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ON GENERALIZED p-ADIC INTEGRATION

Arnt VOLKENBORN

Introduction

Integration of p-adic valued functions has first been considered by F. Tomás [9] and F. Bruhat [2]. A function f defined on the valuation ring $\mathbb{Z}_p$ of the rational p-adic field $\mathbb{Q}_p$ with values in $\mathbb{Q}_p$ is integrable in this sense, if there exists a p-adic number I such that for all $\mathcal{E} > 0$, all great m and all representative systems $\{b_i\}$ of Z_p/P^m (P is the prime ideal in $\mathbb{Z}_p$)

$$(+) \qquad \left|\sum_{i=1}^{p^m} \frac{1}{p^m} f(b_i) - I\right| < \mathcal{E} .$$

This integral being translation-invariant is far too restrictive for analytic and arithmetical purposes : Integrable functions in this sense must be differentiable with derivative zero. Now Leopoldt-Kubota's definition of p-adic L-functions as a mean value indicates the great arithmetical importance of non-translation-invariant p-adic integration. It turns out that a general concept and theory of generalized p-adic integration not only leads to a simpler presentation of some of the properties of p-adic L-functions and Leopoldt's Bernoulli numbers, but serves as a conveniant tool in the investigation and definition of new p-adic analytic functions similar - but of course not to the same extent - to classical function theory. To indicate the concepts and some results of such a theory is the purpose of this note.

The first step in generalizing the Tomás - Bruhat - integral consists in restricting the approximation (+) to a fixed canonical representative system (for each m) only. But in order to write the p-adic L-functions as an integral in a proper way one has to generalize (+) a second time : the completeness of the argument groups of the functions has to be omitted. This is the reason why we start with the notion of an integral for certain functions on a class of topological groups with values in complete topological fields. This leads in particular to the "m-th p-adic integral" on $\mathbb{Q}_p$ with values in an algebraically closed and complete extension $\tilde{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, which makes all locally-analytic functions (e.g. all Laurent-series) integrable. Specialization to the "1-st p-adic integral" then gives rise to ordinary p-adic integration.

A) P-adic integration.

1. Integration for certain classes of topological groups with values in complete topological fields.

Let Ω be a complete topological field.

Definition 1. $(G,(U_n)_{n\geq 0})$ is called a group of finite type, if G is a topological Hausdorff - group and $(U_n)_{n\geq 0}$ is a countable base of open precompact subgroups U_n such that $U_n \supset U_{n+1}$ for all n.

We fix a sequence $\mathcal{R} = (\mathcal{R}_n)_{n\geq 0}$ of representative systems $\mathcal{R}_n$ of G/U_n and regard functions f defined on open, precompact subsets $K \subset G$ of the form

$$K = \dot{\bigcup_{i=1}^{k}} x_i U_m \qquad (x_i \in K \cap \mathcal{R}_m \,,\; m \geq 0)$$

Definition 2. A function $f : K \longrightarrow \Omega$ is called $\mathcal{R}$-integrable on K (or a $\mathcal{R}$-regulated function on K) if the sequence of Riemann sums

$$(1) \qquad \Big(\frac{1}{(U_o : U_n)} \sum_{x\in \mathcal{R}_n \cap K} f(x)\Big)_n$$

is convergent in Ω. Then we call

$$(1') \qquad {}^{\mathcal{R}}\!\!\int_K f(u)du = \lim_{n\to\infty} \frac{1}{(U_o : U_n)} \sum_{x\in \mathcal{R}_n \cap K} f(x)$$

the $\mathcal{R}$-integral of f on K.

We have the following property :

If f is $\mathcal{R}$-integrable on three of the four open precompact sets K, K', $K\cup K'$, $K\cap K'$, so f is $\mathcal{R}$-integrable on the fourth, and we get

$$\mathcal{R}\!\!\int_{K\cup K'} f(u)du + \mathcal{R}\!\!\int_{K\cap K'} f(u)du = \mathcal{R}\!\!\int_{K} f(u)du + \mathcal{R}\!\!\int_{K'} f(u)du.$$

