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ON SOME STRUCTURAL DESIGN PROBLEMS

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1. INTRODUCTION

Some structural design problems may be stated as

$$\begin{aligned} (\mathcal{P}) \quad & \text{Inf}_{u \in C} J(u) \\ & \lambda(u) = \lambda_1 \end{aligned}$$

where u is the thickness of the structure, $J(u)$ its weight, $\lambda(u)$ its fundamental frequency of vibration, λ_1 the fundamental frequency of the structure with uniform thickness and C is the convex of constraints on u .

Here we consider the problems (\mathcal{P}) when J and λ have the following properties

$$\begin{aligned} & J \text{ is convex on } C \\ & \lambda \text{ is pseudoconcave on } C. \end{aligned}$$

We shall state necessary and sufficient conditions of optimality for an abstract problem (\mathcal{P}) . And we shall apply these results in structural design. We refer to [4] [5] for the proof of these results and for more details.

2. PROBLEM (\mathcal{P}) : ABSTRACT CASE.

Let E a locally convex Hausdorff space, C a convex of E , J and λ

two real valued functions defined on C . We consider the two following problems

$$(P) \quad \begin{array}{l} \text{Inf}_{u \in C} J(u) \\ \lambda(u) = \lambda_1 \end{array}$$

$$(Q) \quad \begin{array}{l} \text{Inf}_{u \in C} J(u) \\ \lambda(u) \geq \lambda_1 \end{array}$$

where $\lambda_1 = \lambda(u_1)$ $u_1 \in C$.

Proposition 2.1. Assume

$$(2.1) \quad J \text{ pseudoconvex on } C$$

$$(2.2) \quad \lambda \text{ continuous on } C$$

$$(2.3) \quad \exists u_\alpha \in C \text{ such that } \lambda(u_\alpha) < \lambda_1 \text{ and}$$

$$J(u_\alpha) < J(u) \quad \forall u \in C, u \neq u_\alpha$$

then the problems (P) and (Q) are equivalent (i.e. if u is a solution of (P) u is a solution of (Q) and reciprocally).

Let's recall the definition of a pseudoconvex function. J is a pseudoconvex on C if J is Gateaux-differentiable on C and if

$$\forall (u, v) \in C \times C, J'(u) \cdot (v - u) \geq 0 \implies J(v) \geq J(u).$$

The reader can find the properties of pseudoconvex and quasiconvex functions in MANGASARIAN [6].

We make use of these properties for the proof of Proposition 2.1.

Remark. (Q) is a convex problem.

Proposition 2.2. Assume (2.1),

$$(2.4) \quad \lambda \text{ pseudoconcave on } C \quad \text{and}$$

$$(2.5) \quad \exists u_Y \in C \text{ such that } \lambda(u_Y) > \lambda_1 .$$

Then if $\bar{u} \in C$ and if $\lambda(\bar{u}) \geq \lambda_1$, \bar{u} is a solution of (Q) if and only if

$$\exists \bar{n} \leq 0 \text{ such that}$$

$$(2.6) \quad J'(\bar{u}) \cdot (v - \bar{u}) + \bar{n} \lambda'(\bar{u}) \cdot (v - \bar{u}) \geq 0 \quad \forall v \in C$$

$$(2.7) \quad \bar{n}(\lambda(\bar{u}) - \lambda_1) = 0 .$$

The proof of the necessary condition is based on the results of HALKIN [2] and (2.4)-(2.5) assumptions.

The proof of the sufficiency is easy.

Corollary 2.1. Assume (2.1)-(2.3)-(2.4)-(2.5). Let $\bar{u} \in C$, $\lambda(\bar{u}) = \lambda_1$ then \bar{u} is a solution of (S) if and only if

$$\exists \bar{n} < 0 \text{ such that } J'(\bar{u}) \cdot (v - \bar{u}) + \bar{n} \lambda'(\bar{u}) \cdot (v - \bar{u}) \geq 0 \quad \forall v \in C .$$

3. THE PROBLEM (S) IN STRUCTURAL DESIGN.

Let Ω be an open bounded connected set in \mathbb{R}^n let **a** and **b** two functions

$$\mathbf{a} :]0, +\infty[\rightarrow]0, +\infty[\quad \mathcal{C}^2, \text{ concave}$$

$$\mathbf{b} :]0, +\infty[\rightarrow]0, +\infty[\quad \mathcal{C}^2, \text{ convex} .$$

Denote by (\cdot, \cdot) the scalar product of $L^2(\Omega)$

$\langle \cdot, \cdot \rangle$ the bilinear canonical pairing over $H_0^1(\Omega) \times H^{-1}(\Omega)$.

