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# THE NORM OF UNIFORM CONVERGENCE ON THE K-ALGEBRAIC MAXIMAL SPECTRUM OF AN ALGEBRA OVER A NON-ARCHIMEDEAN VALUATION FIELD K.

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In non-archimedean function theory, i.e. the theory of several variables over a field k with a non-archimedean valuation, the concept of the norm of uniform convergence on a k-affinoid set (i.e. the analogue of a $\mathbb{C}$-analytic set in the theory of several complex variables) is an important tool. It can be computed intrinsically in terms of the spectrum of maximal ideals of the corresponding k-affinoid algebra (i.e. the analogue of a $\mathbb{C}$-analytic algebra). The aim of this lecture, which is based on joint work with R. Remmert (Münster), is to define a semi-norm of uniform convergence on a large class of k-algebras A without any norm or analytic structure thereby reducing this concept to a purely algebraic one depending only on the algebraic structure of A and of course on the valuation of k.

The point of departure for our considerations is the valuation theoretic notion "spectral norm" introduced in [2] and studied in detail in [5] in order to handle extension problems from a field to algebraic extension fields (always with applications to k-affinoid algebras in mind). Here this notion is generalized in such a way that also integral extensions A of a k-algebra B may be treated. We study the behaviour of the norm of uniform convergence under "going up and down" from A to B and vice versa. Due to the Noether normalization lemma many problems in the theory of k-affinoid algebras may be reformulated as extension problems, where the smaller algebra is a "free" k-affinoid algebra $T_n$ enjoying many "nice" properties. Therefore we can apply the general theory in order to algebraize and thereby simplify the proofs for some of the fundamental results for k-affinoid algebras. This program had been started for k-Banach algebras instead of general k-algebras in [4], but there some arguments taking advantage of the special structure of k-affinoid algebras had to be used.

1.1. Supremum-semi-norm on k-algebras.- Let k be a field with a non-archimedean non trivial valuation. (We do not suppose k to be complete). Let A be a (commutative) k-algebra. We want to derive a semi-norm on A from the given valuation on k. In order to do so, we use the following

Def. 1.1 : (Spectrum of k-algebraic maximal ideals of A)

$$\mathrm{Max}_k\, A := \{x \;;\; x \text{ maximal ideal in } A \text{ and } A/x \text{ algebraic over } k\}$$

Of course $\mathrm{Max}_k\, A$ may be empty, e.g. if A is a transcendant field extension of k. But in many cases the k-algebraic maximal spectrum of A contains substantial information about A.

For $x \in \mathrm{Max}_k\, A$ and $f \in A$ denote by $f(x)$ the image of f under the canonical residue epimorphism $\pi_x : A \to A/x$. Because $A/x$ is an algebraic extension of k it can be provided with the spectral norm belonging to the given valuation on k. (For the concept « spectral norm » see [5] ; we just mention here, that it coincides with the uniquely determined valuation extension, in case k is complete). The spectral norm is invariant under k-Galois automorphisms and therefore it does not matter how $A/x$ is embedded into the algebraic closure $k_a$ of k. Thus we may speak of $|f(x)|$ , where $f \in A$ and $x \in \mathrm{Max}_k\, A$, and are able to introduce

<u>Def</u>. 1.2 : (<u>Semi-norm of uniform convergence on</u> $\mathrm{Max}_k\, A$ <u>or supremum-semi-norm</u>)

$$|f|_{\sup} := \begin{cases} 0 & \text{if } \mathrm{Max}_k\, A = \emptyset, \\ \sup\{|f(x)| \;;\; x \in \mathrm{Max}_k\, A\} & \text{if } \mathrm{Max}_k\, A \neq \emptyset \text{ and } f(\mathrm{Max}_k\, A) \text{ bounded}, \\ \infty & \text{otherwise.} \end{cases}$$

This definition generalizes the concept « spectral norm » . Namely if A is contained in $k_a$, then this definition obviously yields the spectral norm on A. As we shall see later on (cor. to prop. 1.5) under suitable circonstances $|\ |_{\sup}$ may be interpreted again as a spectral norm, then of course over some bigger ground field. - We have already seen that $\mathrm{Max}_k\, A$ may be empty ; also the third case occuring in def. 2 is possible, e.g. take $A = k[X]$. Then $\mathrm{Max}_k\, A \supset \{(X - c)\, k[X];\ c \in k\}$. If one takes $f := X \in A$, then $f(\mathrm{Max}_k\, A) \supset k$, which is clearly not bounded. We shall say, that the supremum-semi-norm on A is not <u>degenerated</u>, if $\mathrm{Max}_k\, A \neq \emptyset$ and $f(\mathrm{Max}_k A)$ is bounded for all $f \in A$. For the applications we have in mind, it is easy to verify, that $|\ |_{\sup}$ is not degenerated.

First let us collect some rather obvious properties of $|\ |_{\sup}$ :

<u>Lemma</u> 1.1 : <u>If</u> $|\ |_{\sup}$ <u>is not degenerated, it is a power-multiplicative non-archimedian</u> k-<u>algebra semi-norm on</u> A , <u>i.e. one has for all</u> $f,g \in A$ , $c \in k$ , $n \in \mathbb{N}$ :

(a) $|f|_{sup} \in \mathbb{R}$, $|f|_{sup} \geqslant 0$, $|0|_{sup} = 0$,

(b) $|f+g|_{sup} \leqslant \max \{|f|_{sup}, |g|_{sup}\}$,

(c) $|cf|_{sup} = |c|\ |f|_{sup}$,

(d) $|fg|_{sup} \leqslant |f|_{sup}\ |g|_{sup}$,

(e) $|1|_{sup} = 1$,

(f) $|f^n|_{sup} = |f|^n_{sup}$ .

All these formulas - with the exception of (e)-remain true also in the degenerated case, if one extends the usual operations on $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$ in an obvious way.

The supremum semi-norm is compatible with k-algebra homomorphisms in the following sense.

Lemma 1.2 : Let $\varphi : B \to A$ be a k-algebra homomorphism between two k-algebras A and B. Then $\varphi$ is a contraction with respect to $|\ |_{sup}$, i.e. $|\varphi(f)|_{sup} \leqslant |f|_{sup}$ for all $f \in B$.

Proof : If $\mathrm{Max}_k A = \emptyset$, there is nothing to show. If $x \in \mathrm{Max}_k A$, then $\varphi$ induces a k-algebra monomorphism $B/\varphi^{-1}(x) \to A/x$. Because $A/x$ is an algebraic field extension of k, also its subring $B/\varphi^{-1}(x)$ must be an algebraic field extension of k. Hence $\varphi^{-1}(x) \in \mathrm{Max}_k B$. Then one has

$$(*)\quad |\varphi(f)|_{sup} = \sup_{x\in \mathrm{Max}_k A} |\varphi(f)(x)| = \sup_{x\in \mathrm{Max}_k A} |f(\varphi^{-1}(x))| \leqslant \sup_{y\in \mathrm{Max}_k B} |f(y)| = |f|_{sup},$$

Q.E.D.

We apply this lemma to show that the supremum-semi-norm of A can easily be derived from the supremum-semi-norm of its prime components.

Lemma 1.3 : Define $\mathfrak{M} := \{\mathfrak{p} ; \mathfrak{p}$ minimal prime ideal of $A\}$ and let $\pi_{\mathfrak{p}} : A \to A/\mathfrak{p}$ denote the canonical residue map for all $\mathfrak{p} \in \mathfrak{M}$ . Then

$$|f|_{sup} = \sup_{\mathfrak{p}\in\mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{sup} .$$

<u>Proof</u> : According to lemma 2 we know $\sup_{\mathfrak{p}\in\mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\sup} \leq |f|_{\sup}$. In order to show the opposite inequality, take $x \in \mathrm{Max}_k\, A$. Then one can find $\mathfrak{p}\in\mathfrak{M}$ such that $x \supset \mathfrak{p}$. From $A/x = (A/\mathfrak{p})/(x/\mathfrak{p})$ we get $\pi_{\mathfrak{p}}(x) = x/\mathfrak{p} \in \mathrm{Max}_k\, A/\mathfrak{p}$ and $|f(x)| = |(\pi_{\mathfrak{p}}(f))(\pi_{\mathfrak{p}}(x))|$, whence $|f(x)| \leq |\pi_{\mathfrak{p}}(f)|_{\sup} \leq \sup_{\mathfrak{p}\in\mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\sup}$. Since this holds for all $x \in \mathrm{Max}_k\, A$, we have found $|f|_{\sup} \leq \sup_{\mathfrak{p}\in\mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\sup}$, Q.E.D.