2. <u>The m-th and the ordinary p-adic integral</u>

We specialize this integral to the m-th p-adic integral (m a positive integer) by setting

$$G = \mathbb{Q}^+ \ , \ U_o = \mathbb{Z}^+ \ , \ U_n = m\,p^n\,\mathbb{Z} \qquad\qquad (n \in \mathbb{N})$$

with a prime p and $\Omega = \tilde{\mathbb{Q}}_p$ an algebraically closed complete extension of the field of the p-adic numbers $\mathbb{Q}_p$. The integral then is based on the following canonical representative system of $\mathbb{Q}/U_n$:

$$R_{m,n} = \dot{\bigcup_{i=0}^{m-1}} (R_n + i\,p^n) \ , \tag{2}$$

$$R_n = \left\{ \sum_{i=r}^{n-1} b_i\,p^i \ \middle|\ 0 \leqslant b_i < p,\ n > r,\ r \in \mathbb{Z} \right\} .$$

As the functions f which we want to integrate are defined on (or can be extended to) compact open subsets K of $\mathbb{Q}_p$, we define the <u>m-th p-adic integral of</u> f <u>on</u> K <u>to be</u>

$$\int_K^{(m)} f(u)du = \lim_{n\to\infty} \frac{1}{mp^n} \sum_{x\in R_n\cap K} f(x) \tag{3}$$

provided the limit exists in $\tilde{\mathbb{Q}}_p$ (in which case f is called m-<u>integrable on</u> K). In the case m=1 we speak of ordinary p-<u>adic integral</u>

$$\int_K f(u)du = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x\in R_n\cap K} f(x) \tag{4}$$

and <u>integrable functions</u> on K, evidently this case can be subsumed under 1., also by taking $G = \mathbb{Q}_p^+$, $U_n = p^n\,\mathbb{Z}_p$ and $\mathcal{R} = (R_n)_{n\geqslant 0}$.

It can be shown easily that m-th integration can be reduced to ordinary integration :

Theorem 1.

Let $f : \mathbb{Z}_p \longrightarrow \acute{Q}_p$ *be a function on the* p-*adic integers*, $m \in \mathbb{N}$. *If the* m *functions* $g_i(u) = f(mu+i)$ $(i=0,\ldots,m-1)$ *are integrable on* $\mathbb{Z}_p$, *then* f *is* m-*integrable on* K, *and the following equation holds* :

$$(5) \qquad \int_{\mathbb{Z}_p}^{(m)} f(u)du = \frac{1}{m} \sum_{i=0}^{m-1} \int_{\mathbb{Z}_p} f(mu+i)du.$$

Thus it is sufficient to investigate the properties of the ordinary p-adic integral. First we note the transformation properties :

Theorem 2.

Let $P = p\,\mathbb{Z}_p$ *be the prime ideal of* $\mathbb{Z}_p$.

a) *If* $f : P^t \longrightarrow \tilde{Q}_p$ $(t \in \mathbb{Z})$ *is integrable on* P^t , *then*

$$(6) \qquad \int_{P^t} f(u)du = \frac{1}{p^t} \int_{\mathbb{Z}_p} f(p^t u)du.$$

b) *Let* ℓ, m *be integers with* $\ell < m$. *If a function* $f : \mathbb{Z}_p \longrightarrow Q_p$ *is differentiable in all integers* i *with* $\ell \leqslant i < m$, *and if* $f(u+r)$ *is integrable on* $\mathbb{Z}_p$ *for some* r $(\ell \leqslant r \leqslant m)$, *then* $f(u+j)$ *is integrable on* $\mathbb{Z}_p$ *for all* $\ell \leqslant j \leqslant m$, *and the following equation holds* :

$$(7) \qquad \int_{\mathbb{Z}_p} f(u+m)du = \int_{\mathbb{Z}_p} f(u+\ell)du + \sum_{i=\ell}^{m-1} f'(i).$$

The last property gives a weak translation-invariance of the integral, if the functions are differentiable in all rational integers with vanishing derivative.

The main results are the following two theorems:

Theorem 3.

