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THE REDUCED WITTRING

by

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These notes give a brief account on a joint work of L. Bröcker and the author. Detailed proofs will appear in the Journal of Algebra.

Let K be a real (= formally real) field, $X = X(K)$ the topological space of all orderings of K [5, p.63], and $W(K)$ the Wittring of the nondegenerated bilinearforms over K. By $W_t(K)$ we denote the torsion-subgroup of $W(K)$, which is known also to be its nilradical [6].

Let $C(X,\mathbb{Z})$ be the ring of all continuous functions $X \to \mathbb{Z}$ ($\mathbb{Z}$ provided with the discrete topology). Then we get a homomorphism sign : $W(K) \to C(X,\mathbb{Z})$ defined by $(\mathrm{sign}(\rho))(P) := \mathrm{sign}_P(\rho)$ = signature of ρ at P. The following basic result is due to Pfister [6].

<u>THEOREM 1</u> : <u>The sequence</u> $0 \to W_t(K) \to W(K) \xrightarrow{\mathrm{sign}} C(X,\mathbb{Z})$ <u>is exact</u>.

So it must be considered as a main task in the theory of reduced Wittrings to characterize the elements of sign $W(K)$ among the functions $f \in C(X,\mathbb{Z})$. In order to state the main result the notion of a preorder has to be introduced. A subset $T \subset K$ is called a <u>preorder</u> of K iff the following conditions are satisfied :

i) $T + T \subset K$, $T.T \subset K$ ii) $K^2 \subset T$ iii) $T \cap -T = \{0\}$.

A preorder T is the intersection of all orderings in which it is contained. Given a preorder T, the subset $X_T := \{P \supset T \mid P \text{ ordering of } K\}$ is a closed subspace of X. Clearly, we have the restriction-homomorphism Res : $C(X,Z) \to C(X_T,\mathbb{Z})$. Denote by $W_T(K)$ the image of Res.sign : $W(K) \to C(X_T,\mathbb{Z})$. Choose any ordering $P_o \supset T$. Set $P_o^x = P \setminus \{0\}$, $T^x = T \setminus \{0\}$; P_o^x and T^x are subgroups of K^x. As with $W(K)$, we find an epimorphism $\mathbb{Z}[P_o^x/T^x] \to W_T(K)$. Furthermore the mapping $X_T \hookrightarrow \mathrm{char}\,(P_o^x/T^x)$, $P \mapsto (aT^x \mapsto \mathrm{sgn}_P(a))$ is a topological embedding of X_T into the (Pontrjagin-) character-group of P_o^x/T^x.

<u>PROPOSITION</u>. <u>For a preorder</u> T <u>the following statements are equivalent</u> :

i) $\mathbb{Z}[P_o^x/T^x] \to W_T(K)$ <u>is an isomorphism</u>,

ii) $X_T \to \mathrm{char}(P_o^x/T^x)$ <u>is a homeomorphism</u>,

iii) $T + Ta = T \cup Ta$ <u>for all</u> $a \in K$, <u>such that</u> $a \notin -T$.

A preorder which satisfies the equivalent conditions of the last proposition, is called a <u>fan</u> (in French : éventail). Fans turn out to be of great importance in other contexts, too [1], [3].

THEOREM 2. <u>A function</u> $f \in C(X,\mathbb{Z})$ <u>lies in sign</u> $W(K)$ <u>iff</u>

$$\sum_{P \supset T} f(P) \equiv 0 \mod \tfrac{1}{2}\,(K^{x} : T^{x})$$

<u>for all fans</u> T <u>with</u> $(K^{x} : T^{x}) < \infty$.

The description of sign $W(K)$ in $C(X,\mathbb{Z})$ was also attacked by R. Brown [4] and settled for the case that K admits only finitely many real places. For the general case he was led to a conjecture which (in his terminology) states that all formally real fields are exact. From theorem 2 one can derive:

THEOREM 3. <u>All formally real fieds are exact</u>.

The proof of theorem 2 heavily depends on two local-global principles for reduced quadratic forms, one of which has essentially been proved in [2]. Furthermore, the generalized theory of reduced Wittrings [1] is extensively used, i.e. $W(K)$ is factorized by forms $\langle 1, -t \rangle$, where $0 \neq t$ and t belongs to an arbitrary but fixed preorder T. This point of view turns out to be fundamental even for the study of ordinary reduced Wittrings.

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