$U = \{u \in L^\infty(\Omega) ; \exists \alpha(u) > 0 \text{ such that } u(x) \geq \alpha(u) \text{ a.e. in } \Omega\}$

$$(3.1) \quad C = \{u \in L^\infty(\Omega) ; 0 < \alpha \leq u(x) \leq \beta \text{ a.e. in } \Omega\} \quad \alpha, \beta \text{ given with } 0 < \alpha < 1 < \beta < +\infty$$

a and **b** the maps from U to U defined by

$$a(u)(x) = \mathbf{a}(u(x)) , \quad b(u)(x) = \mathbf{b}(u(x)) .$$

A_u the isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ defined by

$$\langle A_u w, \phi \rangle = \int_{\Omega} a(u)(x) \nabla w(x) \cdot \nabla \phi(x) \, dx, \quad (w, \phi) \in H_0^1(\Omega)^2$$

B_u the operator of $L^2(\Omega)$ defined by

$$B_u w = b(u) w.$$

3.1. The spectral problem.

It is known that the first eigenvalue $\lambda(u)$ of

$$A_u w = \lambda B_u w$$

is simple and that the associated eigenvector $w(u)$ satisfies

$$w(u; x) \geq 0 \quad \text{a.e. in } \Omega$$

$$\lambda(u) = \inf_{\substack{w \in H_0^1(\Omega) \\ w \neq 0}} \frac{\langle A_u w, w \rangle}{\langle B_u w, w \rangle} = \frac{\langle A_u w(u), w(u) \rangle}{\langle B_u w(u), w(u) \rangle}$$

We suppose that $w(u)$ is normalized in $L^2(\Omega)$.

Using the Implicit Function Theorem we can prove that the function

$$u \in U \longrightarrow \lambda(u)$$

is differentiable on U and that

$$(3.2) \quad \lambda'(u) \cdot v = \frac{\int_{\Omega} \{a'(u) \cdot v(x) |\nabla w(u; x)|^2 - \lambda(u) b'(u) \cdot v(x) w^2(u; x)\} \, dx}{\int_{\Omega} b(u) w^2(u; x) \, dx}.$$

From the concavity of a and $(-b)$, it follows that :

$$(3.3) \quad \lambda(v) \leq \lambda(u) + \lambda'(u) \cdot (v-u) \frac{\int_{\Omega} b(u)(x) w^2(u; x) \, dx}{\int_{\Omega} b(v)(x) w^2(u; x) \, dx} \quad \forall (u, v) \in U \times U.$$

Using this inequality we get

Theorem 3.1. λ is pseudo concave on U . λ is upper semicontinuous on C for the weak* topology of $L^\infty(\Omega)$.

3.2. Statement of the problem and results.

Ω is the form of the structure, u is the thickness and therefore the weight is

$$J(u) = \int_{\Omega} u(x) \, dx .$$

Let u_1 be the function in C defined by

$$u_1(x) = 1 \quad \forall x \in \Omega ,$$

and

$$\lambda_1 = \lambda(u_1) .$$

We shall consider the following optimisation problem :

minimize $J(u)$ subject to $\lambda(u) = \lambda_1$ and $u \in C$ i.e.

$$(P) \quad \begin{array}{l} \text{Inf} \\ \lambda(u) = \lambda_1 \\ u \in C \end{array} J(u) .$$

Let u_{α} be the function defined on C by

$$(3.4) \quad u_{\alpha}(x) = \alpha \quad \forall x \in \Omega .$$

This function satisfies

$$J(u_{\alpha}) < J(u) \quad \forall u \in C , u \neq u_{\alpha} .$$

Introduce the problem

$$(Q) \quad \begin{array}{l} \text{Inf} \\ u \in C \\ \lambda(u) \leq \lambda_1 \end{array} J(u) .$$