For the case of integral monomorphisms Lemma 1.2 can be improved considerably.

<u>Lemma</u> 1.4 : <u>Let</u> $\varphi : B \longrightarrow A$ <u>be an integral</u> k-<u>algebra monomorphism.</u>
<u>Then one has</u> :

(a) $\varphi$ <u>is an isometry with respect to</u> $|\ \ |_{\sup}$

(b) $|f|_{\sup} \leq \max_{i=1}^{n} |b_i|_{\sup}^{1/i}$ <u>for all</u> $f \in A$, <u>where</u> $f^n + \varphi(b_1)\, f^{n-1} + \ldots + \varphi(b_n) = 0$ <u>is an equation of integral dependence of</u> f <u>over</u> $\varphi(B)$,

(c) $|\ \ |_{\sup}$ <u>is not degenerated on</u> A <u>if and only if it is not degenerated on B.</u>

<u>Proof</u> : Ad(a) : If $\mathrm{Max}_k\, B = \emptyset$ then due to lemma 1.2 we have $|\varphi(f)|_{\sup} \leq |f|_{\sup} = 0$ for all $f \in B$. Therefore we may assume $\mathrm{Max}_k\, B \neq \emptyset$. Take $y \in \mathrm{Max}_k\, B$. Because $\varphi$ is integral and injective, there is a maximal ideal x of A lying over y, i.e. $\varphi^{-1}(x) = y$. Again $\varphi$ induces an integral monomorphism from B/y into A/x. B/y is an algebraic field extension of k according to our assumption and A/x is integral over B/y is an algebraic field extension of k. In other words : the map $x \to \varphi^{-1}(x)$ from $\mathrm{Max}_k\, A$ to $\mathrm{Max}_k\, B$ is surjective. Therefore one has equality in the formula (*) occuring in the proof of lemma 1.2 and $\varphi$ is an isometry.

Ad(b) : For all $x \in \mathrm{Max}_k\, A$ one has $0 = f(x)^n + (\varphi(b_1))(x)\, f(x)^{n-1} + \ldots + \varphi(b_n)(x) = f(x)^n + b_1(\varphi^{-1}(x))\, f(x)^{n-1} + \ldots + b_n(\varphi^{-1}(x))$. This equation in $k_a$ implies $|f(x)| \leq \max_{i=1}^{n} |b_i(\varphi^{-1}(x))|^{1/i} \leq \max_{i=1}^{n} |b_i|_{\sup}^{1/i}$. Since this holds for all $x \in \mathrm{Max}_k\, A$ we get $|f|_{\sup} \leq \max_{i=1}^{n} |b_i|_{\sup}^{1/i}$.

Ad(c) : If $b(\mathrm{Max}_k\, B)$ is bounded for all $b \in B$, then due to (b) also $f(\mathrm{Max}_k\, A)$ is bounded for all $f \in A$. The converse is true due to (a). Furthermore from the proof (a) we see that $\mathrm{Max}_k\, B = \emptyset$ if and only if $\mathrm{Max}_k\, A = \emptyset$ , Q.E.D.

In order to be able to transform the inequality (b) into an equality, which then allows to compute $|\ |_{\sup}$ on A in terms of $|\ |_{\sup}$ on the smaller algebra B, we must impose some additional assumptions on A and B.

Prop. 1.5 : Let $\varphi : B \to A$ be an integral torsionfree k-algebra monomorphism between two k-algebras A and B, where A is reduced and B is an integrally closed domain. Then one has :

(a) $|f|_{\sup} = \max_{i=1}^{n} |b_i|_{\sup}^{1/i}$ for all $f \in A$, where $f^n + \varphi(b_1) f^{n-1} + \ldots + \varphi(b_n) = 0$ is an equation of integral dependence for f over $\varphi(B)$ of minimal degree.

(b) The maximum modulus principle holds for A (i.e. for all $f \in A$ there is an $x \in \mathrm{Max}_k A$ such that $|f(x)| = |f|_{\sup}$), if an only if it holds for B.

(c) $|\ |_{\sup}$ is a norm on A , if and only if it is a norm on B.

(d) $|\ |_{\sup}$ is a faithfull B-module norm on A (i.e. $|\varphi(b).f|_{\sup} = |b|_{\sup} |f|_{\sup}$ for all $b \in B$ and $f \in A$), if and only if $|\ |_{\sup}$ is a valuation on B.

Proof : Ad(a) : According to assertion (b) of the preceeding lemma we only have to show $|f|_{\sup} \geq \max_{i=1}^{n} |b_i|_{\sup}^{1/i}$. Choose $m \in \mathbb{N}$ with $1 \leq m \leq n$ such that $|b_m|_{\sup}^{1/m} = \max_{i=1}^{n} |b_i|_{\sup}^{1/i}$. Put $p := X^n + b_n X^{n-1} + \ldots + b_n \in B[X]$. If $|b_m|_{\sup}$ is finite, for all $\eta \in R$ with $\eta < 1$ we can find $y \in \mathrm{Max}_k B$ such that $|b_m(y)| \geq \eta |b_m|_{\sup}$. If $|b_m|_{\sup}$ is infinite one has to modify the following lines slightly in a obvious manner. (If the maximum modulus principle holds for B, one can find such an $y \in \mathrm{Max}_k B$ even for $\eta = 1$.) Put $p[y] := X^n + b_1(y) X^{n-1} + \ldots + b_n(y) \in k_a[X]$. Choose $\alpha \in k_a$ such that $p[y](\alpha) = 0$ and such that $|\alpha| = \max_{i=1}^{n} |b_i(y)|^{1/i}$. This is possible, because $p[y]$ has all its roots $\alpha_1, \ldots, \alpha_n$ in $k_a$ and because one has $\max_{i=1}^{n} |\alpha_i| = \max_{i=1}^{n} |b_i(y)|^{1/i}$ according to the first proposition in § 3 of [5], which may be applied even if the spectral norm on $k_a$ is not a valuation. Now it suffices to construct a k-algebraic maximal $x \in \mathrm{Max}_k A$ such that $|f(x)| = |\alpha|$. Namely, then we have

$$(*) \qquad |f|_{\sup} = \sup_{z \in \mathrm{Max}_k A} |f(z)| \geq |f(x)| = |\alpha| = \max_{i=1}^{n} |b_i(y)|^{1/i}$$

$$|b_m(y)|^{1/m} \geq \eta^{1/m} |b_m|_{\sup}^{1/m} \geq \eta \max_{i=1}^{n} |b_i|_{\sup}^{1/i} .$$

Since this holds for all $\eta < 1$, we find $|f|_{sup} \geq \max_{i=1}^{n} |b_i|_{sup}^{1/i}$. (And if $\eta$ could be chosen equal to 1, then we even get $|f|_{sup} \geq |f(x)| \geq \max_{i=1}^{n} |b_i|_{sup}^{1/i} \geq |f|_{sup}$. Therefore the maximum modulus principle holds for A, if it holds for B.) It remains to construct $x \in \mathrm{Max}_k A$ such that $|f(x)| = |\alpha|$. Replacing B by $\varphi(B)$ we may assume $B \subset A$ and $\varphi = \mathrm{id}$. Furthermore we may assume $A = B[f]$. Namely, if we can show the assertion for the special case $A = B[f]$ and the given element f, then we also get the general case, because A is integral over $B[f]$ and therefore according to lemma 1.4 it does not matter, whether one computes the supremum-semi-norm of f in the algebra A or in the possibly smaller algebra $B[f]$. Therefore we assume $A = B[f]$. Choose a fixed embedding of $B/y$ into $k_a$. The mapping $b \to b(y)$ yields a k-algebra homomorphism onto some subfield $k'$ of $k_a$. Extend this homomorphism to a k-algebra epimorphism