All locally - analytic p-adic functions on compact open sets $K \subset \mathbb{Q}_p$ are integrable. In particular Laurent-series

(8) $$f = \sum_{k=-\infty}^{\infty} a_k X^k \qquad (a_k \in \tilde{\mathbb{Q}}_p)$$

converging on a circle $|u| = p^{-t}$ $(t \in \mathbb{Z})$ can be integrated termwise :

(9) $$\int_{|u|=p^{-t}} f(u)du = \sum_{k=-\infty}^{\infty} a_k \int_{|u|=p^{-t}} u^k du .$$

The p-adic numbers

$$\int_{|u|=p^{-t}} u^k du \qquad (k \in \mathbb{Z})$$

involved in the series are essentially the generalized ordinary p-Bernoulli numbers (see part B!). This theorem relies heavily on the following estimation of the integral.

Following Schöbe [8] we define an ultranorm on the space L_t of all Laurent-series (8) converging on $|u|=p^{-t}$ $(t \in \mathbb{Z})$ by

$$\|f\|_{p^{-t}} = \sup_{k\in\mathbb{Z}} |a_k| \, p^{-kt} ;$$

then L_t is a Banach-space.

Theorem 4.

For all functions $f \in L_t$ there holds the estimation

(10) $$\left| \int_{|u|=p^{-t}} f(u)du \right| \leq p^{t+1} \|f\|_{p^{-t}} .$$

Hence the integral on $|u|=p^{-t}$ is a continuous linear functional on the Banach-space L_t .

Moreover, for power series which converge on P^t one has the following approximation of the integral by the Riemann sums :

$$(11) \qquad \left| \int_{P^t} f(u)du - \frac{1}{p^n} \sum_{x \in R_n \cap P^t} f(x) \right| \leqslant p^{2(t+1)-n} \|f\|_{p^{-t}} .$$

B. Applications.

1. Generalized Bernoulli numbers

Leopoldt's Bernoulli numbers introduced in [6] can be written with the aid of the m-th integral : For each congruence character χ with conductor m_χ the k-th Leopoldt-Bernoulli number can be defined by

$$(12) \qquad B_\chi^k = \int_{\mathbb{Z}_p}^{(hm_\chi)} \chi(u)\, u^k\, du \qquad (k \geqslant 0) ,$$

where h is an arbitrary natural number. Here the function χ is extended to a function on $\mathbb{Z}_p$ by setting $\chi(x) = 0$, if $(x,m_\chi) \neq 1$ or $x \in \mathbb{Z}_p \setminus \mathbb{Z}$.

Modified p-Leopoldt-Bernoulli numbers can also be obtained for negative integers by the definition

$$(13) \qquad B_\chi^{p,k} = \int_{|u|=1}^{(\overline{m}_\chi)} \chi(u)\, u^k\, du \qquad (k \in \mathbb{Z}) ,$$

where $\overline{m}_\chi = [m_\chi, q]$ with $q = p$ or 4 according to $p \neq 2$ or $p = 2$. For $k \geqslant 0$ they are essentially the Leopoldt-Bernoulli numbers :

$$(14) \qquad B_\chi^{p,k} = (1 - \chi(p)\, p^{k-1})\, B_\chi^k \qquad (k \geqslant 0).$$

For all $k \in \mathbb{Z}$ one has the p-adic series

$$(15) \qquad B_\chi^{p,k} = \sum_{s=0}^{\infty} \binom{k}{s} E_\chi^{k-s} B^s (\overline{m}_\chi)^{s-1}$$

with

$$E_\chi^{k-s} = \sum_{\substack{r=1 \\ (r,p)=1}}^{\overline{m}-1} \chi(r)\, r^{k-s}$$

and the ordinary Bernoulli numbers

$$B^s = \int_{\mathbb{Z}_p} u^s\, du .$$

Using the properties of the integral, especially the analogue of theorem 4 for the m-th integral, one can derive the Staudt-Clausen theorem and the Kummer congruences for these numbers.

If $\chi = \varepsilon$ is the principal character, we obtain modified ordinary p-Bernoulli numbers by

$$B^{p,k} = \int_{|u|=1} u^k \, du \qquad (k \in \mathbb{Z}).$$

For $k \geqslant 0$ they are essentially the ordinary Bernoulli numbers :

$$B^{p,k} = (1-p^{k-1})\, B^k .$$

For all $k \in \mathbb{Z}$ these numbers have the following p-adic series

$$B^{p,k} = \sum_{s=0}^{\infty} \binom{k}{s} E^{k-s} B^s p^{s-1}$$

with

$$E^{k-s} = \sum_{r=1}^{p-1} r^{k-s} \in \mathbb{Z}_p \cap \mathbb{Q} .$$

As mentioned in theorem 3 the p-Bernoulli numbers occur in explicit integration of Laurent-series (8) :

(16) $$\int_{|u|=p^{-t}} f(u)\, du = \sum_{k=-\infty}^{\infty} a_k\, p^{t(k-1)}\, B^{p,k} .$$

It may be mentioned that a lot of arithmetical identities involving Bernoulli numbers can be derived by using the properties of the p-adic integral.