Then using the previous results it is easy to prove

Theorem 3.2. Assume

$$(3.5) \quad \lambda(u_{\alpha}) < \lambda_1$$

$$(3.6) \quad \exists u_{\gamma} \in C \text{ such that } \lambda(u_{\gamma}) > \lambda_1$$

then i) (P) and (Q) are equivalent ;

ii) (\mathcal{P}) has at least one solution ;

iii) $\bar{u} \in C$ is a solution of (\mathcal{P}) if $\lambda(\bar{u}) = \lambda_1$ and

$$(3.7) \quad \exists \bar{\eta} < 0 \text{ such that}$$

$$J(v) - J(\bar{u}) + \bar{\eta} \lambda'(u) \cdot (v - \bar{u}) \geq 0 \quad \forall v \in C ;$$

iv) the set $S(\mathcal{P})$ of solutions of (\mathcal{P}) is convex and $\{w(u); u \in S(\mathcal{P})\}$ is reduced to one function.

Remark. The condition (3.7) is equivalent to (3.8) $\exists e > 0$ such that

$$g(\bar{u}; x) \leq e \quad \text{a.e. in } \Omega_\beta = \{x \in \Omega; \bar{u}(x) < \beta\}$$

and

$$g(\bar{u}; x) \geq e \quad \text{a.e. in } \Omega_\alpha = \{x \in \Omega; \bar{u}(x) > \alpha\}$$

where

$$g(\bar{u}; x) = \bar{a}'(u(x)) |\nabla w(u; x)|^2 - \lambda_1 \bar{b}'(u(x)) w^2(u; x) .$$

3.3. Examples.

The previous results can be applied to numerous examples and in particular to

Example 1 $a(u) = u \quad b(u) = 1$

Example 2 $a(u) = u \quad b(u) = u + \delta$ where δ is a constant positive

Example 3 $a(u) = 1 \quad b(u) = \frac{1}{u^2}$.

We refer to [1] and [3] for the motivation of these problems.

In these examples the function u_α defined by (3.4) satisfies (3.5) and the function u_γ defined by

$$u_\gamma(x) \equiv \gamma \quad \text{where} \quad 1 < \gamma \leq \beta$$

satisfies (3.6) and we can apply Theorem 3.2. In particular in Example 3, (3.8) gives $\exists e > 0$ such that

$$\frac{w^2(\bar{u};x)}{\bar{u}^3(x)} \leq \epsilon \quad \text{a.e. in } \Omega_\beta$$

$$\frac{w^2(\bar{u};x)}{\bar{u}^3(x)} \geq \epsilon \quad \text{a.e. in } \Omega_\alpha$$

and this implies that, in this example (\mathcal{G}^0) has only one solution \bar{u} . Moreover if $\Omega =]0,1[$ we can construct this solution.

Remark. We have used the same method to study problems of the following form

$$\begin{aligned} & \text{Sup } \lambda(u) . \\ J(u) &= J_1 \\ & u \in C \end{aligned}$$

REFERENCES.

- [1] J.L.P. ARMAND - Applications of the theory of optimal control of distributed parameter systems to structural optimization. N.A.S.A., C.R. 2044, Juin 1972.
- [2] H. HALKIN - Calculus of variations classical and modern. C.I.M.E., 1966, p.177-192.
- [3] E.J. HAUG, K.C. PAN and T.D. STREETER - A computational method for optimal structural design : II continuous problems. Int. J. num. Math. Eng. 9, 1975, p. 649-667.
- [4] C. JOURON - Analyse théorique et numérique de quelques problèmes d'optimisation intervenant en théorie des structures. Publication Mathématique d'Orsay, n°210-76-62, 1976.
- [5] C. JOURON - Sur un problème d'optimisation où la contrainte porte sur la fréquence fondamentale. Article à paraître à la R.A.I.R.O. série Rouge.
- [6] O.L. MANGASARIAN - Nonlinear Programming. Mc. Graw-Hill, 1969.