$\sigma: B[X] \longrightarrow k'(\alpha)$ by defining $\sigma(\sum_{i=0}^{j} b_i X^i) := \sum_{i=0}^{j} b_i(y)\alpha^i$. Obviously one gets $\sigma(p) = p[y](\alpha) = 0$ and therefore $\sigma$ induces a k-algebra epimorphism $\bar{\sigma}: B[X]/p.B[X] \to k'(\alpha)$. Clearly $\bar{\sigma}(\bar{X}) = \alpha$ where $\bar{X}$ denotes the residue class $X + p.B[X]$. Assume that we know already

$$(**) \qquad p.B[X] = \{ q \in B[X] \ ; \ q(f) = 0 \} .$$

Then there is a canonical isomorphism $\tau: B[f] \longrightarrow B[X]/p.B[X]$, which maps f onto $\bar{X}$. Combining $\bar{\sigma}$ and $\tau$ we get a k-algebra epimorphism $\bar{\sigma} \circ \tau : B[f] \to k'(\alpha)$, such that f is mapped onto $\alpha$. The kernel x of this map is a k-algebraic maximal ideal of $B[f]$ and one sees $|f(x)| = |\alpha|$. Thus equation (**) remains to be verified. In order to do so, first we assume that A is without zerodivisors. Then the quotient field Q(A) of A is an algebraic extension of the quotient field Q(B) of B. Because B is integrally closed, well-known results from Algebra (cf. [8], chap. V.§3) assert, that $p \in B[X]$ is also the minimal ireducible polynomial of f over Q(B) and that p divides in $B[X]$ all polynomials $q \in B[X]$ annihilating f. Thus we have verified (**) for the special case, that A is an integral domain. Thus it remains to reduce the general case "A is reduced" to this special case. Define $\mathfrak{M}$ to be the set of all minimal prime ideals of A. Then we know $(0) = \bigcap_{\mathfrak{p} \in \mathfrak{M}} \mathfrak{p}$ and $\mathfrak{p} \cap B = (0)$ for all $\mathfrak{p} \in \mathfrak{M}$ according to the theorem of Cohen-Seidenberg (cf. [8], chap. V, §3, th. 6). For $\mathfrak{p} \in \mathfrak{M}$ denote by $\pi_{\mathfrak{p}} : A \to A/\mathfrak{p}$ the canonical residue map. $\pi_{\mathfrak{p}}$ induces then an integral embedding of B into $A/\mathfrak{p}$ for all $\mathfrak{p} \in \mathfrak{M}$. Put $f_{\mathfrak{p}} := \pi_{\mathfrak{p}}(f)$. Let $q \in B[X]$

such that $q(f) = 0$. Then a fortiori $q(f_{\mathfrak{p}}) = 0$. Let $p_{\mathfrak{p}}$ be the minimal irreductible polynomial of $f_{\mathfrak{p}}$ over $Q(B)$. According to what we have already proved for the special case, where A has no zero divisors, $p_{\mathfrak{p}} \in B[X]$ and $q(f_{\mathfrak{p}}) = 0$ implies that $p_{\mathfrak{p}}$ divides q in $B[X]$. In particular $p_{\mathfrak{p}}$ divides p in $B[X]$ for all $p \in \mathfrak{m}$ . $Q(B)[X]$ is factorial, $p \in Q(B)[X]$ admits only a finite number of non-associated prime divisors. Because all $p_{\mathfrak{p}}$ are monic polynomials, two of them are associated only if they are equal. Therefore the set $\{p_{\mathfrak{p}} ; \mathfrak{p} \in \mathfrak{m}\}$ must be finite. Define $m \in B[X]$ to be the product of these finitely many polynomials $\in B[X]$. Because the factors of m are non-associated primes all of which divide q, also m divides q in $Q(B)[X]$ and then even in $B[X]$. In particular m divides p in $B[X]$. If we can show $m = p$, then we have verified (**). Because $p_{\mathfrak{p}}$ divides m, we get $m(f_{\mathfrak{p}}) = 0$, or equivalently $m(f) \in \mathfrak{p}$ . This holds for all $\mathfrak{p} \in \mathfrak{m}$ and therefore we get $m(f) \in \bigcap_{\mathfrak{p} \in \mathfrak{m}} \mathfrak{p} = (0)$. Hence $m(f) = 0$ is an equation dependence for f over B, whence (degree of m) $\geq$ (degree of p). On the other hand we know already that p and m are monic polynomials and that m divides p. Therefore $m = p$ and hence (**) is verified. And that equation was all we needed to finish the proof of (a) also for the general (i.e. the reduced) case.

Ad(b) : While proving assertion (a) we also proved that the maximum modulus principle for B implies the maximum modulus principle for A. The converse is true due to lemma 1.4.

Ad(c) : Assume that $|\,|_{\sup}$ is a norm on B, i.e. $|\,|_{\sup}$ is not degenerated and $|b|_{\sup} = 0$ implies $b = 0$ for all $b \in B$. We have to show that then also $|\,|_{\sup}$ on A is a norm. Take $f \in A$ with $f \neq 0$ and let $f^n + \varphi(b_1)\, f^{n-1} + \ldots + \varphi(b_n) = 0$ be an equation of integral dependence for f over $\varphi(B)$ of minimal degree. Because A is reduced, there exists an index m with $1 \leq m \leq n$ such that $b_m \neq 0$. According to assumption also $|b_m|_{\sup} \neq 0$ . Hence also $|f|_{\sup} = \max_{i=1}^{n} |b_i|_{\sup}^{1/i} \geq |b_m|_{\sup}^{1/m} > 0$. Using assertion (c) of lemma 1.4 we conclude, that $|\,|_{\sup}$ is a norm on A. - The converse is clear, due to (a) and (c) of lemma 1.4.

Ad(d) : Assume that $|\,|_{\sup}$ is a valuation on B. We have to show $|\varphi(b).f|_{\sup} = |b|_{\sup}.|f|_{\sup}$ for all $b \in B$ and all $f \in A$. If $b = 0$ or $f = 0$ there is nothing to show. Therefore let $b \neq 0$ and $f \neq 0$ and let $p(f) = f^n + \varphi(b_1)\, f^{n-1} + \ldots + \varphi(b_n) = 0$ be an equation of integral dependence for f over $\varphi(B)$ of minimal degree. Then $(\varphi(b)f)^n + \varphi(bb_1)(\varphi(b)f)^{n-1} + \ldots + \varphi(b^n b_n) = 0$ is an equation of integral dependence of $\varphi(b)$ f over $\varphi(B)$ and there is no other equation $q(\varphi(b).f) = 0$ with $q \in B[X]$, $q$ monic and (degree of $q$) $< n$ , because then one would get an equation $q^*(f) = 0$ with some $q^* \in Q(B)[X]$ and (degree of $q^*$) = (degree of $q$) $< n$ = (degree of $p$) in contradiction to the fact that $p$ must divide $q^*$ in $Q(B)[X]$ (cf. the proof of assertion (a)). Thus we have found an equation of integral dependence of minimal degree for $\varphi(b).f$ and may compute $|\varphi(b)f|_{\sup}$ according to (a):

$|\varphi(b)\ f|_{\sup} = \max\limits_{i=1}^{n} |b^i b_i|_{\sup}^{1/i} = |b|_{\sup} \max\limits_{i=1}^{n} |b_i|_{\sup}^{1/i} = |b|_{\sup} |f|_{\sup}$. Again the converse follows immediately from assertion (a) of lemma 1.4. Thus we have finished the proof of (d) and thereby completed the proof for prop. 1.5.

