{\otimes 2}$ is a modification of $P_1 \oplus M^{\otimes 2}$ and hence by Remark i), a modification of P. Using a) and b) we may compute $c(P_2 \oplus M^{\otimes 2})$ and then we will get

$$c(P_2 \oplus M^{\otimes 2}) = 2+3z-3(M.M).$$

Thus $P' = P_2 \oplus M^{\otimes 2}$ has all the properties we wanted to achieve. This finishes the proof of the theorem.

COROLLARY 1. (Murthy) *If $X \subset \mathbb{A}^n$, X a smooth surface which is birationally ruled, then X is a set-theoretic complete intersection.*

PROOF : For $n \leq 4$ see [4].

PROPOSITION. *If A is birational to a product of curves then A satisfies* $(\star)$.

PROOF : Let A be birational to $C_1 \times C_2$ where C_i are smooth projective curves. We may also assume that C_i's have positive genus ; if not A is birationally ruled and so A satisfies $(\star)$ trivially. Let Y be a smooth projective completion of $X = \operatorname{Spec} A$. Then we have a birational morphism $\pi : Y \longrightarrow C_1 \times C_2$, by uniqueness of minimal models. Let Z denote the union of exceptional curves of Y. Then Z is the union of rational curves. So the natural map $A_0(X) \longrightarrow A_0(X-Z)$ is an isomorphism. Also $\operatorname{Pic} X \longrightarrow \operatorname{Pic}(X-Z)$ is a surjection. Thus we need only prove $(\star)$ for X an affine open subset of $C_1 \times C_2$.

Now, since $A_0(X)$ is divisible, we may write any zero cycle $z = 2t$. Also, since X is affine, we may write t as a sum of points of X. So it suffices to prove that for any point $p \in X$, $2p = (L.L)$ in $A_0(X)$ where $L \in \operatorname{Pic} X$. Write $p = (p_1,p_2) \in C_1 \times C_2$. Then $M_1 = p_1 \times C_2$ and $M_2 = C_1 \times p_2$ are divisors on $C_1 \times C_2$. $(M_1.M_2) = p$ and $(M_i.M_i) = 0$ for $i = 1,2$ in $A_0(C_1 \times C_2)$. Then $(M_1 \otimes M_2.M_1 \otimes M_2) = 2p$ in $A_0(C_1 \times C_2)$. Restricting $M_1 \otimes M_2$ to X, we get the desired result.

COROLLARY 2. *If $X \subset \mathbb{A}^n$, is a smooth surface birational to a product of curves then X is a set-theoretic complete intersection.*

PROOF : When $n \leq 4$, this was proved by M.P. Murthy [4].

REMARK. Spencer Bloch has shown me that if X is a smooth affine surface birational to an abelian surface, then it satisfies $(\star)$. So our theorem applies and it is also a set–theoretic complete intersection.

§2. Divisors.

This section grew out of an attempt to decide whether stably trivial modules over a 3–fold are trivial or not. Unfortunately, the following theorem that I prove is inconclusive.

For a module M, $\mu(M)$ will denote the minimum number of generators of M.

THEOREM 2. *Let* $Y \subset X = \operatorname{Spec} A$ *be a divisor on a smooth variety* X *of dimension* n *over an algebraically closed field. Assume* $n \geq 3$. *Let* I *be the defining ideal of* Y *in* X. *Then there exists an ideal* $I' \subset I$ *such that*

i) rad I' = rad I;

ii) $\mu(I'/I'^2) \leq n-1$;

iii) *if* $n = 3$, *there exists a stably trivial module of rank* 2 *mapping onto* I';

iv) *if all stably trivial* (rank 2) *modules on all affine 3–folds over an algebraically closed field are trivial then we have an* I' *satisfying* i) *above with* $\mu(I') = n-1$, *for any* $n \geq 3$.

PROOF : We will first prove the theorem in the crucial case of $n = 3$. The proof is a judicious application of Ferrand construction [7].

To avoid confusion, let L denote the element in Pic A corresponding to the divisor Y. That is, L is a module isomorphic to I. Choose a general homomorphism $f : L \longrightarrow A$ so that, $J' = f(L)+I$ is a local complete intersection ideal of height 2. Thus, we have the following Koszul resolution for J' :

$$(\star) \qquad 0 \longrightarrow L^2 \longrightarrow L \oplus I \longrightarrow J' \longrightarrow 0$$

[L^n denotes $L \otimes \ldots \otimes L$, n times].

Since J' is a local complete intersection ideal of height 2, J'/J'^2 is a projective module of rank 2 over the one–dimensional ring A/J'. So by Serre's theorem [9], we can find a surjective homomorphism, $J'/J'^2 \longrightarrow L^{-6} \otimes A/J'$. Thus we have an exact sequence,

$$\text{(a)} \qquad 0 \longrightarrow K/J'^2 \longrightarrow J'/J'^2 \longrightarrow L^{-6} \otimes A/J' \longrightarrow 0$$

where $J'^2 \subset K \subset J'$, K an ideal of A. It is easy to check that K is also a local complete intersection ideal of height 2. So by the above reasoning, we can get another exact sequence

$$\text{(b)} \qquad 0 \longrightarrow J/K^2 \longrightarrow K/K^2 \longrightarrow A/K \longrightarrow 0\,.$$

Again J is a local complete intersection ideal of height 2 with $K^2 \subset J \subset K$. So rad $J =$ rad $K =$ rad $J' \supset I$.

Claim : $\mathrm{Ext}^1_A(J,L^{-4}) \simeq A/J$.