2. Special functions

Using the m-th p-adic integral we succeed in defining many - well known and new - functions; only a selection can be given here.

a) The logarithm

Leopoldt's definition of the logarithm [7] as the limit

$$\log h = \lim_{n\to\infty} \frac{1}{p^n} (h^{p^n} - 1) \qquad (h \in \tilde{\mathbb{Q}}_p \, ; |h-1| < 1)$$

can also be written as a m-th integral

(17) $$\log h = (h-1) \int_{\mathbb{Z}_p}^{(m)} h^u \, du \qquad \text{for all } m \in \mathbb{N}.$$

All properties of log can then easily be derived from this presentation.

b) Bernoulli power series

Bernoulli polynomials and - power series belonging to a congruence character χ with conductor m_χ can be defined by

$$B_\chi^n(z) = \int_{\mathbb{Z}_p}^{(m_\chi)} \chi(u)\ (u+z)^n\ du \qquad (n \in \mathbb{N}) ,$$

(18)
$$B_\chi(z,s) = \int_{\mathbb{Z}_p}^{(m_\chi)} \chi(u)\ (1+zu)^s\ du ,$$

the latter function being defined for $z,s \in \tilde{\mathbb{Q}}_p$ with

$$|z| < p^{-\frac{1}{p-1}} , \quad |s| \leq 1.$$

We obtain a relation between the two functions for all $n \in \mathbb{N}$ and $z \neq 0$ by

(19)
$$B_\chi(z,n) = z^n\ B_\chi^n(\tfrac{1}{z}) .$$

There are many functional equations, for instance

$$B_\varepsilon(z,s) = (1-z)^s \quad B_\varepsilon(\tfrac{1}{z-1} , s) \quad ;$$

(20)
$$B_\chi(z,s) = \chi(-1) \quad B_\chi(-z,s) \quad \text{for } \chi \neq \varepsilon ,$$

where ε denotes the principal character.

Furthermore it can be shown

Theorem 5. The mapping

$$z \longrightarrow z\ B_\varepsilon(tz,s)$$

is an automorphism of the unit circle $|z| \leq 1$ in $\tilde{\mathbb{Q}}_p$ for all parameters $t \in \tilde{\mathbb{Q}}_p$, $s \in \tilde{\mathbb{Q}}_p$, fullfilling the conditions

$$|t| < p^{-\frac{1}{p-1}} , \quad |s| \leq 1.$$

c. Leopoldt L-functions

T. Kubota and H.W. Leopoldt [5] discovered a p-adic L-function belonging to a character χ with a conductor m . Following [5] we represent a p-adic unit u in the form

$$u = \omega(u) \langle u \rangle \quad \text{with } \langle u \rangle \in (1+q)^{\mathbb{Z}_p}$$

and a character ω of order p-1(resp.2) and conductor q, where q=p or 4 according to $p \neq 2$ or $p = 2$.

With the projection $\langle u \rangle$ on the 1-units the L-function can be written as

$$(21) \qquad L(s,\chi) = \frac{1}{s-1} \int_{|u|=1}^{(\bar{m}_\chi)} \chi(u) \langle u \rangle^{1-s} \, du \qquad (p \neq 2) ,$$

where $\bar{m}_\chi = [m_\chi, q]$. For p = 2 this definition has to be modified slightly.

Many properties of these functions can be obtained by the corresponding properties of the integral, for instance the determination of the values at the rational integer points $s = 1 - k \neq 1$, $k \equiv 0 \bmod p-1$:

$$(22) \qquad L(1-k,\chi) = - \frac{B_\chi^{p,k}}{k} \quad ,$$

where $B_\chi^{p,k}$ are the modified p-Leopoldt-Bernoulli numbers.

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