Remark : Assume that $|\ |_{\sup}$ is a valuation on B. Then it can be extended to a valuation on the quotient field Q(B). If we assume furthermore that A is an integral domain, the quotient field Q(A) exists and is an algebraic extension field of Q(B) . Q(A) can also be described as the quotient ring of A with respect to the mulitplicative system $\varphi(B)\setminus\{0\}$. According to assertion (d) $|\ |_{\sup}$ is a faithfull B-module norm, hence it can be extended to a Q(B)-algebra norm on $Q(A) = A_{\varphi(B)\setminus\{0\}}$. According to statement (a) this norm is nothing else than the spectral norm of the field extension Q(A) over Q(B). To put it in some other way : The supremum-norm on Q(A) considered as a Q(B)-algebra yields, if restricted to A, the supremum-norm on A considered as a k-algebra.

Cor. 1.6 : Under the assumptions of prop. 1.5 assume furthermore that the maximum modulus principle holds for B. Then for every $f \in A$ with $|f| \neq 0$ there are $c \in k$ and $m \in \mathbb{N}$ such that $|cf^m|_{\sup} = 1$.

Proof : For all $b \in B$ there is some $y \in \mathrm{Max}_k B$ such that $|b|_{\sup} = |b(y)|$. Hence $|B|_{\sup} \subset |k_a|$. The set $|k_a|$ is invariant under taking roots. Therefore $\{|b|_{\sup}^{1/i}\ ;\ b \in B\setminus\{0\} \text{ and } i \in \mathbb{N}\} \subset |k_a|$. According to assertion (a) of the preceeding prop. also $|f|_{\sup} \in |k_a|$ for all $f \in A$. Then there is an element $d \in k$ and $m \in \mathbb{N}$ such that $|d|^{1/m} = |f|_{\sup}$ . Define $c := d^{-1} \in k$. Then $1 = |cf^m|_{\sup}$ , Q.E.D.

1.2. Application to k-affinoid algebras. - Let k be a field, complete under a non-archimedean non-trivial valuation. We want to apply the results of the preceeding section to a special class of k-algebras : to the category of k-affinoid algebras (for the definition and properties of k-affinoid algebras see [2] and [7]). Denote by $T_d$, $d \in \mathbb{N} \cup \{0\}$, the "free" k-affinoid algebras, i.e.

$$T_d = \Big\{ \sum_{\nu_1,\dots,\nu_d \geq 0} a_{\nu_1\dots\nu_d} X_1^{\nu_1}\dots X_d^{\nu_d} \; ; \; a_{\nu_1\dots\nu_d} \in k \text{ and}$$

$$a_{\nu_1\dots\nu_d} \to 0 \text{ for } \nu_1+\dots+\nu_d \to \infty \Big\}.$$

For $t = \sum a_\nu X^\nu \in T_d$ put $|t| := \max_\nu |a_\nu|$. This is the so-called Gauss-norm on $T_d$. and one knows $|t| = \sup_{x \in \mathrm{Max}\, T_d} |t(x)|$ for all $t \in T_d$. Applying the rather general considerations of section 1.1 we get a new proof for the following result of non-archimedean function theory saying that some of the facts which are more or less easy to verify for $T_d$ remain valid also for general reduced k-affinoid algebras.

Prop. 1.7 (cf. [2]) : Let A be a reduced k-affinoid algebra. Defining $|f| := \sup_{x \in \mathrm{Max}\, A} |f(x)|$, $f \in A$, one gets a power-multiplicative k-algebra norm on A fullfilling the maximum modulus principle. If $\varphi : T_d \to A$ is an integral torsion-free k-algebra monomorphism for some $d \in \mathbb{N} \cup \{0\}$, and if $f^n + \varphi(t_1) f^{n-1} + \dots + \varphi(t_n)$ is an equation of integral dependence for f over $\varphi(T_d)$ of minimal degree, then

$$|f| = \max_{i=1}^{n} |t_i|^{1/i}$$

and $|\;|$ is a faithfull $T_d$-algebra norm.

Proof : Because all maximal ideals of an k-affinoid algebra have finite co-dimension over k, we see that $|f| = |f|_{\sup}$ for all $f \in A$ and of course $|t| = |t|_{\sup}$ for all $t \in T_d$. Because $T_d$ is integrally closed and the Gauss norm is even a valuation, assertions (b) and (d) of prop. 1.5 yield the validity of the second statement in this proposition. According to the Noether normalization lemma, for all k-affinoid algebras one can find $d \in \mathbb{N} \cup \{0\}$ and a finite k-algebra monomorphism $\varphi : T_d \to A$. In general we do not know, whether $\varphi$ is torsionfree. Therefore take $\mathfrak{p} \in \mathfrak{M}$ where $\mathfrak{M}$ is the set of minimal prime ideals of A ; then any normalization map $T_{d(\mathfrak{p})} \to A/\mathfrak{p}$ is a torsionfree finite k-algebra monomorphism. Applying prop. 1.5 to $A/\mathfrak{p}$ we see, that $| \;|_{sup}$ on $A/\mathfrak{p}$ is a power-multiplicative k-algebra norm fullfilling the maximum modulus principle, because for $| \;|_{sup}$ on $T_{d(\mathfrak{p})}$ the maximum modulus principle holds. $\mathfrak{M}$ is finite and A can be embedded into $\bigoplus_{\mathfrak{p}\in\mathfrak{M}} A/\mathfrak{p}$ Now it is easy to deduce the first statement of the proposition from lemma 1.3.

Remark : The condition, that A has to be reduced, cannot be omitted because $| \;|_{sup}$ being power-multiplicative is a norm only on reduced k-affinoid algebras.

2.1. k-Banach algebras. - So far our study object had been the k-algebra A as an purely algebraic object. The norm or semi-norm we constructed on A was only some other way of paraphrasing the structure of the set of all k-algebraic maximal ideals of A. From now on we assume that we are given a norm $\| \;\|$ on A and ask, how are $\| \;\|$ and $| \;|_{sup}$ interrelated. For simplicity we restrict ourselves to the case, where the ground field k is complete and A is a k-Banach algebra. Then one has the following preliminary result.

Lemma 2.1 : If A is a k-Banach algebra with norm $\| \;\|$ and if $x \in \mathrm{Max}_k A$, then $A/x$ provided with the residue class norm $\| \;\|_{res}$ is a k-Banach algebra and one has :

$$|f(x)| = \inf_{i\in\mathbb{N}} \|f(x)^i\|_{res}^{1/i} \leq \|f(x)\|_{res} \leq \|f\| .$$

Proof : Because A is a Banach algebra and x is maximal, x is closed. Then $\| \;\|_{res}$, defined by $\|f(x)\|_{res} := \inf\limits_{f(x)=g(x)} \|g\|$ for $f \in A$ is a norm on $A/x$. It is easy to see that $\| \;\|_{res}$ is actually a complete k-algebra norm on $A/x$ with $\|f(x)\|_{res} \leq \|f\|$. Define a further norm $\| \;\|'$ on $A/x$ by $\|f(x)\|' := \inf\limits_{i\in\mathbb{N}} \|f(x)^i\|_{res}^{1/i}$. Then $\| \;\|'$ is a power-multiplicative k-algebra semi-norm on $A/x$ (see e.g. [5], section 1.2) and obviously $\|f(x)\|' \leq \|f(x)\|_{res}$ for all $f \in A$. Because $A/x$ is a field, $\| \;\|'$ is even a norm. k is complete and $A/x$ can be embedded into $k_a$. Then the spectral norm is the only power-multiplicative k-algebra norm on $A/x$ and therefore it coincides with $\| \;\|'$. Thus we found $|f(x)| = \inf\limits_{i\in\mathbb{N}} \|f(x)^i\|_{sup}^{1/i}$ , Q.E.D.