Since J is a local complete intersection ideal of height 2, by local checking, one can see that $\mathrm{Ext}^1_A(J,L^{-4})$ is a projective module of rank one over A/J. So to prove the claim it suffices to prove that $\mathrm{Ext}^1_A(J,L^{-4}) \otimes A/J' \simeq A/J'$ since rad $J =$ rad J'. One has

$$\mathrm{Ext}^1(J,L^{-4}) \simeq \overset{2}{\Lambda}(\mathrm{Hom}(J/J^2,A/J)) \otimes L^{-4}\,.$$

[See e.g. [10]]. Since one has a natural filtration

$$0 \longrightarrow K^2/KJ \longrightarrow J/KJ \longrightarrow J/K^2 \longrightarrow 0\,,$$

and J/JK is a projective module of rank 2 over A/K, we see that,

$$\overset{2}{\Lambda}(J/J^2) \otimes A/K \simeq J/K^2 \otimes K^2/KJ\,.$$

But

$$K^2/KJ \simeq K/J \otimes K/J \approx A/K \otimes A/K \approx A/K$$

from (b). Thus

$$\overset{2}{\Lambda}(J/J^2) \otimes A/K \approx J/K^2 \otimes A/K \approx \overset{2}{\Lambda}(K/K^2)$$

from (b). A similar computation done with (a) will yield,

$$\overset{2}{\Lambda}(K/K^2) \otimes A/J' \simeq \overset{2}{\Lambda}(J'/J'^2) \otimes L^{-6}\,.$$

Putting these together, one will get

$$\begin{aligned}\mathrm{Ext}^1(J,L^{-4})\otimes A/J' &\simeq \overset{2}{\Lambda}(K/K^2)\otimes L^{-4}\otimes A/J' \\ &\simeq \overset{2}{\Lambda}(J'/J'^2)^*\otimes L^6\otimes L^{-4} \\ &\simeq \overset{2}{\Lambda}(J'/J'^2)^*\otimes L^2 \\ &\simeq \mathrm{Ext}^1(J',L^2).\end{aligned}$$

But $(\star)$ implies $\mathrm{Ext}^1(J,L^2) \simeq A/J'$, proving the claim. Thus, by Serre's construction [8] we get an exact sequence,

$$0 \longrightarrow L^{-4} \longrightarrow P \longrightarrow J \longrightarrow 0$$

where P is an A–projective module of rank 2. Computing the chern classes, one has

$$c_1(P) = L^{-4} \text{ and } c_2(P) = [A/J] = 4[A/J'] = 4(c_1(L).c_1(L)).$$

Thus $c(P) = c(L^{-2}\oplus L^{-2})$. By [2], this implies that P is stably isomorphic to $L^{-2}\oplus L^{-2}$. Tensoring the above exact sequence by L^2 and noting that $L \simeq I$, we get an exact sequence

$$0 \longrightarrow L^{-2} \longrightarrow P\otimes L^2 \longrightarrow I^2J \longrightarrow 0 .$$

If we take $I' = I^2J$, then $\mathrm{rad}\, I' = \mathrm{rad}\, I$, since $\mathrm{rad}\, J \subset I$. Thus we have part iii) of the theorem, as well as part i) for $n = 3$. By [5], $P\otimes L^2\otimes A/I'$ is free and thus we havee ii) for $n = 3$. iv) is now obvious for $n = 3$.

Now, to do the general case, let $\dim A = n > 3$. Chosse a sufficiently general map,

$$\varphi : \bigoplus_{1}^{n-3} L^{-2} \xrightarrow{\varphi} A ,$$

L as before, so that $B = A/\mathrm{Im}\,\varphi$ is a smooth 3–dimensional affine ring and I_1 = image of I in B is a locally principal ideal of B. From the earlier part, we can find an ideal J_1 of B such that there exists an exact sequence of B–modules

(c) $$0 \longrightarrow L^{-4}\otimes B \longrightarrow Q \longrightarrow J \longrightarrow 0$$

with J a local complete intersection ideal of B containing I_1 up to radical and Q a B–projective module of rank 2, stably isomorphic to $(L^{-2}\oplus L^{-2})\otimes B$. Let J = inverse image of J_1 in A and let $I' = I^2.J$. We will show that I' has all the properties asserted in the theorem. Since $\operatorname{rad} J_1 \supset I_1$, it is clear that $\operatorname{rad} I' = \operatorname{rad} I$. By [5],

$$Q \otimes B/I_1 \simeq (L^{-2} \oplus L^{-2}) \otimes B/I_1 .$$

So we may find an element $f \in A$, $f \equiv 1 \pmod{I}$ such that

$$Q \otimes B_f \simeq (L^{-2} \oplus L^{-2}) \otimes B_f .$$

Notice that by our choice of f,

$$I'/I'^2 \simeq I'_f/I'^2_f .$$

The map from $Q \otimes B_f \longrightarrow J_{1f}$ can be lifted to a map $(L^{-2} \oplus L^{-2}) \otimes A_f \xrightarrow{\psi} J_f$. Also $\operatorname{im} \varphi \subset J_f$ and $\operatorname{im} \varphi \otimes A_f + \operatorname{im} \psi = J_f$. So we get a surjective map, $\oplus_1^{n-1} L_f^{-2} \longrightarrow J_f$; thus a surjective map

(d) $$\overset{n-1}{\underset{1}{\oplus}} A_f \longrightarrow I_f^2 . J_f = I'_f .$$

So $\mu(I'/I'^2) = \mu(I'_f/I'^2_f) \le n-1$. This proves ii).

If the hypothesis in iv) were satisfied then we could have chosen $f = 1$. Then (d) implies $I'_f = I'$ is $n-1$ generated. This completes the proof of the theorem.

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