Applying this lemma to all $x \in \mathrm{Max}_k A$ we get :

Cor. 2.2 : If A is a k-Banach algebra with norm $\| \;\|$, then for all $f \in A$ one has :

$$|f|_{sup} \leq \|f\|.$$

If A is not complete, this statement may fail to be true. E.g. take $A = k[X]$ provided with the Gauss norm and $f := X$, then $\|X\| = 1$, whereas $|f|_{sup} = \infty$, as we have seen already in section 1.1. But on a k-Banach algebra $|f|_{sup} < \infty$ for all $f \in A$ and hence $|\ |_{sup}$ degenerates only if there are no k-algebraic maximal ideals on A. If we exclude this case by imposing on A the stronger condition, that $|\ |_{sup}$ is a norm or equivalently that $\bigcap_{x \in \mathrm{Max}_k A} x = (0)$, then this purely algebraic condition has topological consequences for A, namely one gets the following result, which is a slight generalization of theorem (1.1) of [6] :

Prop. 2.3 : *Let A be a k-Banach algebra. Assume that $|\ |_{sup}$ is a norm on A. Then every k-algebra homomorphism $\varphi$ from an arbitrary k-Banach algebra B into A is continuous.*

Proof : For all $x \in \mathrm{Max}_k A$ consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & A \\ \beta \downarrow & & \downarrow \alpha \\ B/\varphi^{-1}(x) & \xrightarrow{\bar{\varphi}} & A/x \end{array}$$

where $\alpha$ and $\beta$ are the canonical residue epimorphisms. Because $A/x$ is algebraic over k, so is $B/\varphi^{-1}(x)$. Hence $\varphi^{-1}(x) \in \mathrm{Max}_k B$. Provide $A/x$ and $B/\varphi^{-1}(x)$ with the spectral norm. $\bar{\varphi}$ is then an isometry. $\alpha$ and $\beta$ are contractions according to lemma 2.1. Thus we know already that $\bar{\varphi} \circ \beta$ is continuous. In order to show that also $\varphi$ is continuous, we take advantage of the closed-graph-theorem : Let $b_n$, $n \in \mathbb{N}$, be a sequence in B such that $b_n \to 0$ and $\varphi(b_n) \to f \in A$. If we can show $f = 0$, then we know, that $\varphi$ must be continuous.
Then we get $f(x) = \alpha(f) = \alpha(\lim \varphi(b_n)) = \lim (\alpha\varphi)(b_n) = \lim (\bar{\varphi} \circ \beta)(b_n) = (\bar{\varphi} \circ \beta)(\lim b_n) = (\bar{\varphi} \circ \beta)(0) = 0$. This is true for all $x \in \mathrm{Max}_k A$, hence $|f|_{sup} = 0$, which implies $f = 0$, Q.E.D.

Cor. 2.4 : *If A is a k-Banach algebra such that $|\ |_{sup}$ is a norm on A, then all complete k-algebra norms on A are equivalent.*

In Cor. 2.2 we only have an estimation for $|\ |_{sup}$. Now we want to compute $|\ |_{sup}$ in terms of $\|\ \|$. This can be done if we add topological conditions to the assumptions of prop. 1.5.

Prop. 2.2 : *Let $\varphi : B \to A$ be an integral torsionfree k-algebra monomorphism between two k-algebra A and B, where A is reduced and B is an integrally closed* domain. Assume furthermore that A is a k-Banach algebra with norm $\|\ \|$ and that $\varphi$ is continuous, if B is provided with the topology induced by $|\ |_{sup}$. Then one has for all $f \in A$ :

$$\inf_{i \in \mathbb{N}} \|f^i\|^{1/i} = |f|_{sup} = \max_{i=1}^{n} |b_i|_{sup}^{1/i}$$

where $f^n + \varphi(b_n) f^{n-1} + \ldots + \varphi(b_n) = 0$ is an equation of integral dependence for f over $\varphi(B)$ of minimal degree.

Proof : We only have to show $\|f\|_r = |f|_{sup}$, where $\|f\|_r := \inf_{i \in \mathbb{N}} \|f^i\|^{1/i}$. From Lemma 2.2 one deduces immediately $|f|_{sup} \leq \|f\|_r$. In order to show the opposite inequality we shall use prop. 1.5. Because $\|\ \|_r$ is a power-multiplicative k-algebra semi-norm on A, (actually it is not hard to show, that $\|\ \|_r$ is even a norm) one gets $\|f\|_r \leq \max_{i=1}^{n} \|\varphi(b_i)\|_r^{1/i} \leq \max_{i=1}^{n} \|\varphi(b_i)\|^{1/i}$. Because $\varphi$ is continuous, there is a real constant $C > 1$ such that $\|\varphi(b)\| \leq C |b|_{sup}$ for all $b \in B$ and a fortiori $\|\varphi(b)\|^{1/i} \leq C |b|_{sup}^{1/i}$ for all $i \in \mathbb{N}$. Thus we have shown $\|f\|_r \leq C \max_{i=1}^{n} |b_i|_{sup}^{1/i} = C |f|_{sup}$ according to prop. 1.5. Because both $\|\ \|_r$ and $|\ |_{sup}$ are power-multiplicative, this implies $\|f\|_r \leq |f|_{sup}$, Q.E.D.

Remark 1 : In spite of the fact that $|\ |_{sup}$ depends only on the algebraic structure of the k-algebra A (and, of course, on the valuation of k), $|\ |_{sup}$ nevertheless coincides with $\|\ \|_r$, which is derived from the given Banach norm on A. This is a result similar to prop. 2.3 asserting that, (roughly speaking) the topological or normtheoretic structure of A is already determined by the underlying algebraic structure.

Remark 2 : The assumption "$\varphi$ is continuous" can be omitted, if $|\ |_{sup}$ is a complete norm on B. Namely then B provided with $|\ |_{sup}$ is a k-Banach algebra and $|\ |_{sup}$ on A is a norm according to prop. 1.5(d). Hence we may apply prop. 2.3.

For normed algebras A the subsets $\mathring{A} := \{f \in A\ ;\ \{\|f^i\|\ ;\ i \in \mathbb{N}\}$ is bounded$\}$ of power-bounded elements and $\check{A} := \{f \in A\ ;\ f^n \to 0$ for $i \to \infty\}$ of topologically nilpotent elements play an important rôle. It is rather easy to see without any particular assumptions on A, that $f \in \check{A}$ if and only if $\inf_{i \in \mathbb{N}} \|f^i\|^{1/i} < 1$ and that $f \in \mathring{A}$

implies $\inf_{i \in \mathbb{N}} \|f^i\|^{1/i} \leq 1$. In the special situation described in prop. 2.5, the latter implication is even an equivalence and one gets a simple description of $\mathring{A}$ and $\check{A}$ in terms of $| \ |_{sup}$, namely :

<u>Cor.</u> 2.6 : <u>Under the hypothesis of prop.</u> 2.5, <u>for all</u> $f \in A$ <u>the following statements are equivalent</u> :

(a) $f$ <u>is topologically nilpotent</u>,

(b) $\inf_{i \in \mathbb{N}} \|f^i\|^{1/i} < 1$ ,

(c) $|f|_{sup} < 1$.

<u>In the maximum modulus principle holds for</u> B, <u>then also the following statement is equivalent to</u> (a), (b) <u>and</u> (c) :

(d) $|f(x)| < 1$ <u>for all</u> $x \in \mathrm{Max}_k A$.

<u>Furthermore, for all</u> $f \in A$ <u>also the following statements are equivalent</u>

(a') $f$ <u>is power-bounded</u>,

(b') $\inf_{i \in \mathbb{N}} \|f^i\|^{1/i} \leq 1$,

(c') $|f|_{sup} \leq 1$.

<u>Proof</u> : As already mentioned (a) and (b) are equivalent. The preceeding prop. yields the equivalence of (b) and (c). Under the additional assumption, also (d) and (c) are equivalent according to prop. 1.5(b). - (b') follows from (a') as already indicated and the equivalence of (c') and (b') is contained in prop. 2.5. Hence it suffices to show, that (c') implies (a'). Use the notations of prop. 2.5. Then from (c') we get : $|b_i|_{sup} \leq 1$ for $i = 1,\dots,n$ and therefore $|b|_{sup} \leq 1$ for all $b \in P[b_1 ,\dots, b_n]$, where P is the prime ring of B; i.e. the smallest subring of B containing 1. Because $\varphi$ is continuous, also $R := \varphi(P)\,[\varphi(b_1),\dots,\varphi(b_n)]$ is bounded under $\| \ \|$. From the equation of integral dependence for f one easily derives (by induction on $j \in \mathbb{N}$) : $f^j \in \sum_{i=0}^{n-1} Rf^i$. Hence f is power-bounded with respect to $\| \ \|$, Q.E.D.

2.2. <u>Application to</u> k-<u>affinoid algebras</u>. - If we specialize the results of the preceeding section to k-affinoid, we get :

Prop. 2.7 : Let A be a reduced k-affinoid algebra. Then every k-algebra homomorphism from an arbitrary k-Banach algebra into A is continuous. In particular : there is no other (up to topological equivalence) Banach algebra structure on a reduced k-affinoid algebra A.

Proof : According to prop. 1.7 $|\,|_{sup}$ is a norm on A and hence we may apply prop. 2.3, which gives the derived result.

If B is an integrally closed k-affinoid algebra such that $|\,|_{sup}$ is a complete norm on B (this is the case e.g. for $B = T_d$, $d \in \mathbb{N} \cup \{0\}$, or for $B = L_d$, where $L_d$ is the algebra of strictly convergent Laurent series in d indeterminates over k), then due to remark 2 following prop. 2.5 we may apply prop. 2.5, whenever $\varphi : B \to A$ is a finite k-algebra monomorphism into a k-affinoid algebra A without zero-divisors. Due to the Noether normalization lemma and prop. 2.5 this means, that at least for the special case, where A has no zero-divisors we have already proved the following.

Prop. 2.8 : Let A be a reduced k-affinoid algebra. Then for every complete k-algebra norm $\|\,\|$ on A and all $f \in A$ one has

$$|f|_{sup} = \inf_{i \in \mathbb{N}} \|f^i\|^{1/i}.$$

$f \in A$ is power-bounded, if and only if $|f|_{sup} \leq 1$. $f \in A$ is topologically nilpotent, if and only if $|f(x)| < 1$ for all $x \in \operatorname{Max} A$.

Proof : We have to reduce the general case, i.e. A is only a reduced algebra, to the special case, that A does not admit zero divisors $\neq 0$. Because A is noetherian, the set $\mathfrak{M}$ of minimal prime ideals is finite, hence $A^{\#} := \bigoplus_{\mathfrak{p} \in \mathfrak{M}} A/\mathfrak{p}$ is a noetherian A-module. Provide $A^{\#}$ with the norm $\|(\alpha_{\mathfrak{p}})_{\mathfrak{p} \in \mathfrak{M}}\|^{\#} := \max_{\mathfrak{p} \in \mathfrak{M}} \|\alpha_{\mathfrak{p}}\|_{res}$, where $\|\alpha_{\mathfrak{p}}\|_{res}$ denotes the residue class norm of $\alpha_{\mathfrak{p}} \in A/\mathfrak{p}$. Then $A^{\#}$ is a noetherian k-Banach algebra and $\pi(A)$ is closed in $A^{\#}$, where $\pi(f) := (\pi_{\mathfrak{p}}(f))_{\mathfrak{p} \in \mathfrak{M}}$ and $\pi_{\mathfrak{p}} : A \to A/\mathfrak{p}$ are the canonical residue class epimorphisms. Apply prop. 2.7 to the reduced k-affinoid algebras A and $\pi(A)$. Then one sees that $\pi : A \to \pi(A)$ is topological, in particular there is some constant $C \in \mathbb{R}$ such that $\|f\| \leq C\|\pi(f)\|^{\#}$. In order to verify the first assertion of the lemma it suffices to show : $\inf_{i \in \mathbb{N}} \|f^i\|^{1/i} \leq |f|_{sup}$. Due to $\|f\| \leq C\|\pi(f)\|^{\#}$ it is enough to prove $\inf_{i \in \mathbb{N}} \|\pi(f)^i\|^{\# 1/i} \leq |f|_{sup}$ for all $f \in A$. Because $A/\mathfrak{p}$ has no zero divisors $\neq 0$, prop. 2.5 tells us that $\inf_{i \in \mathbb{N}} \|\pi_{\mathfrak{p}}(f)^i\|_{res}^{1/i} = |\pi_{\mathfrak{p}}(f)|_{sup}$ for all $\mathfrak{p} \in \mathfrak{M}$. Because $\mathfrak{M}$ is finite and $A^{\#}$ is the ring theoretic sum of the $A/\mathfrak{p}$ we get $\inf_{i \in \mathbb{N}} \|\pi(f)^i\|^{\# 1/i} = \inf_{i \in \mathbb{N}} \max_{\mathfrak{p} \in \mathfrak{M}} \|\pi_{\mathfrak{p}}(f)^i\|_{res}^{1/i} =$

$\max_{\mathfrak{p}\in\mathfrak{M}} \inf_{i\in\mathbb{N}} \|\pi_{\mathfrak{p}}(f)^i\|_{res}^{1/i} = \max_{\mathfrak{p}\in\mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{sup}$. Now the first assertion follows from lemma 1.3. - To verify the last two assertions of this proposition we consider the following sequence of equivalent statements : For $f \in A$ one has $|f(x)| < 1$ for all $x \in \mathrm{Max}_k A$ if and only if $|(\pi_{\mathfrak{p}}(f))(\pi_{\mathfrak{p}}(x))| < 1$ for all $x \in \mathrm{Max}_k A$ and all $\mathfrak{p} \in \mathfrak{M}$ with $\mathfrak{p} \in x$. This is equivalent to "$\pi_{\mathfrak{p}}(f)$ is topologically nilpotent for all $\mathfrak{p} \in \mathfrak{M}$" according to cor. 2.6. As one can see easily from the proof of the first assertion, this is the same as "f is topologically nilpotent". The verification of the characterization for power bounded elements is carried out in the same way.

3.1. Banach function algebras. - The main result of § 2, i.e. prop. 2.5 allows to compute $| \;|_{sup}$ on A in terms of the given complete norm on A, but it does not answer the question, wether $| \;|_{sup}$ induces also the Banach topology on A. Banach algebras having this property shall be distinguished by the following definition :

Def. 3.1 : A k-algebra A is called a "Banach function algebra", if $| \;|_{sup}$ is a complete norm on A.

This condition means, that not only $| \;|_{sup}$ is not degenerated on A but also that A provided with $| \;|_{sup}$ is a k-Banach algebra. Then A may be interpreted as an algebra of functions on $\mathrm{Max}_k$ A (which is not empty) provided with the norm of uniform convergence on $\mathrm{Max}_k$ A. The question wether A is a Banach function algebra depends only on the algebraic structure of A. If A is a k-Banach algebra with some given norm, we may ask : Does the fact, that A is also a Banach function algebra, influence the given norm ?

Lemma 3.1 : A k-Banach algebra A is a Banach function algebra, if and only if $| \;|_{sup}$ is equivalent to the given norm on A.

Proof : The if-part is obvious. The only-if-part follows from cor. 2.4.

Lemma 3.2 : If A is a Banach function algebra, then $| \;|_{sup}$ is the only power multiplicative complete k-algebra norm on A.

Proof : It is not hard to verify, that two power multiplicative k-algebra norms inducing the same topology must coincide (see §2 of [5]). Therefore there is at most one power multiplicative complete k-algebra norm on A due to cor. 2.4. On the other hand $| \;|_{sup}$ is power multiplicative and complete, if A is a Banach function algebra, Q.E.D.

Given two k-algebras A and B, such that A is an integral extension of B, as in the previous sections we are interested in deriving information about $| \;|_{sup}$ on A from properties of $| \;|_{sup}$ on B and vice versa. Here, where we are concerned with

Banach function algebras, it is natural to ask : If B is a Banach function algebra, is this property inherited by A and vice versa ? As one would expect, "going down" is easier than "going up". Thus we treat the easy case first.

Lemma 3.3 : Let $\varphi : B \to A$ be an integral k-algebra monomorphism between two k-algebras A and B. Assume that A is a Banach function algebra. Then B is a Banach function algebra if and only if $\varphi(B)$ is closed in A.

Proof : Provide $\varphi(B)$ with the restriction of $|\ |_{sup}$ on A to $\varphi(B)$. According to lemma 1.4 $\varphi(B)$ is then isometrically isomorphic to B provided with $|\ |_{sup}$ on B. Hence B is a Banach function algebra if and only if the restriction of $|\ |_{sup}$ on A to $\varphi(B)$ is complete. Because $|\ |_{sup}$ on A is complete, the latter condition is equivalent to the closedness of $\varphi(B)$, Q.E.D.

We want to replace the condition "$\varphi(B)$ is closed" by other assumptions, which are easier to handle :

Cor. 3.4 : Let $\varphi : B \to A$ be a finite k-algebra monomorphism where B is a noetherian k-Banach algebra. Then B is a Banach function algebra if A is a Banach function algebra.

Proof : Provide A with its $|\ |_{sup}$. According to prop. 2.3 then $\varphi$ is continuous This means, that A can be considered as a finite topological B-module. Because B is a noetherian k-Banach algebra, all B-submodules of A are closed, in particular $\varphi(B)$ is closed in A. Now lemma 3.3 gives the assertion.

In order to be able to "go up", i.e. to show that A is a Banach function algebra, if the smaller algebra B has this property, we have to treat the cases char k = p > 0 and char k = 0 differently.

Lemma 3.5 : Let A be a k-Banach algebra with norm $\|\ \|$. Assume that char k = p > 0. Then the following statements are equivalent :

(a) $f \to \inf_{i \in \mathbb{N}} \|f^i\|^{1/i}$ , $f \in A$ , is a norm inducing the same topology as $\|\ \|$,

(b) there is a power multiplicative k-algebra norm on A inducing the same topology as $\|\ \|$.

(c) $A^p$ is closed in A.

Proof : We give a cyclic proof. Obviously (a) implies (b). In order to show that (c) follows from (b) we have to verify that for every sequence $(f_i)_{i \in \mathbb{N}} \in A$, such that $\lim_{i \to \infty} f_i^p =: g \in A$ exists, one has $g \in A^p$. Denote by $|\ |$ the power multiplicative norm, whose existence is assumed in (b). Then we know, that $|f_{i+1} - f_i|^p = |(f_{i+1} - f_i)^p| = |f_{i+1}^p - f_i^p|$ is a zero sequence for $i \to \infty$ . Hence also $(f_i)$ is a

Cauchy sequence in A. Because A is complete, there if $f_0 \in A$ such that $f_i \to f_0$ for $i \to \infty$, whence $f_i^p \to f_0^p$. Because A is Hausdorff, we may conclude $g = f_0^p \in A^p$. Thus we have shown, that (b) implies (c). That (a) follows from (c) shall be shown in two steps. For $i \in \mathbb{N}$ and $f \in A$ define $\|f\|_i := \|f^{p^i}\|^{p^{-i}}$. Then $\| \|_i$ as a k-algebra norm on A. First we claim, that $\| \|_1$ is equivalent to $\| \|$. Indeed : Let $(f_i)_{i\in\mathbb{N}}$ be a Cauchy sequence with respect to $\| \|_1$. Then $\|f_{i+1} - f_i\|_1 \to 0$ for $i \to \infty$, i.e. $\|(f_{i+1} - f_i)^p\|^{p^{-1}} \to 0$ or equivalently $\|f_{i+1}^p - f_i^p\| \to 0$. Because $A^p$ is closed, we can find $f_0 \in A$ such that $\|f_i^p - f_0^p\| \to 0$ and therefore $\|f_i - f_0\|_1 \to 0$ for $i \to \infty$. Hence A is complete also with respect to $\| \|_1$. From the obvious inequality $\|f\|_1 \leq \|f\|$ for all $f \in A$ and Banach's open mapping theorem, we conclude that $\| \|$ and $\| \|_1$ induce the same topology. Thus our claim is justified. Then there must be a positive real constant C such that $\|f\| \leq C\|f\|_1$ for all $f \in A$. Next, we claim that $\|f\|_i \geq C^{-\sum_{j=0}^{i-1} p^{-j}} \|f\|$. For $i = 1$ we have just verified this assertion. Now we proceed by induction on $i$. One has

$$\|f\|_{i+1} = \|f^{p^{i+1}}\|^{p^{-(i+1)}} = (\|(f^p)^{p^i}\|^{p^{-i}})^{p^{-1}} \geq (C^{-\sum_{j=0}^{i-1} p^{-j}} \|f^p\|)^{p^{-1}}$$

according to induction hypothesis. Because the last expression equals $C^{-\sum_{j=1}^{i} p^{-j}} \|f\|_1$, we may continue this chain of inequalities by

$$\|f\|_{i+1} \geq C^{-\sum_{j=1}^{i} p^{-j}} \|f\|_1 \geq C^{-\sum_{j=1}^{i} p^{-j}} \cdot C^{-1}\|f\| = C^{-\sum_{j=0}^{i} p^{-j}} \|f\| ,$$

as claimed. Accordingly we know : $\|f\|_i \geq C^{-p/(p-1)} \|f\|$ for all $i \in \mathbb{N}$. Because

$$\inf_{j\in\mathbb{N}} \|f^j\|^{1/j} = \lim_{j\to\infty} \|f^j\|^{1/j} = \lim_{i\to\infty} \|f^{p^i}\|^{p^{-i}} = \lim_{i\to\infty} \|f\|_i ,$$

we get $\inf_{i\in\mathbb{N}} \|f^i\|^{1/i} \geq C^{-p/(p-1)} \|f\|$. Because obviously $\inf_{i\in\mathbb{N}} \|f^i\|^{1/i} \leq \|f\|$, we finally got (a), Q.E.D.

We shall use this lemma, to show that for char $k > 0$ under suitable algebraic conditions the property of being a Banach function algebra is preserved by finite extensions, more precisely we have :

Prop. 3.6 : Let char $k = p > 0$ and let $\varphi : B \to A$ be a finite torsionfree k-algebra monomorphism, where $B$ is noetherian and integrally closed and where $A$ is reduced and finite over $A^p$. Then $A$ is a Banach function algebra, if $B$ is a Banach function algebra.

Proof : Our goal is to provide $A$ with a complete k-algebra norm $\| \|$ such that $A^p$ is closed. Assume that we have constructed such a norm on A. Then, applying lemma 3.5, we get that $f \to \inf_{i \in \mathbb{N}} \|f^i\|^{1/i}$ , $f \in A$, is a complete k-algebra norm on A. From prop. 2.5 and remark 2 following it we derive, that $| |_{sup}$ is complete on A hence A is a Banach function algebra. Therefore it remains to find a complete k-algebra norm $\| \|$ on A such that $A^p$ is closed. Let $a_1 = 1, a_2, \ldots, a_r$ be a generator system for A over $\varphi(B)$ , i.e. $A = \sum_{i=1}^{r} \varphi(B)\, a_i$. Define $\Phi : rB \to A$ by $\Phi(b_1,\ldots,b_r) = \sum_{i=1}^{r} \varphi(b_i)\, a_i$. Then $\Phi$ is a B-module epimorphism. rB is a noetherian module over the Banach algebra B. Hence Ker $\Phi$ is closed. Provide rB with the norm $\|(b_1,\ldots,b_r)\| := \max_{i=1}^{i} |b_i|_{sup}$ and provide A with the residue norm $\| \|_1$ induced by $\Phi$ , i.e. $\|a\|_1 := \inf \{\|x\| \;;\; x \in \Phi^{-1}(a)\}$. Then A becomes a k-Banach vector-space. In general $\| \|_1$ is not a k-algebra norm, one only has a real constant C such that $\|f.g\|_1 \leq C \|f\|_1 \|g\|_1$ for all $f.g \in A$. Replacing $\| \|_1$ by $\| \|_2$ , defined by $\|f\|_2 := \sup_{g \in A, g \neq 0} \|f.g\|_1 / \|g\|_1$ , we get a k-algebra norm on A inducing the same topology as $\| \|_1$. Hence A provided with $\| \|_2$ is a k-Banach algebra and, of course also a $k^p$-Banach algebra. Applying lemma 3.5 to the k-Banach algebra B, we see that $B^p$ is a closed $k^p$-subalgebra of B. Hence $B^p$ is a $k^p$-Banach algebra and due to lemma 1.4 $B^p$ is even a $k^p$-Banach function algebra. $\varphi | B^p : B^p \to A^p$ is a finite $k^p$-algebra monomorphism. Thus we may carry out the same construction as before, where $A, B, \varphi$ are replaced by $A^p$ , $B^p$ and $\varphi | B^p$ respectively. Thereby we find a $k^p$-Banach algebra norm $\| \|_2'$ on $A^p$. Now the natural injection $i : A^p \to A$ is a $k^p$-algebra homomorphism between two $k^p$-Banach algebras. We may apply prop. 2.3 and get, that i is continuous. According to our assumptions i is finite and $A^p$ is noetherian. Then A is a finite complete topological $A^p$-module over the $k^p$-Banach algebra $A^p$. Hence all $A^p$-submodules of A are closed. In particular the subalgebra $A^p$ of A itself is closed in A, Q.E.D.

If char $k = 0$, then the condition "A is finite over $A^p$" is no longer meaning - full. Removing it, one gets a true statement also for char $k = 0$.

Prop. 3.7 : Let char $k = 0$ and let $\varphi : B \to A$ be a finite torsionfree k-algebra monomorphism, where B is noetherian and integrally closed and A is reduced.

Then A is a Banach function algebra if B is a Banach function algebra.

Proof : Define $\mathring{B} := \{b \in B; |b|_{sup} \leq 1\}$. One verifies easily, that $Q(\mathring{B}) = Q(B)$, where $Q(\mathring{B})$, resp. $Q(B)$, denotes the quotient field of $\mathring{B}$, resp. B. We want to show that $\mathring{B}$ is integrally closed also. Let $f, g \in \mathring{B}$ with $g \neq 0$ such that $u := f/g$ is integral over B, i.e. there are $b_1, \ldots, b_m \in \mathring{B}$ with $u^m + b_1 u^{m-1} + \ldots + b_m = 0$. Because B is integrally closed, we know already that $u \in B$. The integral equation for u implies $|u|_{sup} \leq \max |b_i|_{sup}^{1/i} \leq 1$. Hence $u \in \mathring{B}$ and therefore $\mathring{B}$ is integrally closed. Using prop. 1.5(a) we see that $\mathring{A} := \{f \in A; |f|_{sup} \leq 1\}$ equals the integral closure of B in A. Applying a well known theorem of Algebra (e.g. [8], chap. V, §4, theorem 7, where it is stated only for the case that A has no zero divisors) there is a basis $f_1, \ldots, f_r$ of $A_{B\setminus\{0\}}$ over Q(B) such that $\mathring{A} \subset \sum_{i=1}^{r} \mathring{B} f_i$. Define a free B-submodule F of $A_{B\setminus\{0\}}$ by $F := \sum_{i=1}^{r} B f_i$. Obviously $\mathring{A} \subset \sum_{i=1}^{r} \mathring{B} f_i$ implies $A \subset F$. Provide F with the nom $\|\sum_{i=1}^{r} b_i f_i\| := \max_{i=1}^{r} |b_i|_{sup}$. B is complete according to the assumption and therefore F is a finite complete B-module. Because B is noetherian, every B-submodule of F is closed, in particular A is closed in F and therefore is a k-Banach space. Choose $c \in k$ with $|c| > 1$. We claim $\|f\| < |c| \, |f|_{sup}$ for all $f \in A$. In order to verify this, choose $m \in \mathbb{Z}$ such that $|c|^{m-1} < |f|_{sup} \leq |c|^m$. Then $|c^{-m} f|_{sup} \leq 1$ and $|c|^m < |c| \, |f|_{sup}$. Put $g := c^{-m} f \in A$. Then $g \in \mathring{A}$ and therefore one can find $b_1, \ldots, b_r \in B$ such that $g = \sum_{i=1}^{r} b_i f_i$. Then one has $\|f\| = \|c^m g\| = \|\sum_{i=1}^{r} c^m b_i f_i\| = \max_{i=1}^{r} |c^m b_i|_{sup} \leq |c^m| < |c| \, |f|_{sup}$, whence our claim is justified. Replacing $\| \ \|$ by $\| \ \|_2$ defined by $\|f\|_2 := \sup_{g \in A\setminus\{0\}} \|fg\| / \|g\|$ we get a complete k-algebra norm on A such that $\|f\|_2 \leq C \|f\|$ with some suitable real constant C. Then one has $\|f\|_2 \leq C |c| \, |f|_{sup}$ for all $f \in A$. Together with cor. 2.2 this means, that $\| \ \|_2$ and $| \ |_{sup}$ are equivalent norms on A. Hence $| \ |_{sup}$ is complete, Q.E.D.

If we take the different assumptions of propositions 3.6, 3.7 and cor. 3.4 in their strongest forms, we can combine them into the following.

Theorem 3.8 : Let $\varphi : B \to A$ be a finite torsionfree k-algebra monomorphism, where B is a noetherian integrally closed Banach algebra and where A is a reduced algebra such that A is finite over $A^{\mathrm{char}\ k}$, if char $k > 0$. Then A is a Banach function algebra, if and only if B is a Banach function algebra.

Remark : We do not suppose that the norm on B is multiplicative. It would be interesting to know how the condition "A finite over $A^{\mathrm{char}\ k}$, if char $k > 0$" can be weakened or even removed.

3.2. Application to k-affinoid algebras. - As an easy corollary of the results of section 3.1 we get the following result for k-affinoid algebras :

Prop. 3.9 : Reduced k-affinoid algebras are Banach function algebras, if char $k = 0$ or if $k$ is finite over $k^p$ , where $p = \text{char } k > 0$.

Remark : This result has been proved in [2]. Subsequently it was shown in [1], that the condition on k is superfluous.

Proof : Let $p = \text{char } k > 0$ . If and only if $k$ is finite over $k^p$, also $T_n$ is finite over $T_n^p$. Then, of course, $A$ is finite over $A^p$ for every k-affinoid algebra A. Now let us drop the assumption char $k > 0$ and let us consider the special case, that A has no zero divisors, first. Choose a Noether normalization map $\varphi : T_n \to A$ for some $n \in \mathbb{N} \cup \{0\}$. Then $\varphi$ is a finite torsionfree k-algebra monomorphism. Because the "free" k-affinoid algebras $T_n$ are noetherian integrally closed Banach function algebras, all the assumption of prop. 3.6 or 3.7 are fullfilled and hence A is a Banach function algebra. Now let us return to the general case, where A is only a reduced algebra. Then one can embedd A into the ringtheoretic direct sum of its prime components, i.e. there is a k-algebra monomorphism $i : A \to \bigoplus_{i=1}^{r} A/\mathfrak{p}_i$ , where $\mathfrak{p}_1 , \ldots , \mathfrak{p}_r$ are the minimal primes of A. Define $\|(\alpha_1 , \ldots , \alpha_r)\| = \sum_{i=1}^{r} |\alpha_i|_{\sup}$ for all $\alpha_i \in A/\mathfrak{p}_i$ , $i = 1,\ldots,r$. According to what we have proved already $|\ |_{\sup}$ is complete on $A/\mathfrak{p}_i$ and therefore $\|\ \|$ is a complete power multiplicative k-algebra norm on $\bigoplus_{i=1}^{r} A/\mathfrak{p}_i$ . i is a finite continuous map. Because A is noetherean we may conclude that then $i(A)$ is closed in $\bigoplus_{i=1}^{r} A/\mathfrak{p}_i$. Hence $\|\ \|$ is complete on $i(A)$. Lemma 1.3 gives us, that $\|i(f)\| = |f|_{\sup}$ and therefore $|\ |_{\sup}$ is complete on A, Q.E.D..

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